Some results for a system of NLS arising in optical materials with χ^3 nonlinear response

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Abstract: In this paper, we investigate the nonlinear Schödinger equations with cubic interactions, arising in nonlinear optics. To begin, we prove the existence results for normalized ground state solutions in the L^2-subcritical case and L^2-supercritical case respectively. Our proofs relies on the Concentration-compactness principle, Pohozaev manifold and rearrangement technique. Then, we establish the nonexistence of normalized ground state solutions in the L^2-critical case by finding that there exists a threshold. In addition, based on the existence of the normalized solutions, we also establish the blow-up results are shown by using localized virial estimates, and a new blow-up criterion which is related to normalized solutions.
Keywords: Nonlinear Schrödinger equations; Normalized solutions; Blow-up; χ^3 nonlinear response

1 Introduction

In this paper, we consider the following Cauchy problem for the system of nonlinear Schrödinger equations with cubic interation

$$\begin{cases} iu_t + \Delta u - u + \left(\frac{1}{9}\left|u^2\right| + 2\left|w^2\right|\right)u + \frac{1}{3}\bar{u}w = 0, & x \in \mathbb{R}^n, t \in \mathbb{R}, \\ i\sigma w_t + \Delta w - \mu w + \left(9\left|w^2\right| + 2\left|u^2\right|\right)w + \frac{1}{9}u^3 = 0, & x \in \mathbb{R}^n, t \in \mathbb{R}, \\ (u(x,0), w(x,0)) = (u_0, w_0), \end{cases} \tag{1.1}$$

where $1 \leq n \leq 3$, $u, w : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$, $u_0, w_0 : \mathbb{R}^n \to \mathbb{C}$, and the parameters $\sigma, \mu > 0$.

The system (1.1) comes from nonlinear optics. In nonlinear optics, as light incident on the atom, the electric field of light will make the negative charge do simple harmonic vibration with respect to the positive charge, thus generation a time varying dipole moment.

Dipole moment is a microscopic concept, which can be summed to obtain the macroscopic physical quantity of polarization vector $\vec{P}_{NL}$. That is to say, the electric field in the incident light will polarize the material, and then the polarization amount $\vec{P}_{NL}$ will be generated. For linear optics, $\vec{P}_{NL}$ is proportional to electric field $\vec{E}$, and the two meet $\vec{P}_{NL} = \varepsilon_0 \chi \vec{E}$, where χ is the polarizability of the material. In Franken's opinion, if the light is very strong, the corresponding electric field will be strong, and the strong electric field may cause the response of the material to be nonlinear rather than linear, that is, the polarization $\vec{P}_{NL}$ of the material is no longer in direct proportion to $\vec{E}$, but contains E_1, E_2, E_3 and higher order items. Then it can be written as

$$\vec{P}_{NL} = \varepsilon_0 \left[\chi^{(1)} E_1 + \chi^{(2)} E_2 + \chi^{(3)} E_3 + \cdots\right],$$

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such a simple expression, which can be considered as a simple correction of the response of linear optical materials. The coefficient $\chi^{(j)}$ depends on the electric field frequency $\vec{E}$ and is called the jth optical susceptibility ($j = 1, 2, 3, \cdots$). Therefore quadratic media arise from approximation of the type $\vec{P}_{NL} \sim \chi^{(2)} E_2$, and similarly one can define cubic media as $\vec{P}_{NL} \sim \chi^{(3)} E_3$. In addition, we refer to [[3], [4] , [19], [18], [11]], and references therein, for more insights on physical motivations and physical results.

Let us mention, main difference between $\chi^{(2)}$ and $\chi^{(3)}$ is that in the case of the latter, the cubic nonlinearity is L^2-supercritical, while in the former, the secondary nonlinearity is L^2-subcritical. The so-called non-centrosymmetric crystals and Kerr-materials are typical examples of $\chi^{(2)}$ and $\chi^{(3)}$ materials respectively. These two mediums reflect the possibility of global well posed problem, and the stability/instability properties of solitons are different. For further discussion and strict analysis of solitons in secondary media, see [6] and [15].

We are more concerned with the solutions in the cubic medium. In the so-called cascade nonlinear process, we can get the system in the form of (1.1) related to physics. In particular, we are interested in the qualitative properties of the solutions of (1.1). Under the slowly-varying amplitude approximation and introducing the dimensionless variables, we can obtain the system

$$\begin{cases} iu_t + \Delta u - u + \left(\frac{1}{9}\left|u^2\right| + 2\left|w^2\right|\right) u + \frac{1}{3}\bar{u}w = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \\ i\sigma w_t + \Delta w - \mu w + \left(9\left|w^2\right| + 2\left|u^2\right|\right) w + \frac{1}{9}u^3 = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \end{cases}$$

where $\mu = \left(3 + \frac{\alpha}{\beta}\right)\sigma$. Remark that $\mu = 3\sigma$ is called the mass resonance condition, where the parameters $\sigma, \mu > 0$.

Let us start our rigorous mathmatical discussion about (1.1) in Euclidean space $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $1 \leq n \leq 3$. In $L^2(\mathbb{R}^n)$, the corresponding norm is defined as

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f|^2 dx.$$

Besides, $H^1(\mathbb{R}^n)$ denotes the usual Sobolev space in $\mathbb{R}^n$,

$$H^1(\mathbb{R}^n) := \left\{ f \mid f \in L^2(\mathbb{R}^n), \nabla f \in L^2(\mathbb{R}^n) \right\},$$

endowed with the norm

$$\|f\|_{H^1(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (|\nabla f|^2 + |f|^2) dx\right)^{\frac{1}{2}}.$$

Regarding the system (1.1), the existence of ground states were established by Oliveira and Pastor [14]. Assume $(u_0, w_0) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, in [14], they proved that the system (1.1) is locally well-posed and the solutions to the system (1.1) satisfy conservation laws of mass and energy defined, respectively, by

$$M(u,w) = M(u_0, w_0),$$

and

$$E_\mu(u,w) = E_\mu(u_0, w_0),$$

where

$$M(u,w) = \|u(t)\|^2_{L^2(\mathbb{R}^n)} + 3\sigma\|w(t)\|^2_{L^2(\mathbb{R}^n)}, \tag{1.2}$$

$$E_\mu(u,w) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla u|^2 + |\nabla w|^2 + |u|^2 + \mu|w|^2\right) dx - \int_{\mathbb{R}^n} \left(\frac{1}{36}|u|^4 + \frac{9}{4}|w|^4 + |u|^2|w|^2 + \frac{1}{9}\mathbf{Re}\left(\bar{u}^3 w\right)\right) dx, \tag{1.3}$$

$$N(u,w)=\int_{\mathbb{R}^n}\left(\frac{1}{36}|u|^4+\frac{9}{4}|w|^4+|u|^2|w|^2+\frac{1}{9}\,\mathbf{Re}\left(\bar{u}^3w\right)\right)dx.$$

We rewrite the functional $N(u,w)$ by means of its density, namely

$$N(u,w)=\int_{\mathbb{R}^n}T(u,w)dx, \tag{1.4}$$

where

$$T(u,w)=\left(\frac{1}{36}|u|^4+\frac{9}{4}|w|^4+|u|^2|w|^2+\frac{1}{9}\,\mathbf{Re}\left(\bar{u}^3w\right)\right).$$

The previous conservation law can be formally proved by the usual partial integration, and then it can be proved by the classical regularization argument, see [5].

For the purpose of our paper, we also define the kinetic energy

$$K(u,w):=\|\nabla u\|^2_{L^2(\mathbb{R}^n)}+\|\nabla w\|^2_{L^2(\mathbb{R}^n)}. \tag{1.5}$$

In this paper, due to the dynamics of nonlinear Schrödinger-type equations is strongly related to the notion of ground states. We consider the existence of normalized ground states of the system (1.1). Recall that standing waves are special solutions of the form

$$u(x,t)=e^{i\omega t}P(x),\quad w(x,t)=e^{3i\omega t}Q(x).$$

Plugging into (1.1) we get the system of elliptic equations

$$\begin{cases}\Delta P+\left(\frac{1}{9}P^2+2Q^2\right)P+\frac{1}{3}P^2Q=(\omega+1)P, & x\in\mathbb{R}^n,\\ \Delta Q+\left(9Q^2+2P^2\right)Q+\frac{1}{9}P^3=(\mu+3\sigma\omega)Q, & x\in\mathbb{R}^n.\end{cases} \tag{1.6}$$

Thus we arrive at the conclusion that the solutions to (1.6) exist, provided that

$$\omega>-\min\left\{1,\frac{\mu}{3\sigma}\right\},$$

which was proved in [14]. Moreover, it is easy to check that solutions of (1.6), are called ground state related to (1.6) if it minimizes the action functional

$$S_{\omega,\mu,\sigma}(f,g):=E_\mu(f,g)+\frac{\omega}{2}M_{3\sigma}(f,g),$$

over all nontrivial solutions. And the set of ground states denoted by

$$\mathcal{G}(\omega,\mu,\sigma):=\{(\phi,\psi)\in\mathcal{A}_{\omega,\mu,\sigma}:S_{\omega,\mu,\sigma}(\phi,\psi)\le S_{\omega,\mu,\sigma}(f,g),\forall(f,g)\in\mathcal{A}_{\omega,3\sigma,\sigma}\}\neq\emptyset,$$

where $\mathcal{A}_{\omega,3\sigma,\sigma}$ is the set of all non-trivial solutions to (1.6).

Our task now is to push forward their achievements to prove the normalized solutions of (1.1) in different critical states under the mass resonance condition, namely $\mu=3\sigma$, and (1.6) can be written as

$$\begin{cases}\Delta P+\left(\frac{1}{9}P^2+2Q^2\right)P+\frac{1}{3}P^2Q=(\omega+1)P, & x\in\mathbb{R}^n,\\ \Delta Q+\left(9Q^2+2P^2\right)Q+\frac{1}{9}P^3=3\sigma(\omega+1)Q, & x\in\mathbb{R}^n.\end{cases} \tag{1.7}$$

The existence of normalized traveling solitary waves of problem (1.1) can be formulated as the following problem: for some $c>0$, we solve the problem (1.7) with the normalized condition:

$$\int_{\mathbb{R}^n}\left(|P|^2+3\sigma|Q|^2\right)dx=c, \tag{1.8}$$

where $(P,Q)\in H^1\left(\mathbb{R}^n\right)\times H^1\left(\mathbb{R}^n\right)$ are real functions with a suitable decay at infinity.

In order to obtain the normalized solutions of the Cauchy problem (1.7), $E_\mu(u,w)$ is restricted in proper function space under the mass resonance $\mu=3\sigma$, and we study the critical points of function defined by

$$E_\sigma(P,Q)=\frac{1}{2}\int_{\mathbb{R}^n}\left(|\nabla P|^2+|\nabla Q|^2+|P|^2+3\sigma|Q|^2\right)dx-\int_{\mathbb{R}^n}\left(\frac{1}{36}|P|^4+\frac{9}{4}|Q|^4+|P|^2|Q|^2+\frac{1}{9}\,\mathbf{Re}\left(\bar{P}^3Q\right)\right)dx,$$

on the L^2-sphere

$$S_c=\left\{(P,Q)\in H^1\left(\mathbb{R}^n\right)\times H^1\left(\mathbb{R}^n\right), M(P,Q)=\int_{\mathbb{R}^n}|P|^2+3\sigma\left|Q\right|^2dx=c\right\},$$

for some $c>0$, and the functional of E_σ above the constraint S_c is expressed as follows:

$$E_\sigma|_{S_c}(P,Q)=\frac{1}{2}\int_{\mathbb{R}^n}(|\nabla P|^2+|\nabla Q|^2)dx-\int_{\mathbb{R}^n}\left(\frac{1}{36}|P|^4+\frac{9}{4}|Q|^4+|P|^2|Q|^2+\frac{1}{9}\,\mathbf{Re}\left(\bar{P}^3Q\right)\right)dx.$$

Note that ω is called Lagrange multiplier which plays the main role in our paper. In all normalized solutions, we are mainly concerned with the ground state solutions, that is, the solutions that minimizes the functional among all solutions with the same L^2- norm. Consider the minimization problem

$$m_c:=\inf_{(P,Q)\in S_c}E_\sigma. \tag{1.9}$$

If (P_c,Q_c) be the minimizer of the minimization problem (1.9), $\omega=\omega_c$ be as the Lagrange multiplier. Then (P_c,Q_c) is the ground state solution of (1.7). In particular, we call $(\omega_c,P_c,Q_c)\in\mathbb{R}\times H^1\left(\mathbb{R}^n\right)\times H^1\left(\mathbb{R}^n\right)$ is a normalized solution of the problem (1.7)-(1.8).

