B$\ddot{a}$cklund Transformation of (3+1)-Dimensional Variable Coefficient Potential-YTSF Equation and Related Problems

Minjie Ji and Taogetusang *

[1]College of Mathematics Science, Inner Mongolia Normal University, Huhhot,010022

[2]Center for Applied Mathematics Inner Mongolia,Huhhot 010022,People's Republic of China

Abstract: Based on the Bell polynomial method, the (3+1)-dimensional variable coefficient Potential-YTSF equation is transformed into bilinear form, and the double Bell polynomial B$\ddot{a}$cklund transformation, bilinear B$\ddot{a}$cklund transformation, Lax pair and infinite conservation law of this equation are constructed. Firstly, the Lax integrability of the equation is proved by the B$\ddot{a}$cklund transformation of double Bell polynomials, and the infinite conservation law is constructed. Secondly, the exact solution of the equation is obtained by bilinear B$\ddot{a}$cklund transformation and symbolic computation system Mathematica. Finally, we illustrate their properties by making some graphs of soliton solutions.

Keywords: Bell polynomial method;bilinear B$\ddot{a}$cklund transformation;Lax pair;infinite conservation law

1 Introduction

In nonlinear science, nonlinear partial differential equation is a differential equation whose degree is higher than one. It is an important branch of modern mathematics. Whether in theory or in practical applications, nonlinear partial differential equations are used to describe problems in the fields of mechanics, control engineering, ecological and economic systems, chemical circulation systems, and epidemiology. Nonlinear partial differential equations, also known as nonlinear mathematical physics equations, nonlinear evolution equations. It is a mathematical model of nonlinear phenomena in many modern science and engineering fields such as physical chemistry, biology, atmospheric space science, nonlinear optics and astrophysics. With the development of symbolic computation, scholars have proposed many methods for solving nonlinear problems, such

*Corresponding author.Taogetusang. E-mail address:tgts@imnu.edu.cn

as Hirota bilinear method [1], Bäcklund transformation method [2], homogeneous balance method [3], F-expansion method [4], inverse scattering method [5] and so on.

$$u_{xxxz} + 4u_x u_{xz} + 2u_z u_{xx} - 4u_{xt} + 3u_{yy} = 0 \tag{1}$$

Equation (1) is often used to describe the dynamics of solitons and nonlinear waves in a domain.In reference [6],the self-Bäcklund transformation of equation (1) was constructed by using strong symmetry, and the separated variable solution of the equation was obtained. In reference [7],a new multi-periodic soliton solution of the (3+1)-dimensional Potential-YTSF equation was constructed by using Hirota bilinear form and generalized three-wave test method. Based on the Bell polynomial method,this paper will study the (3+1)-dimensional Potential-Yu-Toda-Sasa-Fukuyama (Potential-YTSF) equation with variable coefficients.

$$h_1(t)u_{xxxz} + h_2(t)u_x u_{xz} + h_3(t)u_z u_{xx} + h_4(t)u_{xt} + h_5(t)u_{yy} = 0 \tag{2}$$

The exact solutions of the double Bell polynomials under Bäcklund transformation,bilinear Bäcklund transformation and bilinear Bäcklund transformation.Where $u = u(x, y, z, t)$,when $h_1(t) = 1, h_2(t) = 4, h_3(t) = 2, h_4(t) = -4, h_5(t) = 3$,equation (2) becomes equation (1).Therefore,it is meaningful to study equation (2).

2 P-Polynomial and Bilinear Form of (3+1)-Dimensional Variable Coefficient Potential YTSF Equation

We introduce the definition of Bell polynomials and related results [8]~[11].We use Bell polynomials to construct the bilinear form of equation (2),and let

$$u = q_x, \tag{3}$$

Where $q = q(x, y, z, t)$.Substituting (3) into equation (2) to obtain

$$h_1(t)q_{xxxxz} + h_2(t)q_{xx}q_{xxz} + h_3(t)q_{xz}u_{xxx} + h_4(t)q_{xxt} + h_5(t)q_{xyy} = 0, \tag{4}$$

When $h_2(t) = h_3(t) = 3h_1(t)$,integrating equation (4) once for x yields

$$h_1(t)q_{xxxz} + 3h_1(t)q_{xx}q_{xz} + h_4(t)q_{xt} + h_5(t)q_{yy} = 0, \tag{5}$$

Then the equation has the following P-polynomial representation

$$E(q) = h_1(t)P_{xxxz}(q) + h_4(t)P_{xt}(q) + h_5(t)P_{yy}(q) = 0. \tag{6}$$

Let $q=2\ln f$,equation (6) be in bilinear form

$$(h_1(t)D_x^3D_z+h_4(t)D_xD_t+h_5(t)D_y^2)f\cdot f=0. \tag{7}$$

Where f is an undetermined function about $x,y,z,t.D_x^3D_z,D_xD_t,D_y^2$ satisfies the D-operator. The definition of D-operator is:

$$D_x^mD_z^n(f\cdot g)=(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'})^m(\frac{\partial}{\partial z}-\frac{\partial}{\partial z'})^nf(x,z)g(x,z)\mid_{x=x',z=z'} \tag{8}$$

$$D_x^mD_t^n(f\cdot g)=(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'})^m(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'})^nf(x,t)g(x,t)\mid_{x=x',t=t'} \tag{9}$$

3 Bi-Bell polynomial B$\ddot{a}$cklund transformation, bilinear B$\ddot{a}$cklund transformation, Lax pair and infinite conservation law for (3+1)-dimensional Potential-YTSF equation with variable coefficients

3.1 Bi-Bell polynomial B$\ddot{a}$cklund transformation for (3+1)-dimensional Potential-YTSF equation with variable coefficients

We use Bell polynomials to construct the bilinear Bäcklund transformation of the (3+1)-dimensional Potential-YTSF equation.Let q,q' be the solution of equation (5),then

$$E(q)=h_1(t)q_{xxxz}+3h_1(t)q_{xx}q_{xz}+h_4(t)q_{xt}+h_5(t)q_{yy}=0, \tag{10}$$

$$E(q')=h_1(t)q'_{xxxz}+3h_1(t)q'_{xx}q'_{xz}+h_4(t)q'_{xt}+h_5(t)q'_{yy}=0, \tag{11}$$

Subtracting and transforming the two forms

$$q=w-v,q'=w+v, \tag{12}$$

we get

$$\begin{aligned} E(q')-E(q)=h_1(t)(q'-q)_{xxxz}+\frac{3}{2}h_1(t)[(q'-q)_{xz}(q'+q)_{xx}+(q'+q)_{xz}(q'-q)_{xx}] \\ +h_4(t)(q'-q)_{xt}+h_5(t)(q'-q)_{yy} \\ =2h_1(t)v_{xxxz}+2h_1(t)[3v_{xz}w_{xx}+3w_{xz}v_{xx}]+2h_4(t)v_{xt}+2h_5(t)v_{yy}=0. \end{aligned} \tag{13}$$

Selecting constraint conditions

$$\mathcal{Y}_{xz}-M\mathcal{Y}_x=\lambda, \tag{14}$$

By differentiating both sides of equation (14) with respect to x,we have

$$w_{xxz} + v_{xx}v_z + v_x v_{xz} - Mv_{xx} = 0. \tag{15}$$

Substituting equations (14) and (15) into equation (13),we get

$$\begin{aligned} E(q') - E(q) &= h_1(t)v_{xxxz} + h_1(t)[3v_{xz}w_{xx} + 3w_{xz}v_{xx}] + h_4(t)v_{xt} + h_5(t)v_{yy} \\ &= [h_1(t)\mathcal{Y}_{xxx} + \alpha\mathcal{Y}_y + \beta\mathcal{Y}_x]_z + [h_4(t)\mathcal{Y}_t - \alpha\mathcal{Y}_z - 3h_1(t)\lambda\mathcal{Y}_x]_x \\ &+ [h_5(t)\mathcal{Y}_y - \beta\mathcal{Y}_z + \mu]_y + 3h_1(t)v_{xx}w_{xz} - 3h_1(t)v_x w_{2xz} - 3h_1(t)v_x^2 v_{xz} = 0. \end{aligned} \tag{16}$$

Therefore,the (3+1)-dimensional variable coefficient Potential-YTSF equation has a double Bell polynomial Bӓcklund transformation

$$\begin{cases} \mathcal{Y}_{xz} - M\mathcal{Y}_x = \lambda, \\ h_1(t)\mathcal{Y}_{xxx} + \alpha\mathcal{Y}_y + \beta\mathcal{Y}_x = 0, \\ h_4(t)\mathcal{Y}_t - \alpha\mathcal{Y}_z - 3h_1(t)\lambda\mathcal{Y}_x = 0, \\ h_5(t)\mathcal{Y}_y - \beta\mathcal{Y}_z + \mu = 0. \end{cases} \tag{17}$$

Where $\lambda, \alpha = \beta, \mu$ is an arbitrary constant.

