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A General Dynamic Programming Approach to the Optimal Water Storage Management for Irrigation

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This paper proposes a dynamic programming approach targeted to solve a natural resource problem of water storage management for irrigation in an environmentally and socially sustainable way. The problem we examine in our formulation, which deals with the control of water storage in tanks, incorporates assumptions that are milder compared to those previously discussed in the literature concerning similar approaches. Specifically, we consider a time periodic optimal control problem whose performance functional to be maximized merely satisfies a contraction assumption (in the sense of Boyd and Wong) weaker than the one usually considered in the pertinent literature. By using an appropriate fixed point theorem, a time periodic value function can be constructed to enable the definition of optimal feedback control strategies. After showing the insufficiency of Banach contraction frameworks, and proving the underlying auxiliary mathematical results, we show our main result under conditions that are weaker than the previous related ones. Simulation results illustrate the performance of our approach.

KEYWORDS
Water storage management, Boyd and Wong contraction, fixed point theory, Performance functional.

1 | INTRODUCTION

Water storage tanks are used in a wide range of applications in which water management requires on-time access. Home water storage, agricultural irrigation, rainwater harvesting, fire fighting, and industrial manufacturing are some of the main general activities that require water storage tanks to rationalize water distribution while ensuring the satisfaction of at least the most critical demand. Water storage tanks are essential to meet the demand whenever the natural flow of available water (rain, natural water reservoirs) does not satisfy demand, being essential to complement the supply with water from tanks previously stored. [1]

Clearly, the natural supply and the demand for water have approximately periodic profiles, and, thus, it is reasonable to formulate the problem of maximizing the water supplied for irrigation in agriculture by controlling its storage in tanks as a periodic optimal control problem. Depending on the supplied water function and the demand, a reward function needs to be defined.

Dynamic Programming (DP) was introduced in [2]. It is essentially a recursive optimization procedure whose convergence is ensured by the satisfaction of the principle of optimality, which entails that the iterative operator is a contraction. Thus, whenever an optimization problem features this property, we can solve it by using dynamic programming, which has been extended to a great variety of control problems, from conventional to impulsive (see [3], [4]) dynamic control systems.

The Banach contraction, [5], is a powerful well-known property that underlies methods to solve nonlinear equations and dynamic programming problems. However, there are instances of dynamic programming problems, especially those related to multistage decision processes in which the cost function is implicit, the Banach contraction property is not satisfied, entailing the need to extend the problem formulation. Moreover, as pointed out in [6], some recent research in economics treats dynamic programming issues with state-dependent discounting that aims to maximize the expected present discounted value of payoffs.

In this type of problem, the discount factor can be greater or equal to 1 with positive probability, which entails that the Banach contraction property fails to be satisfied.

In order to overcome these challenges, in this article, we investigate some weak contractive mappings in order to increase the range of problems to be solved by dynamic programming. Weaker properties for iterative mappings, such as non-expansive mapping introduced in [7], which is similar to the Banach contraction property but allows the modulus of contraction to be equal to one, and the weak φ-contraction provided by Boyd and Wong in [8], which weakens the Banach contraction property by imposing an upper bound on the distance between the operator images of a given pair of points by an upper semi-continuous function, φ, at this pair of points.

The weak φ-contraction map has a particular advantage in solving dynamic programming problems since the convergence of the associated iterative procedure is faster than the one of the Banach contraction iteration. In this paper, we exploit this feature by designing an iterative process (A_n) satisfying the weak φ-contraction property, which we show to converge faster than the one based on the Banach contraction property.

Furthermore, our objective is to illustrate the value function referred to as φ-weak contraction of the reward in the scenario of the infinite time horizon problem with a discount factor of one. Following this, we explore the discourse surrounding time periodic value functions by employing the framework of the Boyd and Wong fixed point theorem.

The organization of this paper is outlined as follows: In Section 2, an array of definitions and pertinent outcomes are presented, which align with the approach developed in this study. Moving to Section 3, the formulation of the water storage management problem is expounded upon, along with the assumptions governing its data. The dynamic programming algorithm is developed in Section 4 followed by the proof of its convergence. A numerical application is then presented in Section 5. Finally, some conclusions and prospective future work are briefly outlined in Section 6.

2 | PRELIMINARY RESULTS

Let $(D, \|\|)$ be a Banach space. An operator $F : D \to D$ is called a Banach contraction if there exists some $\delta \in (0, 1)$ such that

$$\|Fa - Fb\| \leq \delta\|a - b\| \tag{1}$$

for all $a, b \in D$. Let F be a Banach contraction and, for any $a \in D$, define the sequence $\{a_n\}_{n=0,1,...}$ by $a_{n+1} = Fa_n$, where $a_0 = a$. It is a simple matter to see that F has a unique fixed point $a^* \in D$ and that any such sequence $\{a_n\}$ converges to the fixed point $a^*$of F. In general F has a unique fixed point $a^* \in D$ and, for any $a \in D$, the sequence $\{a_n\}$, generated by $a_{n+1} = Fa_n$, converges to a^*, then F is called a Picard operator and the iterative process $a_{n+1} = Fa_n$ is called a Picard iterative process and usually denoted by (P_n). Observe that in (1) we have $\delta < 1$. If (1) holds with $\delta = 1$, the operator F is called a nonexpansive operator.

The generalization of the Banach contraction, often designated by contraction, has been heavily studied in several settings. In [8], the constant δ in (1) is replaced by a function upper semi-continuous from the right, that is:

$$\|Fa - Fb\| \leq \varphi(\|a - b\|\ ) \ \forall a, b \in D, \tag{2}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right, and satisfies $\varphi(t) < t$ for all $t \in D \setminus \{0\}$. This contraction is said to be Boyd and Wong φ-weak contraction. In [8], it is shown that if F is a φ-weak contraction, then T has a unique fixed point $a^* \in D$, that is, for any $a \in D$, $F^n a \to a^*$ as $n \to \infty$.

Remark If a function f is φ_1-weak contraction and a function g is φ_2-weak contraction, then the composition of these two functions is also φ-weak contraction, with $\varphi{=}\varphi_1 \circ \varphi_2$.

