Source term identification of time-fractional diffusion equation under Robin boundary condition ⋆

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Abstract

The source term identification of the time-fractional diffusion equation under Robin boundary condition is studied. This problem is ill-posed. Therefore, we apply Landweber iterative regularization method, Fractional Landweber iterative regularization method, TSVD method, combining TSVD method and Fractional Landweber iterative regularization method, respectively. The comparisons of these four methods are given, which can help us select the most effective method. The error estimates between the regularized approximate solutions and the exact solution are given under the *a priori* and *a posteriori* regularization parameter choice rules. Finally, numerical examples verify the effectiveness of the methods.

Keywords: fractional diffusion equation; Robin boundary condition; source term identification; iterative regularization; error estimates

1. Introduction

In recent years, time-fractional differential equations have received widespread attention due to the memory properties of fractional derivatives. Time-fractional differential equations have advantages in describing hereditary diffusions compare to integer-differential equations. Time-fractional diffusion equation is one of the most important time-fractional differential equations.

The research on the direct problem of time-fractional diffusion equation has been investigated extensively, such as extremum principle [1, 2, 3], finite-difference method [4, 5, 6, 7] and finite element method [8, 9, 10, 11] , etc. In addition, the inverse problem of time-fractional diffusion equation has attracted more and more researchers, the source term identification is a branch of the inverse problem. Many scholars have used different

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methods to study the source term identification: Wei and Wang [12] proposed a modified quasi-boundary value regularization method and obtained two kinds of convergence rates by using an a priori and an a posteriori regularization parameter choice rule, respectively. Zhang and Xu [13] used analytic continuation and Laplace transform to prove the uniqueness of the source term identification. Yang [14] used the Landweber iterative regularization method to identify the source term. Xiong [15] investigated an inverse problem which is highly ill-posed in the two-dimensional setting and constructed some new regularization methods for solving the inverse source problem. Tuan [16] proposed the Tikhonov regularization method to reconstruct the source term and obtained error estimates between the exact solution and its regularized solution. Kirane [17] showed the existence and uniqueness of the solution of the inverse problem by using the properties of the biorthogonal system of functions.

However, most of the above researchers consider the source term identification under Dirichlet or Neumann boundary conditions. Robin boundary condition

$$u(x,t)+\sigma\frac{\partial u}{\partial v}=\varphi(x,t), x\in\partial D$$

is a more general boundary condition. In fact, when damping coefficient $\sigma=0$, the above function is Dirichlet boundary condition; when damping coefficient $\sigma=\infty$, the limit form of the above function is Neumann boundary condition. Therefore, the study of Robin boundary condition is more complex and more meaningful, compared with Dirichlet and Neumann boundary conditions. In this paper, we consider the following time-fractional diffusion equation under Robin boundary condition

$$\begin{cases} D_t^\alpha u(x,t)=\dfrac{\partial^2 u(x,t)}{\partial x^2}+r(t)f(x)+Z(x,t), & 0<x<L, t>0,\\ u(x,0)=\phi(x), & 0\le x\le L,\\ u(0,t)+\alpha_1 u_x(0,t)=\mu_1(t), & 0\le t\le T,\\ u(L,t)+\beta_1 u_x(L,t)=\mu_2(t), & 0\le t\le T, \end{cases} \tag{1.1}$$

where α_1 and β_1 are constants, $D_t^\alpha u(x,t)$ is the Caputo fractional derivative of order $\alpha(0<\alpha<1)$, which is defined by

$$D_t^\alpha u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha},$$

and $\phi(x)$, $\mu_1(t)$ and $\mu_2(t)$ of equation (1.1) need to satisfy the compatibility condition

$$\begin{cases} \phi(0)+\alpha_1 u_x(0,0)=\mu_1(0),\\ u(L,0)+\beta_1 u_x(L,0)=\mu_2(0). \end{cases}$$

We want to investigate is to identify the source term $f(x)$ from additional final value data

$$u(x,T) = g(x), \quad 0 \leq x \leq L. \tag{1.2}$$

Since the measurement is noise-contaminated inevitably, we assume $g^\delta(x)$ be the noisy measurement of $g(x)$ satisfying

$$\| g(x) - g^\delta(x) \| \leq \delta, \tag{1.3}$$

where $\| \cdot \|$ is the L^2 norm and $\delta > 0$ is a noise level.