3.2 Bilinear Bӓcklund Transformation of (3+1)-Dimensional Variable Coefficient Potential-YTSF Equation

In the transformation

$$v = ln\frac{f}{g}, w = lnfg \tag{18}$$

After that,we can get the bilinear form of the Bӓcklund transformation

$$\begin{cases} (D_x D_z - MD_x - \lambda)f \cdot g = 0, \\ (h_1(t)D_x^3 + \alpha D_y + \beta D_x)f \cdot g = 0, \\ (h_4(t)D_t - \alpha D_z - 3h_1(t)\lambda D_x)f \cdot g = 0, \\ (h_5(t)D_y - \beta D_z + \mu)f \cdot g = 0. \end{cases} \tag{19}$$

3.3 Lax Integrability of (3+1)-dimensional Potential YTSF Equation

Here we consider the following transformation

$$v = \ln\varphi, w = q + \ln\varphi \tag{20}$$

Next,according to the double Bell polynomial Bäcklund transformation,the linear system of the equation is obtained

$$\begin{aligned}
&h_1(t)\varphi_{xxx}+3h_1(t)u_x\varphi_x+\alpha\varphi_y+\beta\varphi_x=0,\\
&\varphi q_{xz}+\varphi_{xz}-M\varphi_x-\lambda\varphi=0,\\
&h_4(t)\varphi_t-\alpha\varphi_z+3h_1(t)\lambda\varphi_x=0,\\
&h_5(t)\varphi_y-\beta\varphi_z+\mu\varphi=0.
\end{aligned}\tag{21}$$

The corresponding Lax Operator from equation (21)

$$\begin{aligned}
&L_1(\varphi)=M\varphi_x+q_{xz}\varphi+\varphi_x\varphi_z-\lambda\varphi,\\
&L_2(\varphi)=\varphi_t-\frac{h_1(t)}{h_4(t)}\varphi_{xxx}-\frac{3h_1(t)}{h_4(t)}u\varphi_x-\frac{h_5(t)}{h_4(t)}\partial_z^{-1}\varphi_{yy}.
\end{aligned}\tag{22}$$

It satisfies the compatibility condition $\varphi_{txz}=\varphi_{xzt}$.So equation (22) is the Lax pair of the equation.

3.4 The infinite conservation law of (3+1)-dimensional Potential YTSF equation

Take transformation

$$2\eta=q'_x-q_x\tag{23}$$

Based on equations (23) and (12),we get

$$v_x=\eta,w_x=q_x+\eta\tag{24}$$

Substituting equation (24) into (17) to obtain

$$\begin{aligned}
&q_{xz}+\eta_z+\eta\partial_x^{-1}\eta_z-M\eta=\lambda,\\
&\partial_z[h_1(t)\eta_{xx}+3h_1(t)\eta(q_x+\eta)_x+h_1(t)\eta^3+\alpha\partial_x^{-1}\eta_y+\alpha\eta]\\
&+\partial_x[h_4(t)\partial_x^{-1}\eta_t-\alpha\partial_x^{-1}\eta_z-3h_1(t)\lambda\eta]+\partial_y[h_5(t)\partial_x^{-1}\eta_y-\alpha\partial_x^{-1}\eta_z]=0,
\end{aligned}\tag{25}$$

Where

$$M=-\varepsilon,\lambda=\varepsilon^2,\eta=\varepsilon+\sum_{n=1}^{\infty}\mathcal{F}_n(u,u_x,u_{xx},\cdots)\varepsilon^{-n}\tag{26}$$

Substituting equation (26) into equation (25),we get

$$\mathcal{F}_1=q_{xz}=u_z,\mathcal{F}_2=\mathcal{F}_{1,z}=q_{xzz}=u_{zz}\tag{27}$$

$$\mathcal{F}_{n+1}=\mathcal{F}_{n,z}+\partial_x^{-1}\mathcal{F}_{n,z}+\sum_{k=1}^{n}\mathcal{F}_k(\mathcal{F}_{n-k}+\partial_x^{-1}\mathcal{F}_{n-k,z})\tag{28}$$

Substituting equation (26) into the second equation in equation (25),the infinite conservation law of equation (2) can be expressed as

$$\mathcal{L}_{n,z} + \mathcal{M}_{n,x} + \mathcal{G}_{n,y} + \mathcal{P}_{n,t} = 0(n = 1, 2, 3, \cdots) \tag{29}$$

$\mathcal{L}_n, \mathcal{M}_n, \mathcal{G}_n, \mathcal{P}_n$ can be expressed as

$$\begin{aligned} \mathcal{L}_n = h_1(t)(&\mathcal{F}_{n,xx} + 3\mathcal{F}_{n+1,x} + 3q_{xx}\mathcal{F}_n + 3\mathcal{F}_{n+2} + 3\sum_{k=1}^{n}\mathcal{F}_k\mathcal{F}_{n-k,x} \\ &+ 3\sum_{k=1}^{n+1}\mathcal{F}_k\mathcal{F}_{n+1-k} + \sum_{i+j+k=n}\mathcal{F}_i\mathcal{F}_j\mathcal{F}_k) + \alpha\partial_x^{-1}\mathcal{F}_{n,y} + \alpha\mathcal{F}_n. \end{aligned} \tag{30}$$

$$\mathcal{M}_n = -3h_1(t)\lambda\mathcal{F}_n - \alpha\partial_x^{-1}\mathcal{F}_{n,z}, \mathcal{G}_n = h_5(t)\partial_x^{-1}\mathcal{F}_{n,y} - \alpha\partial_x^{-1}\mathcal{F}_{n,z}, \mathcal{P}_n = h_4(t)\mathcal{F}_n \tag{31}$$

4 Soliton Solutions and Properties of (3+1)-dimensional Potential YTSF Equation

In order to obtain the soliton solution of equation (2),we set

$$\begin{aligned} &f = f(x, y, z, t) = 1 + f_1\varepsilon + f_2\varepsilon^2 + f_3\varepsilon^3 + \cdots, \\ &g = g(x, y, z, t) = 1 + g_1\varepsilon + g_2\varepsilon^2 + g_3\varepsilon^3 + \cdots, \\ &f_i = f_i(x, y, z, t), g_i = g_i(x, y, z, t)(i = 1, 2, 3, \cdots). \end{aligned} \tag{32}$$

Substituting equation (32) into the bilinear equation (19),and let the coefficient of each power of ε be 0.The coefficients of ε is

$$\begin{aligned} &-\lambda f_1 - \lambda g_1 - Mf_{1x} + Mg_{1x} + f_{1xz} + g_{1xz} = 0, \\ &\alpha f_{1y} - \alpha g_{1y} + \alpha f_{1x} - \alpha g_{1x} + h_1(t)f_{1xxx} - h_1(t)g_{1xxx} = 0, \\ &h_4(t)f_{1t} - h_4(t)g_{1t} - \alpha f_{1z} + \alpha g_{1z} - 3\lambda h_1(t)f_{1x} + 3\lambda h_1(t)g_{1x} = 0, \\ &\mu f_1 + \mu g_1 - \alpha f_{1z} + \alpha g_{1z} + h_5(t)f_{1y} - h_5(t)g_{1y} = 0. \end{aligned} \tag{33}$$

