

Pullback \mathcal{D} -attractors for doubly nonlinear parabolic equations*

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Abstract In this paper, we study a class of doubly nonlinear parabolic equations (1.1). The nonlinearity $\beta(u)$ brings great difficulties to the study of the problem. First we show that the problem has a unique solution. Then we prove that the process corresponding to the problem is norm-to-weak continuous. After that, by using Legendre transform, we obtain uniform estimates and asymptotic compactness properties that allow us to ensure the existence of pullback \mathcal{D} -attractors for the associated process to the problem.

Keywords Pullback \mathcal{D} -attractors; Parabolic equations; Norm-to-weak continuous process; Legendre transform

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1 Introduction

We are interested in the long time behavior of doubly nonlinear parabolic equation of the form

$$\begin{cases} \frac{\partial \beta(u)}{\partial t} - \Delta u + f(u) = g(x, t), & x \in \Omega, t \in \mathbb{R} \\ u(x, t)|_{\partial\Omega} = 0, \\ u(x, \tau) = u_\tau(x). \end{cases} \quad (1.1)$$

in a bounded smooth domain Ω , $g(x, t) \in L^2(\tau, T; L^2(\Omega))$. Such equations appear, e.g., in the study of gas filtration (so called porous medium equation). The study of equation of the form (1.1) can be found in [3-5, 10, 13]. It has been extensively studied when $\beta(u) = u, g(x, t) = g(x)$ and the existence of attractors have been proved in ([1, 7, 9, 14, 15, 17, 18]). For more general equation (1.1) with $g(x, t) = g(x)$, the existence of

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attractors are constructed in ([2-4,8,10]), for the non-autonomous case, as far as I know, the existence of attractors has not been studied.

Our aim in this paper is to study the existence of pullback \mathcal{D} -attractors of (1.1), and extend the result of [7] to the non-autonomous case. We make the following assumptions:

$$\beta(s) \in \mathcal{C}^1(\mathbb{R}), \beta(0) = 0, \beta'(s) \geq \beta_0, \beta_0 > 0 \quad s \in \mathbb{R}; \quad (1.2)$$

$$\beta_1|s|^{r+2} - \beta_3 \leq \beta(s)s \leq \beta_2|s|^{r+2} + \beta_3, \quad \beta_1, \beta_2 > 0, \beta_3 \geq 0, r \geq 0; \quad (1.3)$$

$$f(s) \in \mathcal{C}(\mathbb{R}), \gamma_1|s|^q - \gamma_3 \leq f(s)s \leq \gamma_2|s|^q + \gamma_3, \quad s \in \mathbb{R}, \gamma_1, \gamma_2 > 0, \gamma_3 \geq 0, q \geq r + 2; \quad (1.4)$$

There exists a constant $C_0 \geq 0$, such that

$$C_0\beta(s) + f(s) \text{ is increasing.} \quad (1.5)$$

By hypotheses (1.2)-(1.5), β and f are nonlinear functions with polynomial growth of arbitrary order. Here β is more general than in [2-4,10] (where β is linear growth), which is an essential difficulty in proving the existence of attractor. To the problem (1.1), the key points are to obtain the norm-to-weak continuous and compactness of process generated by (1.1). By using Legendre transform and the asymptotic a priori estimate method introduced in [7], we show that the existence of pullback \mathcal{D} -attractor.

This article is organized as follows. In Section 2, we recall some basic concepts about the pullback \mathcal{D} -attractor. In Section 3, we show that the uniqueness of solution and norm-to-weak continuous of process generated by (1.1). In section 4, we verify the asymptotic compactness of the process $U(t, \tau)$ in $L^q(\Omega)$, and prove the existence of the $(L^{r+2}(\Omega), L^q(\Omega))$ pullback \mathcal{D} -attractor under the hypotheses (1.2)-(1.5).

Throughout this paper we use the following notation: $H = L^2(\Omega)$, and the norms in $H_0^1(\Omega)$ and $L^p(\Omega)$ ($1 \leq p \leq \infty$) are denoted by $\|u\|^2 = \int_{\Omega} |\nabla u|^2 dx$ and $|u|_p^p = \int_{\Omega} |u|^p dx$, respectively; $\Omega(u \geq M) = \{x \in \Omega : u(x) \geq M\}$ and $\Omega(u \leq -M) = \{x \in \Omega : u(x) \leq -M\}$; $m(\Omega)$ or $|\Omega|$ denote Lebesgue measure of Ω ; sometimes for special differentiation, we denote the different positive constants by c, c_1, c_2, \dots .

2 Preliminaries

Let X be a complete metric space, and $\{U(t, \tau)\} = \{U(t, \tau) : t \geq \tau\}$ be a two-parameter family of mappings act on $X : U(t, \tau) : X \rightarrow X, t \geq \tau$.