In [9], the notion of weakly contractive mapping is introduced. A mapping $f : D \to D$ is said to be a weakly contractive or ψ-weak contraction, if there exists a continuous function $\psi : [0, \infty) \to [0, \infty)$, with $\psi(0) = 0$, such that, for all $a, b \in D$

$$\|f(a) - f(b)\| \leq \|a - b\| - \psi(\|a - b\|). \tag{3}$$

This definition is a generalization of the Banach contraction principle because it can be regarded as a special case of (3), as it can be seen by choosing $\psi(t) = (1 - \delta)t$. If $\psi(t) = 0$ for all $t \in [0, \infty)$, then f is a non-expansive map.
In [10], the authors resort to this weak contraction to extend the reinforcement learning procedure designated by λ-policy iteration with randomization by applying fixed point theory methods.

To highlight the difference between the φ, the ψ-weak, the Banach contractions, and the nonexpansive operator,To illustrate this point, let's examine the subsequent instance. Consider a function $f : [0, 1] \to [0, 1]$ defined as $f(a) = a - \frac{1}{2}a^2$.

Then, for $a, b \in [0,1]$, such that $t = a - b > 0$,

$$\begin{aligned} \|f(a) - f(b)\| &= \|a - \frac{1}{2}a^2 - (b - \frac{1}{2}b^2)\| \\ &= \|a - b - \frac{1}{2}(a+b)(a - yb\| \\ &= \|[1 - \frac{1}{2}(a+b)](a-b)\| \\ &\leq \|[1 - \frac{1}{2}(a-b)](a-b)\| \\ &\leq (1 - \frac{1}{2}t)t. \end{aligned}$$

Thus, if we define φ by

$$\varphi(t) = t - \frac{1}{2}t^2, \quad t \in [0,1]$$

Then $\varphi(t)$ is continuous on $[0,1]$, hence it is upper semi-continuous from the right on $[0,1]$, $\varphi(t) < t$ for all $t \in [0,1] - \{0\}$, and (2) holds, and therefore f is a φ- weak contraction.

Clearly

$$\begin{aligned} \|f(a) - f(b)\| &= \|a - b - \frac{1}{2}(a+b)(a-b)\| \\ &> \|a - b\| - \frac{1}{2}(a+b)\|(a-b)\|. \end{aligned}$$

Thus:

$$\|f(a) - f(b)\| > \|a - b\| - \frac{1}{2}(a+b)\|(a-b)\|$$

Therefore, f is not a ψ-weak contraction since the last inequality shows that it does not exist a function ψ that satisfies:

$$\|f(a) - f(b)\| \leq \|a - b\| - \psi(\|(a-b)\|)$$

Now, it is easy to check that

$$\|f(a) - f(b)\| \leq \|1 - \frac{1}{2}(a+b)\|\|a - b\|,$$

Thus if we take $a = 0, b = 0$, we get $\delta = \|1 - \frac{1}{2}(a+b)\| = 1$, which contradicts the definition of Banach contraction, but with the same input, we get that f is non-expansive.

In a nutshell, we have the relation discussed in [11]

$$BC \Rightarrow \psi - WC \Rightarrow \varphi - WC \Rightarrow NEX$$

where, BC, $\psi - WC$, $\varphi - WC$, and NEX denote, respectively, Banach Contraction, ψ-weak contraction, φ -weak contraction and non-expansive mapping.

3 | OPTIMAL CONTROL PROBLEM FORMULATION

Following [1], we consider the dynamic of water balance in a tank as illustrated in FIGURE (1) where the water storage management problem can be formulated using a deterministic transition function, with the valve represented by $u(t)$. The goal is to determine the optimal control strategy $u^*(t)$ that maximizes the farmer's feedback of the water released from the tank, subject to the constraints of the system. With the aid of the weak contraction mapping property and non-expansiveness, we can now formulate the optimal control problem and derive the corresponding solution.

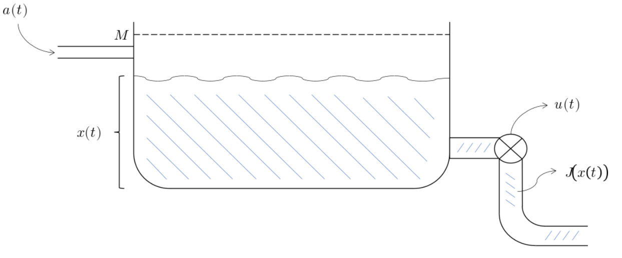

FIGURE 1 The dynamic of water storage system.

Let t denote a unit of time. It can be minutes, hours, or days. Water comes into the tank from a source. The amount of water entering the tank at t is $a(t)$. On the opposite side of the tank, there is a valve to release the water to the farmer. While in [1], the control $u(t)$ that represents the amount of water released by the valve is assumed to take the value 0 or 1, here we consider the case where the valve can be opened gradually and so $u(t) \in [0, 1]$. The amount of water is bounded by a certain quantity $M > 0$. The dynamics of the water in the tank are modeled by the following discrete-time equation :

$$a_{t+1} = \underbrace{\max\{\min\{a_t + L(t), M\} - uJ(\min\{a_t + L(t), M\}), 0\}}_{\omega(a,u)}. \tag{4}$$