The coefficients of ε^2 is

$$\begin{aligned}
&-\lambda f_2-\lambda g_1f_1-\lambda g_2-Mg_1f_{1x}-g_{1z}f_{1x}-Mf_{2x}+Mf_1g_{1x}-f_{1z}g_{1x}\\
&+Mg_{2x}+g_1f_{1xz}+f_{2xz}+f_1g_{1xz}+g_{2xz}=0,\\
&\alpha g_1f_{1y}+\alpha f_{2y}-\alpha f_1g_{1y}-\alpha g_{2y}+\alpha g_1f_{1x}+\alpha f_{2x}-\alpha f_1g_{1x}-\alpha g_{2x}-3h_1(t)g_{1x}f_{1xx}\\
&+3h_1(t)f_{1x}g_{1xx}+h_1(t)g_1f_{1xxx}+h_1(t)f_{2xxx}-h_1(t)f_1g_{1xxx}-h_1g_{2xxx}=0,\\
&h_4(t)g_1f_{1t}+h_4(t)f_{2t}-h_4(t)f_1g_{1t}-h_4(t)g_{2t}-\alpha g_1f_{1z}-\alpha f_{2z}+\alpha f_1g_{1z}+\alpha g_{2z}\\
&-3\lambda h_1(t)g_1f_{1x}-3\lambda h_1(t)f_{2x}+3\lambda h_1(t)f_1g_{1x}+3\lambda h_1(t)g_{2x}=0,\\
&\mu f_2+\mu f_1g_1+\mu g_2-\alpha g_1f_{1z}-\alpha f_{2z}+\alpha f_1g_{1z}+\alpha g_{2z}+h_5(t)g_1f_{1y}+h_5(t)f_{2y}\\
&-h_5(t)f_1g_{1y}-h_5(t)g_{2y}=0.
\end{aligned}\tag{34}$$

The coefficients of ε^3 is

$$\begin{aligned}
&-\lambda f_3-\lambda f_2g_1-\lambda f_1g_2-\lambda g_3-Mg_2f_{1x}-g_{2z}f_{1x}-Mg_1f_{2x}-g_{1z}f_{2x}\\
&-Mf_{3x}+Mf_2g_{1x}-f_{2z}g_{1x}+Mf_1g_{2x}-f_{1z}g_{2x}+Mg_{3x}+g_2f_{1xz}+g_1f_{2xz}\\
&+f_{3xz}+f_2g_{1xz}+f_1g_{2xz}+g_{3xz}=0,\\
&\alpha g_2f_{1y}+\alpha g_1f_{2y}+\alpha f_{3y}-\alpha f_2g_{1y}-\alpha f_1g_{2y}-\alpha g_{3y}+\alpha g_2f_{1x}+\alpha g_1f_{2x}+\alpha f_{3x}\\
&-\alpha f_2g_{1x}-\alpha f_1g_{2x}-\alpha g_{3x}-3h_1(t)g_{2x}f_{1xx}-3h_1(t)g_{1x}f_{2xx}+3h_1(t)f_{2x}g_{1xx}\\
&+3h_1(t)f_{1x}g_{2xx}+h_1(t)g_2f_{1xxx}+h_1(t)g_1f_{2xxx}+h_1(t)f_{3xxx}-h_1(t)f_2g_{1xxx}\\
&-h_1(t)f_1g_{2xxx}-h_1(t)g_{3xxx}=0,\\
&h_4(t)g_2f_{1t}+h_4(t)g_1f_{2t}+h_4(t)f_{3t}-h_4(t)f_2g_{1t}-h_4(t)f_1g_{2t}-h_4(t)g_{3t}-\alpha g_2f_{1z}\\
&-\alpha g_1f_{2z}-\alpha f_{3z}+\alpha f_2g_{1z}+\alpha f_1g_{2z}+\alpha g_{3z}-3\lambda h_1(t)g_2f_{1x}-3\lambda h_1(t)g_1f_{2x}\\
&-3\lambda h_1(t)f_{3x}+3\lambda h_1(t)f_2g_{1x}+3\lambda h_1(t)f_1g_{2x}+3\lambda h_1(t)g_{3x}=0,\\
&\mu f_3+\mu f_2g_1+\mu f_1g_2+\mu g_3-\alpha g_2f_{1z}-\alpha g_1f_{2z}-\alpha f_{3z}+\alpha f_2g_{1z}+\alpha f_1g_{2z}+\alpha g_{3z}\\
&+h_5(t)g_2f_{1y}+h_5(t)g_1f_{2y}+h_5(t)f_{3y}-h_5(t)f_2g_{1y}-h_5(t)f_1g_{2y}-h_5(t)g_{3y}=0.
\end{aligned}\tag{35}$$

First calculate the single solitary wave solution of the equation,we let

$$f_1=m(t)\exp(\eta_1)+\gamma(t),g_1=n(t)\exp(\theta_1)+\rho(t)\tag{36}$$

where $\eta_1=k_1x+l_1y+s_1z+w_1(t),\theta_1=p_1x+q_1y+r_1z+\varpi_1(t)$,substituteing it into equation (33) to get

$$k_1=\frac{1}{2}\sqrt{-\frac{2\alpha}{3h_1(t)}+\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}+\frac{\Theta_1}{3\sqrt[3]{2}h_1(t)h_5(t)}}\tag{37}$$

$$-\frac{1}{2}\sqrt{-\frac{4\alpha}{3h_1(t)}-\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}-\frac{\Theta_1}{3\sqrt[3]{2}h_1(t)h_5(t)}-\frac{2\Theta_3}{h_1(t)h_5(t)\sqrt{\Theta_4}}}$$

$$p_1=\frac{1}{2}\sqrt{-\frac{2\alpha}{3h_1(t)}+\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}+\frac{\Theta_1^{\frac{1}{3}}}{3\sqrt[3]{2}h_1(t)h_5(t)}}$$

$$-\frac{1}{2}\sqrt{-\frac{4\alpha}{3h_1(t)}-\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}-\frac{\Theta_1}{3\sqrt[3]{2}h_1(t)h_5(t)}+\frac{2\Theta_3}{h_1(t)h_5(t)\sqrt{\Theta_4}}}$$

$$r_1=\frac{1}{\alpha^2}[-\frac{\alpha(3M\alpha+\mu)}{4}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_4}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2h_5^2(t)\sqrt{\Theta_4}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}$$

$$-\frac{1}{8}h_1(t)h_5(t)\Theta_4^{\frac{3}{2}}+\frac{1}{4}\alpha h_5(t)\sqrt{\Theta_5}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_5}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$+\frac{\alpha^2h_5(t)\sqrt{\Theta_5}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}+\frac{1}{8}h_1(t)h_5(t)\Theta_5^{\frac{3}{2}}]$$

$$s_1=\frac{1}{\alpha^2}[\frac{\alpha(3M\alpha+\mu)}{4}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_4}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2h_5^2(t)\sqrt{\Theta_4}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}$$

$$-\frac{1}{8}h_1(t)h_5(t)\Theta_4^{\frac{3}{2}}+\frac{1}{4}\alpha h_5(t)\sqrt{\Theta_5}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_5}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$+\frac{\alpha^2h_5(t)\sqrt{\Theta_5}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}+\frac{1}{8}h_1(t)h_5(t)\Theta_5^{\frac{3}{2}}]$$

$$l_1=\frac{1}{\alpha^2}[\frac{3\alpha(M\alpha-\mu)}{4h_5(t)}+\frac{3\alpha^2\lambda h_1(t)\sqrt{\Theta_4}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2h_5(t)\sqrt{\Theta_4}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}h_5(t)}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}$$