Definition 2.1 ([2,9,17]) *A two-parameter family of mappings $\{U(t, \tau)\}$ is said to be a norm-to-weak continuous process in X if*

- (1) $U(t, s)U(s, \tau) = U(t, \tau), \forall t \geq s \geq \tau,$
- (2) $U(\tau, \tau) = Id,$ is the identity operator, $\tau \in \mathbb{R},$
- (3) $U(t, \tau)x_n \rightharpoonup U(t, \tau)x,$ if $x_n \rightarrow x$ in $X.$

Let $B(X)$ is the set of all bounded subsets of X , \mathcal{D} is a nonempty class of parameterised sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset B(X).$

Definition 2.2([2,6,7,9,11,15]) *It is said that $\hat{\mathcal{B}} \in \mathcal{D}$ is pullback \mathcal{D} – absorbing for the process $\{U(t, \tau)\}$ if for any $t \in \mathbb{R}$ and any $\hat{\mathcal{D}} \in \mathcal{D}$, there exists a $\tau_0(t, \hat{\mathcal{D}}) \leq t$ such that $U(t, \tau)D(\tau) \subset B(t)$ for all $\tau \leq \tau_0(t, \hat{\mathcal{D}}).$*

Definition 2.3([2,6,7,9,11,15]) *The process $\{U(t, \tau)\}$ is said to be pullback \mathcal{D} –asymptotically compact if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, and any sequence $\tau_n \rightarrow -\infty$, any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in $X.$*

Definition 2.4([2,6,7,9,11,15]) *The family $\hat{\mathcal{A}} = \{\mathcal{A}(t) : t \in \mathbb{R}\} \subset B(X)$ is said to be a pullback \mathcal{D} – attractor for $U(t, \tau)$ if*

- (1) $\mathcal{A}(t)$ is compact for all $t \in \mathbb{R},$
- (2) $\hat{\mathcal{A}}$ is invariant, i.e., $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geq \tau,$
- (3) $\hat{\mathcal{A}}$ is pullback \mathcal{D} –attracting, i.e.,

$$\lim_{\tau \rightarrow -\infty} \text{dist}((U(t, \tau)D(\tau), \mathcal{A}(t))) = 0, \text{ for all } \hat{\mathcal{D}} \in \mathcal{D}, \text{ and all } t \in \mathbb{R},$$

- (4) if $\{C(t)\}_{t \in \mathbb{R}}$ is another family of closed attracting sets, then $\mathcal{A}(t) \subset C(t)$ for all $t \in \mathbb{R}.$

Let X be a complete metric space and B be a bounded subset of X . The Kuratowski measure of noncompactness $\alpha(B)$ of B is defined by

$$\alpha(B) = \inf\{\delta > 0 \mid B \text{ has a finite open cover of sets of diameter} \leq \delta\}.$$

Definition 2.5([9]) *A process $\{U(t, \tau)\}$ is called pullback ω -D-limit compact if for any $\varepsilon > 0$ and $\hat{\mathcal{D}} \in \mathcal{D}$, there exists a $\tau_0(\hat{\mathcal{D}}, t) \leq t$ such that $\alpha(\bigcup_{\tau \leq \tau_0} U(t, \tau)D(\tau)) \leq \varepsilon.$*

Lemma 2.1([9]) *Assume $\{U(t, \tau)\}$ is pullback ω -D-limit compact, then for any sequence $\{\tau_n\} \subset \mathbb{R}, \tau_n \rightarrow -\infty$ as $n \rightarrow \infty$, and sequence $x_n \in D(\tau_n)$ there exists a convergent subsequence of $\{U(t, \tau_n)x_n\}$ whose limit lies in $\omega(\hat{\mathcal{D}}, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)B(\tau)}.$*

Theorem 2.1([9]) *Suppose that the process $U(t, \tau)$ is norm-to-weak continuous and pullback ω -D-limit compact, $\hat{\mathcal{B}} \in \mathcal{D}$ is a family of pullback \mathcal{D} –absorbing sets for $U(t, \tau).$ Then the family $\hat{\mathcal{A}} = \{\mathcal{A}(t) : t \in \mathbb{R}\} \subset B(X)$ defined by*

$$\mathcal{A}(t) = \omega(\hat{\mathcal{B}}, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)B(\tau)},$$

is a pullback \mathcal{D} -attractor for $U(t, \tau).$

Theorem 2.2([7]) *Let $\{U(t, \tau)\}_{t \geq \tau}$ is norm-to-weak continuous process in $L^p(\Omega),$ $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is pullback \mathcal{D} -absorbing sets in $L^p(\Omega),$ and $U(t, \tau)$ satisfy the*

following two assumptions:

- (1) $\{U(t, \tau)\}_{t \geq \tau}$ is pullback ω - D -limit compact in $L^q(\Omega)$ ($1 \leq q < p$);
- (2) for any $\varepsilon > 0$, there exist $M(\varepsilon, \widehat{\mathcal{B}})$ and $\tau_0 = \tau_0(\varepsilon, \widehat{\mathcal{B}}) \leq t$ such that $\int_{\Omega(|U(t, \tau)u_\tau| \geq M)} |U(t, \tau)u_\tau|^p dx)^{\frac{1}{p}} < \varepsilon$ for any $u_\tau \in B(\tau)$, and $\tau \leq \tau_0$.

Then there exists a pullback \mathcal{D} -attractor $\hat{\mathcal{A}} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ in $L^p(\Omega)$ and

$$\mathcal{A}(t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)B(\tau)}^{L^p(\Omega)},$$

where $\overline{\bigcup_{\tau \leq s} U(t, \tau)B(\tau)}^{L^p(\Omega)}$ denote closure in $L^p(\Omega)$.

By Lemma 2.1, the process $\{U(t, \tau)\}$ is pullback ω - D -limit compact, then $\{U(t, \tau)\}$ is pullback \mathcal{D} -asymptotically compact. In practice, as long as the process is pullback \mathcal{D} -asymptotically compact, then the Theorem 2.1 and the Theorem 2.2 are still hold.

3 Uniqueness of solution and norm-to-weak continuous of process

The existence of weak solution for (1.1) can be obtained by the standard Faedo-Galerkin approximation method(see[1,3,14]). Here we only state the result.