We consider $L(t)$ to be a periodic function subject to perturbations, $a_t \in D$ is the storage volume of the tank immediately before opening the valve at t, $J(a)$ is the volume of water released to the farmer as a result of opening the valve when the storage volume of the tank is a. Set $D = [0, M]$, $U = [0, 1]$. Observe that $\omega : D \times U \to \mathbb{R}$, the transition between two states is deterministic. Our goal is to determine the optimal control strategies $u^*(t)$ that maximize the feedback of farmers, taking into account the constraints of the system. Hence, we define the total reward function obtained from the initial time $\tau = s < 0$ up until the final time $\tau = 0$ as

$$\phi_\rho(s, a) = \sum_{\tau=s}^{t} h(\tau, a(\tau), \rho(\tau, a(\tau))). \tag{5}$$

where $\rho = \{switch\ left,\ switch\ right\}$ is a policy that maps $u = u_t = \rho(t, a)$ of those choices and $h(a, u)$ acquired upon selecting the decision for the given state a, is deemed optimal through ρ^* if:

$$\phi_{\rho^*}(s,a) = \max_{\rho}\{\phi_{\rho}(s,a)\}, \tag{6}$$

applicable to any time instance s and any given state x, the Bellman equation emerges from the optimality principle as follows:

$$\phi_{\rho^*}(s,a) = \max_{u}\{h(s,a,u) + \phi_{\rho^*}(s+1,\omega(s,a,u))\}, \tag{7}$$

together with the final boundary condition

$$\forall a \in D, \quad \phi_{\rho^*}(0,a) = 0. \tag{8}$$

Since the natural supply and the demand for water have approximately periodic profiles, we make the discount factor equal one in (5) because we assume that the performance function will continue to act optimally in the future.
In the proposed setup, we impose the following assumptions on the data of our problem.

(Monotonicity) $\omega(a,u)$ is non-decreasing with respect to a, that is,

$$\omega(a,u) \leq \omega(b,u), \; if \; a \leq b. \tag{9}$$

With the increase in the value of $a \in D$, there is a corresponding increase in the function $\omega(a,u)$. This can be expressed as: $\omega(a,u)$ as a function of a is a φ-weak contraction, that is

$$|\omega(a,u) - \omega(b,u)| \leq \varphi(|a-b|) \; \forall a,b \in D. \tag{10}$$

h is bounded for any bounded a and u.

In 3 we assume ω to be merely a φ-weak contraction and not a strong contradiction. So our assumption 3 is weaker than assumptions 2 and 3 in [1] together.

4 | THE DYNAMIC PROGRAMMING ALGORITHM

Before presenting our main results, we need to introduce some notions related to our problem. We consider a set D of states, a set U of controls, and, for each $a \in D$, a nonempty control constraint $U(a) \subset U$. We denote by $\Re(D)$ the set of real valued functions $v : X \to \mathbb{R}$. For the set of bounded function $B(D)$, we say that $\|v\| \in B(D)$ if $\|v\| < \infty$ and $\|v\|_\infty = \sup_{a\in D} |v(a)|$. Note that $B(D)$ is complete with respect to the topology induced by $\|\cdot\|$, and it can be shown that $B(D) \subseteq \Re(D)$ is closed and convex. Thus, given a sequence $\{v_k\}_{k=1}^{\infty} \subset B(D)$ and $v \in B(D)$, if $v_k \to v^*$ in the sense that $\lim_{k\to\infty} \|v_k - v^*\| = 0$, then $\lim_{k\to\infty} v_k(a) = v^*(a)$, for all $a \in D$.

The Hamilton–Jacobi–Bellman (HJB) equation has been investigated by several authors, e.g., [12, 13, 14]. In parallel, the stochastic DP has also been investigated to characterize and compute optimal control policies also optimal

forest harvesting problem, [15, 16, 17]. However, all these results require the Bellman operator to be a Banach contraction in order to ensure the convergence of the associated iterative procedure. In this paper, we prove the convergence of the DP iterative procedure under the weakest assumptions considered so far. More precisely, we just require the φ-weak contractiveness of the iterative operator.
In this setting, using the formulation in Section 3, the problem of maximizing the performance function reduces to maximize the value function for every day t, using the following Bellman equation with a discount factor equal to one:

$$v_t(a) = \max_u\{h(a,u) + v_{t+1}(\omega(a,u))\}. \tag{11}$$

For each day t, the functional equation (11) defines an operator $F_t : B(D) \to B(D)$, with $v_t(\cdot) = F_t(v_{t+1}(\cdot))$ and

$$F_t v_{t+1}(a) = \max_u\{h(t,a,u) + v_{t+1}(\omega(a,u)))\}, \tag{12}$$

for all $v_t \in B(D)$ and all $a \in D$.

Remark As per the reference [7], it is established that the operator F_t is non-expansive.

4.1 | Solving the problem with the algorithm (A_n)

Before we present our main result, it is important to stress that in the study of non-expansive mappings and their fixed points, the Picard iterative process (P_n) has proven to be ineffective in certain cases. To illustrate this point, consider the following example.

Example Let $F : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$ be defined by $Fa = \dfrac{1}{a}$ for all $a \in [\frac{1}{2}, 1]$. Then T is a nonexpansive mapping and it has a unique fixed point $a^* = 1$. Clearly if we take $a_0 \in [\frac{1}{2}, 1)$, then the Picard operator does not converge to $a^* = 1$. This example highlights the limitations of the Picard iterative process in approximating fixed points of certain non-expansive mappings.

In order to tackle the cases in which the Picard iterative process does not converge to a fixed point, we introduce the following algorithm:

$$(A_n)\begin{cases} a_0 \in X \\ a_{n+1} = (1-\alpha_n)a_n + \varphi(\alpha_n)Fa_n, \ n = 0, 1, 2, \ldots \end{cases}$$

where $\{\alpha_n\}$ is a sequence of positive number in $[0, 1]$, $a_0 \in X$ is arbitrary, and $\varphi : [0, \infty) \to [0, \infty)$ is upper semi-continuous, and satisfies $\varphi(t) < t$ for all $t \in X \setminus \{0\}$.

The following example highlights the importance of considering alternative iterative processes when the Picard iterative process fails to converge and suggests that the (A_n) iteration may be a more suitable approach in certain cases.