$$-\frac{1}{8}h_1(t)\Theta_4^{\frac{3}{2}}+\frac{1}{4}\alpha\sqrt{\Theta_5}+\frac{3\alpha^2\lambda h_1(t)\sqrt{\Theta_5}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$+\frac{\alpha^2h_5(t)\sqrt{\Theta_5}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}h_5(t)}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}+\frac{1}{8}h_1(t)\Theta_5^{\frac{3}{2}}]$$

$$q_1=\frac{1}{\alpha^2}[\frac{3\alpha(\mu-M\alpha)}{4h_5(t)}+\frac{3\alpha^2\lambda h_1(t)\sqrt{\Theta_4}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2h_5(t)\sqrt{\Theta_4}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}h_5(t)}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}$$

$$-\frac{1}{8}h_1(t)\Theta_4^{\frac{3}{2}}+\frac{1}{4}\alpha\sqrt{\Theta_5}+\frac{3\alpha^2\lambda h_1(t)\sqrt{\Theta_5}}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$+\frac{\alpha^2h_5(t)\sqrt{\Theta_5}}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}h_5(t)}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}+\frac{1}{8}h_1(t)\Theta_5^{\frac{3}{2}}]$$

$$w_1'(t)=\frac{1}{\alpha h_4(t)m(t)}[\frac{\alpha m(t)(3M\alpha+\mu)}{4}+\frac{3}{2}\alpha\lambda h_1(t)m(t)\sqrt{\Theta_4}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_4}m(t)}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$+\frac{\alpha^2 h_5^2(t)\sqrt{\Theta_4}m(t)}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}m(t)-\frac{1}{8}h_1(t)h_5(t)\Theta_4^{\frac{3}{2}}m(t)$$

$$+\frac{1}{4}\alpha h_5(t)\sqrt{\Theta_5}m(t)+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_5}m(t)}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2 h_5^2(t)\sqrt{\Theta_5}m(t)}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}m(t)+\frac{1}{8}h_1(t)h_5(t)\Theta_5^{\frac{3}{2}}m(t)-\alpha h_4(t)m'(t)]$$

$$\varpi_1'(t)=\frac{1}{\alpha h_4(t)n(t)}[-\frac{\alpha n(t)(3M\alpha+\mu)}{4}+\frac{3}{2}\alpha\lambda h_1(t)n(t)\sqrt{\Theta_4}+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_4}n(t)}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$+\frac{\alpha^2 h_5^2(t)\sqrt{\Theta_4}n(t)}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_4}n(t)-\frac{1}{8}h_1(t)h_5(t)\Theta_4^{\frac{3}{2}}n(t)$$

$$+\frac{1}{4}\alpha h_5(t)\sqrt{\Theta_5}n(t)+\frac{3\alpha^2\lambda h_1(t)h_5(t)\sqrt{\Theta_5}n(t)}{\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}+\frac{\alpha^2 h_5^2(t)\sqrt{\Theta_5}n(t)}{4\sqrt[3]{4}\Theta_1^{\frac{1}{3}}}$$

$$-\frac{3}{2}\alpha\lambda h_1(t)n(t)\sqrt{\Theta_5}+\frac{1}{8\sqrt[3]{2}}\Theta_1^{\frac{1}{3}}\sqrt{\Theta_5}n(t)+\frac{1}{8}h_1(t)h_5(t)\Theta_5^{\frac{3}{2}}n(t)-\alpha h_4(t)n'(t)]$$

$$\Theta_1=27\Theta_3^2h_1(t)h_5(t)-72\alpha^3\lambda h_1h_5^2+2\alpha^3h_5^3(t)+\sqrt{\Theta_2},\Theta_3=M\alpha^2-\alpha\mu \tag{38}$$

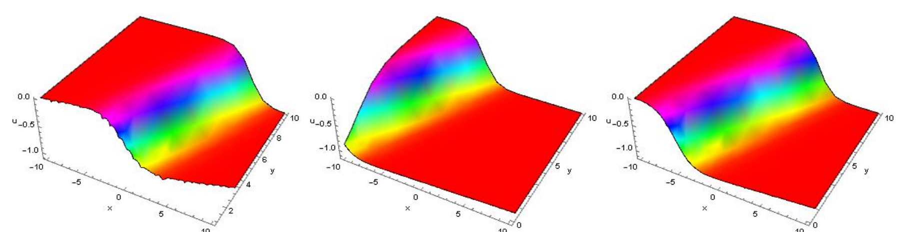
$$\Theta_2=-4h_5^3(t)[12\alpha^2\lambda h_1(t)+\alpha^2h_5(t)]^3+h_5^2(t)[27\Theta_3^2h_1(t)-72\alpha^3\lambda h_1(t)h_5(t)+2\alpha^3h_5^2(t)]^2 \tag{39}$$

$$\Theta_4=-\frac{2\alpha}{3h_1(t)}+\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}+\frac{\Theta_1^{\frac{1}{3}}}{3\sqrt[3]{2}h_1(t)h_5(t)} \tag{40}$$

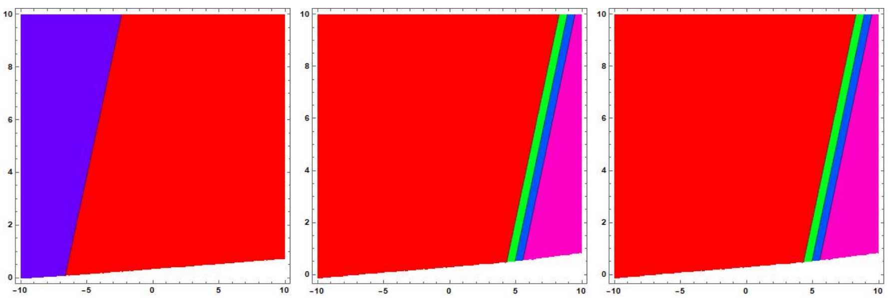
$$\Theta_5=-\frac{4\alpha}{3h_1(t)}-\frac{\sqrt[3]{2}\alpha^2(12\lambda h_1(t)+h_5(t))}{3h_1(t)\Theta_1^{\frac{1}{3}}}-\frac{\Theta_1}{3\sqrt[3]{2}h_1(t)h_5(t)}+\frac{2\Theta_3}{h_1(t)h_5(t)\sqrt{\Theta_4}} \tag{41}$$

Supposing that $f_2=g_2=0, f_i=g_i=0(i=3,4,\cdots)$,the single soliton solution of the variable coefficient (3+1)-dimensional Potential-YTSF equation is

$$u=[\ln(\frac{1+f_1}{1+g_1})]_x=\frac{m(t)\exp(\eta_1)[k_1-\exp(\theta_1)n(t)(p_1-k_1)]-p_1n(t)\exp(\theta_1)(1+\gamma(t))}{(1+n(t)\exp(\theta_1))(1+m(t)\exp(\eta_1)+\gamma(t))} \tag{42}$$

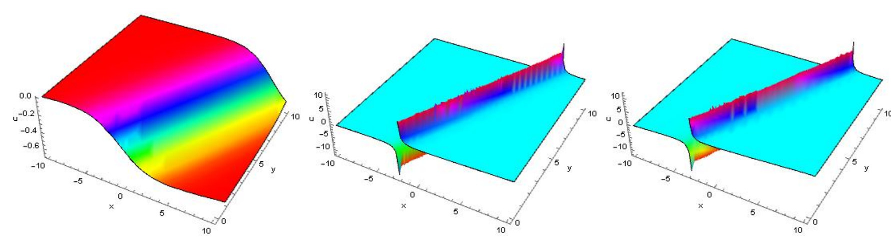


(a)t=0 (b)t=0.5 (c)t=-0.5

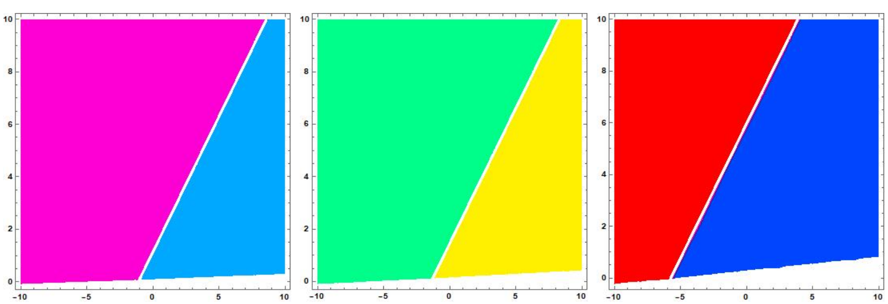


(d) (e) (f)

