Initial boundary value problem of pseudo-parabolic Kirchhoff equations with logarithmic nonlinearity *

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Abstract In this paper, we consider the initial boundary value problem for a pseudo-parabolic Kirchhoff equation with logarithmic nonlinearity. Using the potential well method, we obtain a threshold result of global existence and finite-time blow-up for the weak solutions with initial energy $J(u_0) \le d$. When the initial energy $J(u_0) > d$, we find another criterion for the vanishing solution and blow-up solution. We also establish the decay rate of the global solution and estimate the life span of the blow-up solution. Meanwhile, we study the existence of the ground state solution to the corresponding stationary problem.

Keywords: pseudo-parabolic Kirchhoff equation, global existence, blow-up, logarithmic nonlinearity

1 Introduction

In this paper, we are concerned with the following initial boundary value problem

$$\begin{cases} u_t - \Delta u_t - M(\|\nabla u\|_p^p)\Delta_p u = |u|^{q-1}u\log|u|, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \ge 1)$ is a bounded domain with smooth boundary, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}u)$, $M(s) = a + bs$ with $a > 0$ and $b > 0$, $u_0(x) \in W_0^{1,p}(\Omega)$ with $u_0(x) \ne 0$. The parameters p and q satisfy $1 < 2p - 1 < q < p^* - 1$, where $p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p.

Problem (1.1) belongs to the mixed type of the pseudo-parabolic equation [34] and the Kirchhoff equation [22]. The diffusion coefficient $M(\cdot)$ can express the dependence on the global information in the environment instead of the information at a local location. Another nonlocal mechanism comes from the nonlocal operator $\mathcal{B} = (I - \Delta)^{-1}$, which leads the equation of (1.1) to an equivalent form

$$u_t - \mathcal{B}M(\|\nabla u\|_p^p)\Delta_p u = \mathcal{B}|u|^{q-1}u\log|u|.$$

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The logarithmic nonlinearity appears naturally in the inflationary cosmology and the modern super symmetric field theories [13, 27]. It also appears in the theory of continuous-state branching processes [14, 15], Gravity-mediated super symmetric fracture model [20, 39]. Equations like (1.1) have had a high profile in the study of many mathematical and physical phenomena such as population dynamics, nonlinear elasticity, non-stationary fluid, image recovery, see [1, 6, 7, 35] and the references therein.

With the successive development of remarkable methods, such as the convex method [23, 24], the potential well method [30, 32], especially the functional analysis framework [28] introduced by Lions, more and more excellent works have been done for the problems related to (1.1). Take the parabolic equation with power like source for example, Han et al. [18] studied the parabolic Kirchhoff $u_t - M(\|\nabla u\|_2^2)\Delta u = |u|^{q-1}u$, and gave the global existence and uniqueness, finite time blow-up and asymptotic behavior of solutions with subcritical, critical and supercritical initial energy. The upper and lower bounds of the blow-up time of the solutions were supplemented in [17]. [25] considered the parabolic p-Kirchhoff equation $u_t - M(\|\nabla u\|_p^p)\Delta_p u = |u|^{q-1}u$ and described the impact of the p-Laplacian. Xu et al. [37] investigated the pseudo-parabolic equation $u_t - \Delta u_t - \Delta u = u^p$, and proved the invariance of some sets, global existence, blow-up and asymptotic behavior of solutions with different initial energies. Some other works studied the pseudo-parabolic or thin film equation with non-local power like nonlinearity [4, 5, 31].

Due to its important physical applications and interesting mathematical properties, the logarithmic nonlinearities are attacting more and more attention from researchers. Chen et al. [8, 9] studied the semilinear heat and pseudo-parabolic equation $u_t - k\Delta u_t - \Delta u = u \log u,\ k = 0, 1$. Using the potential well family and the logarithmic Sobolev inequality, they obtained the existence, blow-up at infinity and isolate vacuum of the solution. Their works indicate that while taking blow-up profile in hand, the logarithmic nonlinearity is more close to the one with linear source. Ji et al. [21] found that the logarithmic nonlinearity behaves similar to power like source when considering the existence of periodic solutions, while for the instability of periodic solutions, the effect of the logarithmic nonlinearity is neither like the linear source nor the power like source. On the basis of the above works, [3, 11, 10, 19, 29] discussed extensively the mixed pseudo-parabolic p-Laplacian equation $u_t - \Delta u_t - \Delta_p u = |u|^{p-2}u \log u$. Recently, [12, 16, 33, 36] attempted to ponder the properties of the solutions for parabolic Kirchhoff equation with logarithmic nonlinearity.

Based on the potential well theory, this paper is devoted to discuss the global existence and finite time blow-up for the solutions of (1.1), when the initial energy is subcritical, critical and supercritical. Moreover, we also obtain the decay rate of the global solution and the life span of the finite time blow-up solution. From our proof procedure, we find some impact of the logarithmic nonlinearity. First, the logarithmic nonlinearity may lead to a positive limit of the potential well depth $d(\delta)$ when $\delta \to 0$, which is different from the zero limit for the power like nonlinearity case [5]. Due to this difference, some discussion need to be separated for $0 < J(u_0) \leq d_0$ and $d_0 < J(u_0) < d$, respectively, see the lemmas in Section 2 and Remark 1. Secondly, here we can not use the L^p logarithmic Sobolev inequality. Using this inequality can control the logarithmic nonlinearity $|u|^{q-1}u \log |u|$ by $\|\nabla u\|_{q+1}^{q+1}$, which can not be further controlled by $\|\nabla u\|_p$. Hence, we use the property of Log function, which brings the norm of u in $L^{q+1+\epsilon}(\Omega)$. The condition $q + 1$

being strictly less than p^* is to guarantee the feasibility of the imbedding from $W^{1,p}$ to $L^{q+1+\epsilon}$. Moreover, if comparing the life span in this work and in [5] for the pseudo-parabolic Kirchhoff equation with power like nonlinearity, one can find that the time t_0 in Theorem 5 (ii) is smaller than that in Theorem 1.6 [5], which suggests that the logarithmic nonlinearity do contribute to blowing-up, see Remark 3.

The rest of this paper is arranged as follows. Section 2 states some useful lemmas. Sections 3 and 4 deal with the global existence and the finite blow-up of the solutions of (1.1) in the case of $J(u_0) \le d$ and $J(u_0) > d$, respectively.

2 Preliminary

Definition 2.1 *A function $u(x,t)$ is said to be a weak solution to (1.1) on $\Omega \times [0,T)$, if $u(x,0) = u_0(x) \in W_0^{1,p}(\Omega)$, $u \in L^\infty(0,T;W_0^{1,p}(\Omega))$ with $u_t \in L^2(0,T;H_0^1(\Omega))$ and satisfies*

$$(u_t,\varphi) + (\nabla u_t,\nabla\varphi) + M(\|\nabla u\|_p^p)(|\nabla u|^{p-2}\nabla u,\nabla\varphi) = (|u|^{q-1}u\log|u|,\varphi),$$

for any $\varphi \in W_0^{1,p}(\Omega)$.