Before introducing the main theorems of the paper, we recall the fact that (1.1) is invariant under the scaling

$$u_\lambda(t,x):=\lambda u\left(\lambda^2t,\lambda x\right),\quad w_\lambda(t,x):=\lambda^2w\left(\lambda^2t,\lambda x\right),\quad \lambda>0.$$

By a simple calculation, according to the fact that

$$\|u_\lambda(0)\|_{\dot{H}^\gamma}=\lambda^{\gamma+1-\frac{n}{2}}\|u(0)\|_{\dot{H}^\gamma},\quad \|w_\lambda(0)\|_{\dot{H}^\gamma}=\lambda^{\gamma+1-\frac{n}{2}}\|w(0)\|_{\dot{H}^\gamma},$$

it follows that it leaves the $\dot{H}^\gamma$ -norm of initial data invariant where

$$\gamma_c:=\frac{n}{2}-1.$$

According to the conservation laws of mass and energy, we say the system (1.1) is

$$L^2-\begin{cases}\text{subcritical,} & \text{if } n=1,\\ \text{critical,} & \text{if } n=2,\\ \text{supercritical,} & \text{if } n=3.\end{cases}$$

We first recall the following Gagliardo-Nirenberg(G-N)-type inequalities as in [21]: suppose $(u,w)\in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ and $1\le n\le 3$,

$$N(u,w)\le CK(u,w)^{\frac{n}{2}}M(u,w)^{(2-\frac{n}{2})}, \tag{1.10}$$

we also have the following identity:

$$E_\sigma\,|_{S_c}\ge CN(u,w)^{\frac{2}{n}}-N(u,w). \tag{1.11}$$

Now, we state our main theorems in this paper as follows. Our first goal is to show that some existence results of normalized solutions for the problem (1.7)-(1.8).

Theorem 1.1. (L^2-subcritical case) Suppose $n=1$ and $\mu=3\sigma$, then for any $c>0$, we have

$$m_c := \inf_{(P,Q)\in S_c} E_\sigma < 0,$$

and the infimum is achieved by $(P,Q)\in S_c$ with $\omega>0$. Hence, (ω,P,Q) is a normalized solution of the problem (1.7)-(1.8).

Theorem 1.2.(L^2-supercritical case) Suppose $n=3$ and $\mu=3\sigma$, then for any $c>0$, the problem (1.7)-(1.8) exists a normalized solution (ω,P,Q) for some $\omega>0$.

Remark 1.3. There have been extensive study and application of solutions in cubic media, but only limited to the existence of ground state solutions. In our paper, it is the first time to study the existence of normalized solutions, which is more meaningful from a physical perspective. When we study the existence of normalized solutions. As $n=1$, $E_\sigma\mid_{S_c}$ is bounded from below. As $n=3$, the difficulty we have that the functional is no longer lower bounded, finding a minimum E_σ on S_c is impossible. To overcome it, we make use of Pohozaev identities and the classical Strauss compactness lemma.

Our next results concerns the nonexistence of normalized solutions in L^2-critical case.

Theorem 1.4.(L^2-critical case) Suppose $n=2$ and $\mu=3\sigma$, if there exists a constant $c^* := \frac{1}{2C_{GN}}$, where C_{GN} is the best constant satisfying (1.10), E_σ has no critical point on the constraint S_c for each $c\in(0,c^*]$, i.e., the problem (1.7)-(1.8) does not have normalized solutions for each $c\in(0,c^*]$.

The trick of the proof is to find a threshold value c^* separating the existence and nonexistence of critical points. It is apparent that the threshold value associated with the best constant of (G-N) inequality.

From the mathematical point of view, some global well-posedness, scattering and blow-up results for system (1.1) have been studied in some papers.

When $n=1$, in [17], they established local and global well-posedness results for the associated initial value problem with periodic initial data. When $n=2$, Oliveira and Pastor [14] proved the the system (1.1) is globally well-posed, provided that the initial mass $M(u_0,w_0)$ is sufficiently small such that $M(u_0,w_0) < M(P,Q)$, where (P,Q) is ground state. Besides, they constructed an explicit solution that blows up forward in time.

When $n=3$, they introduced the following scattering conditions

$$E_\mu\left(u_0,w_0\right) M\left(u_0,w_0\right) < \frac{1}{2} E_\sigma(P,Q) M(P,Q), \tag{1.12}$$

$$K\left(u_0,w_0\right) M\left(u_0,w_0\right) < K(P,Q) M(P,Q); \tag{1.13}$$

and the blow-up conditions

$$E_\mu\left(u_0,w_0\right) M\left(u_0,w_0\right) < \frac{1}{2} E_\sigma(P,Q) M(P,Q), \tag{1.14}$$

$$K\left(u_0,w_0\right) M\left(u_0,w_0\right) > K(P,Q) M(P,Q), \tag{1.15}$$

where $(P,Q)\in\mathcal{G}(\omega,3\sigma,\sigma)$, $(u_0,w_0)\in H^1\left(\mathbb{R}^3\right)\times H^1\left(\mathbb{R}^3\right)$. By using (1.12)-(1.14), Ardila, Dinh and Forcella [1] get some results for the scattering and blow-up solutions in radial and non-radial cases. If the initial data satisfies the scattering conditions, the solutions are global and scatters; if it satisfies blow-up conditions, the corresponding solutions blows-up in finite time. In addition, the main results are about formation in the non radial case, suppose the initial data satisfies (1.12) and (1.13), $|\sigma-3|<\eta$ for some $\eta>0$ small enough, the solutions scatter; if (u_0,w_0) is cylindrical symmetry namely $(u_0,w_0)\in\Sigma_3\times\Sigma_3$, where

$$\Sigma_3:=\left\{f\in H^1\left(\mathbb{R}^3\right):f(y,z)=f(|y|,z),zf\in L^2\left(\mathbb{R}^3\right)\right\},$$

with $\left(x=(y,z),y=(x_1,x_2)\in\mathbb{R}^2\right)$ and $z\in\mathbb{R}$. Σ_3 stands for the space of cylindrical symmetric functions with finite variance in the last direction. Then the corresponding solutions to (1.1) blows-up in finite time.

Our second main results are about formation of singularities in finite time for solutions to the system (1.1). Alex et al [7] have demonstrated the blow-up results in radial and cylindrically symmetric. As stated in the previous section, when $n=1$, the Cauchy problem (1.1) is globally well-posed. It follows that we mainly study some blow up results for the normalized solutions exists when $n=2$ and $n=3$. To conclude, we obtain the main Theorems as follows.

Theorem 1.5. Let $\mu,\sigma>0$, $n=2$, suppose that (P,Q) is any normalized ground state of (1.7)-(1.8) with $\mu=3\sigma$, let $(u_0,w_0)\in H^1\left(\mathbb{R}^2\right)\times H^1\left(\mathbb{R}^2\right)$ be the radially symmetric satisfying $E_\mu\left(u_0,w_0\right)<0$ and $M(P,Q)<M(u_0,w_0)$, the corresponding solutions to (1.1) either blows-up forward in finite time, namely, $T^*<\infty$, or it blows-up in finite time in the sense that $T^*=\infty$ and

$$K(u,w)\geq Ct^2. \tag{1.16}$$

for all $t\geq t_0$, where $c>0$ and $t_0\gg 1$ depend only on $\sigma,M\left(u_0,w_0\right)$ and $E_\mu\left(u_0,w_0\right)$. A similar statement apply to negative times.

Theorem 1.6. Let $\mu,\sigma>0$, $n=3$, $(u_0,w_0)\in H^1\left(\mathbb{R}^3\right)\times H^1\left(\mathbb{R}^3\right)$ satisfy either $E_\mu\left(u_0,w_0\right)<0$ or $E_\mu\left(u_0,w_0\right)\geq 0$. We assume

$$K\left(u_0,w_0\right)M\left(u_0,w_0\right)>K(P,Q)M(P,Q),$$

and

$$E_\mu\left(u_0,w_0\right)M\left(u_0,w_0\right)<E_\sigma|_{S_c}(P,Q)M(P,Q), \tag{1.17}$$

where (P,Q) is any normalized ground state of (1.7)-(1.8) with $\mu=3\sigma$. If the initial data satisfy: either (u_0,w_0) is radially symmetric, or $(u_0,w_0)\in\Sigma_3\times\Sigma_3$, the corresponding solutions to (1.1) blows-up in finite time.

Remark 1.7. For the blow-up results, we mainly discuss the (u_0,w_0) is radial symmetry in $n=2$, and cylindrical symmetry in $n=3$. As $n=2$, only when the initial mass $M(u_0,w_0)$ is large enough to be greater than the mass under the normalized solution $M(P,Q)$, can we get the blow up results. As $n=3$, we build the new blow-up conditions with normalized solutions. The proof of it relies instead on an ODE argument, in the same spirit of previous work [7], using localized virial estimates and the negativity property of the Pohozaev functional.

The organizational structure of this paper is as follows. In Section 2 we state preliminary results that will be needed throughout the paper. In Section 3, we prove the existence of normalized solutions for (1.7)-(1.8) by giving the proof of Theorem 1.1 and Theorem 1.2. As $n=1$, by means of Concentration-compactness principle and the monotonicity of m_c, we obtain the existence of normalized solutions. As $n=3$, in addition to unboundness, we also found other difficulties is that (P,Q) is not radial symmetric and the embedding $H^1\left(\mathbb{R}^3\right)\hookrightarrow L^2\left(\mathbb{R}^3\right)$ is not compact. These difficulties will be overcome by using Pohozaev functional, rearrangement method and applying the classical Strauss compactness lemma. In Section 4, we proof the Theorem 1.4, as $n=2$, we find a threshold value c^* related to the sharp constant in (G-N) inequality. It is straightforward to show that if $c\in(0,c^*]$, $m_c=0$ and if $c>c^*$, $m_c=-\infty$, then E_μ has no critical point on the constraint S_c for each $c\in(0,c^*]$, so that we can conclude the nonexistence of normalized solutions for each $c\in(0,c^*]$. In Section 5, we eventually prove the blow-up results, by employing the virial estimates and the blow-up criterion with normalized solutions.