It is well known that the source term identification is ill-posed, so regularization method is required to recover the continuous dependence of the solution. Iterative regularization method [18, 19] is an effective regularization method. Yang [20] investigated an inverse source problem by using Landweber iterative regularization method. Based on Landweber iterative regularization method, Klann [21] proposed Fractional Landweber iterative regularization method, which has advantages in solving nonlinear problems. Meanwhile, TSVD method is simple in construction but an effective regularization method, its advantage is that it can filter out small singular values. In this paper, we put the above three methods together and compare their regularization effects. In addition, we combine TSVD method and Fractional Landweber iterative regularization method, trying to integrate the strengths of two methods. For the above four methods, we have obtained Hölder type error estimates under the *a priori* and *a posteriori* parameter choice rules.

The structure of this paper is as follows. In Section 2, we introduce the direct problem solving process. The ill-posedness of the source term identification and the conditional stability are analyzed in Section 3. In Section 4, we introduce four regularization methods and provide the error estimates under two parameter choice rules. Numerical examples to illustrate the effectiveness of our methods are in Section 5. Finally, we give a brief conclusion in Section 6.

2. Direct problem solving process

We make the following transformation of (1.1)

$$u(x,t) = W(x,t) + V(x,t),$$

where

$$V(x,t) = \frac{\mu_1(t) - \mu_2(t)}{\alpha_1 - \beta_1 - L} x - \frac{(L+\beta_1)\mu_1(t) - \alpha_1\mu_2(t)}{\alpha_1 - \beta_1 - L},$$

and the function $W(x,t)$ is the solution of the following problem with homogeneous boundary conditions

$$\begin{cases} D_t^\alpha W(x,t) = \dfrac{\partial^2 W(x,t)}{\partial x^2} + \widetilde{F}(x,t), & 0 < x < L, t > 0, \\ W(x,0) = \widetilde{\phi}(x), & 0 \le x \le L, \\ W(0,t) + \alpha_1 W_x(0,t) = 0, & 0 \le t \le T, \\ W(L,t) + \beta_1 W_x(L,t) = 0, & 0 \le t \le T, \end{cases}$$

with

$$\begin{aligned} &\widetilde{F}(x,t) = r(t)f(x) + Z(x,t) - D_t^\alpha V(x,t), \\ &\widetilde{\phi}(x) = \phi(x) - V(x,0), \end{aligned}$$

then we obtain the following Sturm-Liouville eigenvalue problem by method of separating variables

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < L, \tag{2.1}$$

$$X(0) + \alpha_1 X'(0) = 0, \qquad X(L) + \beta_1 X'(L) = 0, \tag{2.2}$$

where λ is a constant to be determined. The positive and negative restrictions of the Robin coefficient α_1 and β_1 do not exist in form, but do exist in physics. The reason for these restrictions is: in mechanical problems, the elastic restoring force is always related to the shape variation; in thermal problems, when the system exchanges heat with outside, the heat always flows from high temperature to low temperature[22]. What's more, because of the arbitrariness of the Robin coefficient α_1 and β_1, the Sturm-Liouville eigenvalue problem (2.1)-(2.2) may not have solutions[23].

Inspired by Geng [24], we divide Robin boundary condition into the following four special cases.

Case 1: $\alpha_1 = \beta_1 = 1$. Then the eigenvalues λ_n and eigenfunctions $X_n(x)$ are

$$\begin{aligned} &\lambda_0 = -1, \qquad X_0(x) = e^{-x}, \\ &\lambda_n = \mu_n^2, \qquad X_n(x) = -\mu_n \cos \mu_n x + \sin \mu_n x, \qquad n = 1, 2, \cdots. \end{aligned}$$