Fig.1$h_1(t) = \arctan(t), n(t) = t, m(t) = t^2, \gamma(t) = \sin(t), h_5(t) = \cos(t), r_1 = -1, k_1 = 3$

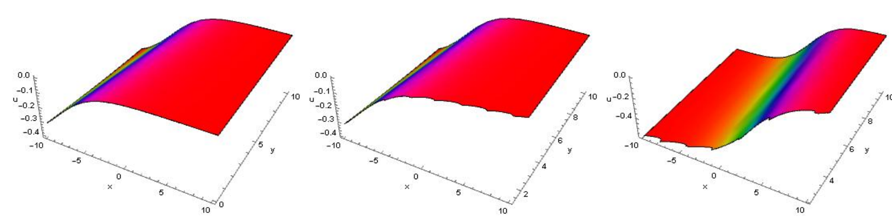


(a)t=0 (b)t=0.5 (c)t=-0.5

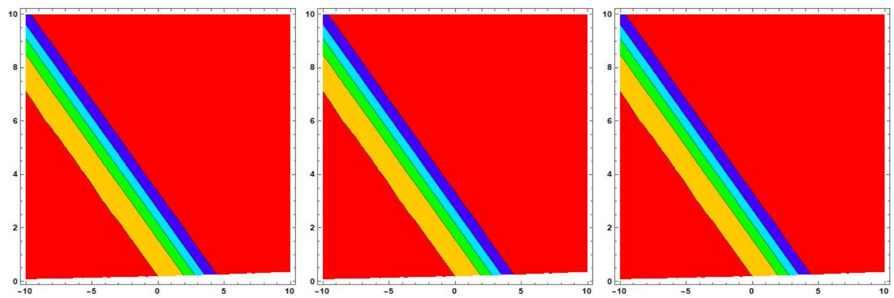


(d) (e) (f)

Fig.2$h_1(t) = \arctan(t), n(t) = \exp(t)\arctan(t), m(t) = t^2\cos(t), \gamma(t) = \sin(t)\cosh(t), h_5(t) = \cos(t), r_1 = -1, k_1 = 3,$



(a)t=0 (b)t=0.5 (c)t=-0.5



(d) (e) (f)

Fig.3$h_4(t)=t^2, n(t)=\sin(t)+\arctan(t), m(t)=t^2+\cos(t), \gamma(t)=t+\sin(t), h_5(t)=\cos(t), r_1=-1, k_1=3,$

Next we calculate the double solitary wave solution of the equation,let

$$f_1=m_1(t)\exp(\eta_1)+m_2(t)\exp(\eta_2)+\gamma(t), f_2=m_{12}(t)\exp(\eta_1+\eta_2) \tag{43}$$

$$g_1=n_1(t)\exp(\theta_1)+n_2(t)\exp(\theta_2)+\rho(t), g_2=n_{12}(t)\exp(\theta_1+\theta_2) \tag{44}$$

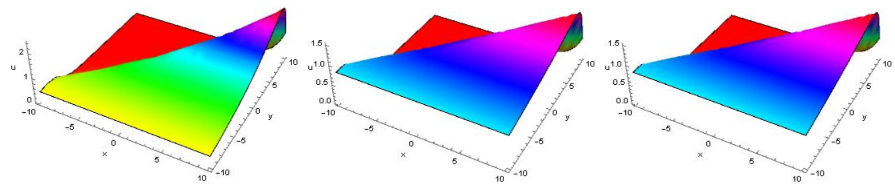
where $\eta_i=k_ix+l_iy+s_iz+w_i(t), \theta_i=p_ix+q_iy+r_iz+\varpi_i(t)(i=1,2)$,substituting equation (45) and (46) into equation (36)

$$\begin{aligned}
&q_1=0, q_2=0, s_1=M, s_2=0, r_1=0, r_2=0, k_2=0, p_1=0, p_2=0, l_2=0, \qquad (45)\\
&\mu=0, \lambda=0, l_1=-k_1-\frac{k_1^3Mh_1(t)m_1(t)\gamma(t)}{h_4(t)[\gamma(t)(m_1'(t)+m_1(t)w_1'(t))-m_1(t)\gamma'(t)]},\\
&\rho(t)=\frac{\gamma(t)\rho'(t)}{\gamma'(t)}, w_2(t)=\int[-\frac{m_2'(t)}{m_2(t)}+\frac{\gamma'(t)}{\gamma(t)}]dt, \varpi_2(t)=\int[-\frac{n_2'(t)}{n_2(t)}+\frac{\gamma'(t)}{\gamma(t)}]dt\\
&n_1(t)=-\frac{\gamma(t)n_1'(t)}{\gamma(t)\varpi_1'(t)-\gamma'(t)}, n_{12}(t)=\frac{n_2(t)n_{12}'(t)\gamma(t)}{\gamma(t)n_2'(t)-n_2(t)\gamma(t)z_1'(t)-n_2(t)\gamma'(t)}\\
&m_{12}(t)=\frac{m_1(t)m_2(t)\gamma(t)m_{12}'(t)}{m_2(t)\gamma(t)m_1'(t)+m_1(t)\gamma(t)m_2'(t)-2m_1(t)m_2(t)\gamma'(t)},\\
&h_5(t)=-\frac{h_4^2(t)[\gamma(t)(m_1'(t)+m_1(t)w_1'(t))-m_1(t)\gamma'(t)]^2}{k_1m_1(t)\gamma(t)[k_1^2Mm_1(t)h_1(t)\gamma(t)+h_4(t)[\Theta_6]]}\\
&\Theta_6=\gamma(t)(m_1'(t)+m_1(t)w_1'(t))-m_1(t)\gamma'(t) \qquad (46)
\end{aligned}$$

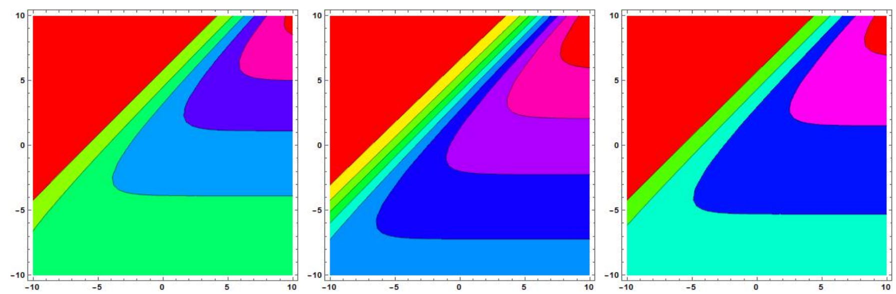
Supposing that $f_3=g_3=0, f_i=g_i=0(i=4,5,\cdots), \varepsilon=1$,the double soliton solution of the (3+1)-dimensional variable coefficient Potential-YTSF equation is

$$u=[\ln(\frac{1+f_1+f_2}{1+g_1+g_2})]_x \tag{47}$$

$$= \frac{k_1 m_1(t)\exp(\eta_1) + \exp(\eta_2)[(k_1+k_2)m_{12}(t)\exp(\eta_1) + k_2 m_2(t)]}{1 + m_1(t)\exp(\eta_1) + m_{12}(t)\exp(\eta_1+\eta_2) + m_2(t)\exp(\eta_2) + \gamma(t)}$$
$$+ \frac{p_1 n_1(t)\exp(\theta_1) + \exp(\theta_2)[\exp(\theta_1)(p_1+p_2)n_{12}(t) + p_2 n_2(t)]}{1 + n_1(t)\exp(\theta_1) + n_{12}(t)\exp(\theta_1+\theta_2) + n_2(t)\exp(\theta_2) + \rho(t)}$$