Example Consider the operator F that maps the interval $[0, 1]$ to itself, defined by $Fa = 1 - a$ for all $a \in [0, 1]$. It is evident that F is a non-expansive operator and possesses a unique fixed point, denoted as $a^* = \frac{1}{2}$. In this case, we can not employ the Picard iterative process (P_n) since it yields the oscillatory sequence,

$$a_0, 1 - a_0, a_0, 1 - a_0, ..$$

While if we apply our iterative (A_n), such that $a^* = Fa^*$ we get,

$$\begin{aligned}
\|a_{n+1} - a^*\| &= \|(1-\alpha)a_n + \varphi(\alpha)Fa_n - a^*\| \\
&\leq \|(1-\alpha)(a_n - a^*) + \varphi(\alpha)Fa_n - \alpha a^*\| \\
&\leq \|(1-\alpha)(a_n - a^*) + \varphi(\alpha)Fa_n - \varphi(\alpha)a^*\| \\
&\leq \|(1-\alpha)(a_n - a^*) + \varphi(\alpha)(Fa_n - a^*)\| \\
&\leq \|(1-\alpha)(a_n - a^*) + \varphi(\alpha)(Fa_n - Fa^*)\| \\
&\leq (1-\alpha)\|a_n - a^*\| + \varphi(\alpha)\|a_n - a^*\| \\
&\leq (1-\alpha+\varphi(\alpha))\|a_n - a^*\|
\end{aligned}$$

.

If we choose $0 < \delta = (1 - \alpha + \varphi(\alpha)) < 1$ and keep moving further we get

$$\|a_{n+1} - a^*\| \leq \delta^n \|a_1 - a^*\|$$

Thus $a_{n+1} \to a^*$ as $n \to \infty$

In order to see the relevance of our algorithm in the applications, we now show the existence and uniqueness solution of the equation (11) by resorting to Boyd and Wong φ-weak contraction.

Theorem 1 *Let $F_t : B(D) \to B(D)$ be an upper semi-continuous operator defined by* (12) *and assume that the following conditions are satisfied:*
(a) The functions h is continuous and bounded;
(b) For all $v_t^1, v_t^2 \in B(D)$: $0 \leq |v_{t+1}^1(a) - v_{t+1}^2(a)| \leq 1$ the following holds:

$$|v_{t+1}^1 - \& v_{t+1}^2(\omega(a,u))| \leq |v_{t+1}^1(a) - v_{t+1}^2(a)| - \& \frac{1}{2}|v_{t+1}^1(x) - v_{t+1}^2(a)|^2, \tag{13}$$

where $a \in D$ and $u \in U$.

Then, the Bellman equation (11) *has a bounded solution*

Proof Let ϵ be any positive number, $a \in D$ and $v_{t+1}^1, v_{t+1}^2 \in B(D)$. Then there exist $u_1, u_2 \in U$ such that

$$F_t v_{t+1}^1(a) < h(a,u_1) + v_{t+1}^1(\omega(a,u_1)) + \epsilon \tag{14}$$

$$F_t v_{t+1}^2(a) < h(a,u_2) + v_{t+1}^2(\omega(a,u_2)) + \epsilon \tag{15}$$

$$F_t v_{t+1}^1(a) \geq h(a, u_2) + v_{t+1}^1(\omega(a, u_2)) \tag{16}$$

$$F_t v_{t+1}^2(a) \geq h(a, u_1) + v_{t+1}^2(\omega(a, u_1)) \tag{17}$$

Let $\varphi : [0, 1] \rightarrow [0, 1]$ be defined by;

$$\varphi(t) = t - \frac{1}{2}t^2$$

Then we can say that (13) is equivalent to

$$|v_{t+1}^1(\omega(a, u)) - v_{t+1}^2(\omega(a, u))| \leq \varphi(\| v_{t+1}^1 - v_{t+1}^2 \|) \tag{18}$$

Therefore, by using (14), (17), and (18), it follows that

$$F_t v_{t+1}^1(a) - F_t v_{t+1}^2(a) \leq v_{t+1}^1(\omega(a, u)) - v_{t+1}^2(\omega(a, u)) + \epsilon$$

$$\leq |v_{t+1}^1(\omega(a, u)) - v_{t+1}^2(\omega(a, u))| + \epsilon$$

$$\leq \varphi(\| v_{t+1}^1 - v_{t+1}^2 \|) + \epsilon$$

Reversing the role of $F_t v_{t+1}^1(a)$ and $F_t v_{t+1}^2(a)$ we also have:

$$F_t v_{t+1}^2(a) - F_t v_{t+1}^1(a) \leq \varphi(\| v_{t+1}^1 - v_{t+1}^2 \|) + \epsilon.$$

By combining the preceding two relations, we get:

$$\| F_t v_{t+1}^1 - F_t v_{t+1}^2 \| \leq \varphi(\| v_{t+1}^1 - v_{t+1}^2 \|) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary we get

$$\| F_t v_{t+1}^1 - F_t v_{t+1}^2 \| \leq \varphi(\| v_{t+1}^1 - v_{t+1}^2 \|).$$

So we deduce that the operator F_t is a φ-weak contraction. Thus, due to Boyd and Wong's fixed point theorem, the operator F_t has a unique fixed point v_t^*, and it is a bounded solution of the functional equation (11). This completes the proof.

4.2 | φ-Weak Contraction for P-Periodic Mappings

Since the performance function has a periodic profile, we need to introduce the following definition.

Definition A mapping F is said to be P-periodic if we have

$$F_{a+P} = F_a,$$

for all values of a in the domain and $P \in \mathbb{N}$.

Now, we discuss time periodic value functions with respect to the fixed-point theorem proposed in ([8]) for φ-Weak Contractions. When F_t is P-periodic, then it can be concluded that the fixed-point theorem is applicable to the φ-contraction $\mathcal{K}_{s,P} = \prod_{t=s}^{s+P-1} F_t$ since

$$\begin{aligned}
\|K_{s,P}(v^0_{s+P}) - K_{s,P}(v^1_{s+P})\| & \\
&\leq \| \prod_{t=s}^{s+P-1} F_t(v^0_{s+P}) - \prod_{t=s}^{s+P-1} F_t(v^1_{s+P})\| \\
&\leq \mathbf{O}_t\varphi_t(\|v^0_{s+P} - v^1_{s+P}\|),
\end{aligned}$$

where $\mathbf{O}_t\varphi_t = \varphi_s \circ \varphi_{s+1} \cdots \circ \varphi_{s+P-1}$. By taking $\varphi = \mathbf{O}_t\varphi_t$, we say that $\mathcal{K}_{s,P}$ is a φ-contraction in the sense of Boyd and Wong. This ensures the presence of an exclusive P-periodic function denoted as $\varsigma_s(a) = \varsigma_s$, which serves as an unchanging point for $\mathcal{K}s, P$:

$$\mathcal{K}s, P(\varsigma_s) = \varsigma_s = \varsigma_{s-P} \quad \text{for all } s \in B(D). \tag{19}$$

4.3 | The Convergence of the Iterative Process (A_n)

We now state and show a result concerning the strong convergence of the (A_n) procedure for φ-weak contraction mappings on a nonempty closed convex subset of a Banach space by using our iterative process (A_n).

