

Periodic and compacton travelling wave solutions of discrete nonlinear Klein-Gordon lattices

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We prove the existence of periodic travelling wave solutions for general discrete nonlinear Klein-Gordon systems, considering both cases of hard and soft on-site potentials. In the case of hard on-site potentials we implement a fixed point theory approach, combining Schauder's fixed point theorem and the contraction mapping principle. This approach enables us to identify a ring in the energy space for non-trivial solutions to exist, energy (norm) thresholds for their existence and upper bounds on their velocity. In the case of soft on-site potentials, the proof of existence of periodic travelling wave solutions is facilitated by a variational approach based on the Mountain Pass Theorem. The proof of the existence of travelling wave solutions satisfying Dirichlet boundary conditions establishes rigorously the presence of compactons in discrete nonlinear Klein-Gordon chains. Thresholds on the averaged kinetic energy for these solutions to exist are also derived.

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I. INTRODUCTION

Travelling wave solutions (TWSs) in lattice dynamical systems have attracted considerable interest due to their fundamental importance in numerous physical contexts such as the energy transfer in biomolecules, the mobility of dislocations in crystalline materials and the propagation of pulses in optical systems, to name a few [1]-[5]. A variety of fundamental models for the description of such phenomena has been studied, including Fermi-Pasta-Ulam-Tsingou, discrete nonlinear Klein-Gordon (DKG) and discrete nonlinear Schrödinger (DNLS) models [1]-[27], as well as, spatially discrete reaction diffusion systems [39]-[46] which are relevant to pattern formation resulting from phase transitions. Discrete systems are appropriate when the scale of decompositions are too small to be effectively described by continuous approximations.

The existence of solitary travelling wave solutions has been rigorously treated with various approaches from nonlinear analysis [6],[7],[8], based on direct and minimax variational methods [11]-[19], reduction to finite dimensional manifolds and normal form techniques [20],[21] and fixed-point methods [22],[23],[26]. For a presentation of several functional-analytic methods and corresponding background for their implementation in nonlinear lattices we refer to [27].

On the other hand, the problem of the existence of *periodic TWSs* is much less explored, in particular for second order in time lattice dynamical systems [28], such as DKG lattices. In the present work, by combining a fixed-point approach and variational methods, we study the existence of periodic TWSs for general DKG systems with anharmonic on-site potentials in one-dimensional lattices.

In particular, we study general DKG systems described by the following set of coupled oscillator equations

$$\frac{d^2 q_n}{dt^2} = \kappa(q_{n+1} - 2q_n + q_{n-1}) - V'(q_n). \quad (1.1)$$

The prime ' stands for the derivative with respect to q_n , the latter being the coordinate of the oscillator at site n evolving in an anharmonic on-site potential $V(q_n)$. Each oscillator interacts with its neighbours to the left and right and the strength of the interaction is regulated by the value of the parameter κ . This system has a Hamiltonian structure related to the energy

$$H = \sum_n \left(\frac{1}{2} p_n^2 + V(q_n) + \frac{\kappa}{2} (q_{n+1} - q_n)^2 \right), \quad (1.2)$$

and it is time-reversible with respect to the involution $p \mapsto -p$. In the case of a finite lattice, the system (1.1) describes the dynamics of an arbitrary number of $N + 1$ oscillators, which are placed equidistantly on the interval $\Omega = (-L, L)$ of length $2L$. The quantity $\kappa = 1/2h^2$ serves as the discretisation parameter, with $h = 2L/N$ defining the lattice spacing. The position of the oscillators is given by the discrete spatial coordinate

$$x_n = -L + nh, \quad n = 0, 1, 2, \dots, N. \quad (1.3)$$

For finite lattices (1.1) we impose *periodic boundary conditions*

$$q_n = q_{n+N}. \quad (1.4)$$

We consider TWSs of the form:

$$q_n(t) = Q(n - ct) = Q(z), \quad (1.5)$$

with a $2L$ -periodic function $Q(z)$, $z = n - ct$, satisfying

$$\int_{-L}^L Q(z) dz = 0, \quad (1.6)$$

where $c \in \mathbb{R} \setminus \{0\}$ is the velocity. The solutions (1.5) satisfy the advance-delay equation

$$c^2 Q''(z) = \kappa(Q(z+1) - 2Q(z) + Q(z-1)) - V'(Q(z)). \quad (1.7)$$

As the system (1.1) possesses the time-reversibility symmetry it suffices to consider $c > 0$.

The presentation of the results is as follows: Section II deals with the existence of periodic TWSs in finite lattices with *hard on-site potentials*. Imposing periodic boundary conditions, the existence of periodic TWSs is proved by a fixed-point approach, based on Schauder's fixed point theorem and contraction mapping principles. The Schauder's fixed point method was made use of in our recent works [34], [35] in order to establish the existence of breather solutions. It is extended herein to equation (1.7) in order to treat the existence of periodic TWSs. This method, combined with a contraction mapping argument, yields the identification of a ring in the energy space where nontrivial solutions may exist, together with the derivation of energy thresholds for their existence, and an upper bound for their velocity. For the latter, the contraction mapping argument is applied on a map defined by a suitable auxiliary linear, non-homogeneous problem stemming from (1.7), whose solvability is guaranteed by the Friedrichs extension theorem. We should remark that while the current literature on the energy threshold problem concerns mainly breather solutions for DNLS models [36],[37] (see also [24],[25],[26],[38] and references therein), to the best of our knowledge, the results of the present work are the first on energy thresholds for TWSs.

In section III we consider the case of *soft on-site potentials*. This time, for the proof of existence of periodic TWSs on finite lattices supplemented with periodic or Dirichlet boundary conditions, we implement a variational approach based on the Mountain Pass Theorem. Notably, our proof of existence of nontrivial solitary TWSs satisfying Dirichlet boundary conditions, also establishes the presence of *discrete compactons* [29–33], as localised waves with compact support (finite wave length) in nonlinear DKG lattices, that is, systems with anharmonic on-site potentials and linear interactions. This has to be distinguished from the numerical and analytical studies of compactons in nonlinear discrete systems with no on-site potential and nonlinear interactions as in [32] and [33], respectively. Again threshold values for the average kinetic energy of TWSs are derived and this time with the aid of the contraction mapping principle. We remark that the derivation of the kinetic energy thresholds is valid for finite as well as infinite lattices. In fact, we present the proof for the latter case where the TWSs satisfy vanishing boundary conditions.

II. HARD ON-SITE POTENTIALS

In this section we consider hard on-site (uniformly convex) potentials $V(x)$ assuming that the following assumption holds on them:

A: $V : \mathbb{R} \rightarrow \mathbb{R}$ is non-negative and at least twice continuously differentiable and is characterised by the following properties:

$$V(0) = V'(0) = 0, \quad V''(0) > 0. \quad (2.1)$$

The unique equilibrium at $x = 0$ is a global minimum of $V(x)$. Further, we assume that V satisfies for some positive constants \bar{m} , α , β , and K , the conditions

$$|V''(x)| \leq \bar{m}|x|^\alpha, \quad \forall x \in \mathbb{R}, \quad (2.2)$$

$$|V'(x_1) - V'(x_2)| \leq K(|x_1|^\beta + |x_2|^\beta)|x_1 - x_2|. \quad (2.3)$$

In what follows, we reformulate the original problem (1.6) and (1.7) as a fixed point problem for a suitably defined operator in a Banach space. Motivated by our recent works [34],[35] applying Schauder's Fixed Point Theorem [8] to prove the existence of breathers in discrete nonlinear KG lattices, we show that the same Theorem is applicable, in order to prove the existence of periodic travelling waves.

We will use the following version of Schauder's Fixed Point Theorem (see e.g. in [8]):

Theorem II.1. *Let G be a non-empty closed convex subset of the Banach space X . Suppose $U : G \rightarrow G$ is a compact map, then U has a fixed point in G .*

A. Existence of periodic TWSs

We start by introducing appropriate function spaces on which our methods will be employed. We consider first the space

$$\mathcal{X}_0 = \left\{ Q \in L_{per}^2(-L, L) \mid \int_{-L}^L Q(s) ds = 0 \right\}, \quad (2.4)$$

that is, the space of $L_{per}^2(-L, L)$, $2L$ -periodic, square integrable functions of zero mean, endowed with the norm,

$$\|Q\|_{\mathcal{X}_0} = \left(\frac{1}{2L} \int_{-L}^L (Q(s))^2 ds \right)^{1/2} = \left(\sum_{k \in \mathbb{Z} \setminus \{0\}} |\hat{Q}_k|^2 dk \right)^{1/2}. \quad (2.5)$$

\hat{Q}_k determines the Fourier-coefficient in the Fourier series expansion:

$$Q(s) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{Q}_k \exp(i\Omega ks), \quad s \in (-L, L), \quad \Omega = \frac{\pi}{L}, \quad (2.6)$$

$$\hat{Q}_k = \frac{1}{2L} \int_{-L}^L Q(s) \exp(-i\Omega ks) ds, \quad \hat{Q}_k = \bar{\hat{Q}}_{-k}, \quad (2.7)$$

of $Q(s)$ and \bar{x} denotes the complex conjugate of x . Evidently, \mathcal{X}_0 is a closed subspace of $L_{per}^2(-L, L)$. We shall also use the spaces

$$\mathcal{X}_1 = \left\{ Q \in H_{per}^1(-L, L) \mid \int_{-L}^L Q(s) ds = 0 \right\}, \quad (2.8)$$

endowed with the scalar product and induced norm

$$(P, Q)_{\mathcal{X}_1} = \int_{-L}^L P'(z) Q'(z) dz, \quad \|Q\|_{\mathcal{X}_1}^2 = \int_{-L}^L Q'(z)^2 dz.$$

It can be easily checked that the Poincaré inequality

$$\int_{-L}^L (Q(z))^2 dz \leq C(L) \int_{-L}^L (Q'(z))^2 dz, \quad (2.9)$$

holds with the optimal constant $C(L) = L^2/\pi^2$ when the Hilbert space \mathcal{X}_1 is used. Of use will also be the space

$$\mathcal{X}_2 = \left\{ Q \in H_{per}^2(-L, L) \mid \int_{-L}^L Q(s) ds = 0 \right\}. \quad (2.10)$$

For \mathcal{X}_2 we shall use the norms

$$\|Q\|_{\mathcal{X}_2}^2 = \frac{1}{2L} \int_{-L}^L ((Q(s))^2 + (DQ(s))^2 + (D^2Q(s))^2) ds = \sum_{k \in \mathbb{Z} \setminus \{0\}} (1 + (k\Omega)^2 + (k\Omega)^4) |\hat{Q}_k|^2. \quad (2.11)$$

It can be readily seen that \mathcal{X}_2 is a closed subspace of $H_{per}^2(-L, L)$. Furthermore, \mathcal{X}_2 is compactly embedded in \mathcal{X}_0 ($\mathcal{X}_2 \Subset \mathcal{X}_0$).