Considering the eigenfunctions

$$\widetilde{X}_n(x) := \begin{cases} \dfrac{1}{\sqrt{b_0}} X_0(x), & n = 0, \\ \dfrac{1}{\sqrt{b_n}} X_n(x), & n = 1, 2, \cdots, \end{cases}$$

where

$$b_0 = \int_0^L X_0^2(x)\,dx = \int_0^L e^{-2x}\,dx = \frac{e^{2L}-1}{2e^{2L}},$$
$$b_n = \int_0^L X_n^2(x)\,dx = \int_0^L \left(-\mu_n\cos\mu_n x + \sin\mu_n x\right)^2\,dx = \frac{(n\pi)^2+L^2}{2L}.$$

One can easily check that $\widetilde{X}_n(x)$ are standard orthogonal function systems in $[0, L]$. Using this basis, we rewrite the analytical solution of (1.1) as

$$u(x,t) = \sum_{n=0}^{\infty}\left[\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\widetilde{F}_n(t-\tau)d\tau + \widetilde{\phi}_n E_{\alpha,1}(-\lambda_n t^\alpha)\right]\widetilde{X}_n(x)$$
$$-\frac{\mu_1(t)-\mu_2(t)}{L}x + \frac{(L+1)\mu_1(t)-\mu_2(t)}{L}, \tag{2.3}$$

where $E_{\alpha,\alpha}(\cdot)$ and $E_{\alpha,1}(\cdot)$ are Mittag-Leffler functions [25], $\widetilde{F}_n(t) = \int_0^L \widetilde{F}(x,t)\widetilde{X}_n(x)\,dx$, $\widetilde{\phi}_n = \int_0^L \widetilde{\phi}(x)\widetilde{X}_n(x)\,dx$.

Case 2: $\alpha_1 = \beta_1 = -1$. Then the eigenvalues λ_n and eigenfunctions $X_n(x)$ are

$$\lambda_0 = -1, \qquad X_0(x) = e^x,$$
$$\lambda_n = \mu_n^2, \qquad X_n(x) = \mu_n\cos\mu_n x + \sin\mu_n x, \qquad n = 1, 2, \cdots,$$

where $\mu_n = \frac{n\pi}{L}$.

Considering the eigenfunctions

$$\widetilde{X}_n(x) := \begin{cases} \sqrt{\dfrac{2}{e^{2L}-1}}X_0(x), & n = 0, \\ \sqrt{\dfrac{2L}{(n\pi)^2+L^2}}X_n(x), & n = 1, 2, \cdots, \end{cases}$$

then we rewrite the analytical solution of (1.1) as

$$u(x,t) = \sum_{n=0}^{\infty}\left[\int_0^T \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\widetilde{F}_n(t-\tau)d\tau + \widetilde{\phi}_n E_{\alpha,1}(-\lambda_n t^\alpha)\right]\widetilde{X}_n(x)$$
$$-\frac{\mu_1(t)-\mu_2(t)}{L}x + \frac{(L-1)\mu_1(t)+\mu_2(t)}{L}. \tag{2.4}$$

Case 3: $\alpha_1 = 1$ and $\beta_1 = -1$ ($\alpha_1 \neq L + \beta_1$). Then the eigenvalues λ_n and eigenfunctions $X_n(x)$ are

$$\lambda_0 = -\mu_0^2, \qquad X_0(x) = -\mu_0\cosh\mu_0 x + \sinh\mu_0 x,$$
$$\lambda_n = \mu_n^2, \qquad X_n(x) = -\mu_n\cos\mu_n x + \sin\mu_n x, \qquad n = 1, 2, \cdots,$$

where μ_0 is the solution of equation $\tanh\mu L = \frac{2\mu}{1+\mu^2}$, μ_n satisfies the equation $\tan\mu L = \frac{2\mu}{1-\mu^2}$.