According to the potential well theory [26, 32, 37], the potential energy functional is

$$J(u) = \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \frac{1}{q+1}\int_\Omega |u|^{q+1}\log|u|dx + \frac{1}{(q+1)^2}\|u\|_{q+1}^{q+1}, \tag{2.1}$$

and the Nehari functional is

$$I(u) = a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p} - \int_\Omega |u|^{q+1}\log|u|dx. \tag{2.2}$$

(2.1) and (2.2) imply that

$$J(u) = \frac{1}{q+1}I(u) + \left(\frac{a}{p} - \frac{a}{q+1}\right)\|\nabla u\|_p^p + \left(\frac{b}{2p} - \frac{b}{q+1}\right)\|\nabla u\|_p^{2p} + \frac{1}{(q+1)^2}\|u\|_{q+1}^{q+1}, \tag{2.3}$$

$$\frac{d}{dt}J(u) = -\|u_t\|_2^2 - \|\nabla u_t\|_2^2, \tag{2.4}$$

$$I(u) = -\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + \|\nabla u\|_2^2). \tag{2.5}$$

For any $\delta > 0$, the modified Nehari functional can be defined as

$$I_\delta(u) = \delta(a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p}) - \int_\Omega |u|^{q+1}\log|u|dx.$$

Then we can define the Nehari manifold and the potential wells

$$\mathscr{N} = \{u \in W_0^{1,p}(\Omega) : I(u) = 0, \|\nabla u\|_p \ne 0\},$$
$$W = \{u \in W_0^{1,p}(\Omega) : J(u) < d, I(u) > 0\}\bigcup\{0\},$$

$$
\begin{aligned}
&V=\{u\in W_0^{1,p}(\Omega): J(u)<d, I(u)<0\},\\
&\mathscr{N}_\delta=\{u\in W_0^{1,p}(\Omega): I_\delta(u)=0, \|\nabla u\|_p\neq 0\},\\
&W_\delta=\{u\in W_0^{1,p}(\Omega): J(u)<d(\delta), I_\delta(u)>0\}\bigcup\{0\},\\
&V_\delta=\{u\in W_0^{1,p}(\Omega): J(u)<d(\delta), I_\delta(u)<0\},
\end{aligned}
$$

where $d(\delta)$ is the depth of the potential well and

$$
d=d(1)=\inf\{J(u): u\in\mathscr{N}\},\qquad d(\delta)=\inf\{J(u): u\in\mathscr{N}_\delta\}. \tag{2.6}
$$

Lemmas 2.1, 2.4 and 2.5 are similar to the lemmas in [3, 5, 9], so we omit their proofs.

Lemma 2.1 *Assume $\lambda>0$ and $u\in W_0^{1,p}(\Omega)$ with $\|\nabla u\|_p\neq 0$, then there hold*
(i) $\lim\limits_{\lambda\to 0} J(\lambda u)=0,\ \lim\limits_{\lambda\to+\infty} J(\lambda u)=-\infty.$
(ii) *There exits an unique $\lambda^*>0$ such that $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda^*}=0$, namely $\lambda^* u\in\mathscr{N}$. Furthermore, $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda^*}>0$ on $(0,\lambda^*)$, $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda^*}<0$ on (λ^*,∞), namely $J(\lambda u)$ takes the maximum at $\lambda=\lambda^*$.*

Lemma 2.2 *For $u\in W_0^{1,p}(\Omega)$ with $\|\nabla u\|_p\neq 0$, $r_\epsilon(\delta)=(\frac{a\delta\epsilon e}{S^{q+1+\epsilon}})^{\frac{1}{q+1+\epsilon-p}}$, where $0<\epsilon<p^*-q-1$, S is the embedding coefficient of the Sobolev inequality $\|u\|_{q+1+\epsilon}\leq S\|\nabla u\|_p$, we have*
(i) *If $0<\|\nabla u\|_p\leq r_\epsilon(\delta)$, then $I_\delta(u)>0$.*
(ii) *If $I_\delta(u)<0$, then $\|\nabla u\|_p>r_\epsilon(\delta)$.*
(iii) *If $I_\delta(u)=0$, then $\|\nabla u\|_p=0$ or $\|\nabla u\|_p>r_\epsilon(\delta)$.*

Proof (i) Using the property of the logarithmic function and the Sobolev embedding inequality, we can get

$$
\int_\Omega |u|^{q+1}\log|u|dx\leq\frac{1}{e\epsilon}\int_\Omega|u|^{q+1+\epsilon}dx\leq\frac{1}{e\epsilon}S^{q+1+\epsilon}\|\nabla u\|_p^{q+1+\epsilon},
$$

which with $0<\|\nabla u\|_p<r_\epsilon(\delta)$ indicate that

$$
\int_\Omega |u|^{q+1}\log|u|dx\leq\frac{1}{e\epsilon}S^{q+1+\epsilon}r_\epsilon(\delta)^{q+1+\epsilon-p}\|\nabla u\|_p^p=a\delta\|\nabla u\|_p^p<a\delta\|\nabla u\|_p^p+b\delta\|\nabla u\|_p^{2p}.
$$

This means $I_\delta(u)>0$.

(ii) can be directly derived from (i).

(iii) If $\|\nabla u\|_p=0$, then $I_\delta(u)=0$. If $I_\delta(u)=0$ and $\|\nabla u\|_p\neq 0$, then

$$
a\delta\|\nabla u\|_p^p<\int_\Omega|u|^{q+1}\log|u|dx\leq\frac{1}{e\epsilon}S^{q+1+\epsilon}\|\nabla u\|_p^{q+1+\epsilon},
$$

namely $\|\nabla u\|_p>r_\epsilon(\delta)$. □

Lemma 2.3 *$d(\delta)$ in (2.6) satisfies*
(i) $d(\delta)\geq\left(\frac{a}{p}-\frac{a\delta}{q+1}\right)r_\epsilon(\delta)+\left(\frac{b}{2p}-\frac{b\delta}{q+1}\right)r_\epsilon(\delta)$, $\lim\limits_{\delta\to+\infty}d(\delta)=-\infty$.
(ii) *$d(\delta)$ is monotonically increased on $0<\delta\leq 1$, monotonically decreased on $\delta>1$ and the maximum is obtained at $\delta=1$. Moreover, there exists a unique $\bar\delta>1$ such that $d(\bar\delta)=0$, and $d(\delta)>0$ for $1\leq\delta<\bar\delta$.*

Proof (i) We can rewrite $J(u)$ as

$$J(u)=\frac{1}{q+1}I_\delta(u)+\left(\frac{a}{p}-\frac{a\delta}{q+1}\right)\|\nabla u\|_p^p+\left(\frac{b}{2p}-\frac{b\delta}{q+1}\right)\|\nabla u\|_p^{2p}+\frac{1}{(q+1)^2}\|u\|_{q+1}^{q+1}.$$

When $u\in\mathscr{N}_\delta$, then $I_\delta(u)=0$ and Lemma 2.2 indicates that $\|\nabla u\|_p>r_\epsilon(\delta)$. Thus

$$J(u)>\left(\frac{a}{p}-\frac{a\delta}{q+1}\right)r_\epsilon(\delta)+\left(\frac{b}{2p}-\frac{b\delta}{q+1}\right)r_\epsilon(\delta).$$

From the definition of $d(\delta)$, we can get

$$d(\delta)\geq\left(\frac{a}{p}-\frac{a\delta}{q+1}\right)r_\epsilon(\delta)+\left(\frac{b}{2p}-\frac{b\delta}{q+1}\right)r_\epsilon(\delta).$$

For any $\delta>0$, if $\lambda u\in\mathscr{N}_\delta$, then $\delta=\frac{\lambda^{q+1-p}\int_\Omega\left(|u|^{q+1}\log|\lambda u|\right)dx}{a\|\nabla u\|_p^p+b\lambda^p\|\nabla u\|_p^{2p}}$ and λ needs to satisfy

$$\lambda>\exp\left\{-\frac{\int_\Omega|u|^{q+1}\log|u|dx}{\|u\|_{q+1}^{q+1}}\right\}. \tag{2.7}$$

When (2.7) holds, from a directly computation, we can derive that

$$\begin{aligned}\frac{d\delta}{d\lambda}=&\frac{\lambda^{q-p}\|u\|_{q+1}^{q+1}}{a\|\nabla u\|_p^p+b\lambda^p\|\nabla u\|_p^{2p}}+\frac{\lambda^q\int_\Omega|u|^{q+1}\log(\lambda|u|)dx((q+1-p)a\|\nabla u\|_p^p+(q-p)\lambda^p b\|\nabla u\|_p^{2p})}{\lambda^p(a\|\nabla u\|_p^p+b\lambda^p\|\nabla u\|_p^{2p})^2}\\&>0,\end{aligned}$$

which means λ and δ have a one to one correspondence and further a positive correlation. Then

$$\lim_{\delta\to0+}\lambda(\delta)=\exp\left\{-\frac{\int_\Omega|u|^{q+1}\log|u|dx}{\|u\|_{q+1}^{q+1}}\right\},\qquad\lim_{\delta\to+\infty}\lambda(\delta)=+\infty.$$