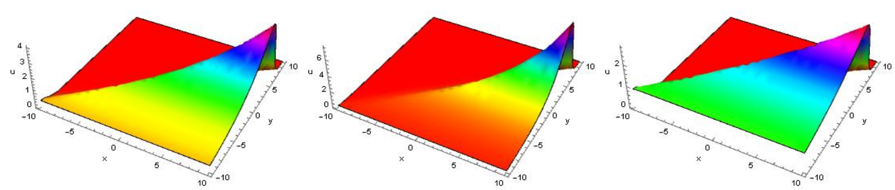


(a)t=5 (b)t=5.5 (c)t=6

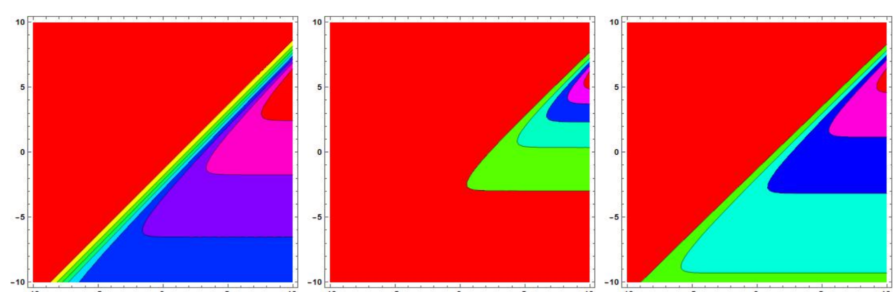


(d) (e) (f)

Fig.4$w_1(t) = \cos(t), h_1(t) = \sin(t), \gamma(t) = \arctan(t), h_4(t) = t^2, m_2(t) = \exp(t), m_{12}(t) = \cosh(t), m_1(t) = t, z = 0, k_1 = 2, M = 4$

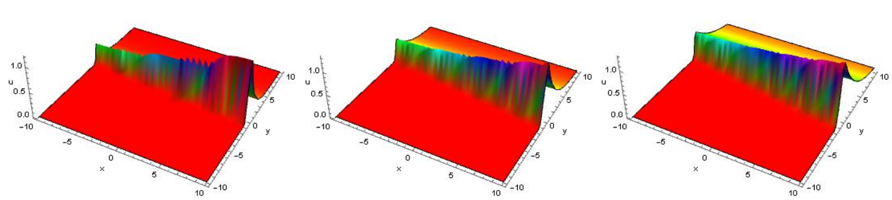


(a)t=2 (b)t=2.3 (c)t=2.8

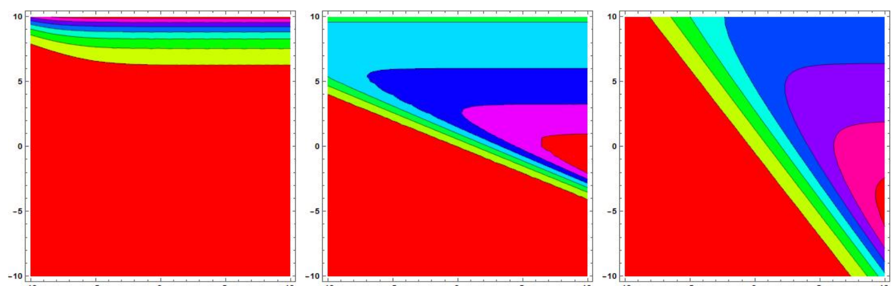


(d) (e) (f)

Fig.5$w_1(t) = t\cos(t), h_1(t) = \sin(t)\cosh(t), \gamma(t) = \arctan(t), h_4(t) = t^2\arctan(t), m_2(t) = \exp(t)\cos(t), m_{12}(t) = t\cosh(t), m_1(t) = t, z = 0, k_1 = 2, M = 4$



(a)t=0 (b)t=0.6 (c)t=1



(d) (e) (f)

Fig.6$w_1(t) = t + \cos(t), h_1(t) = \sin(t) + \cosh(t), \gamma(t) = \arctan(t), h_4(t) = t^2 + \arctan(t), m_2(t) = \exp(t) + \cos(t), m_{12}(t) = t + \cosh(t), m_1(t) = t, z = 0$

Next,we calculate the three-soliton solution of the equation.We let

$$\begin{aligned}
&f_1 = m_1(t)\exp(\eta_1) + m_2(t)\exp(\eta_2) + m_3(t)\exp(\eta_3) + \gamma(t),\\
&f_2 = m_{12}(t)\exp(\eta_1+\eta_2) + m_{13}(t)\exp(\eta_1+\eta_3) + m_{23}(t)\exp(\eta_2+\eta_3),\\
&f_3 = m_{123}(t)\exp(\eta_1+\eta_2+\eta_3),
\end{aligned} \tag{48}$$

$$\begin{aligned}
&g_1 = n_1(t)\exp(\theta_1) + n_2(t)\exp(\theta_2) + n_3(t)\exp(\theta_3) + \rho(t),\\
&g_2 = n_{12}(t)\exp(\theta_1+\theta_2) + n_{13}(t)\exp(\theta_1+\theta_3) + n_{23}(t)\exp(\theta_2+\theta_3),\\
&g_3 = n_{123}(t)\exp(\theta_1+\theta_2+\theta_3)
\end{aligned} \tag{49}$$

where $\eta_i = k_i x + l_i y + s_i z + w_i(t), \theta_i = p_i x + q_i y + r_i z + \varpi_i(t)(i = 1,2,3)$,substituting equations (51) and (52) into equation (38)

$$k_2 = 0, k_3 = -k_1, p_1 = 0, p_2 = 0, p_3 = 0, \tag{50}$$

$$m_2(t) = -\frac{m_1(t)m_{12}(t)\gamma(t)\rho(t)m_2'(t)}{-m_1(t)\gamma(t)\rho(t)m_{12}'(t) + m_{12}(t)[-2m_1(t)\rho(t)\gamma'(t) + \gamma(t)(\Theta_7)]}$$

$$n_2(t) = \frac{n_{123}(t)n_{13}(t)\gamma(t)\rho(t)n_2'(t)}{-n_{123}(t)\gamma(t)\rho(t)n_{13}'(t) + n_{13}(t)[n_{123}(t)\rho(t)\gamma'(t) + \gamma(t)(\Theta_8)]},$$

$$m_3(t) = \frac{m_1(t)m_{12}(t)m_{23}(t)m_3'(t)}{m_{12}(t)m_{23}(t)m_1'(t) - m_1(t)m_{23}(t)m_{12}'(t) + m_1(t)m_{12}(t)m_{23}'(t)}$$

$$n_3(t) = \frac{n_1(t)n_{13}(t)\gamma(t)\rho(t)n_3'(t)}{n_1(t)\gamma(t)\rho(t)n_{13}'(t) - n_{13}(t)[n_1(t)\rho(t)\gamma'(t) + \gamma(t)(\rho(t)n_1'(t) - 2n_1(t)\rho'(t))]}$$

$$n_{12}(t) = -\frac{n_1(t)n_{123}(t)n_{13}(t)\gamma(t)\rho(t)n_{12}'(t)}{-n_1(t)n_{13}(t)\gamma(t)\rho(t)n_{123}'(t) + n_{123}(t)[n_1(t)\gamma(t)\rho(t)n_{13}'(t) + \Theta_9]}$$

$$m_{13}(t) = \frac{m_1(t)m_{12}(t)m_{23}(t)\gamma(t)\rho(t)m_{13}'(t)}{-m_1(t)m_{23}(t\gamma(t)\rho(t)m_{12}'(t) + m_{12}(t)[m_1(t)\gamma(t)\rho(t)m_{23}'(t) + \Theta_{10}])}$$