Considering the eigenfunctions

$$\widetilde{X}_n(x) := \begin{cases} \dfrac{1}{\sqrt{b_0}} X_0(x), & n = 0, \\ \dfrac{1}{\sqrt{b_n}} X_n(x), & n = 1, 2, \cdots, \end{cases}$$

where

$$b_0 = \frac{\mu_0^2}{2}\left(\frac{\sinh 2\mu_0 L}{2\mu_0} + L\right) - \frac{2\mu_0 \cosh 2\mu_0 L - \sinh 2\mu_0 L}{4\mu_0} + \frac{1-L}{2},$$
$$b_n = \frac{\mu_n^2}{2}\left(\frac{\sin 2\mu_n L}{2\mu_n} + L\right) + \frac{2\mu_n \cos 2\mu_n L - \sin 2\mu_n L}{4\mu_n} + \frac{L-1}{2},$$

then we can rewrite the analytical solution of (1.1) as

$$\begin{aligned} u(x,t) = &\sum_{n=0}^{\infty}\left[\int_0^T \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)\widetilde{F}_n(t-\tau)d\tau + \widetilde{\phi}_n E_{\alpha,1}(-\lambda_n t^\alpha)\right]\widetilde{X}_n(x) \\ &+ \frac{\mu_1(t)-\mu_2(t)}{2-L}x - \frac{(L-1)\mu_1(t)-\mu_2(t)}{2-L}. \end{aligned} \tag{2.5}$$

Case 4: $\alpha_1 = -1$ and $\beta_1 = 1$. Then the eigenvalues λ_n and eigenfunctions $X_n(x)$ are

$$\lambda_n = \mu_n^2, \qquad X_n(x) = \mu_n \cos \mu_n x + \sin \mu_n x, \qquad n = 1, 2, \cdots,$$

where μ_n satisfies the equation $\tan \mu L = \frac{2\mu}{\mu^2-1}$.

Considering the eigenfunctions

$$\widetilde{X}_n(x) := \frac{1}{\sqrt{b_n}} X_n(x), \qquad n = 1, 2, \cdots,$$

where

$$b_n = \frac{\mu_n^2}{2}\left(\frac{\sin 2\mu_n L}{2\mu_n} + L\right) - \frac{2\mu_n \cos 2\mu_n L + \sin 2\mu_n L}{4\mu_n} + \frac{L+1}{2},$$

then we can rewrite the analytical solution of (1.1) as

$$\begin{aligned} u(x,t) = &\sum_{n=0}^{\infty}\left[\int_0^T \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)\widetilde{F}_n(t-\tau)d\tau + \widetilde{\phi}_n E_{\alpha,1}(-\lambda_n t^\alpha)\right]\widetilde{X}_n(x) \\ &- \frac{\mu_1(t)-\mu_2(t)}{2+L}x + \frac{(L+1)\mu_1(t)+\mu_2(t)}{2+L}. \end{aligned} \tag{2.6}$$

3. Source term identification and conditional stability

In this section, we analyze the ill-posedness of the source term identification and prove the conditional stability results. In the rest of this paper, we take Case 1 as an example, that is, the solution $u(x,t)$ of Equation (1.1) is given by (2.3). The derivation process of other cases is similar, so we won't repeat it here. From the final value data $u(x,T)=g(x)$, denote $\widetilde{g}(x)=g(x)-\sum_{n=0}^{\infty}[\int_0^T\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)Z_n(T-\tau)d\tau+\widetilde{\phi}_nE_{\alpha,1}(-\lambda_nt^\alpha)]\widetilde{X}_n(x)-V(x,T)$. In order to reconstruct the source term $f(x)$, we only need to solve the following integral equation

$$(Kf)(x)=\int_0^L k(x,\xi)f(\xi)d\xi=\widetilde{g}(x),$$

where

$$k(x,\xi)=\sum_{n=0}^{\infty}\sigma_n\widetilde{X}_n(x)\widetilde{X}_n(\xi),$$

here $\sigma_n=\int_0^T\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)r(T-\tau)d\tau$. Let K^* be the conjugate operator of operator K, it is easy to calculate that $K=K^*$, so K is a self-adjoint operator with singular value σ_n. Then we obtain the source term $f(x)$

$$f(x)=\sum_{n=0}^{\infty}\frac{\widetilde{g}_n}{\sigma_n}\widetilde{X}_n(x). \tag{3.1}$$

In order to facilitate the proof of the subsequent theorem, we give the following lemmas.