Lemma 3.1 Assume that $g(x, t) \in L^2(\Omega)$, β and f satisfying (1.2)-(1.5), $u_\tau(x) \in L^{r+2}(\Omega)$. Then for any initial data $u_\tau(x) \in L^{r+2}(\Omega)$, there exists solution $u(x, t)$ for Eq.(1.1) which satisfies

$$u(x, t) \in C(\tau, T; L^1(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^q(\tau, T; L^q(\Omega)).$$

We now show that the solution is uniqueness and continuous dependence on initial conditions.

Theorem 3.2 Assume that $g(x, t) \in L^2(\Omega)$, $u_\tau(x) \in L^{r+2}(\Omega)$, β and f satisfying (1.2)-(1.5). Then there exists a unique solution of Eq.(1.1)

Proof Suppose that $u(t), v(t)$ be two solution of (1.1) with initial conditions $u_\tau(x), v_\tau(x)$, then

$$\frac{\partial(\beta(u) - \beta(v))}{\partial t} - \Delta(u - v) + f(u) - f(v) = 0,$$

i.e.,

$$\frac{\partial(\beta(u) - \beta(v))}{\partial t} - \Delta(u - v) + (C_0\beta(u) + f(u)) - (C_0\beta(v) + f(v)) = C_0(\beta(u) - \beta(v)).$$

We define the sign function by

$$\text{sign}(\tau) = \begin{cases} 1 & \text{if } \tau > 0, \\ 0 & \text{if } \tau = 0, \\ -1 & \text{if } \tau < 0. \end{cases}$$

Multiplying (3.1) by $\text{sign}(u - v)$ and integrating in Ω , we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\beta(u) - \beta(v)| dx - \int_{\Omega} \Delta(u - v) \text{sign}(u - v) \\ & + \int_{\Omega} [(C_0\beta(u) + f(u)) - (C_0\beta(v) + f(v))] \text{sign}(\beta(u) - \beta(v)) dx \\ & = C_0 \int_{\Omega} |\beta(u) - \beta(v)| dx. \end{aligned}$$

Using (1.6), we get

$$\int_{\Omega} [(C_0\beta(u) + f(u)) - (C_0\beta(v) + f(v))] \text{sign}(\beta(u) - \beta(v)) dx \geq 0.$$

Since $\text{sign}(u - v) = \lim_{\varepsilon \rightarrow 0^+} \frac{u - v}{\varepsilon + |u - v|}$, by dominated convergence theorem, we have

$$\begin{aligned} & - \int_{\Omega} \Delta(u - v) \text{sign}(u - v) dx = - \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \Delta(u - v) \frac{u - v}{\varepsilon + |u - v|} dx \\ & = \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \nabla(u - v) \nabla \left(\frac{u - v}{\varepsilon + |u - v|} \right) dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \varepsilon \frac{|\nabla(u - v)|^2}{(\varepsilon + |u - v|)^2} dx \geq 0. \end{aligned}$$

So

$$\frac{d}{dt} \int_{\Omega} |\beta(u) - \beta(v)| dx \leq C_0 \int_{\Omega} |\beta(u) - \beta(v)| dx.$$

By Gronwall inequality, we get

$$\int_{\Omega} |\beta(u(t)) - \beta(v(t))| dx \leq e^{C_0(t-\tau)} \int_{\Omega} |\beta(u_{\tau}) - \beta(v_{\tau})| dx,$$

From (1.2), we have

$$\int_{\Omega} |u(t) - v(t)| dx \leq \frac{1}{\beta_0} e^{C_0(t-\tau)} \int_{\Omega} |\beta(u_{\tau}) - \beta(v_{\tau})| dx.$$

Which gives continuous dependence on initial conditions and uniqueness of solution in $L^1(\Omega)$.

By Theorem 3.2, we can define the process $\{U(t, \tau)\}_{t \geq \tau}$ in $L^1(\Omega)$ as the following:

$$U(t, \tau)u_{\tau} : L^{r+2}(\Omega) \rightarrow L^1(\Omega),$$

which is continuous in $L^1(\Omega)$.

Since β be a continuous increasing function with $\beta(0) = 0$. We define for $t \in \mathbb{R}$,

$$\psi(t) = \int_0^t \beta(\tau) d\tau.$$

Then the Legendre transform ψ^* is defined by

$$\psi^*(\tau) = \sup_{s \in \mathbb{R}} \{\tau s - \psi(s)\}.$$

Note that

$$\psi^*(\tau) \geq 0, \quad \psi^*(\beta(\tau)) + \psi(\tau) = \tau\beta(\tau), \quad \psi^*(\beta(\tau)) \leq \tau\beta(\tau). \quad (3.1)$$

Theorem 3.3 Assume that the conditions (1.2)-(1.5) are satisfied, $g(x, t) \in L^2(\Omega)$. Then the process $U(t, \tau)$ is norm-to-weak continuous in $L^q(\Omega)$ and $H_0^1(\Omega)$.