In some cases, we will facilitate variational methods. In particular, solutions of (1.7) will be considered as critical points of the action functional $S : \mathcal{X}_1 \rightarrow \mathbb{R}$ given by

$$S(Q) = \int_{-L}^L \left[\frac{1}{2} (cQ'(z))^2 - V(Q(z)) - \frac{1}{2} \kappa [Q(z+1) - Q(z)]^2 \right] dz.$$

Related with the action functional is the (total) energy functional E on \mathcal{X}_1 :

$$E(Q) = \int_{-L}^L \left[\frac{1}{2} (Q'(z))^2 + V(Q(z)) + \frac{\kappa}{2} [Q(z+1) - Q(z)]^2 \right] dz, \quad (2.12)$$

which will be also involved in the derivation of several bounds for the solutions.

Let us denote by $C_{0,*}$ the constant of the embedding $\mathcal{X}_2 \Subset \mathcal{X}_0$, and by $C_{2,*}$ the constant of the embedding $\mathcal{X}_2 \subset L^\infty([-L, L])$. We now proceed to the statement and proof of the result on the existence of periodic TWSs:

Theorem II.2. *Let assumption A hold. If*

$$\frac{4\kappa}{c^2} < \Omega^2, \quad (2.13)$$

then there exists a periodic TWS $q_n(t) = Q(n - ct) \equiv Q(z)$ of (1.1) with $Q \in H^2(-L, L)$ and

$$\|Q\|_{\mathcal{X}_0} \leq \left(\frac{c^2 \Omega^2 - 4\kappa}{\overline{m} C_{3,*}^\beta} \right)^{1/\beta} := \mathcal{R}_{max}, \quad (2.14)$$

where the constant $C_{3,}$ is given by*

$$C_{3,*} := \frac{C_{2,*}}{C_{0,*}}, \quad (2.15)$$

such that

$$Q(z + 2L) = Q(z), \quad \forall z \in \mathbb{R}. \quad (2.16)$$

Proof: We seek $2L$ -periodic functions $Q \in H^2(-L, L)$ satisfying (1.6) and solve Eq. (1.7). For the following discussions it is suitable to re-write Eq. (1.7) as:

$$Q''(z) - \frac{\kappa}{c^2} (Q(z+1) - 2Q(z) + Q(z-1)) = -\frac{1}{c^2} V'(Q(z)). \quad (2.17)$$

Thus only the right-hand side of (2.17) features terms nonlinear in Q . Ultimately, (2.17) shall be expressed as a fixed point equation in Q .

To proceed, we recall first some basic facts for the intersection of Banach spaces [10, Lemma 2.3.1 and Theorem 2.7.1]: For two Banach spaces X and Y , the intersection $X \cap Y$ is a Banach space endowed with the norm

$$\|x\|_{X \cap Y} = \|x\|_X + \|x\|_Y, \quad \text{for all } x \in X \cap Y. \quad (2.18)$$

Clearly, from (2.18), we have that

$$\|x\|_X \leq \|x\|_{X \cap Y}, \quad \text{and} \quad \|x\|_Y \leq \|x\|_{X \cap Y}, \quad \text{for all } x \in X \cap Y. \quad (2.19)$$

Moreover, in the particular case where Y is continuously embedded in X with an embedding constant c_0 , that is $\|x\|_X \leq c_0 \|x\|_Y$, for all $x \in Y$, then, due to (2.18) and (2.19),

$$\|x\|_X \leq \|x\|_{X \cap Y} \leq c_0^* \|x\|_Y, \quad \text{for all } x \in Y, \quad c_0^* = 1 + c_0. \quad (2.20)$$

Hence, if x is in a closed ball of Y of radius r , then due to (2.20), x is in the closed ball of $X \cap Y$ of radius $c_0^* r$, and in the closed ball of the same radius of X . With these preparations we apply the above for $X = \mathcal{X}_0$ and $Y = \mathcal{X}_2$, and we define a convex set of $\mathcal{X}_0 \cap \mathcal{X}_2$, as follows: First, we consider arbitrary elements $P \in \mathcal{X}_0 \cap \mathcal{X}_2$, such that

$$\|P\|_{\mathcal{X}_2} \leq \varrho. \quad (2.21)$$

Since $P \in \mathcal{X}_2 \cap \mathcal{X}_0$ and $\mathcal{X}_2 \Subset \mathcal{X}_0$ with compact embedding, it holds by application of (2.20), that

$$\|P\|_{\mathcal{X}_0} \leq \|P\|_{\mathcal{X}_0 \cap \mathcal{X}_2} \leq C_{0,*} \|P\|_{\mathcal{X}_2} \leq C_{0,*} \varrho, \quad (2.22)$$

Then, we set for simplicity,

$$R = C_{0,*} \varrho, \quad (2.23)$$

and we consider the convex set of $\mathcal{X}_0 \cap \mathcal{X}_2$ as

$$\mathcal{Y}_0 = \{P \in \mathcal{X}_0 \cap \mathcal{X}_2 : \|P\|_{\mathcal{X}_0} \leq R\}. \quad (2.24)$$

Evidentially, the set \mathcal{Y}_0 is well defined and non-empty due to (2.21) and (2.22). As we will prove below, R (and thus, ϱ , due to (2.23)), will be explicitly determined, giving rise to the estimate (2.14) for the TWs.

We relate the left-hand side of (2.17) to the linear mapping: $M : \mathcal{X}_2 \rightarrow \mathcal{X}_0$:

$$M(P) = P''(z) - \frac{\kappa}{c^2}(P(z+1) - 2P(z) + P(z-1)). \quad (2.25)$$

As a next step, we establish the invertibility of this mapping and derive an upper bound for the norm of its inverse.

Applying the operator M to the Fourier elements $\exp(i\Omega ks)$ in (2.6), results in

$$M \exp(i\Omega ks) = \nu_k \exp(i\Omega ks), \quad (2.26)$$

where

$$\nu_k = -\Omega^2 k^2 + 4 \frac{\kappa}{c^2} \sin^2 \left(\frac{\Omega}{2} k \right). \quad (2.27)$$

The hypothesis (2.13) guarantees that $\nu_k \neq 0$, for all $k \in \mathbb{Z} \setminus \{0\}$. Therefore, the mapping M possesses an inverse obeying $M^{-1} \exp(i\Omega ks) = (1/\nu_k) \exp(i\Omega ks)$. For the norm of the linear operator $M^{-1} : \mathcal{X}_0 \rightarrow \mathcal{X}_2$, one gets, by using the hypothesis (2.13), the upper bound:

$$\begin{aligned} \|M^{-1}\|_{\mathcal{X}_0, \mathcal{X}_2} &= \sup_{0 \neq Q \in \mathcal{X}_0} \frac{\|M^{-1}Q\|_{\mathcal{X}_2}}{\|Q\|_{\mathcal{X}_0}} \\ &= \sup_{0 \neq Q \in \mathcal{X}_0} \frac{\left(\frac{1}{2L} \int_{-L}^L ((M^{-1}Q(s))^2 + (DM^{-1}Q(s))^2 + (D^2M^{-1}Q(s))^2) ds \right)^{1/2}}{\|Q\|_{\mathcal{X}_0}} \\ &= \sup_{0 \neq Q \in \mathcal{X}_0} \frac{\left(\frac{1}{2L} \int_{-L}^L \left(|\sum_l' \frac{1}{\nu_l} \hat{Q}_l \exp(i l \Omega s)|^2 + |\sum_l' \frac{i\Omega l}{\nu_l} \hat{Q}_l \exp(i l \Omega s)|^2 + |\sum_l' \frac{(i\Omega l)^2}{\nu_l} \hat{Q}_l \exp(i l \Omega s)|^2 \right) ds \right)^{1/2}}{\|Q\|_{\mathcal{X}_0}} \\ &\leq \sup_{l \in \mathbb{Z} \setminus \{0\}} \frac{\sqrt{1 + (\Omega l)^2 + (\Omega l)^4}}{|\nu_l|} \cdot \sup_{0 \neq Q \in \mathcal{X}_0} \frac{\left(\sum_l' |\hat{Q}_l|^2 \right)^{1/2}}{\|Q\|_{\mathcal{X}_0}} \\ &= \sup_{l \in \mathbb{Z} \setminus \{0\}} \frac{\sqrt{1 + (\Omega l)^2 + (\Omega l)^4}}{|-\Omega^2 l^2 + \frac{4\kappa}{c^2} \sin^2(\frac{\Omega}{2} l)|} \\ &\leq \frac{1 + \Omega^2}{\Omega^2 - \frac{4\kappa}{c^2}}, \end{aligned} \quad (2.28)$$

which verifies the boundedness of M^{-1} ; note that we have used the notation $\sum_l' = \sum_{l \in \mathbb{Z} \setminus \{0\}}$. Obviously, the same estimate (2.28) verifies the boundedness of M^{-1} as an operator $M^{-1} : \mathcal{X}_2 \cap \mathcal{X}_0 \rightarrow \mathcal{X}_0$. On the other hand, we remark for later use, that if we consider only the lower-order terms of the estimate (2.28), we get that

$$\|M^{-1}\|_{\mathcal{X}_0, \mathcal{X}_0} \leq \frac{c^2}{c^2 \Omega^2 - 4\kappa}. \quad (2.29)$$

Associated with the right-hand side of (2.17) we introduce the nonlinear operator $N : \mathcal{Y}_0 \rightarrow \mathcal{Y}_0$, as

$$N(P) = -\frac{1}{c^2} V'(P). \quad (2.30)$$

To verify continuity of the operator N on \mathcal{X}_0 , for any $P \in \mathcal{Y}_0$, (which implies by the definition of \mathcal{Y}_0 that $P \in \mathcal{X}_0 \cap \mathcal{X}_2$), we prove that N is Frechet differentiable with a locally bounded derivative. Using (2.2) we have

$$N'(P) : h \in \mathcal{X}_0 \mapsto N'(P)[h] = -\frac{1}{c^2} V''(P)h \in \mathcal{X}_0. \quad (2.31)$$

