Supplementary Material

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Contained below are three supplementary material sections which support and/or provide additional detail to the work and conclusions presented in the main manuscript: *Characterizing the Ice-Ocean Interface of Icy Worlds: A Theoretical Approach*. Section S1 provides the complete solution to the molecular diffusion equation in the mushy layer – as outlined in Section 2.1.1 of the main text. Section S2 provides a table of all variables used throughout the text as well as their associated symbols and units. Finally, Section S3 provides a table of the ionic species present and their relative abundances in the terrestrial ocean composition we assume throughout the investigation.

S1. Solving the Molecular Diffusion Equation

The evolution of salinity at the ice-mush interface is governed by:

$$\frac{\partial S}{\partial t} = D\frac{\partial^2 S}{\partial z^2} \qquad (S1)$$

Utilizing the similarity variable $\eta = z/2\sqrt{Dt}$ and boundary conditions (Equations 9-10) it can be shown that:

$$\theta = \frac{1}{erfc(\lambda_S)} - \frac{\mathrm{erf}(\eta)}{erfc(\lambda_S)} \qquad (S2)$$

Writing in terms of the original variables *S, z* and *t*:

$$S = S_{oc} + (S_{int} - S_{oc})\left(\frac{1}{erfc(\lambda_S)} - \frac{\mathrm{erf}\left(z/2\sqrt{Dt}\right)}{erfc(\lambda_S)}\right) \qquad (S3)$$

The Stefan condition for this problem can be garnered from the equation for conservation of salt (no salt in the ice phase):

$$\int_{z_m^*(t)}^{\infty} S(z,t)dz = cnst. \qquad (S4)$$

Taking the temporal derivative of this equation and applying the Leibniz integral rule:

$$-S(z_m^*, t)\frac{dz_m^*}{dt} + \int_{z_m^*(t)}^{\infty} \frac{\partial S}{\partial t}dz = 0 \qquad (S5)$$

Substituting $\frac{\partial S}{\partial t} = D\frac{\partial^2 S}{\partial z^2}$, noting $S(z_m^*, t) = S_{int}$ and carrying out the integral gives:

$$S_{int}\frac{dz_m^*}{dt} = -D\frac{\partial S(z = z_m^*)}{\partial z} \qquad (S6)$$

The derivatives are:

$$\frac{dz_m^*}{dt} = \frac{\lambda_S\sqrt{D}}{\sqrt{t}} \qquad (S7)$$

And

$$\frac{\partial S(z = z_m^*)}{\partial z} = \frac{-(S_{int} - S_{oc})}{erfc(\lambda_S)}\frac{\exp(-\lambda_S^2)}{\sqrt{\pi Dt}} \qquad (S8)$$

Substituting back into Equation 17,

$$S_{int}\sqrt{\pi} = \frac{(S_{int} - S_{oc})}{\lambda_S erfc(\lambda_S)} \exp(-\lambda_S^2) \qquad (S9)$$

Or, rearranging

$$\lambda_S erfc(\lambda_S) \exp(\lambda_S^2) = \frac{(S_{int} - S_{oc})}{S_{int}\sqrt{\pi}} \qquad (S10)$$

We assume that the interface is at its melting temperature, T_f, which we take as a linear function of salinity $T_f = T_{mp} - 0.066178S$, where T_{mp} is the melting temperature of pure ice and the freezing point depression coefficient has units of K kg g^{-1}. Solving for *S* and letting T_f lie on a conductive (linear) thermal profile $T(z) = T_S + z(T_{oc} - T_S)/H$ at a depth *H-h*. We have:

$$S_{int} = 15.1106(T_{mp} - T_f) = 15.1106\left(T_{mp} - \left(T_S + (H-h)\frac{(T_{oc} - T_S)}{H}\right)\right) \qquad (S11)$$

Substituting the value of S_{int} into Equation 21 and setting the interface velocities equal ($\dot{z}_m^* = 2\lambda_S\sqrt{D}/\sqrt{t} = 2\lambda_T\sqrt{\kappa_i}/\sqrt{t} = \dot{z}_m$) produces two equations:

$$\lambda_S = \frac{\lambda_T\sqrt{\kappa_i}}{\sqrt{D}} \qquad (S12)$$

and

$$15.1106\left(T_{mp} - \left(T_S + (H-h)\frac{(T_{oc}-T_S)}{H}\right)\right)\sqrt{\pi} = \frac{15.1106\left(T_{mp} - \left(T_S + (H-h)\frac{(T_{oc}-T_S)}{H}\right)\right) - S_{oc}}{\lambda_S erfc(\lambda_S)} \exp(-\lambda_S^2) \qquad (S13)$$

which can be solved for *h*. The equation is linear in *h*, so has one solution.