Theorem 2 *Let E be a nonempty closed convex subset of a Banach space D and $F : E \to E$ be a φ-weak contraction mapping. Suppose that $\{a_n\}$ is defined by the iterative process (A_n) such that $\alpha_n \to \alpha \in (0, 1]$, as n goes to ∞. Consequently, the sequence $\{a_n\}$ exhibits a strong convergence toward the fixed point of operator F.*

Proof Let a^* be the fixed point of F. For each $n \in \{0, 1, ..\}$ we define:

$$c_n = \|a_n - a^*\|.$$

Since $\{c_n\}$ is decreasing, it has a limit $c \geq 0$. Seeking a contradiction, assume that $c > 0$. Using (A_n) we get:

$$\begin{aligned}
\|a_{n+1} - a^*\| &= \|(1 - \alpha_n)a_n + \varphi(\alpha_n)Fa_n - a^*\| \\
&\leq \|(1 - \alpha_n)(a_n - a^*) + \varphi(\alpha_n)Fa_n - \alpha_n a^*\| \\
&\leq \|(1 - \alpha_n)(a_n - a^*) + \varphi(\alpha_n)Fa_n - \varphi(\alpha_n)a^*\| \\
&\leq \|(1 - \alpha_n)(a_n - a^*) + \alpha_n(Fa_n - a^*)\| \\
&\leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_n\varphi(\|a_n - a^*\|).
\end{aligned}$$

Passing to the limit we get:

$$c \leq (1-\alpha)c + \alpha\varphi(c).$$

Hence,

$$c \leq \varphi(c),$$

which is a contradiction. This is because, by definition, we have $\varphi(t) < t$ for all t. This means

$$\lim_{n\to+\infty} \|a_{n+1} - a^*\| = 0.$$

4.4 | Comparison between (A_n) and other alternative approach

Several processes are introduced in [18, 19, 20] to treat the cases in which the Picard iterative process does not converge to a fixed point. Recently, [21] introduces a new iterative process $\{a_n\}$ (consider the faster one for a large class of problems) defined as follows

$$(S_n)\quad \begin{cases} a_0 & \in X \\ b_n & = (1-\beta_n)a_n + \beta_n F a_n \\ e_n & = (1-\gamma_n)a_n + \gamma_n b_n \\ a_{n+1} & = (1-\alpha_n)F e_n + \alpha_n F b_n, \end{cases} \qquad n = 0, 1, \ldots$$

where $\{\alpha_n\}_0^\infty$ and $\{\beta_n\}_0^\infty$ and $\{\gamma_n\}_0^\infty$ are real control sequences taking values in the interval $[0, 1]$.

In [21] the author proves the strong convergence by using the iterative process (S_n). Now, we are going to prove that our iterative process (A_n) is faster than (S_n), for some classes of common examples, but first let us recall the following definition.

Definition [22] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that converge to a^* and b^*. Assume that:

$$l = \lim_{n\to\infty} \frac{|a_n - a^*|}{|b_n - b^*|} \tag{20}$$

- If $l = 0$, then it can be said $\{a_n\}$ converges faster to a^* than $\{b_n\}$ to b^*
- If $0 < l < \infty$, then it can be said $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Example Let $D = \mathbb{R}$, $E = [0, 1]$, $F : E \to E$ be defined by

$$Fa = \frac{a}{2}.$$

It is clear that F satisfies (2) for all $a \in E$ and its fixed point is $a^* = 0$. Let

$$\alpha_n = \beta_n = \gamma_n = \begin{cases} 0, & n = 0, 1, .., 7, \\ \frac{2}{\sqrt{n}}, & n = 8, 9, \ldots \end{cases} \tag{21}$$

The sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ satisfy all the conditions of Theorem 12 in [21]) and Theorem (2). Let $a_0 \neq 0$ be the initial point. For $n = 8, 9, ..,$ we have

$$\begin{aligned} a_{n+1}(S_n) &= (1-\alpha_n)F((1-\gamma_n)a_n \\ &\quad + \gamma_n((1-\beta_n)a_n + \beta_n F a_n) \\ &\quad + \alpha_n F((1-\beta_n)a_n + \beta_n F a_n) \\ &= (1-\frac{2}{\sqrt{n}})\frac{1}{2}[(1-\frac{2}{\sqrt{n}})a_n + \frac{2}{\sqrt{n}}(1-\frac{1}{\sqrt{n}})a_n] \\ &\quad + \frac{2}{\sqrt{n}}\frac{1}{2}((1-\frac{1}{\sqrt{n}})a_n) \\ &= \prod_{i=8}^{n}(\frac{1}{2}-\frac{2}{i}+\frac{2}{i\sqrt{i}})a_8. \end{aligned} \tag{22}$$

On the other hand, for (A_n) we have:

$$\begin{aligned} a_{n+1}(A_n) &= (1-\alpha_n)a_n + \varphi(\alpha_n)F a_n \\ &= (1-\frac{2}{\sqrt{n}})a_n + \frac{1}{2\sqrt{n}}a_n \\ &= (1-\frac{3}{2\sqrt{i}})a_n \\ &= \prod_{i=8}^{n}(1-\frac{3}{2\sqrt{i}})a_8. \end{aligned}$$