Then, since $P \in \mathcal{X}_2 \cap \mathcal{X}_0$ and $\mathcal{X}_2 \subset L^\infty([-L, L])$ with continuous embedding, we have that

$$\begin{aligned}
\|N'(P)[h]\|_{\mathcal{X}_0} &= \left\| -\frac{1}{c^2} V''(P) h \right\|_{\mathcal{X}_0} \\
&= \frac{1}{c^2} \left(\frac{1}{2L} \int_{-L}^L |V''(P(z)) h(z)|^2 dz \right)^{1/2} \leq \frac{1}{c^2} \left(\bar{m}^2 \max_{-L \leq z \leq L} |P(z)|^{2\alpha} \frac{1}{2L} \int_{-L}^L |h(z)|^2 dz \right)^{1/2} \\
&\leq \frac{\bar{m}}{c^2} \max_{-L \leq z \leq L} |P(z)|^\alpha \|h\|_{\mathcal{X}_0} \\
&\leq \frac{\bar{m}}{c^2} C_{2,*}^\alpha \|P\|_{\mathcal{X}_2}^\alpha \|h\|_{\mathcal{X}_0} \leq A(\bar{m}, c^2, C_{2,*}, \varrho) \|h\|_{\mathcal{X}_0},
\end{aligned} \tag{2.32}$$

for the constant

$$A = A(\bar{m}, c^2, C_{2,*}, \varrho) = \frac{\bar{m}}{c^2} C_{2,*}^\alpha \varrho^\alpha,$$

proving the local boundedness of the differential. Then, one concludes that the Frechet derivative is locally bounded as

$$\|N'(P)\|_{\mathcal{L}(\mathcal{X}_0, \mathcal{X}_0)} \leq A. \tag{2.33}$$

With the aid of the locally bounded derivative, we can prove the local Lipschitz continuity of N as follows:

$$\begin{aligned}
\|N(P) - N(Q)\|_{\mathcal{X}_0} &\leq \sup_{S \in [P, Q]} \|N'(S)\|_{\mathcal{L}(\mathcal{X}_0, \mathcal{X}_0)} \|P - Q\|_{\mathcal{X}_0} \\
&\leq A \|P - Q\|_{\mathcal{X}_0}.
\end{aligned} \tag{2.34}$$

As the range of N is concerned, we derive that for any $P \in \mathcal{Y}_0$,

$$\begin{aligned}
\|N(P)\|_{\mathcal{X}_0} = \frac{1}{c^2} \left\| -V'(P) \right\|_{\mathcal{X}_0} &= \frac{1}{c^2} \left(\frac{1}{2L} \int_{-L}^L |-V'(P(z))|^2 du \right)^{1/2} \leq \frac{1}{c^2} \left(\frac{1}{2L} \int_{-L}^L \bar{m}^2 |P(z)|^{2(\beta+1)} dz \right)^{1/2} \\
&\leq \frac{\bar{m}}{c^2} \max_{-L \leq z \leq L} |P(z)|^\beta \|P\|_{\mathcal{X}_0} \\
&\leq \frac{\bar{m}}{c^2} C_{2,*}^\beta \|P\|_{\mathcal{X}_2}^\beta \|P\|_{\mathcal{X}_0} \\
&\leq \frac{\bar{m}}{c^2} C_{2,*}^\beta C_{0,*} \|P\|_{\mathcal{X}_2}^{\beta+1} \\
&\leq \frac{\bar{m}}{c^2} C_{2,*}^\beta C_{0,*} \varrho^{\beta+1} = \frac{\bar{m}}{c^2} \left(\frac{C_{2,*}}{C_{0,*}} \right)^\beta R^{\beta+1}.
\end{aligned} \tag{2.35}$$

Thus, we proved in (2.35), that for any $P \in \mathcal{Y}_0$,

$$\|N(P)\|_{\mathcal{X}_0} \leq \frac{\bar{m} C_{3,*}^\beta}{c^2} R^{\beta+1}, \tag{2.36}$$

with the constant $C_{3,*} = \frac{C_{2,*}}{C_{0,*}}$, as defined in (2.15).

At last, we express the problem (2.17) as a fixed point equation in terms of a mapping $\mathcal{Y}_0 \rightarrow \mathcal{Y}_0$:

$$Q = M^{-1} \circ N(Q) \equiv \mathcal{L}(Q). \tag{2.37}$$

Using (2.29) and (2.35), we have

$$\begin{aligned}
\|\mathcal{L}(Q)\|_{\mathcal{X}_0} &= \|M^{-1}(N(Q))\|_{\mathcal{X}_0} \leq \|M^{-1}\|_{\mathcal{X}_0, \mathcal{X}_0} \|N(Q)\|_{\mathcal{X}_0} \\
&\leq \frac{\bar{m} C_{3,*}^\beta}{c^2 \Omega^2 - 4\kappa} R^{\beta+1} \leq R,
\end{aligned} \tag{2.38}$$

assuring by assumptions (2.13) and (2.14), that indeed,

$$\mathcal{L}(\mathcal{Y}_0) \subseteq \mathcal{Y}_0. \tag{2.39}$$

Furthermore, since it holds that $\mathcal{L}(Q) \in \mathcal{X}_2$ for all $Q \in \mathcal{Y}_0 \subseteq \mathcal{X}_0$, one has $\mathcal{L}(\mathcal{Y}_0) \subseteq \mathcal{X}_2 \cap \mathcal{Y}_0$, and as the embedding of \mathcal{X}_2 in \mathcal{X}_0 is compact, the operator \mathcal{L} is compact. It remains to prove that \mathcal{L} is continuous on \mathcal{Y}_0 : For arbitrary $P_1, P_2 \in \mathcal{Y}_0$, we have

$$\begin{aligned} \|\mathcal{L}(P_1) - \mathcal{L}(P_2)\|_{\mathcal{X}_0} &= \|M^{-1}(N(P_1)) - M^{-1}(N(P_2))\|_{\mathcal{X}_2} \leq \|M^{-1}\|_{\mathcal{X}_0, \mathcal{X}_0} \|N(P_1) - N(P_2)\|_{\mathcal{X}_0} \\ &\leq \frac{Ac^2}{c^2\Omega^2 - 4\kappa} \|P_1 - P_2\|_{\mathcal{X}_0} < \epsilon, \end{aligned} \quad (2.40)$$

if

$$\|P_1 - P_2\|_{\mathcal{X}_0} < \delta = \frac{c^2\Omega^2 - 4\kappa}{Ac^2} \epsilon, \quad (2.41)$$

for any given $\epsilon > 0$, verifying that $\mathcal{L}(Q)$ is continuous on \mathcal{Y}_0 . Thus, all the assumptions of Schauder's fixed point theorem are satisfied, and hence, the fixed point equation $Q = \mathcal{L}(Q)$ has at least one solution. \square

Remark II.1. (*Regularity of travelling waves*). By the Sobolev embeddings, the obtained H^2 -travelling wave solutions Q are C^1 . Therefore, due to Eq. (1.7) it holds that $Q'' \in C^1$ and conclusively $Q \in C^3$, that is, they are classical solutions.

B. Existence of non-trivial TWSs and an energy threshold

In this section, we consider the problem of existence and non-existence of non-trivial TWSs with frequencies satisfying the strengthened condition

$$\Omega^2 > \frac{4\kappa + \overline{m}C_{3,*}^\beta}{c^2}, \quad (2.42)$$

if compared to (2.13). The motivation for assuming (2.42), is explained in Theorem II.3 concluding the section. We start by stating a result on non-existence of non-trivial TWSs with frequencies satisfying (2.42).

Proposition II.1. *Suppose that conditions A and (2.42) hold and that*

$$\|Q\|_{\mathcal{X}_0} \leq R < \left(\frac{c^2\Omega^2 - 4\kappa}{\overline{m}C_{3,*}^\beta} \right)^{1/(1+\beta)} := \mathcal{R}_{crit}. \quad (2.43)$$

Then the equation $Q = \mathcal{L}(Q)$ has only the trivial solution, that is, there exist no non-trivial TWSs.

Proof: If (2.43) holds, we get $\|\mathcal{L}(Q)\|_{\mathcal{X}_0} = \|M^{-1}(N(Q))\|_{\mathcal{X}_0} \leq \|M^{-1}\|_{\mathcal{X}_0, \mathcal{X}_0} \cdot \|N(Q)\|_{\mathcal{X}_0} < 1$. Thus \mathcal{L} is a contraction, and the Contraction Mapping Theorem implies that there is a unique function Q that solves the equation $Q = \mathcal{L}(Q)$. Since $\mathcal{L}(0) = 0$, this unique solution is the trivial one.

Note that due to (2.1) the potential $V(x)$ can be expressed as $V(x) = (1/2)x^2 + W(x)$, with W satisfying (2.2) (see also [35, Eq. (2.7) and Eq. (2.8)]). Furthermore, let us observe that as the (conserved) energy functional (2.12) is coercive, it can be used to bound the $L^2(-L, L)$ -norm of the TWSs as follows

$$\begin{aligned} E(Q) &= \int_{-L}^L \left[\frac{1}{2} (Q'(z))^2 + \frac{1}{2} Q^2(z) + W(Q(z)) + \frac{\kappa}{2} [Q(z+1) - Q(z)]^2 \right] dz \\ &= \int_{-L}^L \left[\frac{1}{2} (Q'(z))^2 + W(Q(z)) + \frac{\kappa}{2} [Q(z+1) - Q(z)]^2 \right] dz + \frac{1}{2} \|Q\|_{L^2(-L, L)}^2, \end{aligned} \quad (2.44)$$

so that

$$\|Q\|_{\mathcal{X}_0} \leq \sqrt{2E}. \quad (2.45)$$

In conclusion, if the energy of the lattice system is less than $E_{crit}(c, \kappa, \overline{m}, \Omega) = \sqrt{2R_{crit}}$, no TWSs of given values for $c, \kappa, \overline{m}, \Omega$ exist. \square

Combining the existence Theorem II.2 and Proposition II.1 we may identify a ring in the phase space \mathcal{X}_0 with quantified radii in which the non-trivial travelling wave solutions with frequencies satisfying the enhanced condition (2.42), exist. This is illustrated in the cartoon of Fig. 1.

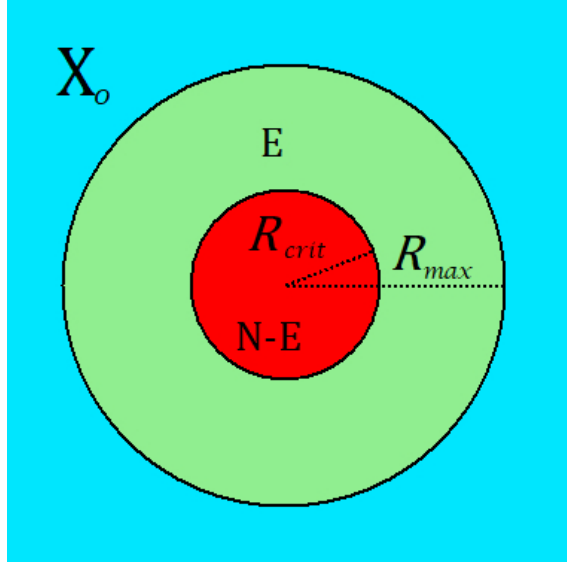


FIG. 1: Illustration of the statement of Theorem II.3: Non-trivial travelling wave solutions exist in the ring $\mathbf{E} = \overline{\mathcal{B}(0, \mathcal{R}_{max})} \setminus \overline{\mathcal{B}(0, \mathcal{R}_{crit})}$ of the space \mathcal{X}_0 [light (green) coloured area (\mathbf{E})]. The radius \mathcal{R}_{crit} of the closed ball $\overline{\mathcal{B}(0, \mathcal{R}_{crit})}$ follows from the contraction mapping Theorem (see Proposition II.1) and defines a threshold for non-existence of non-trivial solutions [in the darker (red) coloured area ($\mathbf{N} - \mathbf{E}$)]. The radius \mathcal{R}_{max} of the closed ball $\overline{\mathcal{B}(0, \mathcal{R}_{max})}$ results from the Schauder's fixed point theorem (see Theorem II.1).