2 Preliminaries

Firstly, let us establish some Pohozaev-type identities for the solutions of (1.7), which will be useful later.
Lemma 2.1 Suppose $2 \le n \le 3$, $(P,Q) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ is a solution of the problem (1.7) with $\mu = 3\sigma$, then the following identities hold:

$$(\omega+1)M(P,Q) + (n-4)N(P,Q) = 0, \tag{2.1}$$

$$-K(P,Q) + nN(P,Q) = 0, \tag{2.2}$$

$$(n-4)K(P,Q) + n(\omega+1)M(P,Q) = 0. \tag{2.3}$$

Proof. Let $(P,Q) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ be a solution of the problem (1.7) with $\mu = 3\sigma$. First we multiply both sides of equation (1.7) by P, the second one by Q, integrate over $\mathbb{R}^n$ and make use of integration by parts, we obtain:

$$\int_{\mathbb{R}^n} \left(-|\nabla P|^2 - (\omega+1)P^2 + \frac{1}{9}P^4 + 2P^2Q^2 + \frac{1}{3}P^3Q \right) dx = 0, \tag{2.4}$$

and

$$\int_{\mathbb{R}^n} \left(-|\nabla Q|^2 - 3\sigma(\omega+1)Q^2 + 9Q^4 + 2P^2Q^2 + \frac{1}{9}P^3Q \right) dx = 0. \tag{2.5}$$

Summing (2.4) and (2.5), we get

$$K(P,Q) + (\omega+1)M(P,Q) - 4N(P,Q) = 0. \tag{2.6}$$

Next, similarly, multiplying the two equations by $x \cdot \nabla P$ and $x \cdot \nabla Q$ yields

$$\int_{\mathbb{R}^n} \left(\frac{(n-2)}{2}|\nabla P|^2 + \frac{n(\omega+1)}{2}P^2 - \frac{n}{36}P^4 + 2Q^2Px \cdot \nabla P + \frac{1}{3}P^2Qx \cdot \nabla P \right) dx = 0, \tag{2.7}$$

and

$$\int_{\mathbb{R}^n} \left(\frac{(n-2)}{2}|\nabla Q|^2 + \frac{3\sigma n(\omega+1)}{2}Q^2 - \frac{9n}{4}Q^4 + 2P^2Qx \cdot \nabla Q + \frac{1}{9}P^3x \cdot \nabla Q \right) dx = 0. \tag{2.8}$$

Then, by using integration by parts, combining (2.7) and (2.8), we deduce

$$\frac{(n-2)}{2}K(P,Q) + \frac{n(\omega+1)}{2}M(P,Q) - nN(P,Q) = 0. \tag{2.9}$$

Finally, (2.1)-(2.3) are consequences of (2.6) and (2.9). The proof of the lemma is thus completed. □

By Lemma 2.2, it is worth introducing the functional $G : H^1(\mathbb{R}^n) \to \mathbb{R}^n$ which is a special form of Pohozeav functional defined by

$$G(u,w) := K(u,w) - nN(u,w). \tag{2.10}$$

The next Lemma is devoted to the proof of virial estimates, which will be crucial for the proof of the Theorem 1.5 and Theorem 1.6.

Lemma 2.2. Let $\mu, \sigma > 0$, $2 \le n \le 3$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a sufficiently smooth and decaying function. Let (u,w) be a $H^1(\mathbb{R}^n)$ solution to (1.1) defined on the maximal time interval $(-T_*, T^*)$. We define:

$$\mathcal{M}_\varphi(t) := 2\,\mathbf{Im}\int_{\mathbb{R}^n} \nabla\varphi(x)\nabla u\bar{u} + \sigma\nabla w\bar{w}(t,x)dx. \tag{2.11}$$

Then we have for all $t \in (-T_*, T^*)$,

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{M}_\varphi(t)}{\mathrm{d}t} = & -\int_{\mathbb{R}^n} \Delta^2\varphi(|u^2| + |w^2|)dx \\
& + 4\,\mathbf{Re}\int_{\mathbb{R}^n} \sum_{1\le k,j\le n} \partial_{kj}^2\varphi(x)\left(\frac{\partial u}{\partial x_k}\frac{\partial \bar{u}}{\partial x_j} + \frac{\partial w}{\partial x_k}\cdot\frac{\partial w}{\partial x_j}\right)dx \\
& - 4\,\mathbf{Re}\int_{\mathbb{R}^n} \Delta\varphi T(u,w)dx.
\end{aligned} \tag{2.12}
$$

where $N(u,w) = \int_{\mathbb{R}^n} T(u,w)dx$.

Proof. By definition (2.11), we can rewrite it as

$$
\begin{aligned}
\mathcal{M}_\varphi(t) &= 2\,\mathbf{Im}\langle \Delta u(t), \varphi\bar{u}(t)\rangle + 2\sigma\,\mathbf{Im}\langle\Delta w(t), \varphi\bar{w}(t)\rangle \\
&:= \mathcal{M}_\varphi^1(t) + \mathcal{M}_\varphi^2(t).
\end{aligned}
$$

For simplicity, we first consider the term on u, and the term on w can be obtained by the same principle. Through simple calculation, for $h \in \mathbb{R}, h \neq 0$ we have

$$
\mathcal{M}_\varphi^1(t+h)) - \mathcal{M}_\varphi^1(t) = -2\,\mathbf{Im}\langle\Delta u(t+h), \varphi[\bar{u}(t+h) - \bar{u}(t)] + \mathbf{Im}\langle\Delta(u(t+h) - u(t)), \varphi\bar{u}(t)\rangle, \tag{2.13}
$$

equivalently,

$$
\begin{aligned}
\mathcal{M}_\varphi^1(t+h) - \mathcal{M}_\varphi^1(t) = & -2\,\mathbf{Im}\langle\Delta(u(t+h) - u(t)), \varphi[\bar{u}(t+h) - \bar{u}(t)]\rangle \\
& -2\,\mathbf{Im}\int_{\mathbb{R}^n} \nabla\varphi\cdot\nabla\bar{u}(t)[u(t+h) - u(t)]dx \\
& -2\,\mathbf{Im}\int_{\mathbb{R}^n} \Delta\varphi\cdot\bar{u}(t)[u(t+h) - u(t)]dx \\
= & 2\,\mathbf{Im}\int_{\mathbb{R}^n} [\bar{u}(t+h) - \bar{u}(t)]\nabla\varphi\cdot\nabla(u(t+h) - u(t))dx \\
& -2\,\mathbf{Im}\int_{\mathbb{R}^n} \nabla\varphi\cdot\nabla\bar{u}(t)[u(t+h) - u(t)]dx \\
& -2\,\mathbf{Im}\int_{\mathbb{R}^n} \Delta\varphi\cdot\bar{u}(t)[u(t+h) - u(t)]dx.
\end{aligned} \tag{2.14}
$$

According to the definition of derivative $\frac{\mathrm{d}\mathcal{M}_\varphi(t)}{\mathrm{d}t} = \lim\limits_{h\to 0}\frac{M\varphi(t+h) - M\varphi(t)}{h}$, we get

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{M}_\varphi^1(t)}{\mathrm{d}t} &= -2(2\,\mathbf{Im}\int_{\mathbb{R}^n} \nabla\varphi\cdot\nabla\bar{u}(t)\cdot u_t(t)dx + \mathbf{Im}\int_{\mathbb{R}^n} \Delta\varphi\cdot\bar{u}(t)u_t(t)dx) \\
&:= -2(2A_1 + A_2).
\end{aligned} \tag{2.15}
$$

By partial integral and fact $\mathbf{Re}(\bar{u}\nabla u) = \frac{1}{2}\nabla\left(|u|^2\right)$,

$$
\begin{aligned}
A_2 = & -\mathbf{Re}\int_{\mathbb{R}^n} \Delta\varphi|\nabla u|^2dx + \int_{\mathbb{R}^n} \frac{1}{2}|u|^2\Delta^2\varphi dx - \mathbf{Re}\int_{\mathbb{R}^n} \Delta\varphi\cdot|u|^2dx \\
& + \mathbf{Re}\int_{\mathbb{R}^n} \Delta\varphi\left(\frac{1}{9}|u|^4 + 2|w|^2|u|^2\right)dx + \frac{1}{3}\,\mathbf{Re}\int_{\mathbb{R}^n} \Delta\varphi\bar{u}^3w.
\end{aligned} \tag{2.16}
$$

For A_1, we have integration by parts:

$$
\begin{aligned}
A_1 = & \mathbf{Re}\int_{\mathbb{R}^n} \nabla\varphi\nabla\bar{u}\Delta u dx + \frac{1}{2}\,\mathbf{Re}\int_{\mathbb{R}^n} \Delta\varphi|u|^2dx \\
& + \mathbf{Re}\int_{\mathbb{R}^n} \nabla\varphi\nabla\bar{u}\left(\frac{1}{9}|u|^2 + |2w|^2\right)udx + \mathbf{Re}\int_{\mathbb{R}^n} \nabla\varphi\nabla\bar{u}\cdot\frac{1}{3}\bar{u}^2wdx.
\end{aligned} \tag{2.17}
$$

On the other hand,

$$\mathbf{Re}\int_{\mathbb{R}^n}(\nabla\varphi,\nabla\bar{u})\Delta u dx = B_1 + B_2, \tag{2.18}$$

where

$$B_1 := -\mathbf{Re}\sum_{1\le k\le n}\int_{\mathbb{R}^n}\partial_k\varphi\left(\nabla\bar{u}\cdot\partial_k\nabla u\right)dx, \tag{2.19}$$

$$B_2 := -\mathbf{Re}\sum_{1\le k\le n}\int_{\mathbb{R}^n}\partial_k u\left(\nabla\bar{u}\cdot\partial_k\nabla\varphi\right)dx. \tag{2.20}$$

Let

$$(H(\varphi)\xi \mid \xi) := \sum_{1\le k,j\le n}\partial^2_{kj}\varphi(x)\xi_j\bar{\xi_k},$$

then (2.20) can be written as

$$B_2 = -\int_{\mathbb{R}^n}(H(\varphi)\nabla u \mid \nabla u)dx. \tag{2.21}$$

Remark the fact $\mathbf{Re}(\nabla\bar{u}\cdot\partial_k\nabla u) = \frac{1}{2}\partial_k|\nabla u|^2$, hence

$$B_1 = \frac{1}{2}\int_{\mathbb{R}^n}\Delta\varphi|\nabla u|^2 dx. \tag{2.22}$$

Combining (2.15) to (2.22),

$$\begin{aligned}\frac{\mathrm{d}\mathcal{M}^1_\varphi(t)}{\mathrm{d}t} =& -2\Bigg[-2\,\mathbf{Re}\int_{\mathbb{R}^n}\sum_{k\le j\le n}\partial^2_{kj}\varphi(x)\frac{\partial u}{\partial x_k}\frac{\partial\bar{u}}{\partial x_j}dx + \frac{1}{2}\int_{\mathbb{R}^n}\Delta^2\varphi|u|^2dx + 2\,\mathbf{Re}\int_{\mathbb{R}^n}\nabla\varphi\nabla\bar{u}\left(\frac{1}{9}|u|^2+2|w|^2\right)u\,dx\\ &+2\,\mathbf{Re}\int_{\mathbb{R}^n}\nabla\varphi\nabla\bar{u}\cdot\frac{1}{3}\bar{u}^2wdx + \mathbf{Re}\int_{\mathbb{R}^n}\Delta\varphi\left(\frac{1}{9}|u|^4+2|u|^2|\omega|^2\right)dx - \frac{1}{3}\,\mathbf{Re}\int_{\mathbb{R}^n}\Delta\varphi\bar{u}^2wdx\Bigg].\end{aligned} \tag{2.23}$$

For the term about w, a routine computation gives rise to

$$\begin{aligned}\frac{\mathrm{d}\mathcal{M}^2_\varphi(t)}{\mathrm{d}t} =& 2\Bigg[2\,\mathbf{Re}\int_{\mathbb{R}^n}(H\varphi\nabla w \mid \nabla w)dx - \frac{1}{2}\,\mathbf{Re}\int_{\mathbb{R}^n}|w|^2\Delta^2\varphi dx - 2\,\mathbf{Re}\int_{\mathbb{R}^n}\nabla\varphi\nabla\bar{w}\left(9|u|^2+2|w|^2\right)w\,dx\\ &-2\,\mathbf{Re}\int_{\mathbb{R}^n}\nabla\varphi\nabla\bar{\omega}\cdot\frac{1}{9}u^3dx - \mathbf{Re}\int_{\mathbb{R}^n}\Delta\varphi\left(9|w|^4+2|u|^2|w|^2\right)dx - \mathbf{Re}\int_{\mathbb{R}^n}\Delta\varphi\cdot\frac{1}{9}u^3\bar{w}dx\Bigg].\end{aligned} \tag{2.24}$$

Thanks to the (2.23) and (2.24), (2.12) is proved.