Proof Let $u_{m\tau}(x) \rightarrow u_\tau(x)$ in $L^{r+2}(\Omega)$, $u_m(t), u(t)$ are the solutions of Eq.(1.1) corresponding to initial data $u_{m\tau}(x), u_\tau(x)$. In (1.1), replace $u(t)$ by $u_m(t)$. Multiply (1.1) by $u_m(t)$ and integrating in Ω , we get

$$\frac{d}{dt} \int_{\Omega} \psi^*(\beta(u_m(t))) dx + |\nabla u_m|_2^2 + (f(u_m), u_m) = (g, u_m).$$

Thanks to Poincaré inequality $\lambda|u|_2^2 \leq |\nabla u|_2^2$, and Cauchy inequality, we have

$$|\int_{\Omega} g(x) u_m dx| \leq \frac{\lambda}{2} |u_m|_2^2 + \frac{1}{2\lambda} |g(x, t)|_2^2 \leq \frac{1}{2} |\nabla u_m|_2^2 + \frac{1}{2\lambda} |g(x, t)|_2^2.$$

So

$$\frac{d}{dt} \int_{\Omega} \psi^*(\beta(u_m(t))) dx + \frac{1}{2} |\nabla u_m|_2^2 + \gamma_1 |u_m|_q^q \leq \gamma_3 |\Omega| + \frac{1}{2\lambda} |g(x, t)|_2^2.$$

Integrating from τ to T , we obtain

$$\begin{aligned} & \int_{\Omega} \psi^*(\beta(u_m(T))) dx + \frac{1}{2} \int_{\tau}^T |\nabla u_m|_2^2 dt + \gamma_1 \int_{\tau}^T |u_m|_q^q dt \\ & \leq \int_{\Omega} \psi^*(\beta(u_{m\tau})) dx + \gamma_3 |\Omega| (T - \tau) + \frac{1}{2\lambda} \int_{\tau}^T |g(x, s)|_2^2 ds. \\ & \leq \int_{\Omega} u_{m\tau} \beta(u_{m\tau}) dx + \gamma_3 |\Omega| (T - \tau) + \frac{1}{2\lambda} \int_{\tau}^T |g(x, s)|_2^2 ds \\ & \leq \beta_2 |u_{m\tau}|_{r+2}^{r+2} + \beta_3 |\Omega| + \gamma_3 |\Omega| (T - \tau) + \frac{1}{2\lambda} \int_{\tau}^T |g(x, s)|_2^2 ds. \end{aligned}$$

$u_{m\tau} \rightarrow u_\tau$ in $L^{r+2}(\Omega)$, so there exists $M > 0$, such that $|u_{m\tau}|_{r+2}^{r+2} \leq M$. We get $u_m(t)$ are bounded in $L^2(\tau, T; H_0^1(\Omega))$ and $L^q(\tau, T; L^q(\Omega))$, there exists weak convergence subsequence $u_{m_k}(t)$ convergence to $v(t)$ in $L^2(\tau, T; H_0^1(\Omega))$ and $L^q(\tau, T; L^q(\Omega))$, obviously, $v(t)$ be a solution of (1.1) satisfies initial value $v(x, \tau) = u_\tau(x)$. By the unique of solution for (1.1), we have $u(t) = v(t)$, i.e., $u_{m_k} \rightharpoonup u(t)$ in $L^2(\tau, T; H_0^1(\Omega))$ and $L^q(\tau, T; L^q(\Omega))$. By Definition 2.1, Theorem 3.3 holds.

Remark 3.4 The process $\{U(t, \tau)\}_{t \geq \tau}$ is norm-to-weak continuous in $L^2(\Omega)$.

4 Pullback \mathcal{D} -attractor in $L^q(\Omega)$

By theorem 3.3, we can define process $\{U(t, \tau)\}_{t \geq \tau}$ as the following:

$$U(t, \tau) : L^{r+2}(\Omega) \rightarrow L^q(\Omega). \quad (4.1)$$

Moreover, we suppose for any $t \in \mathbb{R}$, we have

$$\int_{-\infty}^t e^{\delta s} |g(x, s)|_2^2 ds < \infty, \int_{-\infty}^t \int_{-\infty}^s e^{\delta r} |g(x, r)|_2^2 dr ds < \infty, \quad (4.2)$$

here $\delta = \frac{\gamma_1 q}{\beta_2(r+2)}$.

Lemma 4.1 *Assume that the conditions (1.2)-(1.5) are satisfied and $g(x, t)$ satisfies (4.2), $u(t)$ be a weak solution of (1.1). Then there exists $T > 0$, for any $t - \tau \geq T$, we have the following inequality:*

$$\begin{aligned} |\nabla u(t)|_2^2 + |u(t)|_q^q &\leq c((t - \tau)e^{-\delta(t-\tau)}|u_\tau|_{r+2}^{r+2} + 1 \\ &+ \int_{-\infty}^t e^{\delta(s-t)} |g(x, s)|_2^2 ds + \int_{-\infty}^t \int_{-\infty}^s e^{\delta(r-t)} |g(x, r)|_2^2 dr ds). \end{aligned} \quad (4.3)$$

Proof Multiplying (1.1) by $u(t)$ and integrating over Ω , we obtain

$$\frac{d}{dt} \int_{\Omega} \psi^*(\beta(u)) dx + |\nabla u|_2^2 + (f(u), u) = (g, u).$$

By (1.4), we have

$$\frac{d}{dt} \int_{\Omega} \psi^*(\beta(u)) dx + |\nabla u|_2^2 + \gamma_1 |u|_q^q - \gamma_3 |\Omega| \leq (g, u). \quad (4.4)$$

Thanks to Poincaré inequality and Young inequality, one gets

$$|(g, u)| \leq \frac{\lambda}{2} |u|_2^2 + \frac{1}{2\lambda} |g(x, t)|_2^2 \leq \frac{1}{2} |\nabla u|_2^2 + \frac{1}{2\lambda} |g(x, t)|_2^2,$$