Theorem II.3. *Let the assumption **A** and the condition (2.42) hold. The system (1.1) may possess non-trivial TWSs only if*

$$\mathcal{R}_{crit} = \left(\frac{c^2 \Omega^2 - 4\kappa}{\overline{m} C_{3,*}^\beta} \right)^{1/(1+\beta)} \leq \|Q\|_{\mathcal{X}_0} \leq \mathcal{R}_{max} = \left(\frac{c^2 \Omega^2 - 4\kappa}{\overline{m} C_{3,*}^\beta} \right)^{1/\beta}. \quad (2.46)$$

Proof: On the one hand, Theorem II.2 establishes the existence of solutions when $\|Q\|_{\mathcal{X}_0} \leq \mathcal{R}_{max}$. On the other hand, according to Proposition II.1, if $\|Q\|_{\mathcal{X}_0} < \mathcal{R}_{crit}$ only the trivial solution exists. Thus, a non-trivial solution exists only if

$$\mathcal{R}_{crit} < \|Q\|_{\mathcal{X}_0} \leq \mathcal{R}_{max},$$

that is, when (2.46) is satisfied. For the latter to be valid, we require $\mathcal{R}_{crit} < \mathcal{R}_{max}$, motivating the condition (2.42) on the frequencies of the TWSs. \square

C. Upper bound for the velocity

By using a fixed point approach, we may derive upper bounds for the velocity c of TWSs. For this purpose, we rewrite equation (1.7) in the following operator form:

$$-Q''(z) = \frac{1}{c^2} \{ \kappa ([Q(z) - Q(z-1)] - [Q(z+1) - Q(z)]) + V'(Q(z)) \}. \quad (2.47)$$

We recall some basic auxiliary results, starting with the Friedrichs extension Theorem.

Theorem II.4. *Let $\mathcal{L}_0 : D(\mathcal{L}_0) \subseteq X_0 \rightarrow X_0$ be a linear symmetric operator on the Hilbert space X_0 with its domain $D(\mathcal{L}_0)$ being dense in X_0 . Assume that there exists a constant $c > 0$ such that*

$$(\mathcal{L}_0 v, v)_{X_0} \geq c \|v\|_{X_0}^2 \text{ for all } v \in D(\mathcal{L}_0).$$

Then \mathcal{L}_0 has a self-adjoint extension $\mathcal{L} : D(\mathcal{L}) \subseteq X_1 \subseteq X_0 \rightarrow X_0$ where X_1 denotes the energetic Hilbert space endowed with the energetic scalar product $(v, w)_{X_1} = (\mathcal{L}v, w)_{X_0}$ for all $u, v \in X_1$ and the energetic norm $\|v\|_{X_1}^2 = (\mathcal{L}v, v)_{X_0}$.

Furthermore, the operator equation

$$\mathcal{L}v = f, f \in X_0,$$

has a unique solution $v \in D(\mathcal{L})$. In addition, if $\hat{\mathcal{L}} : X_1 \rightarrow X_1^*$ denotes the energetic extension of \mathcal{L} , then $\hat{\mathcal{L}}$ is the canonical isomorphism from X_1 to its dual X_1^* and the operator equation

$$\hat{\mathcal{L}}v = f, f \in X_1^*,$$

has also a unique solution $v \in X_1$.

With Theorem II.4 in hand, we discuss the left-hand side of Eq. (2.47). It is well known that Theorem II.4 is applicable to the operator $\mathcal{L}_0 : D(\mathcal{L}_0) \subseteq L^2(-L, L) \rightarrow L^2(-L, L)$, $\mathcal{L}_0 Q = -Q''(z)$, with domain of definition, $D(\mathcal{L}_0)$, the space of C^∞ -functions on $(-L, L)$. Since $D(\mathcal{L}_0)$ is dense in \mathcal{X}_0 , and inequality (2.9) holds, the Friedrichs extension of \mathcal{L}_0 is the operator $\mathcal{L} : D(\mathcal{L}) \rightarrow \mathcal{X}_0$ where

$$D(\mathcal{L}) = \{v \in \mathcal{X}_1 : \mathcal{L}v \in L^2(-L, L)\}.$$

Consequently, the equation

$$-Q''(z) = f, \text{ for every } f \in L^2(-L, L), \quad (2.48)$$

has a unique solution in $D(\mathcal{L})$. Thus, we shall consider the right-hand side of Eq. (2.47) as a suitable mapping on $L^2(-L, L)$. For its linear part, we have the following lemma proved in [12, Proposition 1, pg. 268].

Lemma II.1. *The linear operators*

$$A_1[Q(z)] = Q(z+1) - Q(z) = \int_z^{z+1} Q'(s)ds, \quad A_2[Q(z)] = Q(z) - Q(z-1) = \int_{z-1}^z Q'(s)ds,$$

are continuous from \mathcal{X}_1 to $L^2(-L, L) \cap L^\infty(-L, L)$ and $\|A_i Q\|_{L^\infty} \leq \|Q\|_{\mathcal{X}_1}$, $\|A_i Q\|_{\mathcal{X}_0} \leq \|Q\|_{\mathcal{X}_1}$, $i = 1, 2$.

Using Theorem II.4, we treat the following auxiliary linear, non-homogeneous problem

$$-Q''(z) = \frac{1}{c^2} \{ \kappa ([\Psi(z) - \Psi(z-1)] - [\Psi(z+1) - \Psi(z)]) + V'(\Psi(z)) \}. \quad (2.49)$$

for some arbitrary fixed $\Psi \in \mathcal{X}_1$, as an equation of the form (2.48). In particular, we have the following result:

Proposition II.2. *For any $\Psi \in \mathcal{X}_1$, the equation (2.49) has a unique solution $Q \in D(\mathcal{L}) \subset \mathcal{X}_1$.*

Proof: Equation (2.49) can be rewritten in the form

$$-Q''(z) = \frac{1}{c^2} \{ \kappa ([A_2[\Psi(z)] - A_1[\Psi(z)]] + V'(\Psi(z))) \} := \mathcal{F}[\Psi(z)]. \quad (2.50)$$

Due to the continuous embedding $\mathcal{X}_1 \subset L^r(-L, L)$ for any $1 \leq r \leq \infty$ and condition (2.3), we have that for some constant $C = C(K, \beta) > 0$,

$$\|V'(\Psi)\|_{\mathcal{X}_0}^2 = \int_{-L}^L (V'(\Psi(z)))^2 dz \leq K^2 \int_{-L}^L |\Psi(z)|^{2(\beta+1)} dz \leq C \|\Psi\|_{\mathcal{X}_1}^{2(\beta+1)},$$

Then, for the right-hand side of Eq. (2.50) we get

$$\begin{aligned} \|\mathcal{F}[\Psi]\|_{\mathcal{X}_0} &\leq \frac{1}{c^2} \{ \kappa \|A_2[\Psi] - A_1[\Psi]\|_{\mathcal{X}_0} + \|V'(\Psi(z))\|_{\mathcal{X}_0} \} \\ &\leq \frac{1}{c^2} \left\{ 2\kappa \|\Psi\|_{\mathcal{X}_1} + C \|\Psi\|_{\mathcal{X}_1}^{2(\beta+1)} \right\}. \end{aligned} \quad (2.51)$$

Thus, $\mathcal{F}[\Psi] \in L^2(-L, L)$, and by virtue of Theorem II.4 Eq. (2.50) has a unique solution $Q \in D(\mathcal{L})$. \square

We now proceed with the implementation of the fixed point argument. To this aim we consider for some $R > 0$ the closed ball of \mathcal{X}_1 , $\mathcal{B}_R := \{\psi \in \mathcal{X}_1 : \|\psi\|_{\mathcal{X}_1} \leq R\}$. Proposition II.2 shows that the map $\mathcal{T} : \mathcal{X}_1 \rightarrow \mathcal{X}_1$ defined as

$$\mathcal{T}[\Psi] = Q,$$

where Q is the unique solution of the auxiliary problem (2.50), is well defined. Hence we may introduce $\Psi_1, \Psi_2 \in \mathcal{B}_R$ such that $Q = \mathcal{T}[\Psi_1]$ and $P = \mathcal{T}[\Psi_2]$. Then the difference $Y = Q - P$ satisfies the equation

$$\begin{aligned} -Y''(z) &= \mathcal{F}[\Psi_1(z)] - \mathcal{F}[\Psi_2(z)] \\ &= \frac{1}{c^2} \{ \kappa (A_1[\Psi_2(z)] - A_1[\Psi_1(z)] + A_2[\Psi_1(z)] - A_2[\Psi_2(z)]) \} \\ &\quad + V'(\Psi_1(z)) - V'(\Psi_2(z)). \end{aligned} \quad (2.52)$$

From Lemma II.1 the linear operators $A_i : \mathcal{X}_1 \rightarrow L^2(-L, L) \cap L^\infty(-L, L)$ are globally Lipschitz,

$$\begin{aligned} \|A_i \psi_1 - A_i \psi_2\|_{\mathcal{X}_0} &\leq \|\psi_1 - \psi_2\|_{\mathcal{X}_1}, \\ \|A_i \psi_1 - A_i \psi_2\|_{L^\infty} &\leq \|\psi_1 - \psi_2\|_{\mathcal{X}_1}, \quad i = 1, 2. \end{aligned} \quad (2.53)$$

To estimate the difference of the remaining terms in Eq. (2.52), we use (2.3) and the embedding $\mathcal{X}_1 \subset L^\infty(-L, L)$ with its embedding constant C_* ,

$$\begin{aligned} \int_{-L}^L |V'(\Psi_1(z)) - V'(\Psi_2(z))|^2 dz &\leq K^2 \int_{-L}^L (|\Psi_1(z)|^\beta + |\Psi_2(z)|^\beta)^2 |\Psi_1(z) - \Psi_2(z)|^2 dz \\ &\leq K^2 \max_{-L \leq z \leq L} (|\Psi_1(z)|^\beta + |\Psi_2(z)|^\beta) \int_{-L}^L |\Psi_1(z) - \Psi_2(z)|^2 dz \\ &= 2K^2 C_*^{2\beta} R^{2\beta} \|\Psi_1 - \Psi_2\|_{\mathcal{X}_0}^2 \end{aligned} \quad (2.54)$$