Thus from the definition of $d(\delta)$ and Lemma 2.1, we can get

$$\begin{aligned}&\lim_{\delta\to+\infty}d(\delta)\leq\lim_{\delta\to+\infty}J(\lambda u)=\lim_{\lambda\to+\infty}J(\lambda u)=-\infty,\\&\lim_{\delta\to0+}d(\delta)\leq\lim_{\delta\to0+}J(\lambda u)=J\left(\exp\left\{-\frac{\int_\Omega|u|^{q+1}\log|u|dx}{\|u\|_{q+1}^{q+1}}\right\}u\right).\end{aligned}$$

(ii) Assume $0<\delta'<\delta''\leq1$ or $1<\delta''<\delta'$. If $u\in\mathscr{N}_{\delta''}$, namely $\lambda(\delta'')=1$, then $\delta''=\frac{\int_\Omega|u|^{q+1}\log|u|dx}{a\|\nabla u\|_p^p+b\|\nabla u\|_p^{2p}}$ and u must satisfy

$$\int_\Omega|u|^{q+1}\log|u|dx>0. \tag{2.8}$$

Set $v=\lambda(\delta')u$, then from the one to one correspondence of λ and δ, $v\in\mathscr{N}_{\delta'}$. Let $h(\lambda)=J(\lambda(\delta)u)$ with $\lambda(\delta)u\in\mathscr{N}_\delta$, then

$$h'(\lambda)=\lambda^{p-1}a\|\nabla u\|_p^p+\lambda^{2p-1}b\|\nabla u\|_p^{2p}-\int_\Omega\lambda^q|u|^{q+1}\log|\lambda u|dx=\frac{1}{\lambda}I(\lambda u).$$

If $0 < \delta' < \delta'' \leq 1$, since $\lambda(\delta)$ increases as δ increases, then $\exp\left\{-\frac{\int_\Omega |u|^{q+1}\log|u|dx}{\|u\|_{q+1}^{q+1}}\right\} < \lambda(\delta') < \lambda(\delta'') = 1$ and there exist $\delta^* \in (\delta', \delta'')$ and $\lambda^* = \lambda(\delta^*) \in (\lambda(\delta'), 1)$, such that $\lambda^* u \in \mathscr{N}_{\delta^*}$ and

$$
\begin{aligned}
J(u) - J(v) = h(1) - h(\lambda(\delta')) &= \frac{1-\lambda(\delta')}{\lambda^*} I(\lambda^* u) \\
&= \frac{1-\lambda(\delta')}{\lambda^*}\left[a(1-\delta^*)\|\lambda^*\nabla u\|_p^p + b(1-\delta^*)\|\lambda^*\nabla u\|_p^{2p}\right] \\
&> 0.
\end{aligned}
$$

Therefore, for any $u \in \mathscr{N}_{\delta''}$, there exists $v \in \mathscr{N}_{\delta'}$ such that $J(u) > J(v)$, which leads to $d(\delta'') > d(\delta')$. The case for $1 < \delta'' < \delta'$ is similarly and the latter part of (ii) follows from (i). □

Now, we can define

$$
d_0 = \lim_{\delta \to 0+} d(\delta), \tag{2.9}
$$

where $d_0 \geq 0$ from Lemma 2.3.

Lemma 2.4 *For $u \in W_0^{1,p}(\Omega)$, when $d_0 < J(u) < d$, then the sign of $I_\delta(u)$ doesn't change for $\delta_1 < \delta < \delta_2$, where $\delta_1 < 1 < \delta_2$ are the two roots of $d(\delta) = J(u)$; when $J(u) \leq d_0$, then the sign of $I_\delta(u)$ doesn't change for $\delta < \delta_2$, where $\delta_2 > 1$ is the root of $d(\delta) = J(u)$.*

Lemma 2.5 *Assume $u_0 \in W_0^{1,p}$ and u is a weak solution of (1.1).*
(i) If $d_0 < J(u_0) < d$, then $d(\delta) = J(u_0)$ has two roots $\delta_1 < 1 < \delta_2$. If $I(u_0) > 0$, then $u \in W_\delta$, $\delta_1 < \delta < \delta_2$, $0 < t < T$. If $I(u_0) < 0$, then $u \in V_\delta$, $\delta_1 < \delta < \delta_2$, $0 < t < T$.
(ii) If $J(u_0) \leq d_0$, then $d(\delta) = J(u_0)$ has a unique root $\delta_2 \in (1, \bar{\delta})$, where $\bar{\delta}$ is from Lemma 2.3. If $I(u_0) > 0$, then $u \in W_\delta$, $\delta < \delta_2$, $0 < t < T$. If $I(u_0) < 0$, then $u \in V_\delta$, $\delta < \delta_2$, $0 < t < T$.
(iii) If $J(u_0) = d$ and $I(u_0) > 0$, then W is an invariant set. If $J(u_0) = d$ and $I(u_0) < 0$, then V is an invariant set.

Remark 1 *For the power like nonlinearity [5], the limit of $d(\delta)$ as $\delta \to 0$ is zero. However, the logarithmic nonlinearity may lead to a positive limit of $d(\delta)$ when $\delta \to 0$. Such difference results in considering $0 < J(u_0) \leq d_0$ and $d_0 < J(u_0) < d$ separately.*

3 $J(u_0) \leq d$

In this section, we deal with the global existence and the blowing-up of the weak solution to (1.1) under the condition $J(u_0) \leq d$.

Theorem 1 *Let $u_0 \in W_0^{1,p}(\Omega)$ with $J(u_0) < d$ and $I(u_0) > 0$, or with $J(u_0) = d$ and $I(u_0) \geq 0$. Then (1.1) admits a global weak solution u. In addition, when $n = 1, 2$, the global solution is unique; when $n \geq 3$, the global bounded solution is unique.*

Proof To star with, we prove the global existence of solutions by the Galerkin method. The approximate solution $u^m(x,t)$ of (1.1) can be constructed by

$$u^m(x,t)=\sum_{j=1}^{m}\alpha_j^m(t)\phi_j(x), \qquad \alpha_j^m(t)=(u^m,\phi_j), \qquad m=1,2,...,$$

$$(u_t^m,\phi_j)+(\nabla u_t^m,\nabla\phi_j)+M(\|\nabla u^m\|_p^p)(|\nabla u^m|^{p-2}\nabla u^m,\nabla\phi_j)=(|u^m|^{q-1}u^m\log|u^m|,\phi_j), \tag{3.1}$$

$$u^m(x,0)=\sum_{j=1}^{m}\alpha_j^m(0)\phi_j(x)\to u_0(x) \quad \text{in} \quad W_0^{1,p}(\Omega), \tag{3.2}$$

with $\{\phi_j(x)\}_{j=1}^{\infty}$ be the orthogonal base in $W_0^{1,p}(\Omega)$.