By (4.4), we have

$$\frac{d}{dt} \int_{\Omega} \psi^*(\beta(u)) dx + \frac{1}{2} |\nabla u|_2^2 + \gamma_1 |u|_q^q \leq \gamma_3 |\Omega| + \frac{1}{2\lambda} |g(x, t)|_2^2. \quad (4.5)$$

Using Young inequality, we obtain

$$|u|_{r+2}^{r+2} = \int_{\Omega} |u|^{r+2} dx \leq \frac{r+2}{q} |u|_q^q + \frac{q-r-2}{q} |\Omega|.$$

We find that

$$\frac{\gamma_1 q}{r+2} (|u|_{r+2}^{r+2} - \frac{q-r-2}{q} |\Omega|) \leq \gamma_1 |u|_q^q. \quad (4.6)$$

Using (1.3) and (3.1), we get

$$0 \leq \int_{\Omega} \psi^*(\beta(u))dx \leq \int_{\Omega} u\beta(u)dx \leq \beta_2|u|_{r+2}^{r+2} + \beta_3|\Omega|.$$

Hence

$$\gamma_1|u|_q^q \geq \frac{\gamma_1 q}{\beta_2(r+2)} \int_{\Omega} \psi^*(\beta(u))dx - \frac{\gamma_1 q \beta_3}{\beta_2(r+2)}|\Omega| - \frac{\gamma_1 q(q-r-2)}{q(r+2)}|\Omega|.$$

Let $\delta = \frac{\gamma_1 q}{\beta_2(r+2)}$, $c_1 = (\frac{\gamma_1 q \beta_3}{\beta_2(r+2)} + \frac{\gamma_1 q(q-r-2)}{q(r+2)} + \gamma_3)|\Omega|$, by (4.5), we obtain

$$\frac{d}{dt} \int_{\Omega} \psi^*(\beta(u))dx + \delta \int_{\Omega} \psi^*(\beta(u))dx \leq c_1 + \frac{1}{2\lambda}|g(x, t)|_2^2. \quad (4.7)$$

By the Gronwall lemma, for all $t \geq \tau$, one deduces

$$\int_{\Omega} \psi^*(\beta(u(t)))dx \leq e^{-\delta(t-\tau)} \int_{\Omega} \psi^*(\beta(u(\tau)))dx + \frac{c_1}{\delta} + \frac{1}{2\lambda} e^{-\delta t} \int_{-\infty}^t e^{-\delta s} |g(x, s)|_2^2 ds. \quad (4.8)$$

Multiplying (1.1) by u_t and integrating in Ω , we get

$$\int_{\Omega} \beta'(u)u_t^2 dx + \frac{d}{dt}(|\nabla u|_2^2 + \int_{\Omega} F(u)dx) = (g(x, t), u_t),$$

where $F(u) = \int_0^u f(s)ds$. By (1.2), we have

$$|(g(x, t), u_t)| \leq \int_{\Omega} |g(x, t)u_t|dx \leq \frac{1}{2} \int_{\Omega} \beta'(u)u_t^2 dx + \frac{1}{2\beta_0}|g(x, t)|_2^2.$$

Therefore, one has

$$\frac{d}{dt}(|\nabla u|_2^2 + \int_{\Omega} F(u)dx) \leq \frac{1}{2\beta_0}|g(x, t)|_2^2. \quad (4.9)$$

It follows from (1.4) that there exist $\gamma'_1, \gamma'_2 > 0, \gamma'_3 \geq 0$ such that

$$\gamma'_1|s|^q - \gamma'_3 \leq F(s) \leq \gamma'_2|s|^q + \gamma'_3. \quad (4.10)$$

Let $\rho = \min\{\frac{1}{2}, \frac{\gamma_1}{r_2}\}$. Using (4.5), one has

$$\frac{d}{dt} \int_{\Omega} \psi^*(\beta(u))dx + \rho(|\nabla u|_2^2 + \int_{\Omega} F(u)dx) \leq c_2(1 + |g(x, t)|_2^2). \quad (4.11)$$

So, by (4.9) and (4.11)

$$\frac{d}{dt}(e^{\delta t} \int_{\Omega} \psi^*(\beta(u))dx) \leq \delta e^{\delta t} \int_{\Omega} \psi^*(\beta(u))dx + e^{\delta t}(-\rho(|\nabla u|_2^2 + \int_{\Omega} F(u)dx) + c_2(1 + |g(x, t)|_2^2)).$$

Using (4.8), one has

$$\begin{aligned}
& \int_{\tau}^t e^{\delta s} (|\nabla u|_2^2 + \int_{\Omega} F(u) dx) ds \\
& \leq c_3 (e^{\delta \tau} \int_{\Omega} \psi^*(\beta(u_{\tau})) dx + \int_{\tau}^t e^{\delta s} \int_{\Omega} \psi^*(\beta(u(s))) dx ds + e^{\delta t} + \int_{\tau}^t e^{\delta s} |g(x, s)|_2^2 ds) \\
& \leq c_4 ((1 + t - \tau) e^{\delta \tau} \int_{\Omega} \psi^*(\beta(u_{\tau})) dx + e^{\delta t} + \int_{\tau}^t e^{\delta s} |g(x, s)|_2^2 ds + \int_{\tau}^t \int_{\tau}^s e^{\delta r} |g(x, r)|_2^2 dr ds). \tag{4.12}
\end{aligned}$$

