GROUND STATE SOLUTION FOR A PERIODIC P&Q-LAPLACAIN EQUATION INVOLVING CRITICAL GROWTH WITHOUT THE AMBROSETTI-RABINOWITZ CONDITION

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Abstract. We study the ground state solutions for the following p&q-Laplacain equation

$$\begin{cases} -\Delta_p u - \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = \lambda K(x) f(u) + |u|^{q^*-2}u, \ x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases}$$

where $\lambda > 0$ is a parameter large enough, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$ with $r \in \{p, q\}$ denotes the r-Laplacian operator, $1 < p < q < N$ and $q^* = Nq/(N-q)$. Under some assumptions for the periodic potential $V(x)$, weight function $K(x)$ and nonlinearity $f(u)$ without the Ambrosetti-Rabinowitz condition, we show the above equation has a ground state solution.

Key words: ground state solution; p&q-Laplacain equation; compactness-concentration principle; critical; Ambrosetti-Rabinowitz condition.

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1. Introduction and Main Results

In this article, we consider the ground state solution to the p&q-Laplacain equation

$$\begin{cases} -\Delta_p u - \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = \lambda K(x) f(u) + |u|^{q^*-2}u, \ x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter large enough, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$ with $r \in \{p, q\}$ denotes the r-Laplacian operator, $1 < p < q < N$ and $q^* = Nq/(N-q)$. Let's state the assumptions on $V(x)$, $K(x)$ and $f(u)$ as follows

(V) *$V \in C(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic with $\inf_{x\in\mathbb{R}^N} V(x) > 0$;*

(K) *$K \in L^\infty(\mathbb{R}^N)$ and $\inf_{x\in\mathbb{R}^N} K(x) \triangleq K_\infty = \lim_{|x|\to\infty} K(x) > 0$;*

(f_1) *$f \in C^0(\mathbb{R}, \mathbb{R})$ with $f(t) \equiv 0$ for all $t \le 0$ and $f(t)/t^{p-1} \to 0$ as $t \to 0^+$;*

(f_2) *there are $C_0 > 0$ and $s \in (q, q^*)$ such that $|f(t)| \le C_0(1 + |t|^{s-1})$ for all $t > 0$;*

(f_3) *$\lim_{t\to+\infty} F(t)/t^q = +\infty$, where $F(t) = \int_0^t f(s)ds$;*

(f_4) *the map $t \mapsto f(t)/t^{q-1}$ is increasing on $(0, \infty)$.*

Eq. (1.1) usually arises as the stationary version of a general reaction-diffusion equation

$$\partial_t u = \operatorname{div}[D(u)\nabla u] + f(x, u), \ x \in \mathbb{R}^N, \tag{1.2}$$

where u describes a concentration, $D(u) \triangleq \operatorname{div}(|\nabla u|^{p-2} + |\nabla u|^{q-2})$ is the diffusion coefficient and $f(x, u)$ is the reaction term related to source and loss mechanisms. There are several

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applications in biophysics, plasma physics and chemical reaction design, see [5, 16, 4, 7] and the references therein.

To deal with Eq. (1.1), for $r \in \{p, q\}$, we let $W^{1,r}(\mathbb{R}^N) = \{u \in L^r(\mathbb{R}^N) : |\nabla u| \in L^r(\mathbb{R}^N)\}$ denote the usual Sobolev space equipped with the norm

$$\|u\|_{1,r} = \left(\int_{\mathbb{R}^N} (|\nabla u|^r + |u|^r) dx \right)^{\frac{1}{r}}, \ \forall u \in W^{1,r}(\mathbb{R}^N).$$

Define the space

$$W_V^{1,r}(\mathbb{R}^N) = \left\{ u \in D^{1,r}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^r dx < +\infty \right\},$$

where $D^{1,r}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to $\|\cdot\|_{D^{1,r}(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |\nabla \cdot|^r dx)^{1/r}$. Let's introduce the norm on $W_V^{1,r}(\mathbb{R}^N)$ as follows

$$\|u\|_{V,r} = \left(\int_{\mathbb{R}^N} (|\nabla u|^r + V(x)|u|^r) dx \right)^{\frac{1}{r}}, \ \forall u \in W_V^{1,r}(\mathbb{R}^N).$$

Since $V(x) \in C(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic with $\inf_{x\in\mathbb{R}^N} V(x) > 0$ by (V), one shall easily observe that $\|\cdot\|_{V,r}$ is equivalent to $\|\cdot\|_{1,r}$ on $W_V^{1,r}(\mathbb{R}^N)$. Hence, $E \triangleq W_V^{1,p}(\mathbb{R}^N) \cap W_V^{1,q}(\mathbb{R}^N)$ is the natural work space in this paper endowed with the norm

$$\|u\| = \|u\|_{V,p} + \|u\|_{V,q}, \ \forall u \in E.$$

We will establish the existence of ground state solutions for Eq. (1.1) by looking for critical points of the associated functional

$$J_K^\lambda(u) = \frac{1}{p}\|u\|_{V,p}^p + \frac{1}{q}\|u\|_{V,q}^q - \lambda \int_{\mathbb{R}^N} K(x)F(u)dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*}dx, \ \forall u \in E.$$

Combing (K), $(f_1) - (f_2)$ and Lemma 2.1 below, it's simple to verify that $J_K^\lambda \in C^1(E, \mathbb{R})$. Similar to [23, 5], the critical points of J_K^λ are in fact the (weak) solutions of Eq. (1.1). We say that $u \in E$ is a (weak) solution of Eq. (1.1) if for any $\psi \in E$ there holds

$$\begin{aligned} 0 = &\int_{\mathbb{R}^N} \left[|\nabla u|^{p-2}\nabla u \nabla \psi + |\nabla u|^{q-2}\nabla u \nabla \psi + V(x)|u|^{p-2}u\psi + V(x)|u|^{q-2}u\psi \right] dx \\ &- \lambda \int_{\mathbb{R}^N} K(x)f(u)\psi dx - \int_{\mathbb{R}^N} |u|^{p^*-2}u\psi dx. \end{aligned}$$

To search for the ground state solutions, let's introduce the ground state energy and Nehari manifold associated to J_K^λ,

$$m_K^\lambda \triangleq \inf_{u\in\mathcal{N}_K^\lambda} J_K^\lambda(u),$$

where

$$\mathcal{N}_K^\lambda = \left\{ u \in E\backslash\{0\} : \langle (J_K^\lambda)'(u), u\rangle = 0 \right\}.$$

In recent years, there are extensive bibliographies in the study of the quasilinear equation of the p&q-Laplacian type, see e.g. [5, 10, 11, 13, 13, 20, 1, 7, 25, 26, 8, 16, 24, 4, 12, 22, 19, 21, 9, 3, 17, 19, 2] and the references therein.