In what follows, we consider the two cases respectively. For Case 1. $J(u_0)<d$ and $I(u_0)>0$, according to (2.3) and $2p<q+1$, we have $J(u_0)>0$. By the convergence (3.2), there have $J(u^m(x,0))\to J(u_0)<d$ and $I(u^m(x,0))\to I(u_0)>0$. Hence for sufficiently large m, there is $u^m(x,0)\in W$, which with Lemma 2.5 indicates $u^m(x,t)\in W$. Further there holds

$$\begin{aligned} d>J(u^m(x,0))=&J(u^m(x,t))+\int_0^t(\|u_\tau^m\|_2^2+\|\nabla u_\tau^m\|_2^2)d\tau \\ &>\int_0^t(\|u_\tau^m\|_2^2+\|\nabla u_\tau^m\|_2^2)d\tau+\frac{a(q+1-p)}{p(q+1)}\|\nabla u^m\|_p^p \\ &+\frac{b(q+1-2p)}{2p(q+1)}\|\nabla u^m\|_p^{2p}+\frac{1}{(q+1)^2}\|u^m\|_{q+1}^{q+1}, \quad \forall t>0. \end{aligned}$$

Thus

$$\int_0^t(\|u_\tau^m\|_2^2+\|\nabla u_\tau^m\|_2^2)d\tau<d, \quad \|u^m\|_{q+1}^{q+1}<d(q+1)^2,$$

$$\|\nabla u^m\|_p^p<\frac{dp(q+1)}{a(q+1-p)}, \quad \|M(\|\nabla u^m\|_p^p)|\nabla u^m|^{p-2}\cdot\nabla u^m\|_{\frac{p}{p-1}}<C,$$

where C is a constant independent on t. For the logarithmic nonlinearity, using the Sobolev inequality and $\inf\{x^q\log x, x\in(0,1)\}=-\frac{1}{eq}$ with $q>0$, we have

$$\begin{aligned} \int_\Omega\left||u^m|^q\log|u^m|\right|^{\frac{q+\epsilon+1}{q+\epsilon}}dx&=\int_{\{|u^m|\le 1\}}\left||u^m|^q\log|u^m|\right|^{\frac{q+\epsilon+1}{q+\epsilon}}dx+\int_{\{|u^m|>1\}}|(|u^m|^q\log|u^m|)|^{\frac{q+\epsilon+1}{q+\epsilon}}dx \\ &\le\left(\frac{1}{eq}\right)^{\frac{q+\epsilon+1}{q+\epsilon}}\cdot|\Omega|+\left(\frac{1}{e\epsilon}\right)^{\frac{q+\epsilon+1}{q+\epsilon}}\|u^m\|_{q+\epsilon+1}^{q+\epsilon+1} \\ &\le\left(\frac{1}{eq}\right)^{\frac{q+\epsilon+1}{q+\epsilon}}\cdot|\Omega|+\left(\frac{S^{q+\epsilon}}{e\epsilon}\right)^{\frac{q+\epsilon+1}{q+\epsilon}}\cdot\left(\frac{dp(q+1)}{a(q+1-p)}\right)^{\frac{q+\epsilon+1}{p}},\forall t>0, \end{aligned}$$

where $0<\epsilon<p^*-q-1$. According to the above boundedness estimations, there exist $u\in L^\infty(0,\infty;W_0^{1,p}(\Omega))$ and a subsequence of $\{u^m\}_{m=1}^\infty$ (still represented by $\{u^m\}_{m=1}^\infty$), such that

$$u_t^m\rightharpoonup u_t \text{ in } L^2(0,\infty;H_0^1(\Omega)),$$
$$u^m\overset{*}{\rightharpoonup}u \text{ in } L^\infty(0,\infty;W_0^{1,p}(\Omega)),$$

$$u^m \to u \text{ strongly in } C(0,T;L^2(\Omega)),$$
$$|u^m|^{q-1}u^m \cdot \log|u^m| \xrightharpoonup{*} |u|^{q-1}u \cdot \log|u| \text{ in } L^\infty(0,\infty;L^{\frac{q+\epsilon+1}{q+\epsilon}}(\Omega)),$$
$$M(\|\nabla u^m\|_p^p)|\nabla u^m|^{p-2}\cdot\nabla u^m \xrightharpoonup{*} \xi \text{ in } L^\infty(0,\infty;L^{\frac{p}{p-1}}(\Omega)).$$

Similar to the process of [3, 25], we can prove $\xi = M(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u$. Then for fixed j, sending $m \to +\infty$ in (3.1), u is a global weak solution satisfies Definition 2.1.

For Case 2. $J(u_0) = d$ and $I(u_0) \geq 0$, we set $\lambda_s = 1 - \frac{1}{s}$, $s = 1, 2, ...$ and consider (1.1) with the initial data $u(x,0) = \lambda_s u_0(x)$. According to $I(u_0) \geq 0$ and Lemma 2.1, there exists an unique $\lambda^* \geq 1$ such that $I(\lambda^* u_0) = 0$. Notice that $\lambda_s < 1 \leq \lambda^*$, then $I(\lambda_s u_0) > 0$, $J(\lambda_s u_0) < J(u_0) = d$. Due to Case 1. and Lemma 2.5, for any s, there exists a global weak solution $u^s \in L^\infty(0,\infty;W_0^{1,p}(\Omega))$, such that $u^s \in W$ and

$$\begin{aligned} d > J(\lambda_s u_0) =& J(u^s) + \int_0^t (\|u_\tau^s\|_2^2 + \|\nabla u_\tau^s\|_2^2)d\tau \\ &> \int_0^t (\|u_\tau^s\|_2^2 + \|\nabla u_\tau^s\|_2^2)d\tau + \frac{a(q+1-p)}{p(q+1)}\|\nabla u^s\|_p^p \\ &\quad + \frac{b(q+1-2p)}{2p(q+1)}\|\nabla u^s\|_p^{2p} + \frac{1}{(q+1)^2}\|u^s\|_{q+1}^{q+1}, \quad \forall t > 0. \end{aligned}$$

Similar to the estimations and limitations of Case 1., (1.1) has a global weak solution u with $I(u) \geq 0$ and $J(u) \leq d$.

The last step is devoted to the uniqueness. Assume (1.1) has two global weak solution u and v. Setting $w = u - v$ and using the Young inequality, we can get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx \\ \leq& \frac{1}{2}\frac{d}{dt}\int_\Omega w^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \frac{1}{p}\left(M(\|\nabla u\|^p) - M(\|\nabla v\|^p)\right)\left(\|\nabla u\|^p - \|\nabla v\|^p\right) \\ \leq& \int_\Omega \Big(q\,|\theta u + (1-\theta)v|^{q-1}\log|\theta u + (1-\theta)v| + |\theta u + (1-\theta)v|^{q-2}\cdot(\theta u + (1-\theta)v)\Big)\, w^2 dx, \end{aligned}$$

where $\theta \in (0,1)$ and $w(x,0) = 0$. When $n = 1, 2$, we can get the boundedness of u from the estimation of $\|\nabla u\|_p$ and the imbedding inequality. Therefore, like the proof in [5, 3], by the Gronwall inequality and the boundedness of u and v, we can obtain the uniqueness. □

Theorem 2 *Let $u_0 \in W_0^{1,p}(\Omega)$, u is the global solution obtained in Theorem 1.*
(i) If $J(u_0) < d$ and $I(u_0) > 0$, then u decays to zero and

$$\|u\|_2^2 + \|\nabla u\|_2^2 \leq \left[(\|u_0\|_2^2 + \|\nabla u_0\|_2^2)^{1-p} + Ct\right]^{-\frac{1}{p-1}},$$

where C is a positive constant.
(ii) If $J(u_0) = d$ and $I(u_0) > 0$, then u decays to zero and

$$\|u\|_2^2 + \|\nabla u\|_2^2 \leq \left[(\|u(t_0)\|_2^2 + \|\nabla u(t_0)\|_2^2)^{1-p} + C(t-t_0)\right]^{-\frac{1}{p-1}},$$

where $t_0 > 0$ and C is a positive constant.

Proof (i) From Lemma 2.5, if $d_0 < J(u_0) < d$ and $I(u_0) > 0$, then $u(x,t) \in W_\delta$, $\delta_1 < \delta < \delta_2$, where $\delta_1 < 1 < \delta_2$ are two roots of $d(\delta) = J(u_0)$; if $J(u_0) \le d_0$ and $I(u_0) > 0$, then $J(u_0) > 0$ and $u(x,t) \in W_\delta$, $0 < \delta < \delta_2$, where $\delta_2 > 1$ are the root of $d(\delta) = J(u_0)$. So we can choose $\delta_1 < \tilde{\delta} < 1$ such that

$$\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + \|\nabla u\|_2^2) = -I_{\tilde{\delta}}(u) + a(\tilde{\delta} - 1)\|\nabla u\|_p^p + b(\tilde{\delta} - 1)\|\nabla u\|_p^{2p}.$$

Using the Hölder inequality and the Poincaré inequality, we can obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + \|\nabla u\|_2^2) \le b(\tilde{\delta} - 1)\|\nabla u\|_p^{2p} \le C(\tilde{\delta} - 1)(\|u\|_2^{2p} + \|\nabla u\|_2^{2p}),$$

where C is a constant. Noticing the inequality $K_p(a^p + b^p) \ge (a+b)^p$ with non-negative a, b and positive constant K_p, then

$$\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + \|\nabla u\|_2^2) \le C(\tilde{\delta} - 1)\left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^p, \tag{3.3}$$

which implies $\|u\|_2^2 + \|\nabla u\|_2^2 \le \left[\left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)^{1-p} + C(1-\tilde{\delta})(p-1)t\right]^{-\frac{1}{p-1}}$ with C depending on k and p.