$$n_{23}(t) = \frac{n_1(t)n_{123}(t)\gamma(t)\rho(t)n_{23}'(t)}{n_1(t)\gamma(t)\rho(t)n_{123}'(t) + n_{123}(t)[n_1(t)\rho(t)\gamma'(t) + \gamma(t)(-\rho(t)n_1'(t) + n_1(t)\rho'(t))]}$$

$$m_{123}(t) = \frac{m_1(t)m_{23}(t)\gamma(t)\rho(t)m_{123}'(t)}{m_1(t)\gamma(t)\rho(t)m_{23}'(t) - m_{23}(t)[m_1(t)\rho(t)\gamma'(t) + \gamma(t)(-\rho(t)m_1'(t) + m_1(t)\rho(t))]}$$

$$\varpi_3'(t) = \frac{[p_1s_1 + p_3s_1 - k_1(r_1 + r_3)]m_1(t)n_{13}(t)\gamma(t)\rho(t)n_1'(t) + \Theta_{11}}{(p_1s_1 - k_1r_1)m_1(t)n_1(t)n_{13}(t)\gamma(t)\rho(t)}$$

$$\Theta_7 = \rho(t)m_1'(t) + m_1(t)\rho'(t), \Theta_8 = \rho(t)n_{123}'(t) + n_{123}(t)\rho'(t) \tag{51}$$

$$\Theta_9 = n_{13}(t)(-\gamma(t)\rho(t)n_1'(t) - 2n_1(t)\rho(t)\gamma'(t) + n_1(t)\gamma(t)\rho'(t)) \tag{52}$$

$$\Theta_{10} = m_{23}(t)(2\gamma(t)\rho(t)m_1'(t) - 2m_1(t)\rho(t)\gamma'(t) + m_1(t)\gamma(t)\rho'(t)) \tag{53}$$

$$\Theta_{11} = n_1(t)[(-p_1s_1 + k_1r_1)m_1(t)\gamma(t)\rho(t)n_{13}'(t) + \Theta_{12}], \tag{54}$$

$$\Theta_{12} = n_{13}(t)[[(-k_1 + p_3)r_1 + p_1(s_1 - r_3)]m_1(t)\rho(t)\gamma'(t) + \gamma(t)[\Theta_{13}]] \tag{55}$$

$$\Theta_{13} = \rho(t)[(-p_3r_1 + p_1r_3)m_1'(t) + m_1(t)[(-p_3r_1 + p_1r_3)w_1'(t) + (p_3s_1 - k_1r_3)\varpi_1'(t)]] + \Theta_{14} \tag{56}$$

$$\Theta_{14} = (-p_1s_1 - p_3s_1 + k_1(r_1 + r_3))m_1(t)\rho'(t) \tag{57}$$

Supposing that $f_4 = g_4 = 0, f_i = g_i = 0 (i = 5, 6, \cdots), \varepsilon = 1$,the triple soliton solution of the variable coefficient (3+1)-dimensional Potential-YTSF equation is

$$u = [\ln(\frac{1 + f_1 + f_2 + f_3}{1 + g_1 + g_2 + g_3})]_x \tag{58}$$

$$= \frac{k_1m_1(t)\exp(\eta_1) + k_2m_2(t)\exp(\eta_2) + k_3m_3(t)\exp(\eta_3) + \Delta_1}{1 + f_1 + f_2 + f_3}$$

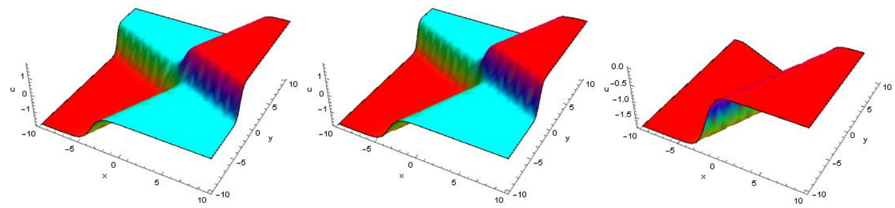
$$+\frac{p_1n_1(t)\exp(\theta_1) + p_2n_2(t)\exp(\theta_2) + p_3n_3(t)\exp(\theta_3) + \Delta_3}{1 + g_1 + g_2 + g_3}$$

$$\Delta_1 = (k_1 + k_2)m_{12}(t)\exp(\eta_1 + \eta_2) + (k_1 + k_3)m_{13}(t)\exp(\eta_1 + \eta_3) + \Delta_2$$

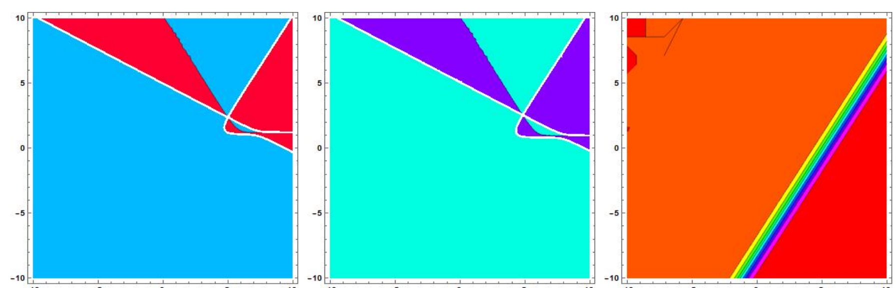
$$\Delta_2 = (k_2 + k_3)m_{23}(t)\exp(\eta_2 + \eta_3) + (k_1 + k_2 + k_3)m_{123}(t)\exp(\eta_1 + \eta_2 + \eta_3)$$

$$\Delta_3 = (p_1 + p_2)n_{12}(t)\exp(\theta_1 + \theta_2) + (p_1 + p_3)n_{13}(t)\exp(\theta_1 + \theta_3) + \Delta_4$$

$$\Delta_4 = (p_2 + p_3)n_{23}(t)\exp(\theta_2 + \theta_3) + (p_1 + p_2 + p_3)n_{123}(t)\exp(\theta_1 + \theta_2 + \theta_3)$$

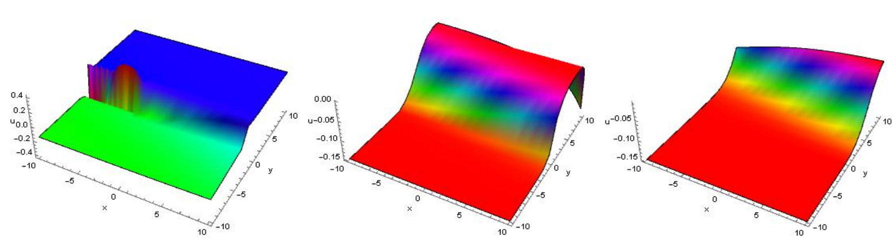


(a)t=0 (b)t=0.5 (c)t=2

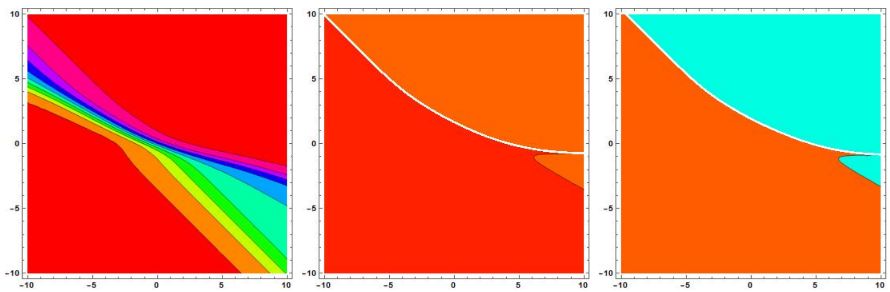


(d) (e) (f)

Fig.7$m_1(t) = \cos(t), w_1(t) = \tan(t), \rho(t) = \sin(t), \gamma(t) = \cos(t), m_{12}(t) = \exp(t), n_1(t) = t^2, m_{23}(t) = t^2, m_3(t) = \arctan(t), z_1(t) = \cosh(t), m_2(t) = t, m_{13}(t) = \sin(t), m_{123}(t) = \arccos(t)$