For $n = 8, 9..$ we get

$$\begin{aligned} \frac{|a_{n+1}(A_n)-a^*|}{|a_{n+1}(S_n)-a^*|} &= \prod_{i=8}^{n}\frac{(1-\frac{3}{2\sqrt{i}})}{(\frac{1}{2}-\frac{2}{i}+\frac{2}{i\sqrt{i}})} \\ &= \prod_{i=8}^{n} 1-\frac{\frac{2}{i\sqrt{i}}+\frac{3}{2\sqrt{i}}-\frac{1}{2}-\frac{2}{i}}{(\frac{1}{2}-\frac{2}{i}+\frac{2}{i\sqrt{i}})} \\ &= \prod_{i=8}^{n} 1-\frac{4-4\sqrt{i}-i\sqrt{i}+3i}{(4-4\sqrt{i}+i\sqrt{i})} \\ &\leq \prod_{i=8}^{n} 1-(\frac{4}{i\sqrt{i}}-\frac{1}{\sqrt{i}}+\frac{3}{\sqrt{i}}) \\ &\leq \prod_{i=8}^{n} 1-\frac{1}{i} = \frac{7}{8}.\frac{8}{9}...\frac{n-1}{n} = \frac{7}{n}. \end{aligned}$$

Thus

$$\lim_{n\to\infty}\frac{|a_{n+1}(A_n)-a^*|}{|a_{n+1}(S_n)-a^*|} = 0.$$

It follows that, in this case, our iterative process (A_n) converges faster than (S_n). Let us discuss another example now:

Example Let $D = \mathbb{R}$, $E = [0, 1]$, $F : E \to E$ be defined by

$$Fa = a - \frac{1}{2}a^2.$$

As we show in section 1, F is φ-weak contraction and its fixed point is $a^* = 0$. Let $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$, after computing we find the following two iterations

$$a_{n+1}(A_n) = \frac{7}{8}a_n - \frac{3}{16}a_n^2,$$
$$a_{n+1}(S_n) = \frac{9}{8}a_n - \frac{57}{64}a_n^2 + \frac{9}{32}a_n^3 - \frac{1}{32}a_n^4.$$

For the starting point $a_0 = 0.9$, our iterative process A_n and the (S_n) iterative process and its behaviors are given in Figure 2.

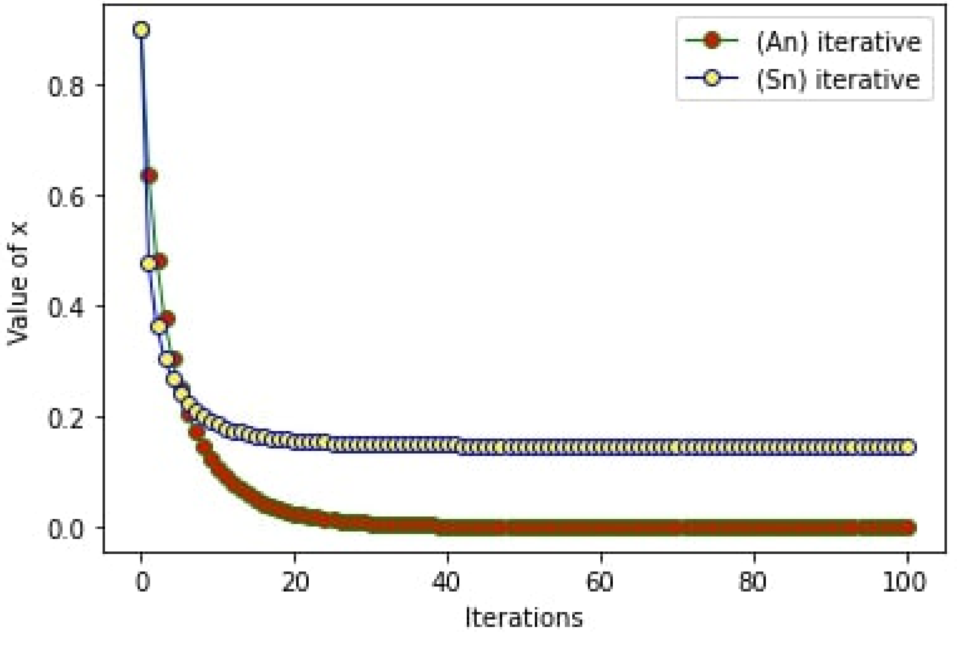

FIGURE 2 Behavior of the iterative processes (S_n) and (A_n) for the functions given in Example 6.3.

5 | APPLICATION

In this section, we study our water management problem of interest. Recall that the value function satisfies Boyd and Wong's φ weak contraction. For t (months) = 1, 2, …, 149, 150, we consider the supplied water to be given by

$$L(t) = 4 + 3\sin\left(\frac{\pi(t-2)}{7.5}\right) + 0.5\sin\left(\frac{\pi(t-2)}{15}\right). \tag{23}$$

The demanded water is expressed as follows:

$$d(t) = \max\{-1,6\sin\left(\frac{\pi(t-2)}{15}\right)\} + 1. \tag{24}$$

The reward function is described as

$$h(a(t), u(t), t) = \begin{cases} K_1 \frac{(1-u)}{(1-\frac{d(t)}{\sqrt{a(t)}})} & if \quad u \geq \frac{d}{\sqrt{a}} \\ K_2(1 - \frac{d}{\sqrt{a}} + u) & if \quad u < \frac{d}{\sqrt{a}}, \end{cases} \tag{25}$$

where $K_1 = K_2 = 1$ are constants.