(ii) From Lemma 2.5, if $J(u_0) = d$ and $I(u_0) > 0$, then $J(u) < d$ and $I(u) > 0$. So we can take any $t_0 > 0$ as the initial time, and repeat the steps of (i) to get

$$\|u\|_2^2 + \|\nabla u\|_2^2 \le \left[(\|u(t_0)\|_2^2 + \|\nabla u(t_0)\|_2^2)^{1-p} + C(1-\tilde{\delta})(p-1)(t-t_0)\right]^{-\frac{1}{p-1}}.$$

□

Remark 2 *If the initial data satisfies $J(u_0) = d$ and $I(u_0) = 0$, then the global solution u obtained in Theorem 1 satisfies $J(u) = d$ and $I(u) = 0$, which means $\|u\|_2^2 + \|\nabla u\|_2^2 = \|u_0\|_2^2 + \|\nabla u_0\|_2^2$, namely the global solution does not decay. In fact, such initial data do exist, it is the ground state solution of (1.1).*

Theorem 3 *There exists a function $u^*(x) \in \mathscr{N}$ such that $J(u^*) = \inf_{u \in \mathscr{N}} J(u) = d$. Further, $u^*(x)$ is the ground state solution of*

$$\begin{cases} -M(\|\nabla u\|_p^p)\Delta_p u = |u|^{q-1}u\log|u|, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.4}$$

namely $u^* \in \Phi \setminus \{0\}$ and $J(u^*) = \inf_{u \in \Phi\setminus\{0\}} J(u)$, where

$$\begin{aligned} \Phi &= \{u \in W_0^{1,p}(\Omega) : J'(u) = 0 \ \textit{in} \ W^{-1,p'}(\Omega)\} \\ &= \{u \in W_0^{1,p}(\Omega) : \langle J'(u), \varphi\rangle = 0, \forall \varphi \in W_0^{1,p}(\Omega)\}. \end{aligned}$$

Proof In the first place, there exists a minimizing sequence $\{u_k\}_{k=1}^{\infty} \in \mathscr{N}$ such that

$$d = \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} \left\{ \frac{a(q+1-p)}{p(q+1)} \|\nabla u_k\|_p^p + \frac{b(q+1-2p)}{2p(q+1)} \|\nabla u_k\|_p^{2p} + \frac{1}{(q+1)^2} \|u_k\|_{q+1}^{q+1} \right\}.$$

The boundedness of $\|u_k\|_{W_0^{1,p}(\Omega)}$ from the above equality induces that there exist a subsequence of $\{u_k\}_{k=1}^{\infty}$ (still denoted by $\{u_k\}_{k=1}^{\infty}$) and $u^*(x) \in W_0^{1,p}(\Omega)$, such that

$$u_k \to u^* \text{ weakly in } W_0^{1,p}(\Omega) \text{ as } k \to \infty,$$
$$u_k \to u^* \text{ strongly in } L^{q+1+\epsilon}(\Omega) \text{ as } k \to \infty,\ q+1+\epsilon < p^*.$$

Next we prove that $u^* \in \mathscr{N}$ and $J(u^*) = d$. On the one hand, the weakly lower semi-continuity of $\|\cdot\|_{W_0^{1,p}}$ implies that

$$a\|\nabla u^*\|_p^p + b\|\nabla u^*\|_p^{2p} \le \liminf_{k\to\infty}(a\|\nabla u_k\|_p^p + b\|\nabla u_k\|_p^{2p}) = \lim_{k\to\infty} \int_\Omega |u_k|^{q+1} \log|u_k| dx. \tag{3.5}$$

On the other hand, there exists $\omega = \theta u_k + (1-\theta)u^*$ with $\theta \in (0,1)$ such that

$$\begin{aligned}
&\int_\Omega |u_k|^{q+1}\log|u_k|dx - \int_\Omega |u^*|^{q+1}\log|u^*|dx \\
=&\int_\Omega ((q+1)|\omega|^{q-1}\omega\log|\omega| + |\omega|^{q-1}\omega)(u_k - u^*)dx \\
\le&\frac{q+1}{eq}|\Omega|^{\frac{q+1}{q}}\|u_k-u^*\|_{q+1} + \frac{q+1}{e\epsilon}\|\omega\|_{q+1+\epsilon}^{q+\epsilon}\|u_k-u^*\|_{q+1+\epsilon} + \|\omega\|_{q+1}^q\|u_k-u^*\|_{q+1} \\
\to&0, \quad \text{as}\, k\to\infty,
\end{aligned}$$

which with (3.5) imply that

$$a\|\nabla u^*\|_p^p + b\|\nabla u^*\|_p^{2p} \le \int_\Omega |u^*|^{q+1}\log|u^*|dx.$$

So we only need to exclude the case of $a\|\nabla u^*\|_p^p + b\|\nabla u^*\|_p^{2p} < \int_\Omega |u^*|^{q+1}\log|u^*|dx$. If it is true, then by Lemma 2.1, there exists an unique $0 < \lambda^* < 1$ such that $\lambda^* u^* \in \mathscr{N}$ and $J(\lambda^* u^*) \ge d$. However,

$$\begin{aligned}
J(\lambda^* u^*) &< \frac{a(q+1-p)}{p(q+1)}\|\nabla u^*\|_p^p + \frac{b(q+1-2p)}{2p(q+1)}\|\nabla u^*\|_p^{2p} + \frac{1}{(q+1)^2}\|u^*\|_{q+1}^{q+1} \\
&\le \liminf_{k\to+\infty} \frac{a(q+1-p)}{p(q+1)}\|\nabla u_k\|_p^p + \frac{b(q+1-2p)}{2p(q+1)}\|\nabla u_k\|_p^{2p} + \frac{1}{(q+1)^2}\|u_k\|_{q+1}^{q+1} \\
&\le d,
\end{aligned}$$

which is a contradiction. Thus $u^* \in \mathscr{N}$ and

$$a\|\nabla u^*\|_p^p + b\|\nabla u^*\|_p^{2p} = \lim_{k\to\infty} a\|\nabla u_k\|_p^p + b\|\nabla u_k\|_p^{2p}.$$

Note the uniform convexity of $W_0^{1,p}(\Omega)$ and the weak convergence of u_k in $W_0^{1,p}(\Omega)$, we can get $u_k \to u^*$ strongly in $W_0^{1,p}(\Omega)$ [2]. Then

$$J(u^*) = \frac{a}{p}\|\nabla u^*\|_p^p + \frac{b}{2p}\|\nabla u^*\|_p^{2p} - \frac{1}{q+1}\int_\Omega |u^*|^{q+1}\log|u|dx + \frac{1}{(q+1)^2}\|u^*\|_{q+1}^{q+1} = \lim_{k\to\infty} J(u_k) = d,$$

which implies $J(u^*) = \inf\limits_{u\in\mathscr{N}} J(u) = d$.