In [5], Cherfils-Il'yasov obtained the existence and nonexistence results for the problem

$$\begin{cases} -\Delta_p u - \Delta_q u + v(x)|u|^{p-2}u + w(x)|u|^{q-2}u = \lambda f(x,u), \ x \in \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

Subsequently, authors in [16] exploited the Morse theory to show that the problem

$$\begin{cases} -\Delta_p u - \mu\Delta_q u = f(x,u), \ x \in \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

has at least three nontrivial solutions, where the classic Ambrosetti-Rabinowitz condition ((AR) condition in short) is replaced by

(F) *if* $\sigma(x,t) \triangleq f(x,t)t - qF(x,t)$, *then there exists* $0 \le \varrho \in L^1(\Omega)$ *such that*

$$\sigma(x,t) \le \sigma(x,s) + \varrho(x) \text{ for a.e. } x \in \Omega, \text{ all } 0 \le s \le t \text{ or all } t \le s \le 0.$$

In [11], He and Li considered the existence and nonexistence of nontrivial solutions for the following p&q-Laplacian problem

$$\begin{cases} -\Delta_p u - \Delta_q u + m|u|^{p-2}u + n|u|^{q-2}u = g(x,u), \ x \in \mathbb{R}^N, \\ u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N), \end{cases}$$

where $m, n > 0$ are constants, $N \ge 3$ and $1 < p < q < N$, $g(x,u)/u^{q-2}$ tends to a positive constant l as $u \to +\infty$ satisfying the following (AR) condition

(AR) *there exists a constant* $\mu \in (q, q^*)$ *such that*

$$0 < \mu G(x,t) = \mu \int_0^t g(x,s)ds \le g(x,t)t, \ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

which was used in some literatures, see [7, 2] for example.

In [4], by introducing the coercive assumptions on $a(x)$ and $b(x)$, the authors proved

$$\begin{cases} -\Delta_p u - \Delta_q u + a(x)|u|^{p-2}u + b(x)|u|^{q-2}u = f(x,u), \ x \in \mathbb{R}^N, \\ u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N), \end{cases}$$

admits a nontrivial solution, where $f(x,t)$ is subcritical and satisfies (F) for $\varrho \in L^1(\mathbb{R}^N)$. Especially, Figueiredo [7] investigated the existence results for Eq. (1.1) with $K(x) \equiv 1$ by imposing the (AR) condition on f.

Inspired by the above mentioned works and their references therein, we try to obtain the ground state solution to the critical p&q-Laplacain equation without the (AR) condition. It's worthy pointing here that the condition (F) is also unnecessary in this paper.

We obtain the following main result.

Theorem 1.1. *Let* (V), (K) *and* $(f_1)-(f_4)$ *hold, then there is a constant* $\lambda^* > 0$ *such that Eq.* (1.1) *has a ground state solution* $u \in E$ *satisfying* $J_K^\lambda(u) = m_K^\lambda = \inf_{v \in E\setminus\{0\}} \max_{t>0} J_K^\lambda(tv)$ *for all* $\lambda > \lambda^*$.

Remark 1.2. Our results generalize [5, 16, 4, 2] to the critical case. Because of the absence of the condition (F), we also improve and replenish the results in [16, 7]. Since we require that f is only continuous, the classical Nehari manifold arguments exploited in [11, 1] do not work in our context.

Motivated by the results in [11], there exists a general form on Theorem 1.1 if $K(x)f(u)$ is replaced by $f(x,u)$ in Eq. (1.1). Precisely, we suppose that $f(x,u)$ satisfies the conditions

(F_1) $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ satisfies the Caratheodory conditions, i.e. for a.e. $x\in\mathbb{R}^N$, $f(x,t)$ is continuous in $t\in\mathbb{R}$ and for each $t\in\mathbb{R}$, $f(x,t)$ is Lebesgue measurable with respect to $x\in\mathbb{R}^N$; $f(x,t)\geq 0$, for $t\geq 0$ and $f(x,t)\equiv 0$, for $t<0$ and all $x\in\mathbb{R}^N$;

(F_2) $f(x,t)=o(t^{p-1})$ as $t\to 0$ uniformly in $x\in\mathbb{R}^N$, there are $C_0>0$ and $s\in(q,q^*)$ such that $|f(x,t)|\leq C_0(1+|t|^{s-1})$ for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$;

(F_3) the map $t\mapsto f(x,t)/t^{q-1}$ is increasing on $(0,\infty)$ for each $x\in\mathbb{R}^N$;

(F_4) there exists a function $\overline{f}(t)\in C(\mathbb{R})$ with $f(x,t)\geq\overline{f}(t)$ for every $(x,t)\in\mathbb{R}^N\times\mathbb{R}$ and meas$\{x\in\mathbb{R}^N: f(x,t)>\overline{f}(t)\}>0$ for any $t>0$ such that $\lim_{|x|\to+\infty}f(x,t)=\overline{f}(t)$ uniformly in bounded t.

By means of the approaches used in Theorem 1.1, we can derive

Corollary 1.3. *Assume that* (V) *and* $(F_1)-(F_4)$, *then there is a constant* $\Lambda>0$ *such that*

$$\begin{cases} -\Delta_p u-\Delta_q u+V(x)(|u|^{p-2}u+|u|^{q-2}u)=\lambda f(x,u)+|u|^{q^*-2}u, \ x\in\mathbb{R}^N,\\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N),\end{cases}$$

possesses a ground state solution for all $\lambda>\Lambda$.

Remark 1.4. One can easily get the proof of Corollary 1.3 after some slight modifications in Theorem 1.1, so we omit the detail proof. Compared with [11], we remove the assumption

(F_5) $\overline{f}\in C^1(\mathbb{R})$ and $(q-1)\overline{f}(t)\leq\overline{f}'(t)t$ for all $t>0$.