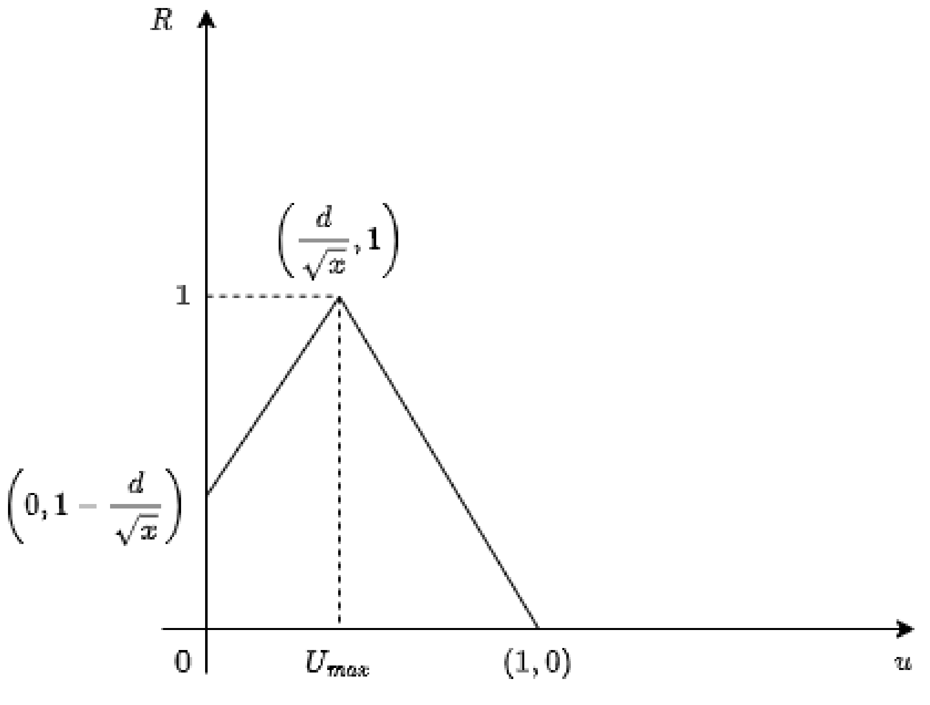

FIGURE 3 Reward function over control function.

The behavior of the tank's storage volume is described by the following dynamics

$$a_{t+1} = \max\{\min\{a(t) + a(t), M\} - u\sqrt{\min\{a(t) + a(t), M\}}, 0\}.$$

Using the proprieties:

$$\max(a, b) = \frac{a + b + |a - b|}{2}, \; \min(a, b) = \frac{a + b - |a - b|}{2}$$

we get

$$
\begin{aligned}
a_{t+1} = \omega(t,a,u) = & \frac{a(t)+L(t)+M}{4} - \frac{|a(t)+L(t)-M|}{4} \\
& - u\sqrt{\frac{a(t)+L(t)+M}{8} - \frac{|a(t)+L(t)-M|}{8}} \\
& + \left| \frac{a(t)+L(t)+M}{4} - \frac{|a(t)+L(t)-M|}{4} \right. \\
& \left. - u\sqrt{\frac{a(t)+L(t)+M}{8} - \frac{|a(t)+L(t)-M|}{8}} \right|.
\end{aligned}
$$

where M is the maximum storage volume, equal to 120 (m^3).

The cost functional for (11) is

$$v_t^1(a) = \max_{u\in[0,1]} \{v_{t+1}^1(\omega(N-1,a,u)) + h(t,a,u)\}.$$

We assume $v_N^1(a) = 0$. The value function $v_{N-1}^1(a)$ can be expressed as

$$v_{N-1}^1(a) = \max_{u\in[0,1]} \{v_N^1(\omega(N-1,a,u)) + h(N-1,a,u)\}$$

$$= \max_{u\in[0,1]} \{h(N-1,a,u)\}$$

The valve signal is given by

$$u = u_{N-1}^*(a) = \frac{d}{\sqrt{a}}.$$

The cost functional using our algorithm,

$$v_t^2(a) = (1-\alpha)v_{t+1}^2(a) + \varphi(\alpha)F_t v_{t+1}^2(a)$$

Since we assume that $v_N^2(a) = 0$, we have

$$
\begin{aligned}
v_{N-1}^2(a) = & (1-\alpha_n)v_N^2(\omega(N-1,a,u)) \\
& + \varphi(\alpha_n) \max_{u\in[0,1]} \{v_N^2(\omega(N-1,a,u)) \\
& + h(t,a,u)\}.
\end{aligned}
\tag{26}
$$

Similar to the nonexpansive case, we assume $v_N(a) = 0$. Figure 4 shows the dynamics of the system and the reward function over time. Figure 5 presents the iterations backward for non-expansive and Boyd and Wong cases. Such a result presents a faster convergence by dynamic programming from Boyd and Wong's case.

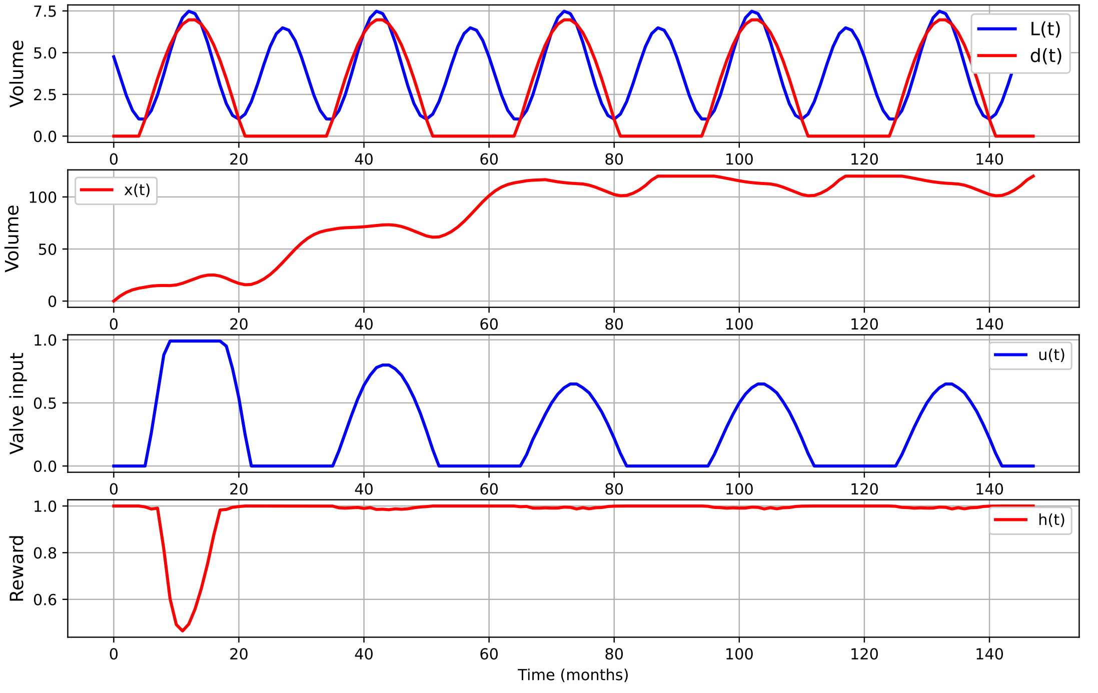

FIGURE 4 Water volume supplied L(t), demanded d(t), state variable a(t), and valve signal u(t).