At last, we prove that u^* is the ground state solution of (3.4). Since $u^* \in \mathscr{N}$, we have $\langle J'(u^*), u^*\rangle = I(u^*) = 0$. According to the Lagrange multiplier method, there exists a constant $\mu \in \mathbb{R}$ such that

$$J'(u^*) - \mu I'(u^*) = 0,$$

which implies $\mu\langle I'(u^*), u^*\rangle = \langle J'(u^*), u^*\rangle = 0$. For any $\varphi \in W_0^{1,p}(\Omega)$, we can deduce that

$$\begin{aligned}\langle I'(u^*), \varphi\rangle =& \frac{d}{d\tau} I\left(u^* + \tau\varphi\right)\Big|_{\tau=0}\\ =& ap\left(|\nabla u^*|^{p-2}\cdot\nabla u^*, \nabla\varphi\right) + 2bp\|\nabla u^*\|_p^p\left(|\nabla u^*|^{p-2}\cdot\nabla u^*, \nabla\varphi\right)\\ &-(q+1)\left(|u^*|^{q-1}\cdot u^*\log|u^*|, \psi\right) - \left(|u^*|^{q-1}\cdot u^*, \psi\right).\end{aligned}$$

Choosing $\varphi = u^*$ in the above equality leads to

$$\langle I'(u^*), u^*\rangle = ap\|\nabla u^*\|_p^p + 2bp\|\nabla u^*\|_p^{2p} - (q+1)\int_\Omega |u^*|^{q+1}\log|u^*|dx - \|u^*\|_{q+1}^{q+1},$$

which together with $I(u^*) = 0$ points that

$$\langle I'(u^*), u^*\rangle = a(p-q-1)\|\nabla u^*\|_p^p + b(2p-q-1)\|\nabla u^*\|_p^{2p} - \|u^*\|_{q+1}^{q+1} < 0.$$

Consequently $\mu = 0$ and $J'(u^*) = 0$, which mean $u^* \in \Phi\setminus\{0\}$. From $\Phi\setminus\{0\} \subset \mathscr{N}$ and $J(u^* = d)$, we get

$$J(u^*) = \inf_{u\in\Phi\setminus\{0\}} J(u) = d,$$

which means that $u^*(x)$ is the ground state solution of (3.4). □

Theorem 4 *Let $u_0 \in W_0^{1,p}(\Omega)$ with $J(u_0) \le d$ and $I(u_0) < 0$. Then the weak solution of* (1.1) *blows up in finite time, namely there exists $T > 0$, such that*

$$\lim_{t\to T^-}\int_0^t (\|u\|_2^2 + \|\nabla u\|_2^2)d\tau = +\infty.$$

Proof We give the proof by contradiction. Assume u is a global solution of (1.1). Let

$$H(t) = \int_0^t (\|u\|_2^2 + \|\nabla u\|_2^2)d\tau + (T^* - t)(\|u_0\|_2^2 + \|\nabla u_0\|_2^2), \quad t \in [0, T^*],$$

where T^* is a sufficiently large time. Then $H(t) \geq 0$ with $t \in [0, T^*]$ and

$$(H'(t))^2 = \left(2\int_0^t ((u_\tau, u) + (\nabla u_\tau, \nabla u))d\tau\right)^2 \leq 4H(t)\left[\int_0^t (\|u_\tau\|_2^2 + \|\nabla u_\tau\|_2^2)d\tau\right], \tag{3.6}$$

$$H''(t) = 2(u_t, u) + 2(\nabla u_t, \nabla u) = -2I(u), \tag{3.7}$$

which lead to $H''(t)H(t) - \frac{q+1}{2}(H'(t))^2 \geq H(t)\left[-2I(u) - 2(q+1)\int_0^t \|u_\tau\|_2^2 + \|\nabla u_\tau\|_2^2 d\tau\right]$. If there exist some $\sigma_1 > 0$ such that

$$-2I(u) - 2(q+1)\int_0^t \|u_\tau\|_2^2 + \|\nabla u_\tau\|_2^2 d\tau > \sigma_1, \tag{3.8}$$

then $[H^{\frac{1-q}{2}}(t)]'' \leq \frac{\sigma_1(1-q)}{2}[H^{\frac{1-q}{2}}(t)]^{\frac{q+1}{q-1}}$, which can result in the finite time blow-up of u.

In what follows, we prove such σ_1 indeed exists. When $J(u_0) \leq 0$, then (2.3) and (2.4) lead to $J(u) \leq 0$ and $I(u) < 0$. Therefore from Lemma 2.2, we can derive $\|\nabla u\|_p > r_\epsilon(1)$ and

$$\begin{aligned}
&-2I(u) - 2(q+1)\int_0^t (\|u_\tau\|_2^2 + \|\nabla u_\tau\|_2^2)d\tau \\
=&-2(q+1)J(u_0) + \frac{2a(q+1-p)}{p}\|\nabla u\|_p^p + \frac{b(q+1-2p)}{p}\|\nabla u\|_p^{2p} + \frac{2}{q+1}\|u\|_{q+1}^{q+1} \\
>&\sigma_1
\end{aligned}$$

with $\sigma_1 = \frac{2a(q+1-p)}{p}r_\epsilon^p(1)$. When $0 < J(u_0) < d$ and $I(u_0) < 0$, then Lemma 2.5 implies that, either $d_0 < J(u_0) < d$ or $0 < J(u_0) \leq d_0$, $I_{\tilde{\delta}}(u) \leq 0$ and $\|\nabla u\|_p > r_\epsilon(\tilde{\delta}) > 0$ with $\tilde{\delta} > 1$ being the larger roots of $J(u_0) = d(\delta)$. Thus from (3.7), we find that

$$H''(t) = 2a(\tilde{\delta}-1)\|\nabla u\|_p^p + 2b(\tilde{\delta}-1)\|\nabla u\|_p^{2p} - 2I_{\tilde{\delta}}(u) \geq 2a(\tilde{\delta}-1)r_\epsilon^p(\tilde{\delta}),$$

which with (3.6) guarantees

$$\|u\|_2^2 + \|\nabla u\|_2^2 \geq H'(t) \geq 2a(\tilde{\delta}-1)r_\epsilon^p(\tilde{\delta})t.$$

Thus there exists $T_* > 0$ and $\sigma_1 > 0$ such that (3.8) is established for $t \geq T_*$. When $J(u_0) = d$ and $I(u_0) < 0$, Lemma 2.5 shows that there exists $t_0 > 0$ such that $I(u(t)) < 0$, $0 < t < t_0$. Then (3.7) leads to $H''(t) > 0$ and $\|u_t\|_2^2 + k\|\nabla u_t\|_2^2 \neq 0$ for $0 < t < t_0$. Therefore $J(u(t_0)) = d - \int_0^{t_0}(\|u_\tau\|^2 + k\|\nabla u_\tau\|^2)d\tau = d_1 < d$. We can choose t_0 as the initial time and complete the proof according to the case $J(u_0) < d$ and $I(u_0) < 0$. $\square$

Theorem 5 *Let $u_0 \in W_0^{1,p}(\Omega)$ with $J(u_0) \leq d$ and $I(u_0) < 0$. Then we have the following life span estimation of the blow-up solution in Theorem 4.*
(i) *If $J(u_0) < 0$, then $T \leq \frac{\|u_0\|_2^2 + \|\nabla u_0\|_2^2}{(1-q^2)J(u_0)}$.*
(ii) *If $0 \leq J(u_0) \leq d$, then $T \leq \frac{4\|u_0\|_2^2 + 4\|\nabla u_0\|_2^2}{(q-1)^2\left(\frac{a(q+1-p)}{p(q-1)}\|\nabla u(t_0)\|_p^p + \frac{b(q+1-2p)}{2p(q-1)}\|\nabla u(t_0)\|_p^{2p} - J(u_0)\right)} + t_0$, where t_0 satisfies $2(q+1)J(u_0) < \min\limits_{t\in[t_0,T)}\left(\frac{2a(q+1-p)}{p}\|\nabla u(t)\|_p^p + \frac{b(q+1-2p)}{p}\|\nabla u(t)\|_p^{2p} + \frac{2}{q+1}\|u(t)\|_{q+1}^{q+1}\right)$.*

Remark 3 *We omit the proof for Theorem 5. It can be proved analogue to Theorem 1.6 [5]. Compared to the life span in [5] for the pseudo-parabolic Kirchhoff equation with power like nonlinearity, the time t_0 in Theorem 5 (ii) is smaller than that in Theorem 1.6 [5], which suggests that the logarithmic nonlinearity do contribute to blowing-up.*

4 $J(u_0) > d$

In this section, we investigate the conditions that ensure the global existence or finite time blowing-up of solution to (1.1). Inspired by the ideas in [18, 25, 37, 38], we need to introduce some new notations. For a positive constant $\sigma > d$, we let

$$J^\sigma = \{u \in W_0^{1,p}(\Omega) : J(u) < \sigma\} \quad \text{and} \quad \mathcal{N}^\sigma = \mathcal{N} \cap J^\sigma.$$

It is easy to find that

$$\mathcal{N}^\sigma = \{u \in W_0^{1,p}(\Omega) : I(u) = 0, \left(\frac{a}{p} - \frac{a}{q+1}\right) \|\nabla u\|_p^p + \left(\frac{b}{2p} - \frac{b}{q+1}\right) \|\nabla u\|_p^{2p} + \frac{1}{(q+1)^2}\|u\|_{q+1}^{q+1} < \sigma\}.$$

Then we define

$$\lambda_\sigma = \inf\{\|u\|_2^2 + \|\nabla u\|_2^2 : u \in \mathcal{N}^\sigma\}, \quad \Lambda_\sigma = \sup\{\|u\|_2^2 + \|\nabla u\|_2^2 : u \in \mathcal{N}^\sigma\},$$

which have been discussed in the following theorems.