In fact, by (4.9), we obtain

$$\begin{aligned}
& \frac{d}{dt} ((t - \tau) e^{\delta t} (|\nabla u|_2^2 + \int_{\Omega} F(u) dx)) \\
& = (1 + \delta(t - \tau)) e^{\delta t} (|\nabla u|_2^2 + \int_{\Omega} F(u) dx) + (t - \tau) e^{\delta t} \frac{d}{dt} (|\nabla u|_2^2 + \int_{\Omega} F(u) dx) \tag{4.13} \\
& \leq c_5 ((1 + t - \tau) e^{\delta t} (|\nabla u|_2^2 + \int_{\Omega} F(u) dx) + (t - \tau) e^{\delta t} |g(x, t)|_2^2).
\end{aligned}$$

For any $t - \tau \geq 1$, integrating from τ to t , we have

$$\begin{aligned}
& |\nabla u(t)|_2^2 + \int_{\Omega} F(u(t)) dx \\
& \leq c_5 ((1 + \frac{1}{t - \tau}) e^{-\delta t} \int_{\tau}^t e^{\delta s} (|\nabla u(s)|_2^2 + \int_{\Omega} F(u(s)) dx) ds + e^{-\delta t} \int_{\tau}^t e^{\delta s} |g(x, s)|_2^2 ds) \\
& \leq c_6 ((t - \tau) e^{-\delta(t - \tau)} \int_{\Omega} \psi^*(\beta(u_{\tau})) dx + 1 \\
& + e^{-\delta t} (\int_{\tau}^t e^{\delta s} |g(x, s)|_2^2 ds + \int_{\tau}^t \int_{\tau}^s e^{\delta r} |g(x, r)|_2^2 dr ds)) \tag{4.14} \\
& \leq c_7 ((t - \tau) e^{-\delta(t - \tau)} |u_{\tau}|_{r+2}^{r+2} + (t - \tau) e^{-\delta(t - \tau)} + 1 \\
& + e^{-\delta t} (\int_{\tau}^t e^{\delta s} |g(x, s)|_2^2 ds + \int_{\tau}^t \int_{\tau}^s e^{\delta r} |g(x, r)|_2^2 dr ds)) \\
& \leq c_7 ((t - \tau) e^{-\delta(t - \tau)} |u_{\tau}|_{r+2}^{r+2} + (t - \tau) e^{-\delta(t - \tau)} + 1 \\
& + e^{-\delta t} (\int_{-\infty}^t e^{\delta s} |g(x, s)|_2^2 ds + \int_{-\infty}^t \int_{-\infty}^s e^{\delta r} |g(x, r)|_2^2 dr ds)).
\end{aligned}$$

We find that exists $T > 0$, for any $t - \tau > \max\{1, T\}$,

$$\begin{aligned}
& |\nabla u(t)|_2^2 + \int_{\Omega} F(u(t)) dx \leq c ((t - \tau) e^{-\delta(t - \tau)} |u_{\tau}|_{r+2}^{r+2} + 1 \\
& + \int_{-\infty}^t e^{\delta(s - t)} |g(x, s)|_2^2 ds + \int_{-\infty}^t \int_{-\infty}^s e^{\delta(r - t)} |g(x, r)|_2^2 dr ds). \tag{4.15}
\end{aligned}$$

By (4.10), we obtain Lemma 4.1.

Let \mathcal{R} be the set of all functions $\rho : \mathbb{R} \rightarrow (0, +\infty)$ such that $\lim_{t \rightarrow +\infty} t e^{\delta t} \rho^{r+2}(t) = 0$, denote by \mathcal{D} the class of all families $\widehat{D} = \{D(t) : t \in \mathbb{R}\}$ such that $D(t) \subset \overline{B}(\rho(t))$ for some $\rho(t) \in \mathcal{R}$, $\overline{B}(\rho(t))$ the closed ball in $L^{r+2}(\Omega)$ with radius $\rho(t)$. Let

$$\rho_0(t) = [2c(1 + \int_{-\infty}^t e^{\delta(s - t)} |g(x, s)|_2^2 ds + \int_{-\infty}^t \int_{-\infty}^s e^{\delta(r - t)} |g(x, r)|_2^2 dr ds)]^{\frac{1}{q}}. \tag{4.16}$$

$\overline{B}_q(\rho_0(t))$ denote close ball in $L^q(\Omega)$ with radius $\rho_0(t)$. Obviousy $\overline{B}_q(\rho_0(t))$ be a family of bounded pullback \mathcal{D} -absorbing sets for the process $\{U(t, \tau)\}_{t \geq \tau}$ generated by (1.1) in $L^q(\Omega)$.

From (4.3), we also get that there exists a family of bounded pullback \mathcal{D} -absorbing sets in $L^2(\Omega)$ and $H_0^1(\Omega)$, therefore, the process generated by (1.1) is pullback ω - D -limit compact in $L^2(\Omega)$. By theorem 2.1, we have the following theorem.

Theorem 4.2 *Assume that the conditions (1.2)-(1.5) are satisfied, $g(x, t)$ satisfies (4.2). Then the process $U(t, \tau)$ generated by (1.1) exists a pullback \mathcal{D} -attractor in $L^2(\Omega)$.*

In the following, we will give the asymptotic a priori estimate of $\{U(t, \tau)\}_{t \geq \tau}$ with respect to $L^q(\Omega)$ norm, which play a crucial role in the proof of the pullback \mathcal{D} -attractor in $L^q(\Omega)$

Theorem 4.3 *Assume that the conditions (1.2)-(1.5) are satisfied, $g(x, t)$ satisfies (4.2), $q > r + 2$. Then the process $U(t, \tau)$ generated by (1.1) exists a pullback \mathcal{D} -attractor in $L^q(\Omega)$.*

Proof We know from Theorem 4.2 that the process $\{U(t, \tau)\}_{t \geq \tau}$ is pullback ω - D -limit compact in $L^2(\Omega)$. Next we will prove that the process satisfies (2) of Theorem 2.2.