We note that, to the best knowledge of us, the result in Theorem 1.1 seems to be new. The proof of Theorem 1.1 will be obtained by exploiting variational procedures. The main difficulties in the proof can be stated as follows. (I) Because $f(t)$ does not satisfy the (AR) condition, we have to obtain the boundedness of the (C) sequence in an unusual way. (II) The lack of compactness due to the critical Sobolev exponent and the whole space $\mathbb{R}^N$ urges us to take some delicate analysis. (III) the Nehari manifold technique, which was utilized in [11], is no longer applicable since $f(t)\in C^0$ leads to that $\mathcal{N}_K^\lambda$ is not a C^1-manifold. As we will see later, the above facts prevent us applying the variational method in a standard way to prove Theorem 1.1.

To complete this section, we sketch our proof of Theorem 1.1.

Firstly, we'll verify that the functional J_K^λ has a Mountain-Pass geometry around $0\in E$ and then the (C) sequence $\{u_n\}\subset E$ can be obtained. To show that $\{u_n\}$ is bounded, we make full use of the concentration-compactness principle developed by P.-L. Lions [15] to get across this desired result. In the meanwhile, we have to pull the Mountain-Pass energy down to some critical value because of the appearance of the critical term. Secondly, since J_K^λ and $(J_K^\lambda)'$ are not translation-invariant, we study the existence of ground state solution of the limiting problem

$$\begin{cases} -\Delta_p u-\Delta_q u+V(x)(|u|^{p-2}u+|u|^{q-2}u)=\lambda K_\infty f(u)+|u|^{q^*-2}u, \ x\in\mathbb{R}^N,\\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N),\end{cases}\tag{1.3}$$

which plays a significant role in this paper. To this end, we obtain the following result

Proposition 1.5. *Suppose that* $(f_1) - (f_4)$, *then Eq.* (1.3) *admits a ground state solution for some sufficiently large* $\lambda > 0$.

Finally, with the help of Proposition 1.5 and the Non-Vanishing in Lemma 2.2 below, we can finish the proof of Theorem 1.1.

The article is organized as follows. In Section 2, we introduce some preliminary results and present the proof of Theorem 1.1 in Section 3.

Notations. Throughout this paper we shall frequently use the following notations:

- C and C_i $(i = 1, 2, \cdots)$ for various positive constants;
- $L^s(\mathbb{R}^N)$ $(1 \leq s \leq +\infty)$ is the usual Lebesgue space with the standard norm $|u|_s$;
- The best Sobolev constant

$$S \triangleq \{|\nabla u|_q^q : u \in D^{1,q}(\mathbb{R}^N) \text{ and } |u|_{q^*} = 1\}; \tag{1.4}$$

- " $\to$ " and " $\rightharpoonup$ " denote the strong and weak convergence in the related function space, respectively;
- For any $\rho > 0$ and any $x \in \mathbb{R}^N$, $B_\rho(x) \triangleq \{y \in \mathbb{R}^N : |y - x| < \rho\}$.

Let $(X, \|\cdot\|_X)$ be a Banach space with its dual space $(X^{-1}, \|\cdot\|_*)$, and Ψ be its functional on X. The Cerami sequence at a level $c \in \mathbb{R}$ ($(C)_c$ sequence in short) corresponding to Ψ assumes that $\Psi(x_n) \to c$ and $(1 + \|x_n\|_X)\|\Psi'(x_n)\|_* \to 0$ as $n \to \infty$, where $\{x_n\} \subset X$.

2. Preliminaries

In this section, we introduce some technical lemmas which will be used later.

Lemma 2.1. *(see e.g. [1]) The work space E is continuously embedded in $L^s(\mathbb{R}^N)$ for any $s \in [p, q^*]$ and compactly embedded in $L^s_{loc}(\mathbb{R}^N)$ for all $s \in [1, q^*)$.*

As a direct consequence of Lemma 2.1, for all $u \in E$, there exists a constant $C > 0$ such that

$$|u|_s \leq C\|u\|, \text{ where } p \leq s \leq q^*. \tag{2.1}$$

It is easy to see that the following lemmas hold, see e.g. [15] for details.

Lemma 2.2. *Let $\{\rho_n\} \subset L^1(\mathbb{R}^N)$ be a bounded sequence and $\rho_n \geq 0$, then there exists a subsequence, still denoted by ρ_n, such that one of the following two possibilities occurs:*

(i) (Vanishing) $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n dx = 0$ *for all* $R > 0$;

(ii) (Non-Vanishing) *there are* $\beta > 0$ *and* $R < +\infty$ *such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n dx = \beta.$$

Lemma 2.3. *Let $1 < \tau \leq +\infty$, $1 < s \leq +\infty$ with $\tau^* = N\tau/(N - \tau)$ if $\tau < N$. Suppose that $\{u_n\}$ is bounded in $L^s(\mathbb{R}^N)$, $\{|\nabla u_n|\}$ is bounded in $L^\tau(\mathbb{R}^N)$ and*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^s dx = 0.$$

Then $u_n \to 0$ in $L^\alpha(\mathbb{R}^N)$ for $\alpha \in (s, \tau^*)$.

The following critical point theorem established by Costa and Miyagaki in [6] will be used to search for the existence of solutions.

Theorem 2.4. *Let $(X, \|\cdot\|_X)$ be a real Banach space and assume that $\Psi \in C^1(X, \mathbb{R})$ satisfies the condition*

$$\max\{\Psi(0), \Psi(e)\} \le \alpha < \inf_{\|u\|_X \le \rho} \Psi(u)$$

for some $\alpha > 0$, $\rho > 0$ and some $e \in E$ with $\|e\|_X > \rho$. Let

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Psi(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\},$$

then there exists a $(C)_c$ sequence $\{u_n\}$ for Ψ.

3. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1 in detail. Firstly, we shall discuss the conditions $(f_1) - (f_4)$: By (f_1), one has

$$f(t) = f(t^+), \ \forall t \in \mathbb{R}, \tag{3.1}$$

where $t^+ = \max\{0, t\}$. Using (f_4), there holds,

$$\begin{aligned} 0 < F(t) = \int_0^t f(s)ds = \int_0^t \frac{f(s)}{s^{q-1}} s^{q-1} ds \le \frac{f(t)}{t^{q-1}} \int_0^t s^{q-1} ds \\ \le \frac{1}{q} f(t)t, \ \forall t \in (0, +\infty). \end{aligned} \tag{3.2}$$

We claim that

$$f(t)t - qF(t) \text{ is increasing on } t \in (0, +\infty). \tag{3.3}$$

Indeed, for all $0 < t_1 < t_2 < +\infty$, by (f_4), one has

$$\begin{aligned} qF(t_2) - qF(t_1) = q\int_{t_1}^{t_2} \frac{f(s)}{s^{q-1}} s^{q-1} ds \le q\frac{f(t_2)}{t_2^{q-1}} \int_{t_1}^{t_2} s^{q-1} ds \\ = \frac{f(t_2)}{t_2^{q-1}} (t_2^q - t_1^q) \le f(t_2)t_2 - f(t_1)t_1 \end{aligned}$$

indicating the desired result. Combing $(f_1) - (f_2)$ and (3.2), for every $\epsilon > 0$ and $s \in (q, q^*)$, there exists a constant $C_\epsilon > 0$ such that

$$0 \le \max\{f(t)t, F(t)\} \le \epsilon|t|^p + C_\epsilon|t|^s, \ \forall t \in \mathbb{R}. \tag{3.4}$$

Next, we verify that the variational functional J_K^λ possesses a Mountain-Pass geometry.

Lemma 3.1. *Suppose that (K) and $(f_1) - (f_3)$, for each fixed $\lambda > 0$, J_K^λ satisfies*

(i) *there exist constants $\alpha, \rho > 0$ such that $J_K^\lambda(u) \ge \alpha$ with $\|u\| = \rho$;*

(ii) *there exists a function $e \in E$ with $\|e\| > \rho$ such that $J_K^\lambda(e) < 0$.*

Proof. (i) Since $K(x) \in L^\infty(\mathbb{R}^N)$, for all $u \in E$ and $\epsilon = \frac{1}{2|K|_\infty} > 0$, by (2.1), one has

$$\begin{aligned} J_K^\lambda(u) &\geq \frac{1}{q}\|u\|_{V,q}^q + \frac{1}{2p}\|u\|_{V,p}^p - C|u|_s^s - C|u|_{q^*}^{q^*} \\ &\geq \min\left\{\frac{1}{q}, \frac{1}{2p}\right\}(\|u\|_{V,q}^q + \|u\|_{V,p}^p) - C\|u\|^s - C\|u\|^{q^*} \\ &\geq \min\left\{\frac{1}{q}, \frac{1}{2p}\right\}(\|u\|_{V,q}^p + \|u\|_{V,p}^p) - C\|u\|^s - C\|u\|^{q^*} \\ &\geq \min\left\{\frac{1}{q}, \frac{1}{2p}\right\}C_p^{-1}\|u\|^p - C\|u\|^s - C\|u\|^{q^*}, \end{aligned} \tag{3.5}$$

if $\|u\|_{V,q} \leq \|u\| \leq 1$ in the third inequality, where we have used the following fact

$$(a+b)^p \leq C_p(a^p + b^p), \ \forall a, b > 0.$$

Because $p < s < q^*$, one can find a small $\rho \in (0,1)$ in (3.5) to get the Point (i).

(ii) Choosing $v \in E\backslash\{0\}$ with $\|v\| = 1$, by (3.4), we have

$$\begin{aligned} J_K^\lambda(tv) &\leq \frac{t^q}{q} + \frac{t^p}{p} - \lambda\int_{\mathbb{R}^N} K(x)F(tv)dx - \frac{t^{q^*}}{q^*}|v|_{q^*}^{q^*} \\ &\to -\infty \text{ as } t \to +\infty, \end{aligned}$$

where we have used (2.1) and (3.1). By taking $e \triangleq t_0 v$ with a sufficiently large $t_0 > 0$, one can obtain the Point (ii). □

By Lemma 3.1 and Theorem 2.4, a (C) sequence of the functional J_K^λ at the level

$$c_K^\lambda \triangleq \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_K^\lambda(\gamma(t)) \geq \alpha > 0 \tag{3.6}$$

can be constructed, where the set of paths can be defined as

$$\Gamma_K^\lambda \triangleq \big\{\gamma \in C\big([0,1], E\big) : \gamma(0) = 0, J_K^\lambda(\gamma(1)) < 0\big\}.$$

In other words, there exists a sequence $\{u_n\} \subset E$ such that

$$J_K^\lambda(u_n) \to c_K^\lambda \text{ and } (1 + \|u_n\|)\|(J_K^\lambda)'(u_n)\|_* \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Motivated by [23, Theorem 4.1], we can obtain the following result.

Lemma 3.2. *Suppose that (K) and $(f_1) - (f_3)$, then*

$$c_K^\lambda = m_K^\lambda = d_K^\lambda \triangleq \inf_{u\in E\backslash\{0\}}\max_{t>0} J_K^\lambda(tu), \forall\lambda > 0. \tag{3.8}$$

Proof. The proof is standard, we present it for the reader's convenience. For all $u \in \mathcal{N}_K^\lambda\backslash\{0\}$ we deduce that $\xi(t) \triangleq J_K^\lambda(tu) > 0$ if $t > 0$ is small enough and $\xi(t) < 0$ if $t > 0$ is large enough. By using (f_4), there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_K^\lambda$. Moreover, the set $\mathcal{N}_K^\lambda$ separates E into two components. Combing (f_1) and (f_2), the component containing the origin also contains a small ball around the origin. Since $\langle (J_K^\lambda)'(t_u u), t_u u\rangle \geq 0$ for every $t \in [0, t_u]$, $J_K^\lambda(u) \geq 0$ for all u in this component. Thereby, every $\gamma \in \Gamma_K^\lambda$ has to cross $\mathcal{N}_K^\lambda$

and $m_K^\lambda \le c_K^\lambda$. Using the fact that $\xi(t) < 0$ if $t > 0$ is large enough for all $u \in E\backslash\{0\}$, one has $c_K^\lambda \le d_k^\lambda$. We finish the proof of this lemma by verifying $d_K^\lambda \le m_K^\lambda$.