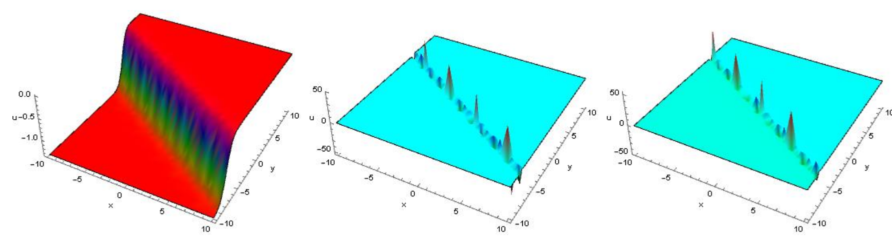


(a)t=0 (b)t=0.3 (c)t=0.5

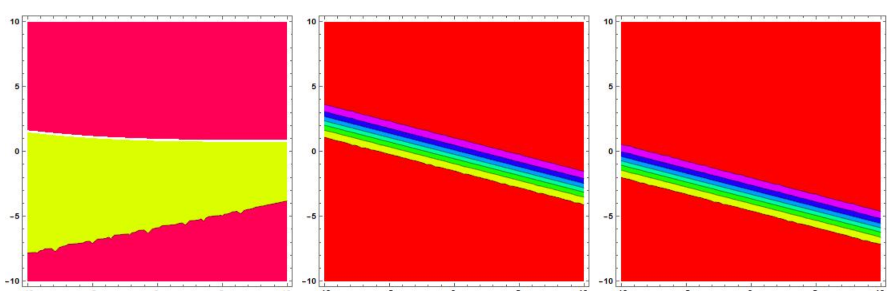


(d) (e) (f)

Fig.8$m_1(t) = \cos(t)\arctan(t), w_1(t) = t\tan(t), \rho(t) = t\sin(t), \gamma(t) = \cos(t), m_{12}(t) = t^2\exp(t), n_1(t) = t^2\sin(t), m_{23}(t) = t^2, m_3(t) = \arctan(t), z_1(t) = \exp(t)\cosh(t), m_2(t) = t\exp(t), m_{13}(t) = \sin(t), m_{123}(t) = \arccos(t)$



(a)t=6 (b)t=8 (c)t=8.3



(d) (e) (f)

Fig.9$m_1(t) = \cos(t) + \arctan(t), w_1(t) = t + \tan(t), \rho(t) = t + \sin(t), \gamma(t) = \cos(t), m_{12}(t) = t^2 + \exp(t), n_1(t) = t^2 + \sin(t), m_{23}(t) = t^2, m_3(t) = \arctan(t), \varpi_1(t) = \exp(t) + \cosh(t), m_2(t) = t + \exp(t), m_{13}(t) = \sin(t), m_{123}(t) = \arccos(t)$

5 Conclusion

Ref. [6] obtained the exact solution of equation (1) by using the self-B$\ddot{a}$cklund transformation. In [7], the exact solution of (3+1)-dimensional constant coefficient equation (1) is obtained by traveling wave transformation method. Reference [13] obtained the exact solution of the constant coefficient equation (1) by the generalized projective Ricatti method. Reference [14] constructed a new form of solution of the constant coefficient equation (1) by using the KdV equation and its various solutions to generate various solutions of the nonlinear evolution equation. In this paper, based on the Bell polynomial method, the (3+1)-dimensional variable coefficient Potential-YTSF equation is transformed into P-polynomial, and the double Bell polynomial B$\ddot{a}$cklund transformation and bilinear form B$\ddot{a}$cklund transformation, Lax pair and infinite conservation law are obtained by P-polynomial.The exact solution of the equation is obtained by using the bilinear B$\ddot{a}$cklund transformation and symbolic computation system Mathematica.By solving different parameters containing t, a part of the three-dimensional graph and contour map are obtained,and the physical meaning of the understanding is interpreted through these different graphs.

Acknowledgement

The author deeply appreciates the useful and constructive comments made by the anonymous reviewers, which help to further improve this article. This work was supported by the National Natural Science Foundation of China (grant no.11361040), Inner Mongolia Autonomous Region Natural Science Foundation Project (grant no.2020LH01008),Inner Mongolia Autonomous Region Natural Science Foundation Project (granted no.2020LH01008),inner Mongolia Normal University basic research business special funds (grant no.2022JBZD011),China Inner Mongolia Normal University Graduate Research and Innovation Fund Project (grant no.CXJJS20089).

Data Availability

All data generated or analyzed during this paper are included in this published article.

Declarations

Conflict of interest

The authors declare that there is no conflict of interests regarding the research effort and the publication of this study.

References

[1] Wazwaz A M.Multiple soliton solutions and multiple singular soliton solutions for (2+1)-dimensional shallow water wave equations[J].Phys Lett A,2009,373:2927-2930.

[2] Liu J G,Li Y Z,Wei G M.Auto-Bäcklund transformation and soliton-typed solutions of the generalized Variable-Coefficient KP equation[J].Chin Phys Lett,2006,23(7):1670-1673.

[3] Fan E,Zhang H.Anote on the homogeneous balance method[J].Phys Lett A,1998,246:403-406.

[4] Zhang S.The periodic wave solutions for the (2+1) dimensional Konopelchenko-Dubrovsky equations.Chaos Soliton Fract,2006,30:1213-1220.

[5] Ablowitz M J,Clarkson P A.Solitons,Nonlinear Evolution Equations and Inverse Scattering Transgorm[J].London:Cambridge University Press,1990.

[6] Yan Z Y.New families of nontravelling wave solutions to a new (3+1)-dimensional potential-YTSF equation[J].Phys Lett A,2003,318:78-83.

[7] Wei Huan,Yang Lianwu,Liu Jianguo.New multi-periodic soliton solutions of (3+1)-dimensional Potential-Yu-Toda-Sasa-Fukuyama equation[J].Acta Mathematica Scientia,2018,38A(6):1193-1204.

[8] Xiao Zisong,Fan Engui.Bell polynomial solution of soliton equation[D].Fudan University Master's degree thesis,2012

[9] Hao Shanshan,Sun Fuwei.Research on variable coefficient soliton equation based on Bell polynomial method[D].Master's thesis of North China University of Technology,2014.

[10] Song Lili,Pu Zhilin.Solitary and rogue wave solutions of several nonlinear evolution equations and their dynamic behavior[D].Doctoral Dissertation of Sichuan Normal University,2018.

[11] Han Pengfei,Taogetusang.Novel hybrid-type solutions for the (3+1)-dimensional generalized BogoyavlenskyCKonopelchenko equation with time-dependent coefficients[J].Nonlinear Dynamics,2022,107:1163-1177.

[12] Cheng Qiaoxin,Ma Hongcai.N-soliton solutions and soliton molecular solutions of several partial differential equations[D].Donghua University Master's Degree Thesis,2021.

[13] Zhang T X,Xuan H N,Zhang D F,Wang C J.Non-travelling wave solutions to a (3+1)-dimensional potential-YTSF equation and a simplified model for reacting mixtures.Chaos Soliton Fract,2007,34:1006-1013.

[14] Somg L N,Zhang H Q.A new variable coefficient Korteweg-de Vries equation-based sub-equation method and its application to the (3+1)-dimensional potential-YTSF equation.Appl Math Comput,2007,189:560-566.

[15] Liu J G,Zeng Z F.Multiple soliton solutions,soliton-type solutions and rational solutions for the (3+1)-dimensional potential-YTSF equation.Indian J Pure App Math,2014,45:989-1002.

[16] Li Z T,Dai Z D.Exact periodic cross-kink wave solutions and breather type of two-solitary wave solutions for the (3+1)-dimensional potential-YTSF equation.Comput Math Appl,2011,61:1939-1945.

[17] Liu na.B$\ddot{a}$cklund transformation and multi-soliton solutions for the(3+1)-dimensional BKP equation with Bell polynomials and symbolic computation[J].Nonliear Dynamics,2015,82(1-2):311-318.