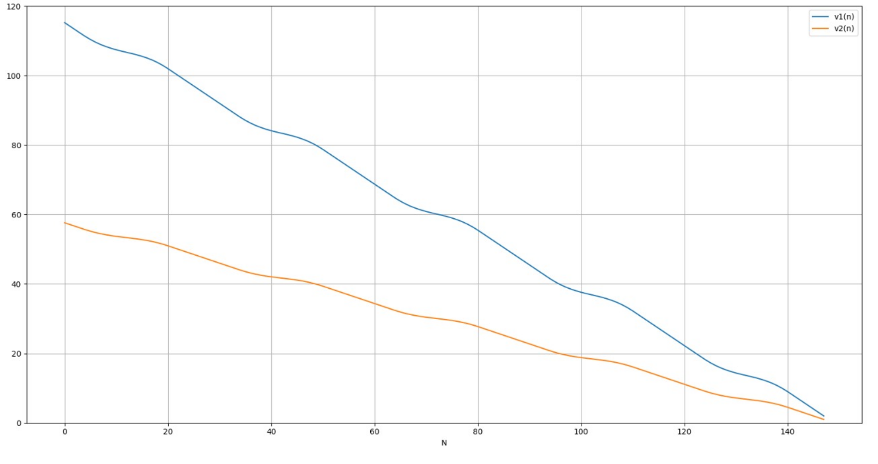

FIGURE 5 The value iteration for nonexpansive case ($v_1(n)$) and for Boyd and Wong case ($v_2(n)$). The graphs show that Boyd and Wong Value function converges faster than the nonexpansive value function

6 | CONCLUSION

The key idea of this article is to show how in a straightforward real-life model concerning optimal irrigation management, the need for weak contraction - in this case, the φ-weak contraction - naturally emerges. These problems are not tackled by dynamic programming frameworks that require the Bellman operator to be a contraction in some Banach space. We present results on iterative schemes and propose one that is faster than those previously developed (at least, those we are aware of) to solve the Bellman equation in the case of weak-contractiveness. An example illustrates the application of our proposed iterative procedure, presenting faster computation for the Boyd and Wong contraction case. Such results might be crucial for computational feasibility, depending on the application. In future work, we are seeking to investigate applications using our results, which were not conceivable before.

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conflict of interest

The authors declare no potential conflict of interests.

References

references

[1] Unami K, Mohawesh O, Fadhil RM. Time periodic optimal policy for operation of a water storage tank using the dynamic programming approach. Applied Mathematics and Computation 2019 jul;353:418–431.

[2] Bellman R. The theory of dynamic programming. Bulletin of the American Mathematical Society 1954;60(6):503–515.

[3] Arutyunov A, Jaćimović V, Pereira F. Second order necessary conditions for optimal impulsive control problems. Journal of Dynamical and Control Systems 2003;9(1):131–153.

[4] Fraga SL, Pereira FL. Hamilton-Jacobi-Bellman Equation and Feedback Synthesis for Impulsive Control. IEEE Transactions on Automatic Control 2012 jan;57(1):244–249.

[5] Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund math 1922;3(1):133–181.

[6] Toda AA. Perov's contraction principle and dynamic programming with stochastic discounting. Operations Research Letters 2021;49(5):815–819.

[7] Browder FE. Nonexpansive nonlinear operators in a Banach space. Proceedings of the National Academy of Sciences of the United States of America 1965;54(4):1041.

[8] Boyd DW, Wong JS. On nonlinear contractions. Proceedings of the American Mathematical Society 1969;20(2):458–464.

[9] Alber YI, Guerre-Delabriere S. Principle of weakly contractive maps in Hilbert spaces. In: New results in operator theory and its applications Springer; 1997.p. 7–22.

[10] Belhenniche A, Benahmed S, Pereira FL. Extension of λ-PIR for weakly contractive operators via fixed point theory. Fixed Point Theory 2021 jul;22(2):511–526.

[11] Rhoades B. Some theorems on weakly contractive maps. Nonlinear Analysis: Theory, Methods & Applications;47(4):2683–2693.

[12] Kirk DE. Optimal control theory: an introduction. Courier Corporation; 2004.

[13] Frankowska H. Lower Semicontinuous Solutions of Hamilton–Jacobi–Bellman Equations. SIAM Journal on Control and Optimization 1993 jan;31(1):257–272.

[14] Yong J, Zhou XY. Dynamic Programming and HJB Equations. In: Stochastic Controls New York, NY: Springer New York; 1999.p. 157–215.

[15] Unami K, Yangyuoru M, Alam B, Md AH, Kranjac-Berisavljevic G. Stochastic control of a micro-dam irrigation scheme for dry season farming. Stochastic Environmental Research and Risk Assessment 2013;27(1):77–89.

[16] Ieda M. An implicit method for the finite time horizon Hamilton–Jacobi–Bellman quasi-variational inequalities. Applied Mathematics and Computation 2015 aug;265:163–175.

[17] Bensoussan A. Impulse Control in Discrete Time 2008;15(3):439–454. https://doi.org/10.1515/GMJ.2008.439.

[18] Mann WR. Mean value methods in iteration. Proceedings of the American Mathematical Society 1953;4(3):506–510.

[19] R P Agarwal DO, Sahu DR. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. Journal of Nonlinear and Convex Analysis 2007;8:61–79.

[20] Ishikawa S. Fixed points by a new iteration method. Proceedings of the American Mathematical Society 1974;44(1):147–150.

[21] Sintunavarat W. An iterative process for solving fixed point problems for weak contraction mappings. In: Proceedings of the International MultiConference of Engineers and Computer Scientists, vol. 2; 2017. p. 15–17.

[22] Berinde V. Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators. Fixed Point Theory and Applications 2004;2004(2):1–9.