To show $d_K^\lambda \le m_K^\lambda$, we claim that

$$pt^q - qt^p + q - p \ge 0 \text{ and } qt^{q^*} - q^*t^q + q^* - q \ge 0,\ \forall t > 0, \tag{3.9}$$

and

$$g_u(t) \triangleq \frac{1-t^q}{q} f(u)u + F(tu) - F(u) \ge 0,\ \forall t > 0 \text{ and } u \in E\backslash\{0\}. \tag{3.10}$$

Obviously, (3.9) is clear. It's easy to compute that

$$\begin{aligned}\frac{\partial}{\partial_t} g_u(t) &= f(tu)u - t^{q-1}f(u)u = t^{q-1}u^q\left[\frac{f(tu)}{(tu)^{q-1}} - \frac{f(u)}{u^{q-1}}\right] \\ &\begin{cases} \ge 0, & \text{if } t \in [1,+\infty), \\ \le 0, & \text{if } t \in (0,1], \end{cases}\end{aligned}$$

where we have used (f_4) and (3.1). Therefore, $g_u(t)$ is decreasing on $t \in (0,1]$ and increasing on $t \in [1,+\infty)$ for all $u \in E\backslash\{0\}$. Then, $g_u(t) \ge \min_{t>0} g_u(t) = g_u(1) = 0$. Combing (3.9) and (3.11), we have

$$\begin{aligned} & J_K^\lambda(u) - J_K^\lambda(tu) - \frac{1-t^q}{q}\langle (J_K^\lambda)'(u), u\rangle \\ = \ & \frac{pt^q - qt^p + q - p}{pq}\int_{\mathbb{R}^N}\big[|\nabla u|^p + V(x)|u|^p\big]dx + \frac{qt^{q^*} - q^*t^q + q^* - q}{qq^*}\int_{\mathbb{R}^N}|u|^{q^*}dx \\ & + \lambda\int_{\mathbb{R}^N}K(x)\left[\frac{1-t^q}{q}f(u)u + F(tu) - F(u)\right]dx \\ \ge \ & 0. \end{aligned} \tag{3.11}$$

Given a $u \in \mathcal{N}_K^\lambda$, by (3.11), we can conclude that $J_K^\lambda(u) \ge J_K^\lambda(tu)$ for all $t > 0$ which yields that $d_K^\lambda \le m_K^\lambda$. The proof is complete. □

Because of the appearance of the critical term, we have to obtain the following result.

Lemma 3.3. *Suppose that* (K) *and* $(f_1) - (f_3)$, *then* $\lim_{\lambda\to+\infty} c_K^\lambda = 0$. *In particular, there exists a constant* $\lambda^* > 0$ *such that*

$$c_K^\lambda < c^* \triangleq \frac{q^*-q}{q^*q} S^{\frac{q^*}{q^*-q}},\ \forall \lambda > \lambda^* \tag{3.12}$$

Proof. In view of the proof of Lemma 3.1-(ii), one derives $\lim_{t\to+\infty} J_K^\lambda(tv) = -\infty$ and then there is a constant $t_\lambda > 0$ such that $\max_{t>0} J_K^\lambda(tv) = J_K^\lambda(t_\lambda v)$, So, $\langle (J_K^\lambda)'(t_\lambda v), t_\lambda v\rangle = 0$,

$$t_\lambda^q\|v\|_{V,q}^q + t_\lambda^p\|v\|_{V,p}^p = \lambda\int_{\mathbb{R}^N}K(x)f(t_\lambda v)t_\lambda v dx + t_\lambda^{q^*}\int_{\mathbb{R}^N}|v|^{q^*}dx. \tag{3.13}$$

Dividing t_λ^q on both sides of (3.13), then we can apply (f_5) to show that t_λ is bounded with respect to λ. Up to a subsequence if necessary, there is a constant $t_0 \in [0,+\infty)$ such that

$t_\lambda \to t_0$ as $\lambda \to +\infty$. We claim that $t_0 \equiv 0$. If not, we can suppose that $t_0 > 0$ and then

$$\lim_{\lambda \to +\infty} \left(\lambda \int_{\mathbb{R}^N} K(x) f(t_\lambda v) t_\lambda v dx + t_\lambda^{p^*} \int_{\mathbb{R}^N} |v|^{p^*} dx \right) = +\infty$$

which together with (3.13) yields a contradiction. Hence, $t_0 \equiv 0$ holds, that is, $\lim_{\lambda \to +\infty} t_\lambda \to 0$. Let $\gamma_0(t) = tv$, then $\gamma_0 \in \Gamma_K^\lambda$ and by (3.1),

$$0 < c_K^\lambda \le \max_{t>0} J_K^\lambda(tv) = J_K^\lambda(t_\lambda v) \le \frac{t_\lambda^q}{q} + \frac{t_\lambda^p}{p} \to 0 \text{ as } \lambda \to +\infty$$

showing the desired result. The proof is complete. □

Lemma 3.4. *Suppose that (K) and $(f_1) - (f_4)$, then every sequence $\{u_n\} \subset E$ satisfying (3.7) is bounded in E for all $\lambda > \lambda^*$.*

Proof. Let $\{u_n\} \subset E$ be a sequence satisfying (3.7). In view of (3.2), one has

$$\begin{aligned} c_K^\lambda + o_n(1) &= J_K^\lambda(u_n) - \frac{1}{q} \langle (J_K^\lambda)'(u_n), u_n \rangle \\ &= \frac{q-p}{pq} \int_{\mathbb{R}^N} \big[|\nabla u_n|^p + V(x)|u_n|^p \big] dx + \frac{\lambda}{q} \int_{\mathbb{R}^N} K(x) \big[f(u_n)u_n - qF(u_n) \big] dx \\ &\quad + \frac{q^* - q}{qq^*} \int_{\mathbb{R}^N} |u_n|^{q^*} dx \\ &\ge \frac{q-p}{pq} \|u_n\|_{V,p}^p \end{aligned}$$

showing that $\{\|u_n\|_{V,p}\}$ is bounded. To end the proof, it suffices to conclude that $\{\|u_n\|_{V,q}\}$ is bounded. Arguing it by the contradiction, we can suppose that $\|u_n\|_{V,q} \to \infty$ as $n \to \infty$. Set $v_n = u_n / \|u_n\|_{V,q}$, then $\|v_n\|_{V,q} \equiv 1$ for each $n \in \mathbb{N}$. By Lemma 2.2, one of the following alternatives occurs:

Vanishing: $\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n|^q dx = 0$ for all $R > 0$;

Non-Vanishing: there are $\beta > 0$ and $R < +\infty$ such that

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n|^q dx = \beta.$$

In what follows, we'll verify that $\{v_n\}$ satisfies neither Vanishing nor Non-Vanishing. This is a contradiction. Thus, $\{\|u_n\|_{V,q}\}$ is bounded.