Hence, for the right-hand side of (2.52) we get

$$\|\mathcal{F}[\Psi_1(z)] - \mathcal{F}[\Psi_2(z)]\|_{\mathcal{X}_0} \leq M \|\Psi_1 - \Psi_2\|_{\mathcal{X}_1}, \quad (2.55)$$

where the constant M is given by

$$M = \frac{2}{c^2} (\kappa + K C_* R^\beta). \quad (2.56)$$

Next, by multiplying (2.52) in the $L^2(-L, L)$ -scalar product and using the Cauchy-Schwarz inequality and Young's inequality, we get the estimate

$$\begin{aligned} \|Y\|_{\mathcal{X}_1}^2 &\leq \|\mathcal{F}[\Psi_1(z)] - \mathcal{F}[\Psi_2(z)]\|_{\mathcal{X}_0} \|Y\|_{\mathcal{X}_0} \\ &\leq M \sqrt{C(L)} \|\Psi_1 - \Psi_2\|_{\mathcal{X}_1} \|Y\|_{\mathcal{X}_1} \\ &\leq \frac{1}{2} \|Y\|_{\mathcal{X}_1}^2 + \frac{M^2 C(L)}{2} \|\Psi_1 - \Psi_2\|_{\mathcal{X}_1}^2. \end{aligned} \quad (2.57)$$

Note that the Poincaré inequality (2.9) has been used. From (2.57) we derive

$$\|Y\|_{\mathcal{X}_1}^2 = \|\mathcal{T}[\Psi_1] - \mathcal{T}[\Psi_2]\|_{\mathcal{X}_1}^2 \leq M^2 C(L) \|\Psi_1 - \Psi_2\|_{\mathcal{X}_1}^2. \quad (2.58)$$

We conclude that if the Lipschitz constant satisfies

$$M \sqrt{C(L)} < 1, \quad (2.59)$$

then the map $\mathcal{T} : \mathcal{B}_R \rightarrow \mathcal{B}_R$ is a contraction. Hence the map \mathcal{T} satisfies the assumptions of the Banach Fixed Point Theorem and has a unique fixed point. By the assumptions we have that $\mathcal{F}(0) = 0$. Therefore we deduce that if (2.59) holds, then the unique fixed point is the trivial one. *Thus nontrivial solutions exist only if (2.59) is violated, that is, when*

$$M \sqrt{C(L)} > 1. \quad (2.60)$$

Regarding the upper bound for the velocity we summarise in

Theorem II.5. *An upper bound for the velocity c of nontrivial periodic TWSs $q_n(t) = Q(n - ct) = Q(z)$ of prescribed norm $\|Q\|_{\mathcal{X}_1} \leq R$ to the system (1.1), on the periodic lattice $-L \leq n \leq L$, is given by*

$$c^2 < 2(\kappa + K C_* R^\beta) C(L). \quad (2.61)$$

a. Remarks on the physical significance of the estimates for the TWSs and their velocity. The estimates on the TWSs proved in Theorems II.2, II.3 and II.5 implicate a coherent dependence on the lattice parameters, the frequency Ω and R and the velocity c . Motivated by the discussion of [35, Section A, pg. 9], we aim to discuss the potential physical relevance of these estimates:

1. For fixed \bar{m} and κ , we observe that

$$\begin{aligned}\lim_{\Omega \rightarrow \infty} \mathcal{R}_{max} &= \lim_{\Omega \rightarrow \infty} \mathcal{R}_{crit} = \infty, \text{ for fixed } c, \\ \lim_{c \rightarrow \infty} \mathcal{R}_{max} &= \lim_{c \rightarrow \infty} \mathcal{R}_{crit} = \infty, \text{ for fixed } \Omega.\end{aligned}$$

Both limits are physically relevant in the sense that in the limit of arbitrary large frequency or velocity, a type of “energy” of the solution, measured herein in the norm of \mathcal{X}_0 , should become also arbitrarily large. This behavior can be relevant to energy localization phenomena [47] or the notion of quasi-collapse [48]. Note that in the second limit as $c \rightarrow \infty$, the growth of the norm in \mathcal{X}_0 , implies due to the Poincaré inequality (2.9) (or due to Sobolev embeddings), the growth of the kinetic energy of the TWSs, which is consistent with the growth of c .

2. The above coherent dependence on the lattice parameters is also evident in the derived upper bound for the velocity (2.61). Theorem II.5 justifies that TWs of given “energy” measured by the norm of \mathcal{X}_1 can evolve with velocity satisfying the upper-bound (2.61).

III. SOFT ON-SITE POTENTIALS

In this section we study the KG lattice with soft on-site potentials of the form

$$V(x) = -\frac{\omega_0^2}{2}x^2 + \frac{a}{p+1}x^{p+1}, \quad a > 0, \quad p > 1. \quad (3.1)$$

The standard quartic double-well potential is obtained for $p = 3$. We prove the existence of periodic TWSs on finite lattices with imposed periodic boundary conditions and compacton TWSs on finite lattices with Dirichlet boundary conditions, utilising the Mountain Pass Theorem.

A. Periodic TWSs: Existence by the Mountain Pass Theorem

If $p + 1 = 2r > 0$, the on-site potentials $V(x)$ possess the reflection symmetry $V(x) = V(-x)$. For such potentials we first consider periodic TWSs satisfying (1.6), that is, solutions $Q(z)$ performing oscillations about $Q = 0$ so that the associated energy $E > 0$. Hence, the Poincaré inequality applies. These periodic TWSs, as solutions of (1.7), are critical points of the action functional $S : \mathcal{X}_1 \rightarrow \mathbb{R}$ given by

$$S(Q) = \int_{-L}^L \left[\frac{c^2}{2} (Q'(z))^2 + \frac{\omega_0^2}{2} Q^2(z) - \frac{a}{p+1} Q^{p+1}(z) - \frac{\kappa}{2} [Q(z+1) - Q(z)]^2 \right] dz.$$

We get

$$\begin{aligned}\langle S'(Q), P \rangle &= \int_{-L}^L [c^2 Q'(z) P'(z) + \omega_0^2 Q(z) P(z) - a Q^p(z) P(z) \\ &\quad + \kappa [Q(z+1) - 2Q(z) + Q(z-1)] P(z)] dz, \quad P, Q \in \mathcal{X}_1,\end{aligned} \quad (3.2)$$

where $\langle \cdot, \cdot \rangle$ is the standard duality bracket between \mathcal{X}_1 and its dual \mathcal{X}_1^* ; by the definition of the derivative S' , for any $Q \in \mathcal{X}_1$, the functional $S'(Q) : \mathcal{X}_1 \rightarrow \mathbb{R}$ is a linear functional acting on any $P \in \mathcal{X}_1$ as $S'(Q)[P] = \langle S'(Q), P \rangle$ [7].

To prove the existence of TWSs we facilitate the Mountain Pass Theorem (MPT). We recall [[7], Definition 4.1, p. 130] (Palais-Smale (PS) condition) and [[7], Theorem 6.1, p. 140] (Mountain Pass Theorem (MPT) of Ambrosetti-Rabinowitz [6]).

Definition III.1. Let X be a Banach space and $\mathbf{E} : X \rightarrow \mathbb{R}$ be C^1 . We say that \mathbf{E} satisfies condition (PS) if, for any sequence $\{u_n\} \in X$ such that $|\mathbf{E}(u_n)|$ is bounded and $\mathbf{E}'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ in X^* , there exists a convergent subsequence.

Theorem III.1. Let $\mathbf{E} : X \rightarrow \mathbb{R}$ be C^1 and satisfy (A) $\mathbf{E}(0) = 0$, (B) $\exists \rho > 0, \alpha > 0$: $\|u\|_X = \rho$ implies $\mathbf{E}(u) \geq \alpha$, (C) $\exists u_1 \in X$: $\|u_1\|_X \geq \rho$ and $\mathbf{E}(u_1) < \alpha$. Define

$$\Gamma = \{\gamma \in C^0([0, 1], X) : \gamma(0) = 0, \gamma(1) = u_1\}. \quad (3.3)$$

Let $F_\gamma = \{\gamma(t) \in X : 0 \leq t \leq 1\}$ and $\mathcal{L} = \{F_\gamma : \gamma \in \Gamma\}$. If \mathbf{E} satisfies (PS), then

$$\beta := \inf_{F_\gamma \in \mathcal{L}} \sup\{\mathbf{E}(v) : v \in F_\gamma\} \geq \alpha, \quad (3.4)$$

is a critical value of the functional \mathbf{E} .

We proceed by proving the validity of the assumptions of the MPT:

Lemma III.1. Assume either

(i) $\omega_0^2 \geq 4\kappa$,

or

(ii) $\omega_0^2 < 4\kappa$ and $c^2 > C(L)(4\kappa - \omega_0^2)$.

Then the functional S satisfies the PS-condition.

Proof: Assume $S(Q_m)$ is bounded, i.e. $|S(Q_m)| \leq M$ for all $m \in \mathbb{N}$, and

$$S'(Q_m) \rightarrow 0, \text{ as } m \rightarrow \infty \text{ in } \mathcal{X}_1^*. \quad (3.5)$$

Recall also that

$$|\langle S'(Q_m), Q_m \rangle| \leq \|S'(Q_m)\|_{\mathcal{X}_1^*} \|Q_m\|_{X_1}, \quad (3.6)$$

by the standard inequality for the duality bracket [9, Sec. 21.5, pg. 251]. Then, using (3.5), we deduce that for any $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that

$$\|S'(Q_m)\|_{\mathcal{X}_1^*} \leq \epsilon, \text{ for } m > N(\epsilon). \quad (3.7)$$

Applying (3.7) for $\epsilon \leq 1$ and using the inequality (3.6), we deduce that

$$|\langle S'(Q_m), Q_m \rangle| \leq \|Q_m\|_{X_1}, \text{ for } m > N(\epsilon). \quad (3.8)$$

Hence, for chosen $b \in (1/(p+1), 1/2)$, we may deduce from (3.8), that

$$b|\langle S'(Q_m), Q_m \rangle| \leq \|Q_m\|_{X_1} < \|Q_m\|_{X_1} + 1, \text{ for } m > N(\epsilon). \quad (3.9)$$

We will use (3.9) to prove that Q_m is bounded in \mathcal{X}_1 , as follows: First, we derive the inequality

$$\begin{aligned} 1 + M + \|Q_m\|_{X_1} &\geq M - b \langle S'(Q_m), Q_m \rangle \geq S(Q_m) - b \langle S'(Q_m), Q_m \rangle \\ &= \left(\frac{1}{2} - b\right) \int_{-L}^L \left[c^2 (Q'_m(z))^2 + \omega_0^2 Q_m^2(z) - \kappa [Q_m(z+1) - Q_m(z)]^2 \right] dz \\ &\quad - \frac{a}{p+1} (1 - (p+1)b) \int_{-L}^L Q_m^p(z) dz \\ &\geq \left(\frac{1}{2} - b\right) [c^2 \|Q'_m\|_{L^2}^2 + (\omega_0^2 - 4\kappa) \|Q_m\|_{L^2}^2]. \end{aligned} \quad (3.10)$$