By (1.3) and (1.5), we find that there exists $M_1 > 0, \forall |u| \geq M_1$ such that

$$f(u)u \geq \frac{\gamma_1}{2}|u|^q, \quad \frac{\beta_1}{2}|u|^{r+1} \leq |\beta(u)| \leq 2\beta_2|u|^{r+1}. \quad (4.17)$$

Let $M_2 = \max\{1, \frac{\beta_1}{2}|M_1|^{r+1}\}$, $|u| \geq M_1$, then $|\beta(u)| \geq M_2$. Multiply (1.1) with $|(\beta(u) - M_2)_+|^{\frac{q}{r+1}-2}(\beta(u) - M_2)_+$, we get

$$\begin{aligned} & \frac{r+1}{q} \frac{d}{dt} \int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}} dx + \int_{\Omega} \nabla u \nabla (|(\beta(u) - M_2)_+|^{\frac{q}{r+1}-2}(\beta(u) - M_2)_+) dx \\ & + \int_{\Omega} f(u) |(\beta(u) - M_2)_+|^{\frac{q}{r+1}-2}(\beta(u) - M_2)_+ dx \\ & = \int_{\Omega} g(x, t) |(\beta(u) - M_2)_+|^{\frac{q}{r+1}-2}(\beta(u) - M_2)_+ dx. \end{aligned} \quad (4.18)$$

Where $(\beta(u) - M_2)_+$ denote the positive part of $(\beta(u) - M_2)$, that is

$$(\beta(u) - M_2)_+ = \begin{cases} \beta(u) - M_2, & \beta(u) \geq M_2, \\ 0, & \beta(u) < M_2. \end{cases}$$

Thus we have

$$\begin{aligned}
& \int_{\Omega} \nabla u \nabla (|(\beta(u) - M_2)_+|^{\frac{q}{r+1}-2} (\beta(u) - M_2)_+) dx \\
&= \int_{\Omega(\beta(u) \geq M)} \nabla u \nabla (|(\beta(u) - M_2)|^{\frac{q}{r+1}-1}) dx \\
&= (\frac{q}{r+1} - 1) \int_{\Omega(\beta(u) \geq M)} \beta'(u) |(\beta(u) - M_2)|^{\frac{q}{r+1}-2} |\nabla u|^2 dx \\
&\geq 0,
\end{aligned}$$

$$\begin{aligned}
& \int_{\Omega} f(u) |(\beta(u) - M_2)_+|^{\frac{q}{r+1}-2} (\beta(u) - M_2)_+ dx \\
&\geq \frac{\gamma_1}{2} \int_{\Omega} |u|^{q-1} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}-1} dx \\
&\geq c_8 \int_{\Omega} |\beta(u)|^{\frac{q-1}{r+1}} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}-1} dx \\
&\geq \frac{c_8}{2} \int_{\Omega} |\beta(u)|^{\frac{q-1}{r+1}} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}-1} dx + \frac{c_8}{2} \int_{\Omega} |\beta(u)|^{\frac{r}{r+1}} |\beta(u)|^{\frac{q-r-1}{r+1}} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}-1} dx \\
&\geq \frac{c_8}{2} M_2^{\frac{q-r-2}{r+1}} \int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}} dx + \frac{c_8}{2} \int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{2(q-r-1)}{r+1}} dx
\end{aligned}$$

and

$$\begin{aligned}
& |\int_{\Omega} g(x, t) |(\beta(u) - M_2)_+|^{\frac{q}{r+1}-2} (\beta(u) - M_2)_+ dx| \\
&\leq \int_{\Omega} |g(x, t)| |(\beta(u) - M_2)_+|^{\frac{q-r-1}{r+1}} dx \\
&\leq \frac{c_8}{2} \int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{2(q-r-1)}{r+1}} dx + \frac{1}{2c_8} \int_{\Omega(\beta(u) \geq M_2)} |g(x, t)|^2 dx.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}} dx + c_9 \int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}} dx \\
&\leq c_{10} \int_{\Omega(\beta(u) \geq M_2)} |g(x, t)|^2 dx.
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
& \frac{d}{dt} [(t - \tau) e^{c_9 M_2^{\frac{q-r-2}{r+1} t}} \int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}} dx] \\
&\leq e^{c_9 M_2^{\frac{q-r-2}{r+1} t}} \int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}} dx + c_{10} (t - \tau) e^{c_9 M_2^{\frac{q-r-2}{r+1} t}} \int_{\Omega(\beta(u) \geq M_2)} |g(x, t)|^2 dx.
\end{aligned} \tag{4.20}$$