Theorem 6 $$\lambda_\sigma \geq \begin{cases} \left[\dfrac{a\epsilon e}{\beta^{q+1+\epsilon}}\kappa^{p-\theta(q+1+\epsilon)} + \dfrac{b\epsilon e}{\beta^{q+1+\epsilon}}\kappa^{2p-\theta(q+1+\epsilon)}\right]^{\frac{2}{(1-\theta)(q+1+\epsilon)}}, & p-\theta(q+1+\epsilon) > 0, \\ \left[\dfrac{a\epsilon e}{\beta^{q+1+\epsilon}}\tilde{\kappa}^{p-\theta(q+1+\epsilon)} + \dfrac{b\epsilon e}{\beta^{q+1+\epsilon}}\tilde{\kappa}^{2p-\theta(q+1+\epsilon)}\right]^{\frac{2}{(1-\theta)(q+1+\epsilon)}}, & 2p-\theta(q+1+\epsilon) < 0, \\ \left[\dfrac{a\epsilon e}{\beta^{q+1+\epsilon}}\tilde{\kappa}^{p-\theta(q+1+\epsilon)} + \dfrac{b\epsilon e}{\beta^{q+1+\epsilon}}\kappa^{2p-\theta(q+1+\epsilon)}\right]^{\frac{2}{(1-\theta)(q+1+\epsilon)}}, & p \leq \theta(q+1+\epsilon) \leq 2p. \end{cases}$$

and

$$\Lambda_\sigma \leq 2|\Omega|^{\frac{p-2}{p}}\tilde{\kappa}^2,$$

where $0 < \epsilon < p^* - q - 1$, θ *satisfies* $\theta(\frac{1}{2} - \frac{1}{p} + \frac{1}{n}) = \frac{1}{2} - \frac{1}{q+1+\epsilon}$, $\tilde{\kappa} = \min\left\{\left(\frac{p\sigma(q+1)}{a(q+1-p)}\right)^{1/p}, \left(\frac{2p\sigma(q+1)}{b(q+1-2p)}\right)^{1/2p}\right\}$, κ *is the unique positive solution of* $f(y) = d$ *with*

$$f(y) = \frac{b(q+1-2p)}{2p(q+1)}y^{2p} + \frac{a(q+1-p)}{p(q+1)}y^p + \frac{S_1^{q+1}}{(q+1)^2}y^{q+1}, \quad y \in \mathbb{R}. \tag{4.1}$$

Proof We first give the upper bound and lower bound of $\|\nabla u\|_p$. On the one hand, from (2.6), we have

$$d = \inf_{u \in \mathcal{N}} J(u)$$

$$
\begin{aligned}
&= \inf_{u\in\mathcal{N}} \left[\frac{a(q+1-p)}{p(q+1)}\|\nabla u\|_p^p + \frac{b(q+1-2p)}{2p(q+1)}\|\nabla u\|_p^{2p} + \frac{1}{(q+1)^2}\|u\|_{q+1}^{q+1}\right] \\
&\le \inf_{u\in\mathcal{N}} \left[\frac{a(q+1-p)}{p(q+1)}\|\nabla u\|_p^p + \frac{b(q+1-2p)}{2p(q+1)}\|\nabla u\|_p^{2p} + \frac{S_1^{q+1}}{(q+1)^2}\|\nabla u\|_{p}^{q+1}\right] \\
&= \inf_{u\in\mathcal{N}} f(\|\nabla u\|_p),
\end{aligned}
$$

where S_1 is the embedding coefficient of the Sobolev inequality $\|u\|_{q+1} \le S_1\|\nabla u\|_p$, $f(\cdot)$ is given in (4.1). Since $f(\cdot)$ is strictly increasing on $[0, +\infty)$ and $f(0) = 0$, there exists an unique κ such that $f(\kappa) = d$. Then for any $u \in \mathcal{N}$, there is

$$\|\nabla u\|_p \ge \kappa > 0. \tag{4.2}$$

On the other hand, if $u \in \mathcal{N}^\sigma$, there holds

$$\|\nabla u\|_p \le \tilde{\kappa} = \min\left\{\left(\frac{p\sigma(q+1)}{a(q+1-p)}\right)^{1/p}, \left(\frac{2p\sigma(q+1)}{b(q+1-2p)}\right)^{1/2p}\right\}. \tag{4.3}$$

In what follows, we derive the lower bound of λ_σ and the upper bound of Λ_σ. By the Gagliardo–Nirenberg inequality [2], we get

$$\|u\|_{q+1+\epsilon} \le \beta\|u\|_2^{(1-\theta)}\|\nabla u\|_p^\theta,$$

where β is a positive constant and $\theta(\frac{1}{2} - \frac{1}{p} + \frac{1}{n}) = \frac{1}{2} - \frac{1}{q+1+\epsilon}$. Then it follows from the above inequality that for any $u \in \mathcal{N}$ and $0 < \epsilon < p^* - q - 1$

$$a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p} \le \int_\Omega |u|^{q+1}\log|u|dx \le \frac{1}{\epsilon e}\|u\|_{q+1+\epsilon}^{q+1+\epsilon} \le \frac{1}{\epsilon e}\beta^{q+1+\epsilon}\|u\|_2^{(1-\theta)(q+1+\epsilon)}\|\nabla u\|_p^{\theta(q+1+\epsilon)},$$

which says

$$a\|\nabla u\|_p^{p-\theta(q+1+\epsilon)} + b\|\nabla u\|_p^{2p-\theta(q+1+\epsilon)} \le \frac{1}{\epsilon e}\beta^{q+1+\epsilon}\|u\|_2^{(1-\theta)(q+1+\epsilon)}. \tag{4.4}$$

For the lower bound of λ_σ, we divide into three cases to discuss.

Case 1: $p - \theta(q+1+\epsilon) > 0$, then using (4.2) and (4.4), we have

$$
\begin{aligned}
\lambda_\sigma &= \inf_{u\in\mathcal{N}^\sigma}\left\{\|u\|_2^2 + \|\nabla u\|_2^2\right\} \\
&\ge \inf_{u\in\mathcal{N}}\left\{\|u\|_2^2 + \|\nabla u\|_2^2\right\} \\
&\ge \inf_{u\in\mathcal{N}}\left[\frac{a\epsilon e}{\beta^{q+1+\epsilon}}\|\nabla u\|_p^{p-\theta(q+1+\epsilon)} + \frac{b\epsilon e}{\beta^{q+1+\epsilon}}\|\nabla u\|_p^{2p-\theta(q+1+\epsilon)}\right]^{\frac{2}{(1-\theta)(q+1+\epsilon)}} \\
&\ge \left[\frac{a\epsilon e}{\beta^{q+1+\epsilon}}\kappa^{p-\theta(q+1+\epsilon)} + \frac{b\epsilon e}{\beta^{q+1+\epsilon}}\kappa^{2p-\theta(q+1+\epsilon)}\right]^{\frac{2}{(1-\theta)(q+1+\epsilon)}}.
\end{aligned}
$$