□

Remark 2.3. From the above, we have the following conclusions:

(1) If φ is radially symmetric, $2\le n\le 3$, using the define $|x| = r$ we have

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_\varphi(t) =& -\int_{\mathbb{R}^n}\Delta^2\varphi(x)\left(|u|^2+|w|^2\right)dx + 4\int_{\mathbb{R}^n}\frac{\varphi'(r)}{r}\left(|\nabla u|^2+|\nabla w|^2\right)dx\\ &+4\int_{\mathbb{R}^n}\left(\frac{\varphi''(r)}{r^2}-\frac{\varphi'(r)}{r^3}\right)\left(|x\cdot\nabla u|^2+|x\cdot\nabla w|^2\right)dx\\ &-4\int_{\mathbb{R}^n}\Delta\varphi(x)T(u,w)dx.\end{aligned} \tag{2.25}$$

(2)If both $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and $(u,w) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ are radial, we get

$$\begin{aligned} \frac{d}{dt}\mathcal{M}_\varphi(t) = &- \int_{\mathbb{R}^2} \Delta^2\varphi(x)\left(|u|^2+|w|^2\right)dx + 4\int_{\mathbb{R}^2} \varphi''(r)\left(|\nabla u|^2+|\nabla w|^2\right)dx \\ &- 4\int_{\mathbb{R}^2} \Delta\varphi(x)T(u,w)dx. \end{aligned} \tag{2.26}$$

(3) Denote $x = (y,z)$ with $y = (x_1,x_2) \in \mathbb{R}^2$ and $z \in \mathbb{R}$. Let ψ be a sufficiently smooth and decaying function and $\varphi(x) = \psi(y) + z^2$. If $(u,w) \in \Sigma_3 \times \Sigma_3$ for all $t \in (-T_-, -T_+)$, where

$$\Sigma_3 := \left\{ f \in H^1\left(\mathbb{R}^3\right) : f(y,z) = f(|y|,z), zf \in L^2\left(\mathbb{R}^3\right) \right\},$$

then we have

$$\begin{aligned} \frac{d}{dt}\mathcal{M}_\varphi(t) = &- \int_{\mathbb{R}^3} \Delta_y^2\psi(y)\left(|u|^2+|v|^2\right)(t,x)dx + 4\int_{\mathbb{R}^3} \psi''(\rho)\left(|\nabla_y u|^2+|\nabla_y v|^2\right)(t,x)dx \\ &+ 8\left(\|\partial_z u(t)\|^2_{L^2(\mathbb{R}^3)} + \|\partial_z v(t)\|^2_{L^2(\mathbb{R}^3)}\right) - 8N(u(t),v(t)) \\ &- 4\int_{\mathbb{R}^3} \Delta_y\psi(y)T(u,v)(t,x)dx. \end{aligned} \tag{2.27}$$

Proof. (1)If φ is radially symmetric and $|x| = r$, then using the fact that

$$\partial_j = \frac{x_j}{r}\partial_r, \quad \partial^2_{jk} = \left(\frac{\delta_{jk}}{r} - \frac{x_jx_k}{r^3}\right)\partial_r + \frac{x_jx_k}{r^2}\partial_r^2,$$

where $\delta_{jk} = \begin{cases} 1, \text{if} & j = k, \\ 0, \text{if} & j \neq k, \end{cases}$ we have

$$\frac{\partial^2\varphi}{\partial x_{jk}} = \frac{\partial^2\varphi}{\partial r^2}\cdot\frac{x_kx_j}{r^2} + \frac{\partial\varphi}{\partial r}\left(\frac{\delta_{jk}}{r} - \frac{x_jx_k}{r^3}\right).$$

Combined with the above formulas, we have

$$\begin{aligned} &\mathbf{Re}\int_{\mathbb{R}^n} \sum_{1\le k,j\le n} \partial^2_{k_j}\varphi(x)\left(\partial_k u\partial_j\bar{u} + \partial_k w\partial_j\bar{w}\right)dx \\ =&\mathbf{Re}\int_{\mathbb{R}^n} \sum_{1\le k,j\le n} \left[\varphi''(r)\cdot\frac{x_kx_j}{r^2} + \varphi'(r)\left(\frac{\delta_{jk}}{r} - \frac{x_jx_k}{r^3}\right)\right](\partial_k u\partial_j\bar{u} + \partial_k w\partial_j\bar{w})dx \\ =&\mathbf{Re}\int_{\mathbb{R}^n} \sum_{1\le k,j\le n} \left[\frac{\varphi'(r)}{r}\delta_{jk}\bar{u} + \left(\frac{\varphi''(r)}{r^2} - \frac{\varphi'(r)}{r^3}\right)x_jx_k\right](\partial_k u\partial_j\bar{u} + \partial_k w\partial_j\bar{w})dx \\ =&\int_{\mathbb{R}^n} \frac{\varphi'(r)}{r}(|\nabla u|^2 + |\nabla w|^2) + \left(\frac{\varphi''(r)}{r^2} - \frac{\varphi'(r)}{r^3}\right)(|x\cdot\nabla u|^2 + |x\cdot\nabla w|^2)dx. \end{aligned}$$

(2)Because φ is radial, we can directly use (2.25), and because (u,w) is radial, there is $|x|^2 = r^2$. (2.26) can be obtained by a simple calculation.
(3)From the choice of the function $\varphi(x) = \psi(y) + z^2$, we have the (2.27).

□

3 Existence of normalized solutions

In this section, we prove the main Theorems 1.1-1.2. More precisely, we will establish the following results:

3.1 L^2-subcritical growth case

If $n = 1$, by (1.11), we have

$$E_\sigma \left|_{S_c} \geq CN(u,w)^2 - N(u,w),\right.$$

and obtain $E_\sigma \left|_{S_c}\right.$ is bounded below, we call it is L^2-subcritical.

Lemma 3.1.1. Assume $n = 1$, the functional $E_\sigma \left|_{S_c}\right.$ is coercive and bounded from the below, and

$$-\infty < m_c := \inf_{(u,w)\in S_c} E_\sigma < 0. \tag{3.1.1}$$

Proof. Because of (1.11), if $n = 1$

$$E_\sigma \left|_{S_c} \geq CN(u,w)^2 - N(u,w),\right.$$

we can get the $E_\sigma \left|_{S_c}\right.$ is coercive, and then $m_c > -\infty$, it has a lower bound, that is, a minimum.

On the other hand, to prove the lemma, we introduce a map as in [8]. Define the map: $s * (u,w) := e^{\frac{s}{2}}(u(t, e^s x), w(t,(e^s x))$, if $(u,w) \in S_c$ yields

$$M(s*(u,w)) = \int_{\mathbb{R}} e^s(|u(t,e^sx)|^2 + 3\sigma\, |w(t,e^sx)|^2)dx = \int_{\mathbb{R}} |u(t,x)|^2 + 3\sigma\, |w(t,x)|^2\, dx = c,$$

namely $s * (u,w) \in S_c$,

$$\begin{aligned}
E_\sigma |_{S_c}\, (u,w) =& \frac{1}{2}\int_{\mathbb{R}} e^s \left(e^{2s}\, |\nabla u\,(t,e^sx)|^2 + e^{2s}\, |\nabla w\,(t,e^sx)|^2\right) dx \\
&- \int_{\mathbb{R}} \left(\frac{1}{36}e^{2s}|u|^4 + \frac{9}{4}e^{2s}|w|^u + e^{2s}|u|^2|w|^2 + \frac{1}{9}e^{2s}|u|^3|w|\right) dx \\
=& \frac{1}{2}\int_{\mathbb{R}} \left(e^{2s}|\nabla u|^2 + e^{2s}|\nabla w|^2\right) dx \\
&- \int_{\mathbb{R}} e^s \left(\frac{1}{36}|u|^4 + \frac{9}{4}|w|^u + |u|^2|w|^2 + \frac{1}{9}|u|^3|w|\right) dx,
\end{aligned}$$

when $s \to -\infty$, $E_\sigma |_{S_c}\, (u,w) \to 0^-$. It results that $m_c < 0$ for any $c > 0$. ☐

Lemma 3.1.2. Assume $n = 1$ and $c_1 > 0$, $c_2 > 0$ satisfy $c_1 + c_2 = c$, we have

$$m_c < m_{c_1} + m_{c_2}, \tag{3.1.2}$$

where

$$m_c := \inf_{(u,w)\in S_c} E_\sigma.$$

Proof. First, we prove that the inequality $m_{\tau^2 c} < \tau^2 m_c$, for all $c > 0, \tau > 1$ holds. Take the minimizing sequence $(u_j, w_j) \in S_c$, such that

$$E_\sigma(u_j, w_j) \to \inf_{(u,w)\in S_c} E_\sigma = m_c, \quad E_\sigma(\tau u_j, \tau w_j) \to m_{\tau^2 c}.$$

We get

$$\begin{aligned}
m_{\tau^2 c} &\leq E_\sigma |_{S_c}\, (\tau u_j, \tau w_j) \\
&= \frac{1}{2}\int_{\mathbb{R}} \tau^2 \left(|\nabla\,(u_j)|^2 + |\nabla\,(\tau w_j)|^2\right) dx \\
&- \int_{\mathbb{R}} \tau^4 \left(\frac{1}{36}\,|u_j|^4 + \frac{9}{4}\,|w_j|^4 + |u_j|^2\,|w_j|^2 + \frac{1}{9}\,|u_j|^3\,|w_j|\right) dx \leq \tau^2 m_c.
\end{aligned}$$

Next, we will prove that the equal does not hold. In fact, we find that $m_{\tau^2 c} = \tau^2 m_c$ is equivalent to $N(u_j, w_j) \to 0$ as $j \to \infty$. If the statement holds, we have

$$0 > m_c = \lim_{j\to\infty} E_\sigma\left|_{S_c}\right. (u_j, w_j) \geq \liminf_{j\to\infty} \frac{1}{2} K(u_j, w_j) \geq 0,$$

this leads to a contradiction. Then we should show $m_c < m_{c_1} + m_{c_2}$. In fact, without losing generality, it may be assumed that $c_1 \geq c_2$. If $c_1 > c_2$, we get

$$m_c = m_{\frac{c}{c_1} c_1} < \frac{c}{c_1} m_{c_1} = m_{c_1} + \frac{(c - c_1)}{c_1} m_{c_1} = m_{c_1} + \frac{c_2}{c_1} m_{\frac{c_1}{c_2} c_2} < m_{c_1} + m_{c_2},$$

and if $c_1 = c_2$,

$$m_c = m_{2c_1} < 2m_{c_1} = m_{c_1} + m_{c_2}.$$

□

Proof of Theorem 1.1: Since the functional E_σ has a lower bound on S_c and is coercive, the minimizing sequence $\{(u_j, w_j)\} \in S_c$ is taken. It can be seen from lemma 3.1.1 that it is bounded. By applying the Concentration-compactness principle, we can get that there is one of three cases where the subsequence (still called (u_j, w_j)) satisfies vanishing, dichotomy or compactness.