Archie's law is employed to estimate the molecular diffusion in a porous medium and it is assumed that transport processes (diffusion and advection) will be limited by the critical porosity of ocean/brine derived ices ($\phi_c = 0.05$ [*K M Golden et al.*, 2007]):

$$D = k_S\phi_c^m \qquad (S14)$$

Where k_S is the molecular diffusivity of salt in water and *m* is a cementation exponent, here=2, which describes how ion transport is limited by porosity.

S2. Variables Used in the Text

Symbol	Definition	Units
$\boldsymbol{\alpha}$	1D Advection Coefficient	kg m^{-3} s^{-1}
$\boldsymbol{\beta}$	Density (Salinity) Coefficient	ppt^{-1}
$\boldsymbol{br}^{\uparrow,\downarrow}$	Vertical Brine Velocity	m s^{-1}
$\boldsymbol{c_i}$	Ice Heat Capacity	J kg^{-1} K^{-1}
$\boldsymbol{D}$	Salt Diffusivity	m^{2} s^{-1}
$\boldsymbol{\eta}$	Similarity Variable	-
$\boldsymbol{\eta_m}$	Similarity Variable at z_m^*	-
$\boldsymbol{\Gamma}$	Freezing Point Depression Coefficient	K kg g^{-1}
$\boldsymbol{g}$	Acceleration Due to Gravity	m s^{-2}
$\boldsymbol{G}$	Thermal Gradient	K m^{-1}
$\boldsymbol{G'}$	Thermal Gradient Perturbation	K m^{-1}
$\boldsymbol{h}$	Mushy Layer Equilibrium Thickness	m
$\boldsymbol{h'}$	Thickness Perturbation	m
$\boldsymbol{H}$	Ice Shell Thickness	m
$\boldsymbol{k_s}$	Salt Diffusivity in Pure Water	m^{2} s^{-1}
$\boldsymbol{\kappa_i}$	Thermal Diffusivity of Ice	m^{2} s^{-1}
$\boldsymbol{\kappa_{br}}$	Thermal Diffusivity of Brine	m^{2} s^{-1}
$\boldsymbol{L}$	Latent Heat of Fusion	J kg^{-1}
$\boldsymbol{Le}$	Lewis Number	-
$\boldsymbol{\lambda_T, \lambda_S}$	Stefan Problem Variables	-
$\boldsymbol{m}$	Cementation Exponent	-
$\boldsymbol{\mu}$	Kinematic Viscosity	m^{2} s^{-1}
$\boldsymbol{\phi}$	Liquid Fraction	-
$\boldsymbol{\phi_c}$	Critical Porosity	-
$\boldsymbol{\Pi}$	Permeability	m^{2}
$\boldsymbol{Ra_c}$	Critical Rayleigh Number	-
$\boldsymbol{\rho_{sw}}$	Ocean Density	kg m^{-3}
$\boldsymbol{S}$	Salinity	ppt
$\boldsymbol{S_{int}}$	Interface Salinity	ppt
$\boldsymbol{S_{oc}}$	Ocean Salinity	ppt
$\boldsymbol{\Delta S_j}$	$S_{int} - S_{oc}$	ppt
$\boldsymbol{t}$	Time	s
$\boldsymbol{T}$	Temperature	K
$\boldsymbol{T_m}$	Melting/Freezing Temperature	K
$\boldsymbol{T_{oc}}$	Ocean Temperature	K
$\boldsymbol{T_s}$	Surface Temperature	K
$\boldsymbol{\theta}$	Dimensionless Salinity	-
$\boldsymbol{v_m}$	Mush-Ocean Freezing Front Velocity	m s^{-1}
$\boldsymbol{v_m^*}$	Ice-Mush Freezing Front Velocity	m s^{-1}
$\boldsymbol{\Delta V}$	$v_m - v_m^*$	m s^{-1}
$\boldsymbol{z}$	Vertical Coordinate	m
$\boldsymbol{z_m}$	Mush-Ocean Interface	m
$\boldsymbol{z_m^*}$	Ice-Mush Interface	m

S3. Terrestrial Ocean Composition

Species	Terrestrial Seawater (mol/kg)
Na^+	4.69×10^{-1}
K^+	1.02×10^{-2}
Ca^{2+}	1.03×10^{-2}
Mg^{2+}	5.28×10^{-2}
Cl^-	5.46×10^{-1}
SO_4^{2-}	2.82×10^{-2}
Total Salt (ppt)	34

Table S1 – Ocean composition. List of ion species and relative abundances for terrestrial seawater [*Dickson and Goyet*, 1994].