If the Vanishing occurs, then $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for every $s \in (q, q^*)$ by Lemma 2.3. Let $t_0 \triangleq S^{\frac{q^*}{q(q^*-q)}} > 0$, then we can apply (K) and $(f_1) - (f_2)$ to obtain

$$\lim_{n\to\infty} \lambda \int_{\mathbb{R}^N} K(x) F(t_0 v_n) dx = 0, \ \forall \lambda > 0. \tag{3.14}$$

Proceeding as the proof of Lemma 3.2, there is a constant $t_n \in (0,1)$ such that $J_K^\lambda(t_n u_n) = \max_{t \in (0,1)} J_K^\lambda(t u_n)$ and $\langle (J_K^\lambda)'(t_n u_n), t_n u_n \rangle = 0$. For some sufficiently large $n \in \mathbb{N}$, $\frac{t_0}{\|u_n\|_{V,q}} \in$

$(0,1)$ since $\|u_n\|_{V,q}\to\infty$ as $n\to\infty$. By (3.3) and (3.7), we have

$$
\begin{aligned}
J_K^\lambda(t_0v_n) &= J_K^\lambda\big(\frac{t_0}{\|u_n\|_{V,q}}u_n\big)\le \max_{t\in(0,1)} J_K^\lambda(tu_n)=J_K^\lambda(t_nu_n)=J_K^\lambda(t_nu_n)-\frac{1}{q}\langle (J_K^\lambda)'(t_nu_n),t_nu_n\rangle\\
&=\frac{q-p}{pq}t_n^p\int_{\mathbb{R}^N}\big[|\nabla u_n|^p+V(x)|u_n|^p\big]dx+\frac{\lambda}{q}\int_{\mathbb{R}^N}K(x)\big[f(t_nu_n)t_nu_n-qF(t_nu_n)\big]dx\\
&\quad+\frac{q^*-q}{qq^*}t_n^{q^*}\int_{\mathbb{R}^N}|u_n|^{q^*}dx\\
&\le\frac{q-p}{pq}\int_{\mathbb{R}^N}\big[|\nabla u_n|^p+V(x)|u_n|^p\big]dx+\frac{\lambda}{q}\int_{\mathbb{R}^N}K(x)\big[f(u_n)u_n-qF(u_n)\big]dx\\
&\quad+\frac{q^*-q}{qq^*}\int_{\mathbb{R}^N}|u_n|^{q^*}dx\\
&=J_K^\lambda(u_n)-\frac{1}{q}\langle (J_K^\lambda)'(u_n),u_n\rangle=J_K^\lambda(u_n)+o_n(1)\\
&=c_K^\lambda+o_n(1).
\end{aligned}\tag{3.15}
$$

On the other hand, by using (1.4) and (3.14), we obtain

$$
\begin{aligned}
J_K^\lambda(t_0v_n)&=\frac{t_0^q}{q}\|v_n\|_{V,q}^q+\frac{t_0^p}{p}\|v_n\|_{V,p}^p-\lim_{n\to\infty}\lambda\int_{\mathbb{R}^N}K(x)F(t_0v_n)dx-\frac{t_0^{q^*}}{q^*}|v_n|_{q^*}^{q^*}\\
&\ge\frac{t_0^q}{q}-\frac{t_0^{q^*}}{q^*S^{q^*/q}}+o_n(1)\\
&=\frac{q^*-q}{q^*q}S^{\frac{q^*}{q^*-q}}+o_n(1).
\end{aligned}\tag{3.16}
$$

Obviously, there is a contradiction to Lemma 3.3 by (3.15) and (3.16). Thus, the Vanishing cannot occur.

If the Non-Vanishing occurs, then there exists a sequence $\{y_n\}\subset\mathbb{R}^N$ such that

$$\int_{B_R(y_n)}|v_n|^q dx=\frac{\beta}{2}>0. \tag{3.17}$$

Without loss of generality, we can suppose that there exist an $\overline{R}>R$ and $\{\overline{y}_n\}\subset\mathbb{Z}^N$ such that

$$\int_{B_{\overline{R}}(\overline{y}_n)}|v_n|^q dx\ge\frac{\beta}{2}>0. \tag{3.18}$$

Indeed, for each $n\in\mathbb{N}$, there is $\{\overline{y}_n\}\subset\mathbb{Z}^N$ such that

$$B_R(y_n)\subset B_{R+\sqrt{N}}(\overline{y}_n),$$

which together with (3.17) gives (3.18) if $\overline{R}\triangleq R+\sqrt{N}$. Setting $\overline{v}_n(x)=v_n(x+\overline{y}_n)$, there exists a nontrivial function $\overline{v}\in E$ such that $\overline{v}_n\rightharpoonup\overline{v}$ in E. Let $\Omega=\{x\in\mathbb{R}^N:\overline{v}\neq0\}$, then

$m(\Omega) > 0$ and $|\overline{u}_n| \to +\infty$ as $n \to \infty$ in Ω. Since $V(x)$ is $\mathbb{Z}^N$-periodic, combing (K) and (f_3), one has

$$\begin{aligned}\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{K(x)F(u_n)}{\|u_n\|_{V,q}^q}dx &\geq K_\infty\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{F(u_n)}{\|u_n\|_{V,q}^q}dx \\ = K_\infty\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{F(\overline{u}_n)}{\|\overline{u}_n\|_{V,q}^q}dx &\geq K_\infty\liminf_{n\to\infty}\int_{\Omega}\frac{F(\overline{u}_n)}{|\overline{u}_n|^q}|\overline{v}_n|^q dx = +\infty.\end{aligned} \tag{3.19}$$

Since $J_K^\lambda(u_n) \to c_K^\lambda$ as $n \to \infty$ and $\|u_n\|_{V,p}$ is bounded, by means of (3.19), we have

$$0 = \limsup_{n\to\infty}\frac{J_K^\lambda(u_n)}{\|u_n\|_{V,q}^q} \leq \frac{1}{q} - \lambda\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{K(x)F(u_n)}{\|u_n\|_{V,q}^q}dx = -\infty,$$

a contradiction. The proof of this lemma is complete. □

As stated in the Introduction, in the process of looking for ground state solutions of Eq. (1.1), we have to establish the existence of ground state solutions of Eq. (1.3). To achieve this purpose, we need the associated functional $J_\infty^\lambda : W_V^{1,p}(\mathbb{R}^N) \cap W_V^{1,q}(\mathbb{R}^N) \to \mathbb{R}$ of Eq. (1.3) defined by

$$J_\infty^\lambda(u) = \frac{1}{q}\|u\|_{V,q}^q + \frac{1}{p}\|u\|_{V,p}^p - \lambda K_\infty\int_{\mathbb{R}^N}F(u)dx - \frac{1}{p^*}\int_{\mathbb{R}^N}|u|^{p^*}dx$$

and the ground state energy

$$m_\infty^\lambda \triangleq \inf_{u\in\mathcal{N}_\infty^\lambda} J_\infty^\lambda(u),$$

where

$$\mathcal{N}_\infty^\lambda = \big\{u \in E\backslash\{0\} : \langle (J_\infty^\lambda)'(u), u\rangle = 0\big\}.$$

Now, we give the proof of Proposition 1.5 as follows.