To begin with, we claim that the vanishing case cannot occur. In fact, according to [12], if $(u_j w_j) \to 0$, $\liminf_{j\to\infty} E_\sigma (u_j w_j) \geq 0$ can be obtained, in contradiction with Lemma 3.1.1. Secondly, we claim that the dichotomy case cannot occur. Suppose there are two bounded sequences $\{(u_{n_1}, w_{n_1})\}, \{(u_{n_2}, w_{n_2})\}$ satisfying dichotomy, then

$$m_c = \lim_{n\to\infty} E_\sigma|_{S_c} (u_j, w_j) \geq \limsup_{n\to\infty} ((u_{n_1}, w_{n_1}) + E_\sigma|_{S_c} (u_{n_2}, w_{n_2})) \geq m_{c_1} + m_{c_2}$$

which is inconsistent with Lemma 3.1.2.

Hence we have the compactness case hold, and the statements in Theorem 1.1 are proved.

3.2 L^2-supercritical growth case

In order to get Theorem 1.2, from (1.11), we have

$$E_\sigma \left|_{S_c}\right. \geq CN(u, w)^{\frac{2}{3}} - N(u, w).$$

Obviously, the energy functional has no lower bound, to overcome the difficulty, we will study the minimization problem in the Pohozaev manifold. By Lemma 2.2, the Pohozaev functional is defined by

$$G(u, w) := K(u, w) - 3N(u, w), \tag{3.2.1}$$

and the corresponding set

$$\mathcal{P} = \left\{(P, Q) \in H^1\left(\mathbb{R}^3\right) \times H^1\left(\mathbb{R}^3\right) \mid G(P, Q) = 0\right\}.$$

be the Pohozaev manifold. We also define

$$V(c) := S_c \cap \mathcal{P}, \tag{3.2.2}$$

and

$$\mathcal{K}(c) := \left\{(P, Q) \in S_c, \text{ s.t. } E'_\sigma(P, Q)|_{S_c} = 0\right\}. \tag{3.2.3}$$

Lemma 3.2.1. Let $n = 3$, $(P, Q) \in H^1\left(\mathbb{R}^3\right) \times H^1\left(\mathbb{R}^3\right)$ be such that $G(P, Q) = 0$, then

$$\inf_{(P,Q)\in\mathcal{K}(c)} E_\sigma = \inf_{(P,Q)\in V_c} E_\sigma.$$

Proof. According to the (3.2.1) and $n=3$, we get

$$G(P,Q)=\int_{\mathbb{R}^3}(|\nabla P|^2+|\nabla Q|^2)dx-3\int_{\mathbb{R}^3}(\frac{1}{36}|P|^4+\frac{9}{4}|Q|^4+|P|^2|Q|^2+\frac{1}{9}P^3Q)dx.$$

Furthermore,

$$\begin{aligned}\langle G'(P,Q),(P,Q)\rangle&=2\int_{\mathbb{R}^3}(|\nabla P|^2+|\nabla Q|^2)dx-\int_{\mathbb{R}^3}(\frac{1}{3}|P|^4+27|Q|^4+12|P|^2|Q|^2+\frac{4}{3}P^3Q)dx\\&=2K(P,Q)-12N(P,Q),\end{aligned}$$

and, if $(P,Q)\in V(c)$,

$$\langle G'(P,Q),(P,Q)\rangle=-2K(P,Q)\neq 0, \tag{3.2.4}$$

shows that $\mathcal{P}$ is locally smooth. Besides, $E_\sigma\big|_{V(c)}=\frac{1}{2}K(P,Q)-N(P,Q)=\frac{1}{6}K(P,Q)\geq 0$, which means that $(0,0)$ is a strict minimizer. Indeed, any critical point of E_σ constrained to $V(c)$ is a critical point of E_μ. Assume $(P_0,Q_0)\in V(c)$ is a critical point of E_μ constrained to $V(c)$. There exists a Lagrange multiplier λ such that $E'_\sigma(P_0,Q_0)=\lambda G'(P_0,Q_0)$. By taking L^2-inner product on both sides with (P_0,Q_0), we get

$$\langle E'_\sigma(P_0,Q_0),(P_0,Q_0)\rangle_{L^2}=\lambda\langle G'(P_0,Q_0),(P_0,Q_0)\rangle_{L^2},$$

in view of (3.2.4), $\lambda=0$ and $E'_\sigma(P_0,Q_0)=0$, which confirms the claim. □

Through Lemma 3.2.1, in order to prove Theorem 1.2, we will prove the existence of the minimum value of the problem (3.2.3).

Proof of Theorem 1.2: For each $(P,Q)\in V_c$, $(P,Q)\neq(0,0)$, we obtain $G(P,Q)\leq 0$. There exists $k\in[0,1]$ such that $(kP,kQ)\in V(c)$. Namely, if $G(P,Q)=0$, let $k=1$. If $G(P,Q)<0$,

$$\begin{aligned}G(kP,kQ)&=k^2\left\{\int_{\mathbb{R}^3}(|\nabla P|^2+|\nabla Q|^2)dx-k^2\int_{\mathbb{R}^3}(\frac{1}{12}|P|^4+\frac{27}{4}|Q|^4+3|P|^2|Q|^2+\frac{1}{3}P^3Q)\,dx\right\}\\&:=k^2F_{P,Q}(k),\end{aligned}$$

and $F_{P,Q}(0)=K(P,Q)>0$, $F_{P,Q}(1)=G(P,Q)<0$. The intermediate value Theorem leads us to the conclusion: there exists some $k\in[0,1]$, such that $F_{P,Q}(k)=0$, in other words $G(kP,kQ)=0$. Next, we will prove the strong convergence of the minimizing sequence , then we can get the existence of the minimum. Take a minimizing sequence $(P_j,Q_j)\in V(c)$ for

$$m:=\inf\{E_\sigma(P,Q):(P,Q)\in V(c)\},$$

such that

$$\liminf_{j\to\infty}E_\sigma\big|_{V(c)}(P_j,Q_j)=m.$$

Since $(P_j,Q_j)\in V(c)$,

$$E_\sigma\big|_{V(c)}(P_j,Q_j)=\frac{1}{6}K(P_j,Q_j).$$

It's not hard to get $m\geq 0$ and (P_j,Q_j) is bounded. In particular, $(P_j,Q_j)\rightharpoonup(P,Q)$ in $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$.

In proving compactness, the embedding $H^1(\mathbb{R}^3)\hookrightarrow L^2(\mathbb{R}^3)$ is not compact, we need to use Strauss Lemma. Replace (P_j,Q_j) with (P_j^*,Q_j^*), where P_j^* and Q_j^* are symmetric decreasing rearrangements of P_j and Q_j, respectively. On the other hand, the Pólya-Szegö inequality and the convex inequality of gradient $\|\nabla f^*\|_{L^2(\mathbb{R}^3)}\leq\|\nabla|f|\|_{L^2(\mathbb{R}^3)}\leq\|\nabla f\|_{L^2(\mathbb{R}^3)}$(see [12]) show that

$$E_\sigma\big|_{V(c)}\left(P_j^*,Q_j^*\right)\leq E_\sigma\big|_{V(c)}(P_j,Q_j).$$

Furthermore, combining the Hardy-Littlewood inequality

$$\int_{\mathbb{R}^3} |PQ| dx \leq \int_{\mathbb{R}^3} P^* Q^* dx,$$

with (see [9] for details)

$$\int_{\mathbb{R}^3} P^2 Q^2 dx \leq \int_{\mathbb{R}^3} (P^*)^2 (Q^*)^2 dx \text{ and } \int_{\mathbb{R}^3} \left|P^3 Q\right| dx \leq \int_{\mathbb{R}^3} (P^*)^3 Q^* dx,$$

we obtain

$$\begin{aligned}
G\left(P_j^*, Q_j^*\right) &= \int_{\mathbb{R}^3} (\left|\nabla P_j^*\right|^2 + \left|\nabla Q_j^*\right|^2) dx - \int_{\mathbb{R}^3} \left(\frac{1}{12}\left|P_j^*\right|^4 + \frac{27}{4}\left|Q_j^*\right|^4 + 3\left|P_j^*\right|^2\left|Q_j^*\right|^2 + \frac{1}{3}(P_j^*)^3 Q_j^*\right) dx \\
&\leq \int_{\mathbb{R}^3} (|\nabla |P_j||^2 + |\nabla |Q_j||^2) dx - \int_{\mathbb{R}^3} \left(\frac{1}{12}|P_j|^4 + \frac{27}{4}|Q_j|^4 + +3\left|P_j\right|^2\left|Q_j\right|^2 + \frac{1}{3}(P_j)^3 Q_j\right) dx \\
&\leq \int_{\mathbb{R}^3} (|\nabla P_j|^2 + |\nabla Q_j|^2) dx - \int_{\mathbb{R}^3} \left(\frac{1}{12}\left|P_j\right|^4 + \frac{27}{4}\left|Q_j\right|^4 + 3\left|P_j\right|^2\left|Q_j\right|^2 + \frac{1}{3}(P_j)^3 Q_j\right) dx \\
&= G(P_j, Q_j) = 0.
\end{aligned}$$

Let $k_j \in (0,1]$, such that $G(k_j P_j^*, k_j Q_j^*) = 0$, namely, $(k_j P_j^*, k_j Q_j^*) \in V(c)$. We have

$$E_\sigma \left|_{V(c)} \left(k_j P_j^*, k_j Q_j^*\right) = k_j^2 E_\sigma \right|_{V(c)} \left(P_j^*, Q_j^*\right) \leq E_\sigma \left|_{V(c)} \left(P_j, Q_j\right).\right.$$

Since we obtain the minimizing sequence $\left(k_j P_j^*, k_j Q_j^*\right)$ of the radial decreasing functions, denoted by $(\bar{P}_j, \bar{Q}j)$. Because the sequence is bounded, there must be weakly convergent subsequences (P_*, Q_*) , namely $(\bar{P}_j, \bar{Q}_j) \rightharpoonup (P_*, Q_*)$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. We are done if we show that the convergence is strong.

In order to prove strong convergence, applying the Strauss's compactness lemma([20]), if $u \in L^2\left(\mathbb{R}^3\right)$, is radially decreasing, it can establish that

$$|u(x)| \leq C|x|^{-\frac{3}{2}} \|u\|_{L^2(\mathbb{R}^3)},$$

which can get the compactness of the injection $H_{rd}^1\left(\mathbb{R}^3\right) \hookrightarrow L^4\left(\mathbb{R}^3\right)$, where

$$H_{rd}^1\left(\mathbb{R}^3\right) = \left\{u \in H_d^1\left(\mathbb{R}^3\right) : u \text{is radially decreasing}\right\}$$

(see [4] for more details). Consequently, up to a subsequence, $(\bar{P}_j, \bar{Q}_j) \to (P_*, Q_*)$ strongly in $L^4\left(\mathbb{R}^3\right)$. It also shows (P_*, Q_*) is radially symmetric and nonnegative.

Next, since

$$\int_{\mathbb{R}^3} (\frac{1}{12}\left|\bar{P}_j\right|^4 + \frac{27}{4}\left|\bar{Q}_j\right|^4 + 3\left|\bar{P}_j\right|^2\left|\bar{Q}_j\right|^2 + \frac{1}{3}(\bar{P}_j)^3 \bar{Q}_j) dx \to \int_{\mathbb{R}^3} (\frac{1}{12}\left|P_*\right|^4 + \frac{27}{4}\left|Q_*\right|^4 + 3|P_*|^2\left|Q_*\right|^2 + \frac{1}{3}(P_*)^3 Q_*) dx,$$

and $G\left(\bar{P}_j, \bar{Q}_j\right) = 0$, $j \to \infty$. We deduce that

$$G\left(P_*, Q_*\right) \leq \liminf_{j\to\infty} G\left(\bar{P}_j, \bar{Q}_j\right) = 0.$$

Similarly, there exists $\bar{k} \in (0,1]$, such that $G(\bar{k}P_*, \bar{k}Q_*) = 0$, namely, $(\bar{k}P_*, \bar{k}Q_*) \in V(c)$. Thus we get

$$m \leqslant E_\sigma \left|_{V(c)} \left(\bar{k}P_*, \bar{k}Q_*\right) = \bar{k}^2 E_\mu \right|_{V(c)} \left(P_*, Q_*\right) \leq \liminf_{j\to\infty} E_\sigma \left|_{V(c)} \left(\bar{P}_j, \bar{Q}_j\right) = m.\right.$$

It is easy to conclude that $\left(\bar{k}P_*, \bar{k}Q_*\right)$ is a minimum and $\bar{k} = 1$, so we can get that $(P_*, Q_*) \in V(c)$ and $E_\sigma \left|_{V(c)} (P_*, Q_*) = m\right.$, Theorem 1.2 is proved.