Case 2: $2p - \theta(q+1+\epsilon) < 0$, then using (4.3) and (4.4), we have

$$\lambda_\sigma = \inf_{u\in\mathcal{N}^\sigma}\left\{\|u\|_2^2 + \|\nabla u\|_2^2\right\}$$

$$\geq \left[\frac{a\epsilon e}{\beta^{q+1+\epsilon}}\tilde{\kappa}^{p-\theta(q+1+\epsilon)} + \frac{b\epsilon e}{\beta^{q+1+\epsilon}}\tilde{\kappa}^{2p-\theta(q+1+\epsilon)}\right]^{\frac{2}{(1-\theta)(q+1+\epsilon)}}.$$

Case 3: $p \leq \theta(q+1+\epsilon) \leq 2p$, then using (4.2), (4.3) and (4.4), we have

$$\begin{aligned}\lambda_\sigma &= \inf_{u\in\mathcal{N}^\sigma}\left\{\|u\|_2^2 + \|\nabla u\|_2^2\right\}\\ &\geq \left[\frac{a\epsilon e}{\beta^{q+1+\epsilon}}\tilde{\kappa}^{p-\theta(q+1+\epsilon)} + \frac{b\epsilon e}{\beta^{q+1+\epsilon}}\kappa^{2p-\theta(q+1+\epsilon)}\right]^{\frac{2}{(1-\theta)(q+1+\epsilon)}}.\end{aligned}$$

For the upper bound of Λ_σ, using the Hölder inequality and (4.3), we have

$$\begin{aligned}\Lambda_\sigma &= \sup_{u\in\mathcal{N}^\sigma}\{\|u\|_2^2 + \|\nabla u\|_2^2\}\\ &\leq \sup_{u\in\mathcal{N}_\sigma} 2|\Omega|^{\frac{p-2}{p}}\|\nabla u\|_p^2\\ &\leq 2|\Omega|^{\frac{p-2}{p}}\tilde{\kappa}^2.\end{aligned}$$

□

Theorem 7 *Assume $u_0 \in W_0^{1,p}(\Omega)$ with $J(u_0) > d$.*
(i) *If $I(u_0) > 0$, $\|u_0\|_2^2 + \|\nabla u_0\|_2^2 \leq \lambda_{J(u_0)}$, then u exists globally and decays to zero as $t \to \infty$.*
(ii) *If $I(u_0) < 0$, $\|u_0\|_2^2 + \|\nabla u_0\|_2^2 \geq \Lambda_{J(u_0)}$, then u blows up in finite time.*

Proof Let u be a solution of (1.1), and $T(u_0)$ be the maximal existence time of u.

(i) First, we assert that if $I(u_0) > 0$ and $\|u_0\|_2^2 + k\|\nabla u_0\|_2^2 \leq \lambda_{J(u_0)}$, then

$$I(u(t)) > 0, \quad 0 \leq t < T(u_0). \tag{4.5}$$

Otherwise there exists $t_0 \in (0, T(u_0))$ such that

$$I(u(t)) > 0, \quad 0 \leq t < t_0 \quad \text{and} \quad I(u(t_0)) = 0. \tag{4.6}$$

On the one hand, it can be seen from (2.5) and (4.6) that

$$\frac{d}{dt}(\|u\|_2^2 + \|\nabla u\|_2^2) < 0, \quad 0 < t < t_0.$$

Then we have

$$\|u(t_0)\|_2^2 + \|\nabla u(t_0)\|_2^2 < \|u(t_0)\|_2^2 + \|\nabla u(t_0)\|_2^2 \leq \lambda_{J(u_0)}. \tag{4.7}$$

On the other hand, the non-increasing property of $J(u)$ in (2.4) indicates that $J(u(t_0)) < J(u_0)$, which with the definition of J^σ leads to $u(t_0) \in J^{J(u_0)}$. Thus $u(t_0) \in \mathcal{N}^{J(u_0)}$. According to the definition of $\lambda_{J(u_0)}$, we can get

$$\|u(t_0)\|_2^2 + \|\nabla u(t_0)\|_2^2 \geq \lambda_{J(u_0)},$$

which contradicts (4.7). Hence (4.5) is correct.

Next we can verify that $T(u_0)=+\infty$. Using (2.3), (2.4) and (4.5), there holds

$$\begin{aligned} J(u_0)\geq J(u)&=\frac{1}{q+1}I(u)+(\frac{a}{p}-\frac{a}{q+1})\|\nabla u\|_p^p+(\frac{b}{2p}-\frac{b}{q+1})\|\nabla u\|_p^{2p}+\frac{1}{(q+1)^2}\|u\|_{q+1}^{q+1}\\ &>(\frac{a}{p}-\frac{a}{q+1})\|\nabla u\|_p^p+(\frac{b}{2p}-\frac{b}{q+1})\|\nabla u\|_p^{2p}+\frac{1}{(q+1)^2}\|u\|_{q+1}^{q+1}, \end{aligned}$$

which means $\|\nabla u\|_p$ and $\|u\|_{q=1}$ are bounded and further $T(u_0)=+\infty$.

At last, we prove that $u\to 0$ as $t\to\infty$. Define the $\omega-$ limit set of u_0 by $\omega(u_0)=\bigcap_{t\geq 0}\overline{\{u(\cdot,s):s\geq t\}}$. Then for any $\omega\in\omega(u_0)$, we have

$$\|\omega\|_2^2+\|\nabla\omega\|_2^2<\|u_0\|_2^2+\|\nabla u_0\|_2^2\leq\lambda_{J(u_0)},\quad J(\omega)\leq J(u_0).$$

So that $\omega(u_0)\cap\mathcal{N}=\emptyset$. Since $I(u)>0$, then from (2.4), we obtain

$$J(u)>\left(\frac{a}{p}-\frac{a}{q+1}\right)\|\nabla u\|_p^p+\left(\frac{b}{2p}-\frac{b}{q+1}\right)\|\nabla u\|_p^{2p}+\frac{1}{(q+1)^2}\|u\|_{q+1}^{q+1}\geq 0,$$

which with the non-increasing property of $J(u)$ tells us that

$$\lim_{t\to\infty}J(u(t))=c,$$

where c is a constant. Taking any $\omega\in\omega(u_0)$ as the initial data, then the solution $u_\omega(t)$ has $J(u_\omega(t))=c$ for all $t\geq 0$. Using (2.4) again, we achieve $u_\omega(t)\equiv\omega$. which with (2.5) means $I(\omega)=0$. It's a contradiction. Then $\omega(u_0)=0$, namely $u\to 0$ as $t\to\infty$.

(ii) If $I(u_0)<0$, $\|u_0\|_2^2+\|\nabla u_0\|_2^2\geq\Lambda_{J(u_0)}$, then similar to (i), we can get $I(u(t))<0$, $u(t)\in J^{J(u_0)}$ for $0\leq t<T(u_0)$. If $T(u_0)=\infty$, then for any $\omega\in\omega(u_0)$, we conclude that

$$\|\omega\|_2^2+\|\nabla\omega\|_2^2>\Lambda_{J(u_0)},J(\omega)\leq J(u_0).$$

Then $\omega(u_0)\cap\mathcal{N}=\emptyset$. Similar to (i), $\omega(u_0)=\{0\}$. However due to $I(u)<0$, we have

$$a\|\nabla u\|_p^p<a\|\nabla u\|_p^p+b\|\nabla u\|_p^{2p}<\int_\Omega|u|^{q+1}\log|u|dx\leq\frac{1}{e\epsilon}\|u\|_{q+1+\epsilon}^{q+1+\epsilon}<\frac{S^{q+1+\epsilon}}{e\epsilon}\|\nabla u\|_p^{q+1+\epsilon},$$

which means $\|\nabla u\|_p\geq\left(\frac{ae\epsilon}{S^{q+1+\epsilon}}\right)^{\frac{1}{q+1+\epsilon-p}}$. It is a contradiction. Then $T(u_0)<+\infty$. □

Conflicts of Interest

Authors have no conflict of interest to declare.

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