Proof. Obviously, the critical points of J_∞^λ are weak solutions of Eq. (1.3), and vice versa. Similar to Lemma 3.1, we can see that J_∞^λ also admits a (C) sequence at the level

$$c_\infty^\lambda = \inf_{\gamma\in\Gamma_\infty^\lambda}\max_{t\in[0,1]} J_\infty^\lambda(\gamma(t)),$$

where the set of paths is defined as

$$\Gamma_\infty^\lambda \triangleq \big\{\gamma \in C\big([0,1], E\big)|\gamma(0) = 0, J_\infty^\lambda(\gamma(1)) < 0\big\}.$$

Let $\{u_n\} \subset E$ be a $(C)_{c_\infty^\lambda}$ sequence of J_∞^λ, similar to Lemma 3.4, $\{u_n\}$ is bounded in E for all $\lambda > \lambda^*$. Up to a subsequence if necessary, there exists a function $u \in E$ such that $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. Proceeding as Lemma 3.4, the Vanishing cannot happen for $|u_n|_q^q$. Therefore, there are $\tilde{\beta} > 0$, $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and $\tilde{R} < +\infty$ such that

$$\int_{B_{\tilde{R}}(\tilde{y}_n)}|u_n|^q dx \geq \tilde{\beta}.$$

Arguing as in the proof of Lemma 3.4, without loss of generality, we can suppose that the sequence $\{\tilde{y}_n\} \subset \mathbb{Z}^N$ if increasing $\tilde{R} > 0$ large enough.

Set $\tilde{u}_n = u(\cdot + \tilde{y}_n)$, then $\|\tilde{u}_n\| = \|u_n\|$ is bounded and there exists a function $0 \neq \tilde{u} \in E$ such that $\tilde{u}_n \rightharpoonup \tilde{u}$ in E, $\tilde{u}_n \to \tilde{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ with $q \leq s < q^*$ and $\tilde{u}_n \to \tilde{u}$ a.e. in $\mathbb{R}^N$ in the sense of a subsequence. Since J^λ_∞ and $(J^\lambda_\infty)'$ are translation-invariant with respect to $\{\tilde{y}_n\} \subset \mathbb{Z}^N$, $\{\tilde{u}_n\}$ is still a $(C)_{c^\lambda_\infty}$ sequence of J^λ_∞. Proceeding as [2, Lemma 3.4], $(J^\lambda_\infty)'$ is weakly sequentially continuous in E^{-1} and thus $(J^\lambda_\infty)'(\tilde{u}) = 0$. We deduce that $(J^\lambda_\infty)'(\tilde{u}) = 0$ and $\tilde{u} \neq 0$. It follows from (3.2) and the Fatou's lemma that

$$\begin{aligned}
m^\lambda_\infty &\leq J^\lambda_\infty(\tilde{u}) = J^\lambda_\infty(\tilde{u}) - \frac{1}{q}\langle (J^\lambda_\infty)'(\tilde{u}), \tilde{u}\rangle \\
&= \frac{q-p}{pq}\int_{\mathbb{R}^N}\big[|\nabla \tilde{u}|^p + V(x)|\tilde{u}|^p\big]dx + \frac{\lambda}{q}\int_{\mathbb{R}^N} K_\infty\big[f(\tilde{u})\tilde{u} - qF(\tilde{u})\big]dx \\
&\quad + \frac{q^*-q}{qq^*}\int_{\mathbb{R}^N}|\tilde{u}|^{q^*}dx \\
&\leq \liminf_{n\to\infty}\bigg\{\frac{q-p}{pq}\int_{\mathbb{R}^N}\big[|\nabla \tilde{u}_n|^p + V(x)|\tilde{u}_n|^p\big]dx + \frac{\lambda}{q}\int_{\mathbb{R}^N} K_\infty\big[f(\tilde{u}_n)\tilde{u}_n - qF(\tilde{u}_n)\big]dx \\
&\quad + \frac{q^*-q}{qq^*}\int_{\mathbb{R}^N}|\tilde{u}_n|^{q^*}dx\bigg\} \\
&= \liminf_{n\to\infty}\big[J^\lambda_\infty(\tilde{u}_n) - \frac{1}{q}\langle (J^\lambda_\infty)'(\tilde{u}_n), \tilde{u}_n\rangle\big] \\
&= c^\lambda_\infty.
\end{aligned} \tag{3.20}$$

Similar to (3.2), we can exploit (f_4) to conclude that $c^\lambda_\infty = m^\lambda_\infty$ which together with (3.20) gives that $J^\lambda_\infty(\tilde{u}) = m^\lambda_\infty$. The proof is complete. □

Now, we are in a position to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality, there exists a set of positive measure on which $K(x) > K_\infty$. Otherwise, Proposition 1.5 is the special case of Theorem 1.1. By the assumptions on the potential $K(x)$, we infer that $c^\lambda_K < c^\lambda_\infty$. In fact, by Proposition 1.5, the level $c^\lambda_\infty = m^\lambda_\infty$ is attained at a ground state solution u^λ_∞ of the limit Eq. (1.3) for all $\lambda > \lambda^*$. We then deduce that $J^\lambda_K(tu^\lambda_\infty) < J^\lambda_\infty(tu^\lambda_\infty)$ for all $t > 0$, from this we have that

$$\begin{aligned}
c^\lambda_K &= \inf_{u\in E\setminus\{0\}}\max_{t\geq 0} J^\lambda_K(tu) \leq \max_{t\geq 0} J^\lambda_K(tu^\lambda_\infty) = J^\lambda_K(\bar{t}u^\lambda_\infty) < J^\lambda_\infty(\bar{t}u^\lambda_\infty) \\
&\leq \max_{t\geq 0} J^\lambda_\infty(tu^\lambda_\infty) = m^\lambda_\infty = c^\lambda_\infty, \ \forall \lambda > \lambda^*,
\end{aligned}$$

where $\bar{t} > 0$ is unique and satisfies that $\bar{t}u^\lambda_\infty \in \mathcal{N}^\lambda_K$ (see [23, Chapter 4] for details).