4 Nonexistence of normalized solutions

In this section, we will show the nonexistence of normalized solutions in L^2-critical growth case. Suppose $n=2$, by confirming in (1.11), we have

$$E_\sigma\,|_{S_c} \geq CN(u,w) - N(u,w),$$

and obtain $E_\sigma\,|_{S_c}$ is unboundness, we call it L^2-critical. In this case, we cannot conclude that for all $c>0$, the functional restricted to S_c has a lower bound.

While we can find a threshold value c^* separating the existence and nonexistence of critical points, if $0<c\leq c^*$ then E_σ has no critical point constrained on S_c. It s apparent that the threshold is closely related to the best constant of (1.10). To obtain it, we define

$$J(u,w) := \frac{K(u,w)M(u,w)}{N(u,w)}. \tag{4.1}$$

In order to obtain the specific form of the best constant, we introduce the following lemma.

Lemma 4.1 Assume $n=2$, let (P,Q) be a ground state solution of (1.7) with $\mu=3\sigma$, $\omega>0$. Then we have

$$S(P,Q) = N(P,Q), \tag{4.2}$$

$$K(P,Q) = 2S(P,Q), \tag{4.3}$$

$$K(P,Q) = (\omega+1)M(P,Q), \tag{4.4}$$

In particular, we get

$$J(P,Q) = (\frac{4}{\omega+1})S(P,Q). \tag{4.5}$$

Proof. By summing (2.4) and (2.5) we deduce that

$$K(P,Q) + (\omega+1)M(P,Q) = 4N(P,Q).$$

Thus, by the definition of S, we obtain

$$S(P,Q) = \frac{1}{2}K(P,Q) + \frac{(\omega+1)}{2}M(P,Q) - N(P,Q) = N(P,Q).$$

Also, from (2.2) and (2.3) ,one obtains (4.3) and (4.4). In particular, (4.5) is a consequence of (4.3)-(4.5). The proof of the lemma is thus completed. □

It was shown that any ground state solution (P,Q) with $\mu=3\sigma$ optimizes the (G-N) inequality (1.10), that is ,

$$C_{GN} = \frac{(\omega+1)^{1-n/2}}{n^{n/2}(4-n)^{1-n/2}M(P,Q)}. \tag{4.6}$$

when $n=2$, $C_{GN}=\frac{1}{2M(P,Q)}$, we refer to [14] for details on these results.

Lemma 4.2 Let $n=2$, there exists $c^* := \frac{1}{2C_{GN}} = M(P,Q)$, such that
(1) for each $c\in(0,c^*]$, $m_c := \inf_{(u,w)\in S_c} E_\sigma = 0$,
(2) for any $c>c^*$, $m_c\to-\infty$,

Proof. (1) Set $(u_\theta,w_\theta) := \theta(u,w)$, $\theta>0$, then

$$\int_{\mathbb{R}^2} |u_\theta|^2 + 3\sigma\,|w_\theta|^2\,dx = \theta^2 c,$$

namely, for any $(u,w)\in S_1$, $(u_\theta,w_\theta)\in S_{\theta^2}$, we have

$$\begin{aligned}
m_c \leq & E_\sigma(u_\theta,w_\theta)\\
=&\frac{1}{2}\int_{\mathbb{R}^2}\theta^2\left(|\nabla u|^2+|\nabla w|^2\right)dx+\frac{1}{2}\int_{\mathbb{R}^2}\theta^2\left(|u|^2+|w|^2\right)dx\\
&-\int_{\mathbb{R}^2}\theta^4\left(\frac{1}{36}|u|^4+\frac{9}{4}|w|^4+|u|^2|w|^2+\frac{1}{9}\left(\bar{u}^3w\right)\right)dx\\
=&\frac{1}{2}\theta^2K(u,w)+\frac{1}{2}\theta^2M(u,w)-\theta^4N(u,w)).
\end{aligned}$$

It is not hard to get $m_c\leq 0$, as $\theta\to 0$. Hence, $m_c\in(-\infty,0]$ for each $c>0$.

On the other hand, for any $c\in(0,c^*]$ and $(u,w)\in S_c$, we have

$$\begin{aligned}
E_\sigma &= \frac{1}{2}K(u,w)+\frac{1}{2}M(u,w)-N(u,w)\\
&\geq \frac{1}{2}K(u,w)+\frac{1}{2}M(u,w)-C_{GN}K(u,w)M(u,w)\\
&= \frac{1}{2}(1-\frac{c}{c^*})K(u,w)+\frac{1}{2}M(u,w)\geq 0,
\end{aligned}$$

so, $m_c\geq 0$. It is apparent that $m_c=0$ for each $c\in(0,c^*]$.

(2) Set $(P^\theta,Q^\theta):=(\frac{c\theta}{c^*}P(t,\theta x),\frac{c\theta}{c^*}Q(t,\theta x))$, where (P,Q) is the ground state solution, then,

$$\int_{\mathbb{R}^n}|P^\theta|^2+3\sigma\left|Q^\theta\right|^2dx=\frac{c^2\theta^2}{(c^*)^2}c,$$

and if $(P,Q)\in S_1$, $(P^\theta,Q^\theta)\in S_{\frac{\theta^2}{(c^*)^2}}$. Moreover, we get

$$\begin{aligned}
m_c \leq & E_\sigma(P^\theta,Q^\theta)\\
=&\frac{c^2\theta^2}{2(c^*)^2}\int_{\mathbb{R}^2}\left(|\nabla P|^2+|\nabla Q^2\right)dx+\frac{c^2}{2(c^*)^2}\int_{\mathbb{R}^2}\left(|P|^2+|Q|^2\right)dx\\
&-\frac{c^2\theta^4}{2(c^*)^4}\int_{\mathbb{R}^2}\left(\frac{1}{36}|P|^4+\frac{9}{4}|Q|^4+|P|^2|Q|^2+\frac{1}{9}\left(\bar{P}^3Q\right)\right)dx\\
=&\frac{c^2}{2c^*}-(\frac{(\omega+1)c^3\theta^2}{2(c^*)^2})((\frac{c}{c^*})^2-1),
\end{aligned}$$

hence, for any $c>c^*$, when $\theta\to+\infty$, $m_c\to-\infty$. The last equality comes from (4.2)-(4.4). □

Proof of Theorem 1.4: To confirm the conclusion, we first define

$$\mathcal{N}_c:=\left\{(u,w)\in S_c\,|\frac{1}{2}K(u,w)<N(u,w)\right\}.$$

By the proof of Lemma 4.2, we have the following result

$$\begin{cases}\mathcal{N}_c=\varnothing, & 0<c\leq c_*,\\ \mathcal{N}_c\neq\varnothing, & c>c_*.\end{cases}$$

The proof is by contradiction, we just suppose that there exists some $c\in(0,c^*]$, such that E_σ has a minimizer, and some $(P,Q)\in S_c$ such that $(E_\sigma|_{S_c})'(P,Q)=0$. Hence we get

$$\frac{1}{2}K(P,Q)+\frac{1}{2}M(P,Q)=N(P,Q),$$

which implies that $(P,Q)\in\mathcal{N}_c$. This leads to a contradiction, then Theorem 1.4 is proved.

5 Blow up results

In this section, our aim is to show some blow up results. Let us start with the following observation.

Lemma 5.1. [14] Assume $1 \leq n \leq 3$, $(u_0, w_0) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, the Cauthy problem (1.1) admits a unique solution defined in the maximal interval of existence $(-T_*, T^*)$:

$$(u, w) \in C\left((-T_*, T^*); H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)\right),$$

where $T_*, T^* > 0$. In addition, the maximal times of existence obey the blow-up alternative: if $T_* < \infty$ then

$$\lim_{t \to T^*} \|(u(t), w(t))\|_{H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)} = \infty,$$

and similarly for T_*.

Definition 5.2. We say that the solution blows up forward in time it $T^* < \infty$ and backward in time if $T_* < \infty$. In particular, the solution blows up if it blows up forward and backward in time.

The proof of the blow-up results in [14] is based on the virial identity

$$\frac{d}{dt}\mathcal{M}_\varphi(t) = 24E(u_0, w_0) - 4\int_{\mathbb{R}^n} \left(|\nabla u|^2 + |\nabla w|^2\right) dx - 12\int_{\mathbb{R}^n} \left(|u|^2 + 9|w|^2\right) dx, \tag{5.1}$$

where

$$\mathcal{M}_\varphi(t) := 4\,\mathrm{Im}\int_{\mathbb{R}^2} (\bar{u}(t)x \cdot \nabla u(t) + 3\bar{w}(t)x \cdot \nabla w(t))dx.$$

Let $\varphi = |x|^2$, $\mu = 3\sigma$ and $\sigma = 3$ in lemma 2.2, we get the above equations. For the power-type NLS equation, the result based on the convexity argument. Firstly, Glassey [8] was proposed this strategy, for finite variance solutions with negative initial energy. After, Ogawa and Tsutsumi [13] for the removal of the finiteness hypothesis of the variance, but with the addition of the radial assumption. Then, see the paper the [10] for an extension to the cubic NLS up to the mass-energy threshold.

While, if we do not have the mass resonance condition $\mu = 3\sigma$, the (5.1) no holds and the convexity argument is no-more applicable. In our paper, we mainly study the blow up results in the L^2-critical and L^2-supercritical cases. The proof relies on the localized virial estimates and the Pohozaev functional. We point-out that our results not only extends to the whole range that $\mu, \sigma > 0$, but also extends to the cylindrical solutions as supercritical case.

To begin with, we study the L^2-critical case, i.e. $n = 2$. The difference from the previous discussion is that the blow-up alternative is no longer valid due to the critical case. To overcome this difficulty, we choose a function, inspired by [13]. Let

$$\xi(s) := \begin{cases} 2s, & \text{if} \quad 0 \leq s \leq 1, \\ 2\left[s - (s-1)^3\right], & \text{if } 1 < s \leq 1 + 1/\sqrt{3}, \\ \xi'(s) < 0, \text{smooth}, & \text{if } 1 + 1/\sqrt{3} < s < 2, \\ 0, & \text{if } s \geq 2, \end{cases} \tag{5.2}$$

and

$$\chi(r) := \int_0^r \xi(s)\mathrm{d}s = \begin{cases} r^2, & 0 \leq r \leq 1, \\ c, & r \geq 2, \end{cases}$$

where $\chi(r) : [0, +\infty) \to [0, +\infty)$ be a sufficiently smooth function as above. For $R > 1$, we define the radial function

$$\varphi_R(x) = \varphi_R(r) := R^2\chi(r/R). \tag{5.3}$$

In this case, we have the following virial estimates.