Integrating from τ to t , we have

$$\begin{aligned}
& \int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}} dx \\
& \leq \frac{1}{t-\tau} e^{-c_9 M_2 \frac{q-r-2}{r+1} t} \left(\int_{\tau}^t e^{c_9 M_2 \frac{q-r-2}{r+1} s} \int_{\Omega} |(\beta(u(s)) - M_2)_+|^{\frac{q}{r+1}} dx ds \right. \\
& \quad \left. + c_{10} \int_{\tau}^t (s - \tau) e^{c_9 M_2 \frac{q-r-2}{r+1} s} \int_{\Omega(\beta(u(s)) \geq M_2)} |g(x, t)|^2 dx ds \right) \\
& \leq c_{11} \left(\frac{1}{t-\tau} \int_{\tau}^t e^{-c_9 M_2 \frac{q-r-2}{r+1} (t-s)} |u(s)|_q^q ds + \int_{\tau}^t e^{-c_9 M_2 \frac{q-r-2}{r+1} (t-s)} \int_{\Omega(\beta(u(s)) \geq M_2)} |g(x, t)|^2 dx ds \right) \quad (4.21)
\end{aligned}$$

Using (4.3), we obtain

$$\begin{aligned}
& \int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}} dx \\
& \leq c_{12} \left[\frac{e^{-\delta(t-\tau)}}{(c_9 M_2 \frac{q-r-2}{r+1} - \delta)(t-\tau)} |u_{\tau}|_{r+2}^{r+2} + \frac{1}{c_9 M_2 \frac{q-r-2}{r+1} (t-\tau)} + \frac{e^{-\delta t}}{c_9 M_2 \frac{q-r-2}{r+1}} \int_{-\infty}^t e^{\delta s} |g(x, s)|_2^2 ds \right. \\
& \quad \left. + \frac{e^{-\delta t}}{c_9 M_2 \frac{q-r-2}{r+1}} \int_{-\infty}^t \int_{-\infty}^s e^{\delta s} |g(x, r)|_2^2 dr ds \right]. \quad (4.22)
\end{aligned}$$

Obviously, for any $\varepsilon > 0$, $p > r + 2$, there exist $M > 0$, $\tau_0 < t$, for any $M_2 > M$, $\tau < \tau_0$, we have

$$\int_{\Omega} |(\beta(u) - M_2)_+|^{\frac{q}{r+1}} dx < \varepsilon. \quad (4.23)$$

Hence

$$\int_{\Omega(\beta(u) \geq M)} (|\beta(u)| - M)_+^{\frac{q}{r+1}} dx < \varepsilon. \quad (4.24)$$

Repeat the same step above, multiplying (1.1) with $|(\beta(u) + M_2)_-|^{\frac{q}{r+1}-2} (\beta(u) + M_2)_-$, we get

$$\int_{\Omega(\beta(u) \leq -M)} (|\beta(u(t))| - M)^{\frac{q}{r+1}} dx < \varepsilon, \quad (4.25)$$

where $(\beta(u) + M_2)_- = \begin{cases} \beta(u) + M_2, & \beta(u) \leq -M_2, \\ 0, & \beta(u) > -M_2. \end{cases}$

Combining (4.24) and (4.25), we have

$$\begin{aligned}
& \int_{\Omega(|\beta(u)| \geq 2M)} |\beta(u(t))|^{\frac{q}{r+1}} dx \\
& = \int_{\Omega(|\beta(u)| \geq 2M)} (|\beta(u(t))| - M + M)^{\frac{q}{r+1}} dx \\
& \leq c_{13} \left(\int_{\Omega(|\beta(u)| \geq 2M)} (|\beta(u(t))| - M)^{\frac{q}{r+1}} dx + \int_{\Omega(|\beta(u)| \geq 2M)} M^{\frac{q}{r+1}} dx \right) \\
& \leq c_{13} \left(\int_{\Omega(|\beta(u)| \geq M)} (|\beta(u(t))| - M)^{\frac{q}{r+1}} dx + \int_{\Omega(|\beta(u)| \geq M)} (|\beta(u(t))| - M)^{\frac{q}{r+1}} dx \right) \\
& \leq c_{14} \varepsilon.
\end{aligned}$$

Thanks to (4.17), we conclude

$$\int_{\Omega(|u(t)| \geq M)} |u(t)|^q dx < c_{15}\varepsilon.$$

Hence, the (2) of Theorem 2.2 is satisfied, which say that the process is pullback ω - D -limit compact in $L^q(\Omega)$.

For $p = r + 2$, the constant $M_2^{\frac{q-r-2}{r+1}} = 1$, the above prove can not obtain the right hand side of (4.22) tend to 0. For our purpose, we add another condition,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall e \subset \Omega, me < \delta, \forall t \in \mathbb{R}, \int_e |g(x, t)|^2 dx < \varepsilon. \quad (4.26)$$

Using (4.13) and (4.17), we get

$$\frac{\beta_1}{2} M_2^{\frac{q}{r+1}} m(\Omega(|\beta(u)| \geq M_2)) \leq \int_{\Omega(|\beta(u)| \geq M_2)} |\beta(t)|^{\frac{q}{r+1}} dx < +\infty.$$

We find that there exists $M > 0$, for any $M_2 > M$, $m(\Omega(|\beta(u)| \geq M_2)) < \varepsilon$, using (4.19) and the previous proof method, we also get

$$\int_{\Omega(|u(t)| \geq M)} |u(t)|^q dx < c_{16}\varepsilon,$$

which say that the process is pullback ω - D -limit compact in $L^q(\Omega)$, so we have the following theorem.

Theorem 4.4 *Assume that the conditions (1.2)-(1.5) are satisfied, $g(x, t)$ satisfy (4.2) and (4.26), $q = r + 2$. Then the process $U(t, \tau)$ generated by (1.1) exists a pullback \mathcal{D} -attractor in $L^q(\Omega)$.*

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