Let $\{u_n\}$ be a $(C)_{c^\lambda_K}$ sequence of J^λ_K, for every $\lambda > \lambda^*$, passing to a subsequence, there exists a function $u \in E$ such that $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ with $q \leq s < q^*$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. In view of the proof of Lemma 3.4, the Vanishing can never occur for $|u_n|^q_q$. So, there are $\underline{\beta} > 0$, $\{z_n\} \subset \mathbb{R}^N$ and $\underline{R} < +\infty$ such that

$$\int_{B_{\underline{R}}(z_n)} |u_n|^q dx \geq \underline{\beta}.$$

Arguing as in the proof of Lemma 3.4, without loss of generality, we can suppose that the sequence $\{z_n\} \subset \mathbb{Z}^N$ if increasing $\underline{R} > 0$ large enough.

By a similar argument in the proof of Proposition 1.5, to finish the proof, it suffices to prove that $u \neq 0$. For this purpose, we claim that $\{z_n\} \subset \mathbb{Z}^N$ in bounded in $\mathbb{R}^N$. If not, we assume that for a subsequence $|z_n| \to \infty$ as $n \to \infty$. Let's define $v_n \triangleq u_n(\cdot + z_n)$ and then $\|v_n\| = \|u_n\|$ (since $V(x)$ is $\mathbb{Z}^N$-periodic) is bounded in E. Up to a subsequence, there exists a function $v \neq 0$ such that $v_n \rightharpoonup v$ in E. Next, we verify that $(J_\infty^\lambda)'(v) = 0$. In fact, since $B_{|z_n|/2}(0 - z_n) \subset \mathbb{R}^N \backslash B_{|z_n|/2}(0)$, for all $\psi \in E$, we have

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^N} [K(x) - K_\infty] f(u_n) \psi(x - z_n) dx \right| \\
&\leq \int_{B_{\frac{|z_n|}{2}}(0)} [K(x) - K_\infty] |f(u_n)| |\psi(x - z_n)| dx + \int_{\mathbb{R}^N \backslash B_{\frac{|z_n|}{2}}(0)} [K(x) - K_\infty] |f(u_n)| |\psi(x - z_n)| dx \\
&\leq C|K|_\infty |\psi|_{L^q(\mathbb{R}^N \backslash B_{|z_n|/2}(0))} + C|K - K_\infty|_{L^\infty(\mathbb{R}^N \backslash B_{|z_n|/2}(0))} |\psi|_q \to 0, \text{ as } n \to \infty,
\end{aligned}
$$

where we have used $|f(u_n)|_q$ is bounded by $(f_1) - (f_2)$. So, for all $\psi \in E$, we obtain

$$
\begin{aligned}
\langle (J_\infty^\lambda)'(v), \psi \rangle &= \langle (J_\infty^\lambda)'(v_n), \psi \rangle + o_n(1) \\
&= \int_{\mathbb{R}^N} \big[|\nabla v_n|^{q-2} \nabla v_n \nabla \psi + |\nabla v_n|^{p-2} \nabla v_n \nabla \psi + V(x)(|v_n|^{q-2} v_n \psi + |v_n|^{p-2} v_n \psi) \big] dx \\
&\quad - \lambda \int_{\mathbb{R}^N} K(x) f(v_n) \psi dx - \int_{\mathbb{R}^N} |v_n|^{q^*-2} v_n \psi dx \\
&= \langle (J_K^\lambda)'(u_n), \psi(x - z_n) \rangle + \lambda \int_{\mathbb{R}^N} [K(x) - K_\infty] f(u_n) \psi(x - z_n) dx + o_n(1) \\
&= (J_K^\lambda)'(u_n), \psi(x - z_n) \rangle + o_n(1) \to 0, \text{ as } n \to \infty,
\end{aligned}
$$

indicating that $(J_\infty^\lambda)'(v) = 0$, where we have used the fact that $V(x)$ is $\mathbb{Z}^N$-periodic in the third equality. However, by means of $V(x)$ is $\mathbb{Z}^N$-periodic in in (V) and $K(x) \geq K_\infty$ in (K) and Fatou's lemma,

$$
\begin{aligned}
c_K^\lambda + o_n(1) &= J_K^\lambda(u_n) - \frac{1}{q} \langle (J_K^\lambda)'(u_n), u_n \rangle \\
&= \frac{q-p}{pq} \int_{\mathbb{R}^N} \big[|\nabla u_n|^p + V(x)|u_n|^p \big] dx + \frac{\lambda}{q} \int_{\mathbb{R}^N} K(x) \big[f(u_n) u_n - qF(u_n) \big] dx \\
&\quad + \frac{q^* - q}{qq^*} \int_{\mathbb{R}^N} |u_n|^{q^*} dx \\
&\geq \frac{q-p}{pq} \int_{\mathbb{R}^N} \big[|\nabla u_n|^p + V(x)|u_n|^p \big] dx + \frac{\lambda}{q} K_\infty \int_{\mathbb{R}^N} \big[f(u_n) u_n - qF(u_n) \big] dx \\
&\quad + \frac{q^* - q}{qq^*} \int_{\mathbb{R}^N} |u_n|^{q^*} dx \\
&= \frac{q-p}{pq} \int_{\mathbb{R}^N} \big[|\nabla v_n|^p + V(x)|v_n|^p \big] dx + \frac{\lambda}{q} K_\infty \int_{\mathbb{R}^N} \big[f(v_n) v_n - qF(v_n) \big] dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{q^*-q}{qq^*}\int_{\mathbb{R}^N}|v_n|^{q^*}dx\\
&=J_\infty^\lambda(v_n)-\frac{1}{q}\langle(J_\infty^\lambda)'(v_n),v_n\rangle\ge J_\infty^\lambda(v)-\frac{1}{q}\langle(J_\infty^\lambda)'(v),v\rangle+o_n(1)\\
&=J_\infty^\lambda(v)+o_n(1)
\end{aligned}
$$

which contradicts with the fact that $c_K^\lambda < c_\infty^\lambda$. The proof is finished. $\square$

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