Lemma 5.3. Let $n = 2$ and $\mu, \sigma > 0$. Let (u, w) be a radial solution to (1.1) defined on the maximal forward time interval $[0, T^*)$. Let φ_R be as in above and $M_\varphi(t)$ as in (2.11). For some $C > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{\varphi_R}(t) \le 16E_\mu(u(t), w(t)) - 4\int_{\mathbb{R}^2} \left((2-\varphi_R''(r)) - CR^{-1}\left(4-\Delta\varphi_R\right)\right)\left(|\nabla u(t,x)|^2 + |\nabla w(t,x)|^2\right)\mathrm{d}x + o_R(1).$$

Proof. Since both (u_0, w_0) and φ are radially symmetric, by (2.26) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M_{\varphi_R}(t) = &-\int_{\mathbb{R}^2}\Delta^2\varphi_R(x)\left(|u(t,x)|^2 + |w(t,x)|^2\right)\mathrm{d}x + 4\int_{\mathbb{R}^2}\varphi_R''(r)\left(|\nabla u(t,x)|^2 + |\nabla w(t,x)|^2\right)\mathrm{d}x \\ &- 4\int_{\mathbb{R}^2}\Delta\varphi_R T(u,w)\mathrm{d}x \\ =&8G(u(t), w(t)) - 4\int_{\mathbb{R}^2}(2-\varphi_R''(r))\left(|\nabla u(t,x)|^2 + |\nabla w(t,x)|^2\right)\mathrm{d}x \\ &+ 4\int_{\mathbb{R}^2}(4-\Delta\varphi_R)T(u,w)\mathrm{d}x.\end{aligned} \tag{5.4}$$

It is not hard to have $\left\|\Delta^2\varphi_R\right\|_{L^\infty} \lesssim R^{-2}$, combining with the conservation of mass, we get

$$\left|\int_{\mathbb{R}^2}\Delta^2\varphi_R(x)\left(|u(t,x)|^2 + |w(t,x)|^2\right)\mathrm{d}x\right| \lesssim R^{-2}.$$

Indeed , define $M_\mu(u, w) := \int\left(|u|^2 + \mu|w|^2\right)$. For the definition of $G(u, w)$ and E_μ we have

$$\begin{aligned}G(u, w) &= K(u, w) - 2N(u, w) \\ &= 2E_\mu(u, w) - M_\mu(u, w),\end{aligned}$$

and (5.4) can be controlled by

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{\varphi_R}(t) \le 16E_\mu(u(t), w(t)) - 4\int(2-\varphi_R''(r))\left(|\nabla u(t,x)|^2 + |\nabla w(t,x)|^2\right)\mathrm{d}x + 4\int(4-\Delta\varphi_R)T(u,w)\mathrm{d}x + CR^{-2}.$$

Furthermore, by Hölder's inequality and the Cauchy inequality

$$\begin{aligned}\int_{\mathbb{R}^2}T(u,w)dx &= \int_{\mathbb{R}^2}(\frac{1}{36}|u|^4 + \frac{9}{4}|w|^4 + |u|^2|w|^2 + \frac{1}{9}u^3 w)dx \\ &\le \frac{1}{36}\|u\|^4_{L^4(\mathbb{R}^2)} + \frac{9}{4}\|u\|^4_{L^4(\mathbb{R}^2)} + \|u\|^2_{L^4(\mathbb{R}^2)}\|w\|^2_{L^4(\mathbb{R}^2)} + \|u\|^3_{L^4(\mathbb{R}^2)}\|w\|_{L^4(\mathbb{R}^2)} \\ &\lesssim \int_{\mathbb{R}^2}(|u|^4 + |w|^4)dx,\end{aligned}$$

thus we have

$$\int_{\mathbb{R}^2}(4-\Delta\varphi_R)T(u,w)dx \lesssim \int_{\mathbb{R}^2}(4-\Delta\varphi_R)(|u|^4 + |w|^4)dx.$$

In fact, according to a simple calculation, if $|x| \le R$, we have $(4 - \Delta\varphi_R) = 0$. Radial Sobolev embedding and conservation of mass imply that

$$\begin{aligned}\int_{\mathbb{R}^2}\left(4-\Delta\varphi_{R(x)}\right)|u|^4 dx &\le \sup_{|x|\ge R}\left|\left(4-\Delta\varphi_{R(x)}\right)^{\frac{1}{2}}u(t,x)\right|^2\|u(t)\|^2_{L^2(\mathbb{R}^2)} \\ &\lesssim R^{-1}\sup_{|x|\ge R}\left\|\nabla[(4-\Delta\varphi_R(x))^{\frac{1}{2}}u(t)]\right\|_{L^2(\mathbb{R}^2)}\left\|(4-\Delta\varphi_R(x))^{\frac{1}{2}}u(t,x)\right\|_{L^2(\mathbb{R}^2)}\|u(t)\|^2_{L^2(\mathbb{R}^2)} \\ &\lesssim R^{-1}\sup_{|x|\ge R}\left\|\nabla[(4-\Delta\varphi_R(x))^{\frac{1}{2}}u(t)]\right\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

It follows that

$$\left\|\nabla[(4-\Delta\varphi_R(x))^{\frac{1}{2}}u(t)]\right\|^2_{L^2(\mathbb{R}^2)} \lesssim \left\|\nabla(4-\Delta\varphi_R(x))^{\frac{1}{2}}\right\|_{L^\infty(\mathbb{R}^2)} \|u_{(t)}\|^2_{L^2(\mathbb{R}^2)} + \left\|(4-\Delta\varphi_R(x))^{\frac{1}{2}}\nabla u(t)\right\|^2_{L^2(\mathbb{R}^2)}$$
$$\lesssim \left\|(4-\Delta\varphi_{(R)}(x))^{\frac{1}{2}}\nabla u_{(t)}\right\|_{L^2(\mathbb{R}^2)} + 1.$$

The last unequal sign is due to $\left\|\nabla(4-\Delta\varphi_R(x))^{\frac{1}{2}}\right\|_{L^\infty(\mathbb{R}^2)} \lesssim 1$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{\varphi_R}(t) \leq 16E_\mu(u(t),w(t)) - 4\int_{\mathbb{R}^2}\left((2-\varphi_R''(r)) - CR^{-1}(4-\Delta\varphi_R)\right)(|\nabla u(t,x)|^2+|\nabla w(t,x)|^2)\,\mathrm{d}x + CR^{-1} + CR^{-2}.$$

The proof is complete. □

Next, we will prove the Theorem 1.5.

Proof of Theorem 1.5: Let $(u_0,w_0)\in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$ be radially symmetric, (u,w) is the solution to system (1.1) defined on the maximal forward time interval $[0,T^*)$. If $T^*<\infty$, we have done. Next, we will discuss $T^*=\infty$ under the assumption $E_\mu(u_0,w_0)<0$ and $M(P,Q)<M(u_0,w_0)$, and show that there exists a constant $C>0$ such that

$$K(u,w)\geq Ct^2,$$

for all $t\geq t_0$, namely that (1.16) holds true.

According to the conservation of the energy, Hölder's inequality and the (G-N) inequality, we have

$$\begin{aligned}
K(u,w) &\leq K(u,w) + \left(\|u\|^2_{L^2(\mathbb{R}^2)} + \mu\|w\|^2_{L^2(\mathbb{R}^2)}\right)\\
&= 2E_\mu(u_0,w_0) + 2\int\left(\frac{1}{36}|u|^4 + \frac{9}{4}|w|^4 + |u|^2|w|^2 + \frac{1}{9}|u|^3|w|\right)\\
&\leq 2E_\mu(u_0,w_0) + 2C\left(\|u\|^4_{L^4(\mathbb{R}^2)} + \|w\|^4_{L^4(\mathbb{R}^2)}\right)\\
&\leq 2E_\mu(u_0,w_0) + 2C\left(\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \|\nabla w\|^2_{L^4(\mathbb{R}^2)}\right)\left(\|u\|^2_{L^2(\mathbb{R}^2)} + 3\sigma\|w\|^2_{L^2(\mathbb{R}^2)}\right)\\
&= 2E_\mu(u_0,w_0) + 2CK(u,w)M(u_0,w_0),
\end{aligned}$$

for this reason

$$-2E_\mu(u_0,w_0)\leq(2C_{GN}M(u_0,w_0)-1)K(u,w).$$

Let $(2C_{GN}M(u_0,w_0)-1)\geq 0$, by the definition of C_{GN}, we get $M(u_0,w_0)>M(P,Q)$.

On the other hand, let φ_R be as in (4.3) and $\mathcal{M}_\varphi(t)$ as in (2.11). By Lemma 5.3, we have for all $t\in[0,\infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{\varphi_R}(t) \leq 16E_\mu(u(t),w(t)) - 4\int_{\mathbb{R}^2}\left((2-\varphi_R''(r)) - CR^{-1}(4-\Delta\varphi_R)\right)(|\nabla u(t,x)|^2+|\nabla w(t,x)|^2)\,\mathrm{d}x + CR^{-1} + CR^{-2}.$$

The theorem will be proved if we can show that

$$\left((2-\varphi_R''(r)) - CR^{-1}(4-\Delta\varphi_R)\right)\geq 0. \tag{5.5}$$

Let $R>1$ large enough, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{\varphi_R}(t)\leq 8E_\mu(u_0,w_0)<0.$$

Integrating the above estimates for all $t\in[0,\infty)$ we have

$$M_{\varphi_R}(t)\leq M_{\varphi_R}(0)+8E_\mu(u_0,v_0)\,t,$$

namely

$$M_{\varphi_R}(t)\leq 4E_\mu(u_0,v_0)\,t<0,$$

where $t_0 := \frac{|M_{\varphi_R}(0)|}{-4E_\mu(u_0,w_0)}$. As mentioned above, we get

$$
\begin{aligned}
|\ M\varphi_R(t)] &= 2\,\mathbf{Im}\int_{\mathbb{R}^2} \nabla\varphi_R(\nabla u\cdot\bar{u}+\sigma\nabla w\cdot\bar{w})dx. \\
&\le C\,\|\nabla\varphi_R\|_{L^\infty}\left(\|\nabla u\|_{L^2(\mathbb{R}^2)}\|u\|_{L^2(\mathbb{R}^2)}+\sigma\|\nabla w\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)}\right) \\
&\le C\left(\varphi_R,\sigma,M\left(u_0,w_0\right)\right)\sqrt{K(u,w)},
\end{aligned}
$$

for all $t\ge t_0$,

$$
-4E_\mu\left(u_0,w_0\right)t < -M_{\varphi_R}(t) = |M\varphi_R(t)| \le C\left(\varphi_R,\sigma,M\left(u_0\ w_0\right)\right)\sqrt{K(u,w)}.
$$

This shows (1.16). Now, we will confirm the (5.5) holds for this choice of (5.2)-(5.3). Indeed, it is straightforward to show that

$$
\varphi_R'(r) = R\chi'(r/R) = R\xi(r/R), \quad \varphi_R''(r) = \chi''(r/R) = \xi'(r/R),
$$

using the fact for radial function

$$
\Delta\varphi_R(x) = \varphi_R''(r) + \frac{1}{r}\varphi_R'(r),
$$

we can show that

$$
\varphi_R''(r) = \chi''\left(\frac{r}{R}\right) = \xi'\left(\frac{r}{R}\right) = \begin{cases} 2, & 0\le\frac{r}{R}\le 1, \\ 2\left(1-3\left(\frac{r}{R}-1\right)^2\right), & 1<\frac{r}{R}\le 1+\frac{1}{\sqrt{3}}, \\ <0, & 1+\frac{1}{\sqrt{3}}<\frac{r}{R}<2, \\ 0, & \frac{r}{R}\ge 2, \end{cases}
$$

and

$$
\varphi_{R(r)}' = R\xi\left(\frac{r}{R}\right) = \begin{cases} 2r, & 0<\frac{r}{R}\le 1, \\ 2R\left[\frac{r}{R}-\left(\frac{r}{R}-1\right)^3\right], & 1<\frac{r}{R}\le 1+\frac{1}{\sqrt{3}}, \\ \text{smooth}, & 1+\frac{1}{\sqrt{3}}<\frac{r}{R}<2, \\ 0, & \frac{r}{R}\ge 2, \end{cases}
$$

and

$$
\Delta\varphi_R = \varphi_R''(r)+\frac{1}{r}\varphi_R'(r) = \begin{cases} 4, & 0\le\frac{r}{R}\le 1, \\ 2-6\left(\frac{r}{R}-1\right)^2+2\frac{R}{r}\left[\frac{r}{R}-\left(\frac{r}{R}-1\right)^3\right], & 1<\frac{r}{R}\le 1+\frac{1}{\sqrt{3}}, \\ \xi'<0,\text{smooth}, & 1+\frac{1}{\sqrt{3}}<\frac{r}{R}<2, \\ 0, & \frac{r}{R}\ge 2. \end{cases}
$$

As what we have anticipated,
(1) For $0\le r\le R$, we get $2-\varphi_R''(r)=0$ and $4-\Delta\varphi_R=0$;
(2) For $R<r\le 1+(\frac{1}{\sqrt{3}})R$, we have

$$
2-\varphi_R''(r) = 6\left(\frac{r}{R}-1\right)^2,
$$

and

$$
4-\Delta\varphi_R = 2+6\left(\frac{r}{R}-1\right)^2-2+2\frac{R}{r}\left(\frac{r}{R}-1\right)^3 = 2\left(\frac{r}{R}-1\right)^2\left(7-\frac{R}{r}\right) < 2\left(7-\frac{3}{3+\sqrt{3}}\right)\left(\frac{r}{R}-1\right)^2;
$$

(3) For $r>(1+\frac{1}{\sqrt{3}})R$, we can deduce

$$
2-\varphi_R''(r)\ge 2,
$$

and there exists a constant $C>0$, such that

$$4-\Delta \varphi_R \geq C.$$

Thus by choosing $R>1$ sufficiently large, we get that (5.5) is fulfilled. The proof is complete by collecting the above estimates.