If assumption (i) holds, then

$$1 + M + \|Q_m\|_{X_1} \geq \left(\frac{1}{2} - b\right) c^2 \|Q_m\|_{X_1}^2, \quad (3.11)$$

and if assumption (ii) holds, then

$$1 + M + \|Q_m\|_{X_1} \geq \left(\frac{1}{2} - b\right) (c^2 - C(L)(4\kappa - \omega_0^2)) \|Q_m\|_{X_1}^2, \quad (3.12)$$

implying that Q_m is bounded in \mathcal{X}_1 . Hence, there is a subsequence of Q_m (not relabeled) and a $Q \in \mathcal{X}_1$ such that $Q_m \rightarrow Q$ weakly in \mathcal{X}_1 , so that by Sobolev compact embedding, one has the strong convergence $Q_m \rightarrow Q$ in $L^2(-L, L)$ (and in $C([-L, L])$). Using Hölder's inequality and the embedding $\mathcal{X}_1 \subset L^\infty(-L, L)$, we obtain

$$\begin{aligned} \|Q_m - Q\|_{\mathcal{X}_1}^2 &= \frac{1}{c^2} \langle S'(Q_m) - S'(Q), Q_m - Q \rangle \\ &= \frac{1}{c^2} \int_{-L}^L [(\omega_0^2(Q_m(z) - Q(z))^2 - \kappa[Q_m(z+1) - Q(z+1) - (Q_m(z) - Q(z))]^2 \\ &\quad - a(Q_m(z) - Q(z))^{p+1}] dz \\ &\leq \frac{1}{c^2} |\langle S'(Q_m) - S'(Q), Q_m - Q \rangle| \\ &\quad + \frac{1}{c^2} ((\omega_0^2 + 4\kappa)\|Q_m - Q\|_{L^2} + a\|Q_m^p - Q^p\|_{L^2(-L, L)}) \|Q_m - Q\|_{L^2(-L, L)}. \end{aligned} \quad (3.13)$$

The first term on the right-hand side of (3.13) converges to zero because by assumption $\langle S'(Q_m) - S'(Q), Q_m - Q \rangle \rightarrow 0$ as $m \rightarrow \infty$. The last converges to zero by strong convergence. Thus, $\|Q_m - Q\|_{\mathcal{X}_1} = 0$ so that $(Q_m)_{m \in \mathbb{Z}}$ has a strongly convergent subsequence and the proof is finished. \square

Lemma III.2. *The functional S is C^1 on \mathcal{X}_1 .*

Proof: The functional S can be expressed as

$$S(q) = \frac{c^2}{2}(Q, Q) + \Gamma(Q), \quad (3.14)$$

where

$$\Gamma(Q) = \int_{-L}^L \left[\frac{\omega_0^2}{2} Q^2(z) - \frac{a}{p+1} Q^{p+1}(z) - \frac{\kappa}{2} (Q(z+1) - Q(z))^2 \right] dz. \quad (3.15)$$

Continuity of the quadratic term (Q, Q) is obvious. By using The embedding $\mathcal{X}_1 \subset L^\infty(-L, L)$ which implies that $Q^p \in \mathcal{X}_0$ and the Poincaré inequality (2.9) we get the estimate,

$$\|Q^p\|_{\mathcal{X}_0}^2 = \int_{-L}^L |Q(z)|^{2p} dz \leq C(L) \|Q\|_{\mathcal{X}_1}^{2p}. \quad (3.16)$$

Then, we have that

$$|\Gamma(Q)| \leq \frac{\omega_0^2}{2} \|Q\|_{L^2}^2 + \frac{a}{p+1} C(L) \|Q\|_{\mathcal{X}_1}^{p+1} + 2\kappa \|Q\|_{L^2}^2 \leq C(L) \left(\left(\frac{\omega_0^2}{2} + 2\kappa \right) \|Q\|_{\mathcal{X}_1}^2 + \frac{a}{p+1} \|Q\|_{\mathcal{X}_1}^{p+1} \right) < \infty. \quad (3.17)$$

The Gateaux derivative of Γ exists and is given by

$$\begin{aligned} \langle \Gamma'(Q), h \rangle &= \int_{-L}^L [\omega_0^2 Q(z) - aQ^p(z) \\ &\quad + \kappa(A_1[Q(z)] - A_2[Q(z)])] h(z) dz. \end{aligned} \quad (3.18)$$

To prove that Γ' is continuous, we let $\|h\|_{\mathcal{X}_1} \leq 1$ and $Q_m \rightarrow Q$ in \mathcal{X}_1 . Then

$$\begin{aligned} |\langle \Gamma'(Q_m) - \Gamma'(Q), h \rangle| &= \left| \int_{-L}^L [\omega_0^2(Q_m(z) - Q(z)) - a(Q_m^p(z) - Q^p(z)) \right. \\ &\quad \left. + \kappa(A_1[Q_m(z)] - A_2[Q_m(z)] - (A_1[Q(z)] - A_2[Q(z)]))] h(z) dz \right| \\ &\leq \omega_0^2 \|Q_m - Q\|_{L^2}^2 + aC(\|Q_m\|_{L^\infty}, \|Q\|_{L^\infty}) \|Q_m - Q\|_{L^2(-L, L)}^2 + 2\kappa \|Q_m - Q\|_{L^2(-L, L)}^2 \\ &= (\omega_0^2 + aC(\|Q_m\|_{L^\infty}, \|Q\|_{L^\infty}) + 2\kappa) \|Q_m - Q\|_{L^2(-L, L)}^2 \leq C\epsilon, \end{aligned}$$

for m sufficiently large. Hence

$$|\langle \Gamma'(Q_m) - \Gamma'(Q), h \rangle| \rightarrow 0 \text{ as } m \rightarrow \infty,$$

and the proof of the lemma is completed. \square

Obviously, $S(0) = 0$, therefore condition (A) of the MPT is satisfied. For the remaining conditions (B) and (C), the proofs are given as follows:

Proof of (B): We distinguish the cases (i) $\omega_0^2 \geq 4\kappa$, and (ii) $\omega_0^2 < 4\kappa$, $c^2 > C(L)(4\kappa - \omega_0^2)$.

(i) With the aid of the estimate

$$\int_{-L}^L \left[\frac{a}{p+1} Q^{p+1}(z) + \frac{\kappa}{2} [Q(z+1) - Q(z)]^2 \right] dz \leq \frac{a}{p+1} C(L) \|Q\|_{\mathcal{X}_1}^{p+1} + 2\kappa \|Q\|_{L^2(-L,L)}^2,$$

we get the sufficient condition

$$\frac{c^2}{2} \|Q\|_{\mathcal{X}_1}^2 > \frac{a}{p+1} C(L) \|Q\|_{\mathcal{X}_1}^{p+1},$$

for $S(Q)$ being positive. Conclusively, for $\|Q\|_{\mathcal{X}_1}$ small enough, say $\|Q\|_{\mathcal{X}_1} = \rho$, and

$$0 < \rho < \left(\frac{(p+1)c^2}{2aC(L)} \right)^{1/(p-1)} \quad (3.19)$$

there is a $\alpha > 0$ with $S(Q) \geq \alpha$ for all $\|Q\|_{\mathcal{X}_1} = \rho$.

(ii) In this case we use the estimate

$$\int_{-L}^L \left[\frac{a}{p+1} Q^{p+1}(z) + \frac{\kappa}{2} [Q(z+1) - Q(z)]^2 \right] dz \leq \frac{a}{p+1} C(L) \|Q\|_{\mathcal{X}_1}^{p+1} + 2\kappa C(L) \|Q\|_{\mathcal{X}_1}^2.$$

Using the Poincaré inequality we derive the following condition for $S > 0$:

$$\frac{1}{2} (c^2 - (4\kappa - \omega_0^2)C(L)) \|Q\|_{\mathcal{X}_1}^2 > \frac{a}{p+1} C(L) \|Q\|_{\mathcal{X}_1}^{p+1}.$$

Hence, for $\|Q\|_{\mathcal{X}_1}$ small enough, say $\|Q\|_{\mathcal{X}_1} = \rho$, and

$$0 < \rho < \left(\frac{(p+1)}{2aC(L)} (c^2 - (4\kappa - \omega_0^2)C(L)) \right)^{1/(p-1)}, \quad (3.20)$$

there is a $\alpha > 0$ with $S(Q) \geq \alpha$ for all $\|Q\|_{\mathcal{X}_1} = \rho$.

Proof of (C): For $\|Q\|_{\mathcal{X}_0} \neq 0$ we observe that

$$S(tQ) = \int_{-L}^L \left[\frac{c^2}{2} t^2 (Q'(z))^2 + \frac{\omega_0^2}{2} t^2 Q^2(z) - \frac{a}{p+1} t^{p+1} Q^{p+1}(z) - \frac{\kappa}{2} t^2 [Q(z+1) - Q(z)]^2 \right] dz \rightarrow -\infty, \quad (3.21)$$

as $t \rightarrow \infty$. To summarise, by virtue of the MPT we may state

Theorem III.2. *Let either*

(i) $\omega_0^2 \geq 4\kappa$,

or

(ii) $\omega_0^2 < 4\kappa$ and $c^2 > C(L)(4\kappa - \omega_0^2)$.

Then the system (1.1) on the periodic lattice $-L \leq n \leq L$ with an on-site potential (3.1) has at least one nontrivial periodic TWS.

We conclude this section with the remark that *positive (negative)* periodic TWSs associated with oscillations of $Q(z)$ about the minima $\tilde{Q}_{\pm} = \pm(\omega_0^2/a)^{1/(p-1)}$ of the potential $V(Q)$, possessing reflection symmetry $V(Q) = V(-Q)$, can also be treated using the approach above. These TWSs are characterised by oscillations of Q without changes of sign and possess energy $E < 0$. One only needs to consider the system (1.7) expressed in the shifted variable $Q \rightarrow Q \mp \tilde{Q}_{\pm}$. Note that the linear operators $A_{1,2}$ are invariant with respect to the shift operator $Q \rightarrow Q + Q_0$.

B. Thresholds for the average kinetic energy of TWSs of prescribed speed

For the derivation of lower bounds of the average kinetic energy of TWSs we utilise the fixed point method outlined in Section II. In the present case, using Theorem II.4, we treat the following auxiliary linear, non-homogeneous problem

$$-Q''(z) = \frac{\kappa}{c^2} \{ [\Psi(z) - \Psi(z-1)] - [\Psi(z+1) - \Psi(z)] - \omega_0^2 \Psi(z) + a[\Psi(z)]^p \}. \quad (3.22)$$

for some arbitrary fixed $\Psi \in \mathcal{X}_1$, as an equation of the form (2.48). We have the following result.

Proposition III.1. *For any $\Psi \in \mathcal{X}_1$, the equation (3.22) has a unique solution $Q \in D(\mathcal{L}) \subset \mathcal{X}_1$.*

Proof: We reformulate Eq. (3.22) as

$$-Q''(z) = \frac{1}{c^2} \{ \kappa([A_2[\Psi(z)] - A_1[\Psi(z)]] - \omega_0^2 \Psi(z) + a[\Psi(z)]^p \} := \mathcal{F}[\Psi(z)]. \quad (3.23)$$

We may use again the estimate (3.17) for Ψ ,

$$\|\Psi^p\|_{\mathcal{X}_0}^2 = \int_{-L}^L [\Psi(z)]^{2p} dz \leq C(L) \|\Psi\|_{\mathcal{X}_1}^{2p}.$$

Then, for the right-hand side of Eq. (3.23) we get

$$\begin{aligned} \|\mathcal{F}[\Psi]\|_{\mathcal{X}_0} &\leq \frac{1}{c^2} \left\{ \kappa \|A_2[\Psi] - A_1[\Psi]\|_{\mathcal{X}_0} + C(L) \omega_0^2 \|\Psi\|_{\mathcal{X}_1} + C_0 a \|\Psi\|_{\mathcal{X}_1}^{2p} \right\} \\ &\leq \frac{1}{c^2} \left\{ 2\kappa \|\Psi\|_{\mathcal{X}_1} + C_1 \omega_0^2 \|\Psi\|_{\mathcal{X}_1} + C_0 a \|\Psi\|_{\mathcal{X}_1}^{2p} \right\}. \end{aligned} \quad (3.24)$$

Thus, $\mathcal{F}[\Psi] \in L^2[-L, L]$, and due to Theorem II.4, Eq. (3.23) has a unique solution $Q \in D(\mathcal{L})$. \square

We proceed along the lines in Section II C by adapting the steps for the use of the fixed point argument to the present case, that is replacing the hard potential $V(Q)$ in the expressions (2.52) by the soft potential (3.1).

In particular, we need the following estimate of the difference of the power nonlinearity terms

$$\int_{-L}^L |s_1^p - s_2^p|^2 \leq p^2 \int_{-L}^L \left\{ \int_0^1 |\xi|^{p-1} |s_1 - s_2| d\theta \right\}^2, \quad (3.25)$$

where $s_1, s_2 \in \mathbb{R}$, and $\xi = \theta s_1 + (1 - \theta) s_2$, $\theta \in (0, 1)$. Then, we have that

$$\begin{aligned} \int_{-L}^L |[\Psi_1(z)]^p - [\Psi_2(z)]^p|^2 dz &\leq p^2 \|\xi\|_{L^\infty}^{2(p-1)} \int_{-L}^L |\Psi_1(z) - \Psi_2(z)|^2 dz \\ &\leq p^2 \|\xi\|_{L^\infty}^{2(p-1)} C \int_{-L}^L |\Psi_1'(z) - \Psi_2'(z)|^2 dz, \end{aligned} \quad (3.26)$$

for $\xi(z) = \theta \Psi_1(z) + (1 - \theta) \Psi_2(z)$. For the norm $\|\xi\|_{L^\infty}$ we have the estimate

$$\|\xi\|_{L^\infty} \leq \theta \|\Psi_1\|_{L^\infty} + (1 - \theta) \|\Psi_2\|_{L^\infty} \leq \theta C_* \|\Psi_1\|_{\mathcal{X}_1} + (1 - \theta) C_* \|\Psi_2\|_{\mathcal{X}_1}, \quad (3.27)$$

where the constant $C_* > 0$ denotes the optimal constant of the embedding $\mathcal{X}_1 \subset L^\infty(-L, L)$. Therefore, since $\Psi_1, \Psi_2 \in \mathcal{B}_R$, we have that $\|\xi\|_{L^\infty} \leq C_* R$. Thus by using (3.26) and (3.27) we deduce the inequality

$$\|[\Psi_1(z)]^p - [\Psi_2(z)]^p\|_{\mathcal{X}_0} \leq p \sqrt{C} (C_* R)^{p-1} \|\Psi_1 - \Psi_2\|_{\mathcal{X}_1}.$$

Then the Lipschitz constant M in the equation equivalent to (2.56) in II C is determined by

$$M = \frac{1}{c^2} \left(2\kappa + C(L) \omega_0^2 + p \sqrt{C(L)} a C_*^{p-1} R^{p-1} \right). \quad (3.28)$$

Using again the fixed point argument we end up with the following statement: If

$$M \sqrt{C(L)} < 1 \quad (3.29)$$

holds, the unique fixed point is the trivial one. *Nontrivial solutions exist only if*

$$M\sqrt{C(L)} > 1. \quad (3.30)$$

From Eq. (3.28) we derive the following condition for the existence of nontrivial solutions:

$$R^2 > \left[\frac{c^2 - \sqrt{C(L)}(2\kappa + C(L)\omega_0^2)}{C(L)} \right]^{\frac{2}{p-1}} \left(\frac{1}{paC_*^{p-1}} \right)^{\frac{2}{p-1}} := T_{\text{thresh}}. \quad (3.31)$$

We conclude:

Theorem III.5. *Consider the system (1.1) on the periodic lattice of $2L$ particles, $-L \leq n \leq L$ with a soft on-site potential (3.1) with reflection symmetry $V(x) = V(-x)$. Every nontrivial periodic travelling wave solution $q_n(t) = Q(n - ct) = Q(z)$ with speed c satisfying*

$$c^2 > \sqrt{C(L)}(2\kappa + C(L)\omega_0^2) = c_{\text{crit}}^2, \quad (3.32)$$

must have average kinetic energy $T(Q) = \frac{1}{2} \int_{-L}^L U'(z)^2 dz$ satisfying the lower bound

$$T_{\text{thresh}} < 2T(Q). \quad (3.33)$$

The relation (3.33) can be regarded as a threshold value criterion for the average kinetic energy in order that travelling waves of speed $c > c^*$ exist.

C. TWSs in the case of the finite lattice with Dirichlet boundary conditions

In this section we consider solitary TWSs on the finite lattice with Dirichlet boundary conditions. Again, the $N + 1$ -oscillators of the lattice are placed equidistantly on the interval $\Omega = (-L, L)$ of length $2L$, and the boundary conditions read as

$$q_0 = q_N = 0. \quad (3.34)$$

Then the natural phase space is the standard Sobolev space

$$H_0^1(\Omega) := \left\{ Q : \Omega \rightarrow \mathbb{R} : \|Q\|_{H_0^1(\Omega)}^2 = \int_{\Omega} |Q(z)|^2 dz + \int_{\Omega} |Q'(z)|^2 dz < \infty \right\},$$

endowed with the scalar product and induced norm

$$(Q, P)_{H_0^1(\Omega)} = \int_{\Omega} Q(z)P(z)dz + \int_{\Omega} Q'(z)P'(z)dz, \quad \|Q\|_{H_0^1(\Omega)}^2 = \int_{\Omega} Q(z)^2 dz + \int_{\Omega} Q'(z)^2 dz, \quad (3.35)$$

which is the closure of the infinitely differentiable functions with compact support on Ω denoted by $C_0^\infty(\Omega)$, in the norm (3.35). In this setup, we may use the standard compact Sobolev embedding $H_0^1(\Omega) \subset L^2(\Omega)$ and the Poincaré inequality

$$\|Q\|_{L^2(\Omega)}^2 \leq C(L)\|Q'\|_{L^2(\Omega)}^2, \quad (3.36)$$

which implies the equivalence of norm (3.35) with the norm

$$\|Q\|_{H_0^1(\Omega)}^2 = \int_{\Omega} Q'(z)^2 dz. \quad (3.37)$$

It is also useful to recall that the embedding

$$H_0^1(\Omega) \subset L^p(\Omega), \quad 1 \leq p \leq \infty, \quad (3.38)$$

is compact. In this setting, solutions of (1.7) are considered as critical points of the action functional $S : H_0^1(\Omega) \rightarrow \mathbb{R}$ given by

$$S(Q) = \int_{\Omega} \left[\frac{c^2}{2} (Q'(z))^2 + \frac{\omega_0^2}{2} Q^2(z) - \frac{a}{p+1} Q^{p+1}(z) - \frac{\kappa}{2} [Q(z+1) - Q(z)]^2 \right] dz. \quad (3.39)$$

1. Existence of TWSs and the Mountain Pass Theorem

Concerning the existence of solitary TWSs we have the following:

Lemma III.3. *Let $\omega_0^2 > 4\kappa$. Then the functional (3.39) satisfies the PS-condition.*

Proof: Assume $S(Q_m)$ is bounded, i.e. $|S(Q_m)| \leq M$ for all $m \in \mathbb{N}$, and $S'(Q_m) \rightarrow 0$ as $m \rightarrow \infty$ in $(H_0^1(\Omega))^*$. For chosen $b \in (1/(p+1), 1/2)$ and m sufficiently large, we have

$$\begin{aligned}
1 + M + \|Q_m\|_{H_0^1(\Omega)} &\geq S(Q_m) - b < S'(Q_m), Q_m > \\
&= \left(\frac{1}{2} - b\right) \int_{\Omega} \left[c^2 (Q'_m(z))^2 + \omega_0^2 Q_m^2(z) - \kappa [Q_m(z+1) - Q_m(z)]^2 \right] dz \\
&\quad - \frac{a}{p+1} (1 - (p+1)b) \int_{\Omega} Q_m^p(z) dz \\
&\geq \left(\frac{1}{2} - b\right) \left[c^2 \|Q'_m\|_{L^2(\Omega)}^2 + (\omega_0^2 - 4\kappa) \|Q_m\|_{L^2(\Omega)}^2 \right] \\
&\geq \left(\frac{1}{2} - b\right) \underline{M} \|Q_m\|_{H_0^1(\Omega)}^2, \quad \underline{M} = \min \{c^2, \omega_0^2 - 4\kappa\}, \tag{3.40}
\end{aligned}$$

implying that Q_m is bounded in $H_0^1(\Omega)$. Hence, there is a subsequence of Q_m (not relabeled) and a $Q \in H_0^1(\Omega)$ such that $Q_m \rightarrow Q$ weakly in $H_0^1(\Omega)$, and by Sobolev embedding one has strong convergence $Q_m \rightarrow Q$ in $L^2(\Omega)$. Furthermore using the Hölder inequality we obtain

$$\begin{aligned}
\|Q_m - Q\|_{H_0^1(\Omega)}^2 &= \frac{1}{c^2} \langle S'(Q_m) - S'(Q), Q_m - Q \rangle \\
&\quad - \frac{1}{c^2} \int_{\Omega} [(\omega_0^2 (Q_m(z) - Q(z))^2 - \kappa [Q_m(z+1) - Q(z+1) - (Q_m(z) - Q(z))]^2 \\
&\quad - a(Q_m(z) - Q(z))^{p+1}] dz \\
&\leq \frac{1}{c^2} \langle S'(Q_m) - S'(Q), Q_m - Q \rangle + \frac{1}{c^2} ((\omega_0^2 + 4\kappa) \|Q_m - Q\|_{L^2(\Omega)} \\
&\quad + a \|Q_m^p - Q^p\|_{L^2(\Omega)}) \|Q_m - Q\|_{L^2(\Omega)}. \tag{3.41}
\end{aligned}$$