On the other hand, we will study the blow-up results about supercritical growth case with normalized solution. We begin by stating the following Lemma. For a rigorous proof of this lemma the reader is referred to [2] and [16]:

Lemma 5.4. Let I be an open interval with $0 \in I$. Assume $a \in \mathbb{R}, b>0$ and $q>1$. Define $\gamma=(bq)^{-\frac{1}{q-1}}$ and $f(r)=a-r+br^q$, where $r \geq 0$. Let $Z(t)$ be a nonnegative continuous function such that $f \circ Z \geq 0$ on I. Assume $a<\left(1-\frac{1}{q}\right)\gamma$, we get

(i) If $Z(0)<\gamma$, then $Z(t)<\gamma$, for all $t \in I$,

(ii)If $Z(0)>\gamma$, then $Z(t)>\gamma$, for all $t \in I$.

In addition if $a<(1-\delta_1)\left(1-\frac{1}{q}\right)\gamma$ and $Z(0)>\gamma$, for some $\delta_1>0$, then there exists δ_2 , depending only on δ_1 such that $Z(t)>(1+\delta_2)\gamma, \forall t \in I$.

Through the above Lemma, we can build up the following blow-up criterion with normalized solutions.

Lemma 5.5. Let $n=3$ and $(u_0, w_0) \in H^1\left(\mathbb{R}^3\right) \times H^1\left(\mathbb{R}^3\right)$. Suppose that (1.15) and (1.17) hold, where (P, Q) is any normalized ground state solution of (1.7) with $\mu=3\sigma$, then there holds

$$K(u(t), w(t)) M(u(t), w(t))>K(P, Q) M(P, Q). \tag{5.6}$$

Proof. Let $a=2E_\mu\left(u_0, w_0\right)$, $b=2C_{GN} M\left(u_0, w_0\right)^{1/2}$, and $q=3/2$, where as $n=3$, the best constant of (G-N) inequality is

$$C_{GN}=\frac{(\omega+1)^{-\frac{1}{2}}}{3^{\frac{3}{2}} M(P, Q)}.$$

If $G(t)=K(u(t), w(t))$, we obtain $f \circ G \geq 0$, where $f(r)=a-r+br^{3/2}$. Also, by using (4.6) we see that

$$\gamma=\frac{3(\omega+1) M(P, Q)^2}{M\left(u_0, w_0\right)}.$$

By (2.1)-(2.3), we have

$$E_\sigma|_{S_c}(P, Q)=\frac{1}{2} K(P, Q)-N(P, Q)=\frac{1}{2}(\omega+1) M(P, Q).$$

By a simple calculation, we get

$$a<\left(1-\frac{1}{q}\right)\gamma \quad \Leftrightarrow \quad E\left(u_0, w_0\right) M\left(u_0, w_0\right)<E_\sigma|_{S_c}(P, Q) M(P, Q),$$

and

$$R(0)<\gamma \quad \Leftrightarrow \quad K\left(u_0, w_0\right) M\left(u_0, w_0\right)<K(P, Q) M(P, Q).$$

It follows that (5.6) holds. ☐

In addition, we find $\delta_1>0$ sufficiently small, such that

$$E_\mu\left(u_0, w_0\right) M\left(u_0, w_0\right)<(1-\delta_1) E_\sigma|_{S_c}(P, Q) M(P, Q), \tag{5.7}$$

and there exists δ_2 only related to δ_1, such that

$$K(u(t), w(t)) M(u(t), w(t))>(1+\delta_2) K(P, Q) M(P, Q). \tag{5.8}$$

Lemma 5.6. Under the assumption of Theorem 1.6, define (u, w) is the solution to (1.1) with initial data (u_0, w_0) on the maximal time interval $(-T_*, T^*)$, then for $\varepsilon > 0$ sufficiently small, there exists $c = c(\varepsilon) > 0$ such that

$$G(u(t), w(t)) + \varepsilon K(u(t), w(t)) \leq -c, \tag{5.9}$$

for all $t \in (-T_*, T^*)$.

Proof. If $E_\mu < 0$, by conservation of energy, we can get if $\varepsilon = \frac{1}{2}$ and $c = -3E_\mu > 0$, the (5.9) holds. If $E_\mu \geq 0$, under the assumption (1.15) and (1.17), combining (5.7) with (5.8), we obtain the conclusion. □

Lemma 5.7. Let $n = 3$ and $\mu, \sigma > 0$. Let (u, w) be a Σ_3-solution to (1.1) defined on the maximal time interval $(-T_*, T^*)$. And let $\varphi(x) = \psi(y) + z^2$, we have for all $t \in (-T_*, T^*)$,

$$\frac{d}{dt}\mathcal{M}_{\varphi_R}(t) \leq 8G(u(t), v(t)) + CR^{-1}K(u(t), v(t)) + CR^{-2}. \tag{5.10}$$

Proof. It is clearly from the (2.27) in Remark 2.3 that

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_{\varphi_R}(t) \leq 8G(u(t), v(t)) + CR^{-2} - 4\int_{\mathbb{R}^3}(2 - \psi_R''(\rho))\left(|\nabla_y u|^2 + |\nabla_y v|^2\right)(t, x)dx \\ + 4\,\mathbf{Re}\int_{\mathbb{R}^3}(4 - \Delta_y\psi_R(y))\,T(u, v)(t, x)dx.\end{aligned}$$

By the conservation of mass and radial Sobolev embedding with respect to the y-variable, the proof is complete. The reader will find the details in [[1]]. □

Proof of Theorem 1.7: Let $(u_0, w_0) \in H^1\left(\mathbb{R}^3\right) \times H^1\left(\mathbb{R}^3\right)$ be cylindrical symmetric satisfy either $E_\mu < 0$ or if $E_\mu \geq 0$, we assume that (1.15) and (1.17) hold. The Σ_3-data and radial state are similar, it is sufficient to show the cylindrical symmetric. And we only show the $T^* < \infty$, the $T_* < \infty$ is analogous. Prove it by negation, suppose that $T^* = \infty$, by Lemma 5.6, for $\varepsilon > 0$ sufficiently small, there exists $c = c(\varepsilon) > 0$ such that

$$G(u(t), v(t)) + \varepsilon K(u(t), v(t)) \leq -c.$$

On the other hand, by the blow-up criterion with normalized solutions, we have (5.10) holds. Combining (5.9) and (5.10), choosing $R > 1$ sufficiently large, we get

$$\frac{d}{dt}M_{\varphi_R}(t) \leq -4c - 4\varepsilon K(u(t), v(t)),$$

for all $t \in [0, \infty)$. Integrating the above inequality, we see that

$$M_{\varphi_R}(t) \leq -4\varepsilon\int_{t_0}^{t} K(u(s), v(s))ds,$$

for all $t \geq t_0$ with some $t_0 > 0$ sufficiently large. On the other hand, using the Hölder's inequality and conservation of mass, we find that when $t \to t^*$, $M_{\varphi_R}(t) \to -\infty$, hence $K(u, w) \to +\infty$. The solution cannot exist for all time $t \geq 0$. Detailed process references [1].

Through the above description, we have established blow-up criteria (1.15) and (1.17) to prove some blow-up results. We finally remark that by using Pohozaev identity and the best constant C_{GN} for (G-N) inequality, we find that (1.17) can be replaced by Pohozaev functional $G(u, w) < 0$ under the assumption of mass resonance, so that the same blow up results can be obtained. The Corollary is as follows:

Corollary 5.7. Let $n=3$, $\mu=3\sigma$, (P,Q) be a normalized ground state solution related (1.7). Denote

$$\mathcal{A}:=\left\{(u,w)\in H^1\left(\mathbb{R}^3\right)\times H^1\left(\mathbb{R}^3\right)\quad \text{s.t.}\quad \begin{array}{c} E_\mu(u,w)M(u,w)<E_\sigma\,|_{S_c}\,(P,Q)M(P,Q)\\ K(u,w)M(u,w)>K(P,Q)M(P,Q)\end{array}\right\},$$

and

$$\tilde{\mathcal{A}}:=\left\{(u,w)\in H^1\left(\mathbb{R}^3\right)\times H^1\left(\mathbb{R}^3\right)\quad \text{s.t.}\quad \begin{array}{c} E_\mu(u,w)M(u,w)<E_\sigma\,|_{S_c}\,(P,Q)M(P,Q)\\ G(u,w)<0\end{array}\right\}.$$

Then, we get $\mathcal{A}\equiv\tilde{\mathcal{A}}$.

Proof. First, we will show $\mathcal{A}\subseteq\tilde{\mathcal{A}}$. Let $(u,w)\in\mathcal{A}$, if we can show that $G(u,w)<0$, hence $(u,w)\in\tilde{\mathcal{A}}$. According to the definition of G, under the assumption of mass resonance $\mu=3\sigma$, it follows from the Pohozeav identity that

$$E_\sigma\,|_{S_c}\,(P,Q)=\frac{1}{2}(\omega+1)M(P,Q)=\frac{1}{2}N(P,Q)=\frac{1}{6}K(P,Q).$$

It follows that

$$\begin{aligned} G(u,w)M(u,w)&=2E_\mu(u,w)M(u,w)-(\omega+1)M_\mu(u,w)M(u,w)-N(u,w)M(u,w)\\ &<2E_\sigma\,|_{S_c}\,(P,Q)M(P,Q)-N(P,Q)M(P,Q)\\ &=0.\end{aligned}$$

On the other hand, let $(u,w)\in\tilde{\mathcal{A}}$, $G(u,w)<0$. By substituting (4.4) into (4.6) , we get when $n=3$, $C_{GN}=\frac{1}{K(P,Q)^{1/2}M(P,Q)^{1/2}}$, besides we use (1.10) to have

$$K(u,w)<3N(u,w)\leq 3C_{GN}K(u,w)^{3/2}M(u,w)^{1/2}=\frac{K(u,w)^{3/2}M(u,w)^{1/2}}{K(P,Q)^{1/2}M(P,Q)^{1/2}},$$

which implies that

$$K(P,Q)M(P,Q)\leq K(u,w)M(u,w).$$

This completes the proof of Corollary 5.7. $\square$

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