The two expressions on the right-hand side of (3.41) converge to zero; the first one by assumption and the last one by strong convergence. Thus, $\|Q_m - Q\|_{H_0^1(\Omega)} = 0$ so that $(Q_m)_{m \in \mathbb{N}}$ has a strongly convergent subsequence and the proof is finished. \square

Lemma III.4. *Assume $\omega_0^2 > 4\kappa$. Then the functional (3.39) satisfies the conditions for an application of the MPT.*

Proof: For condition (A) we note that $S(0) = 0$. Regarding the remaining conditions (B) and (C) of the MPT we have the following proofs:

Proof of (B): For any $p > 1$, the embedding (3.38) yields

$$\|Q\|_{L^{p+1}(\Omega)}^{p+1} \leq C \|Q'\|_{L^2(\Omega)}^{p+1}. \tag{3.42}$$

Then we get the estimate

$$\begin{aligned}
\int_{\Omega} \left[\frac{a}{p+1} Q^{p+1}(z) + \frac{\kappa}{2} [Q(z+1) - Q(z)]^2 \right] dz &\leq \frac{a}{p+1} C \|Q'\|_{L^2(\Omega)}^{p+1} + 2\kappa \|Q\|_{L^2(\Omega)}^2, \\
&\leq \frac{a}{p+1} C \|Q\|_{H_0^1(\Omega)}^{p+1} + 2\kappa \|Q\|_{L^2(\Omega)}^2, \tag{3.43}
\end{aligned}$$

giving rise to the condition

$$\frac{c^2}{2} \|Q'\|_{L^2(\Omega)}^2 + \frac{\omega_0^2}{2} \|Q\|_{L^2(\Omega)}^2 > \frac{a}{p+1} C \|Q\|_{H_0^1(\Omega)}^{p+1} + 2\kappa \|Q\|_{L^2(\Omega)}^2 \tag{3.44}$$

for S being positive. We infer that if

$$\frac{m}{2} \|Q\|_{H_0^1(\Omega)}^2 > \frac{a}{p+1} C \|Q\|_{H_0^1(\Omega)}^{p+1}, \quad \underline{M} = \min \{c^2, \omega_0^2 - 4\kappa\},$$

is satisfied, then $S(Q) > 0$. That is, for $\|Q\|_{H^1(\Omega)}$ small enough, say $\|Q\|_{H_0^1(\Omega)} = \rho$, and

$$0 < \rho < \left(\frac{(p+1)\underline{M}}{2aC} \right)^{1/(p-1)} \quad (3.45)$$

there is an $\alpha > 0$ with $S(Q) \geq \alpha$ for all $\|Q\|_{H_0^1(\Omega)} = \rho$.

Proof of (C): For $\|Q\|_{\mathcal{X}_0}$ one has

$$S(tQ) = \int_{\Omega} \left[\frac{c^2}{2} t^2 (Q'(z))^2 + \frac{\omega_0^2}{2} t^2 Q^2(z) - \frac{a}{p+1} t^4 Q^{p+1}(z) - \frac{\kappa}{2} t^2 [Q(z+1) - Q(z)]^2 \right] dz \rightarrow -\infty, \quad (3.46)$$

as $t \rightarrow \infty$. \square

Conclusively, by virtue of the MPT we state:

Theorem III.6. *The system (1.1) on the finite lattice with Dirichlet boundary conditions (3.34) and an on-site potential (3.1) has at least one nontrivial solitary TWS.*

a. Thresholds for the average kinetic energy of TWSs of prescribed speed: finite lattice with Dirichlet boundary conditions and infinite lattice with vanishing conditions at infinity. While in this paper we do not establish the existence of TWSs in the case of the infinite lattice, we nevertheless may prove the existence of suitable energy thresholds for such solutions if they exist, by implementing again a fixed point approach. Such thresholds also hold in the case of the finite lattice with Dirichlet boundary conditions, whose existence was shown in Theorem III.6. For the brevity of the presentation we focus only on the case of the infinite lattice with the vanishing boundary conditions

$$\lim_{|n| \rightarrow \infty} q_n = 0, \quad (3.47)$$

associated with the energy level $E = 0$. Then the natural phase space is the standard Sobolev space

$$H^1(\mathbb{R}) := \left\{ Q : \mathbb{R} \rightarrow \mathbb{R} : \|Q\|_{H^1(\mathbb{R})}^2 = \int_{\mathbb{R}} |Q(z)|^2 dz + \int_{\mathbb{R}} |Q'(z)|^2 dz < \infty \right\},$$

endowed with the scalar product and induced norm

$$(Q, P)_{H^1(\mathbb{R})} = \int_{\mathbb{R}} Q(z)P(z)dz + \int_{\mathbb{R}} Q'(z)P'(z)dz, \quad \|Q\|_{H^1(\mathbb{R})}^2 = \int_{\mathbb{R}} Q(z)^2 dz + \int_{\mathbb{R}} Q'(z)^2 dz.$$

In this case, we may use the standard continuous Sobolev embedding $H^1(\mathbb{R}) \subset L^2(\mathbb{R})$ and the inequality

$$\|Q\|_{L^2(\mathbb{R})}^2 \leq C \|Q\|_{H^1(\mathbb{R})}^2. \quad (3.48)$$

In order to derive an energy threshold criterion for the existence of solitary TWSs we conveniently write Eq. (1.7) as

$$\begin{aligned} -Q''(z) + \frac{\omega_0^2}{c^2} Q(z) &= \frac{1}{c^2} \{ \kappa ([Q(z) - Q(z-1)] - [Q(z+1) - Q(z)]) + a[Q(z)]^p \} \\ &= \mathcal{F}[Q(z)], \end{aligned} \quad (3.49)$$

so that the left side defines a strongly monotone operator on L^2 . Hence, Theorem II.4 can be applied. For every $Q \in C_0^\infty(\mathbb{R})$, setting this time $\mathcal{L}_0 Q = -Q''(z) + \omega_0^2/c^2 Q(z)$, one observes that

$$(\mathcal{L}_0 Q, Q)_{L^2(\mathbb{R})} = \|Q'\|_{L^2(\mathbb{R})}^2 + \frac{\omega_0^2}{c^2} \|Q\|_{L^2(\mathbb{R})}^2 \geq \frac{\omega_0^2}{c^2} \|Q\|_{L^2(\mathbb{R})}^2. \quad (3.50)$$

Obviously, the left-hand side of (3.50) defines an equivalent norm on $H^1(\mathbb{R})$, since

$$\omega_1^2 \|Q\|_{H^1(\mathbb{R})}^2 \leq (\mathcal{L}_0 Q, Q)_{L^2(\mathbb{R})} \leq \omega_2^2 \|Q\|_{H^1(\mathbb{R})}^2, \quad \text{for every } Q \in H^1(\mathbb{R}), \quad (3.51)$$

with $\omega_1^2 = \min \{1, \omega_0^2/c^2\}$ and $\omega_2^2 = \max \{1, \omega_0^2/c^2\}$. With these preparations we can apply Theorem II.4 with $\mathcal{X}_1 = H^1(\mathbb{R})$ and $D(\mathcal{L}_0) = H^2(\mathbb{R})$, and repeat the main lines of proofs of Propositions III.1 and III.5 with the following similarities and modifications: Lemma II.1 remains unchanged due to the continuous embedding (3.36) and the proofs of the respective counterparts of Proposition III.1 and Theorem III.5 are almost identical, however some constants will quantitatively change. Due to (3.36), here $C(L) = 1$. Moreover, the optimal constant C_* will be replaced by the optimal constant $C_{1,*}$ resulting from the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$. Then, the constant M given in (3.28) is modified as

$$M_1 = \frac{1}{c^2} \left(2\kappa + paC_{1,*}^{p-1} R^{p-1} \right). \quad (3.52)$$

Furthermore, due to (3.50) and (3.51), the estimate (2.57) becomes

$$\begin{aligned} \omega_1^2 \|Y\|_1^2 &\leq \|Y'\|_{L^2(\mathbb{R})}^2 + \omega_0^2 \|Y\|_{L^2(\mathbb{R})}^2 \leq \|\mathcal{F}[\Psi_1(z)] - \mathcal{F}[\Psi_2(z)]\|_0 \|Y\|_0 \\ &\leq M_1 \|\Psi_1 - \Psi_2\|_1 \|Y\|_1, \end{aligned}$$

and therefore

$$\|Y\|_1^2 \leq \frac{1}{2} \|Y\|_1^2 + \frac{M_1^2}{2\omega_1^2} \|\Psi_1 - \Psi_2\|_1^2. \quad (3.53)$$

From (3.53), we derive that

$$\|Y\|_1^2 = \|\mathcal{T}[\Psi_1] - \mathcal{T}[\Psi_2]\|_1^2 \leq \frac{M_1^2}{\omega_1^2} \|\Psi_1 - \Psi_2\|_1^2. \quad (3.54)$$

Hence, in the case of vanishing boundary conditions we have

Theorem III.7. *Consider the system (1.1) on the infinite lattice with vanishing boundary conditions. Every (non-trivial) solitary TWS $q_n(t) = Q(n - ct) = Q(z)$ with speed c satisfying*

$$c^2 > \frac{\kappa + 2\omega_0^2}{\omega_1^2} = c_{\text{crit}}^2, \quad (3.55)$$

must have average kinetic energy $T(Q) = \frac{1}{2} \int_{-L}^L Q'(z)^2 dz$ satisfying the lower bound

$$2T(Q) > T_{\text{thresh}} := \left[\frac{c^2 \omega_1^2 - (\kappa + 2\omega_0^2)}{paC_{1,*}^{p-1}} \right]^{\frac{2}{p-1}}. \quad (3.56)$$

In the case of the finite lattice, we have the following

Corollary III.1. *The result of Theorem III.7 remains valid in the case of the finite lattice supplemented with Dirichlet boundary conditions (3.34) (modulo the modifications of constants of Sobolev embeddings).*

We remark that further useful quantifications of the norm and energy thresholds can be provided as explicit values of the optimal constants C_* and $C_{1,*}$ of the Sobolev embeddings used in our proofs (see in [49]).

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The article has no associated data.

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Authors Contributions Statement

All authors contributed equally to the study conception, design and writing of the manuscript. Material preparation,

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