Lemma 3.1 [26]: Let $r(t)\in C[0,T]$ satisfies $r(t)>0$, $t\in[0,T]$, for any λ_n satisfying $\lambda_n\geq\lambda_1>0$ and $0<\alpha<1$, there exist positive constants C_1 and C_2 which depend on α, T and λ_1, such that

$$\frac{C_1}{\lambda_n}\leq\sigma_n=\int_0^T\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)r(T-\tau)d\tau\leq\frac{C_2}{\lambda_n},n=1,2,\cdots.$$

Lemma 3.2 [21]: Let $0<\beta<\frac{1}{\|K\|^2}$, $m>0$, $\gamma\in[0,1]$, $\mu\in[0,1]$, when $\gamma>\mu/2$,

$$\sup_{0<\sigma_n\leq\sigma_1}|(1-(1-\beta\sigma_n^2)^m)^\gamma\sigma_n^{-\mu}|\leq\beta^{\frac{\mu}{2}}m^{\frac{\mu}{2}}.$$

Lemma 3.3 [20]: As constants $\beta>0$, $p>0$, $s>0$ and $0<\beta\frac{C_1^2}{s^2}<1$, we have

$$\left(1-\beta\frac{C_1^2}{s^2}\right)^m s^{-\frac{p}{2}}\leq\left(\frac{p}{\beta C_1^2}\right)^{\frac{p}{4}}(m+1)^{-\frac{p}{4}},$$

$$\left(1-\beta\frac{C_1^2}{s^2}\right)^{m-1} s^{-\frac{p}{2}-1}\leq\left(\frac{p+2}{2\beta C_1^2}\right)^{\frac{p+2}{4}}m^{-\frac{p+2}{4}}.$$

Lemma 3.4 [27]: For $0<\gamma<1$ and $k\geq1$, define $p_k(\gamma)=\sum_{i=0}^{k-1}(1-\gamma)^i$ and $r_k(\gamma)=1-\gamma p_k(\gamma)=(1-\gamma)^k$. Then,

$$p_k(\gamma)\gamma^\mu\leq k^{1-\mu},0\leq\mu\leq1,$$

$$r_k(\gamma)\gamma^v\leq\theta_v(k+1)^{-v},$$

where

$$\theta_v = \begin{cases} 1, & 0 \le v \le 1, \\ v^v, & v > 1. \end{cases}$$

According to Lemma 3.1, we notice that

$$\frac{1}{\sigma_n} = \frac{1}{\int_0^T \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) r(T-\tau) d\tau} \ge \frac{\lambda_n}{C_2} \to \infty, n \to \infty.$$

It means that the small disturbance for the data $g(x)$ will lead to a huge change in the source term $f(x)$. Therefore, the recovery of the source term $f(x)$ from the measured data $g^\delta(x)$ is an ill-posed problem.

Theorem 3.1: Let source term $f(x)$ satisfies the following *a priori* bound

$$\| f(x) \|_p = \left(\sum_{n=0}^{\infty} \left(\frac{\lambda_n}{L} \right)^p \mid f_n \mid^2 \right)^{\frac{1}{2}} \le E, \tag{3.2}$$

where p and E are both positive constants, then we obtain

$$\| f(x) \| \le \left(\frac{L}{C_1} \right)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \| \widetilde{g}(x) \|^{\frac{p}{p+2}} .$$

Proof: Applying the Hölder inequality, we obtain

$$\begin{aligned} \| f(x) \|^2 &= \sum_{n=0}^{\infty} \frac{\widetilde{g}_n^2}{\sigma_n^2} = \sum_{n=0}^{\infty} \frac{\widetilde{g}_n^{\frac{4}{p+2}}}{\sigma_n^2} \widetilde{g}_n^{\frac{2p}{p+2}} = \sum_{n=0}^{\infty} \frac{\widetilde{g}_n^{\frac{4}{p+2}}}{\sigma_n^{\frac{4}{p+2}}} \sigma_n^{-\frac{2p}{p+2}} \widetilde{g}_n^{\frac{2p}{p+2}} \\ &\le \left(\sum_{n=0}^{\infty} \frac{\widetilde{g}_n^2}{\sigma_n^2} \sigma_n^{-p} \right)^{\frac{2}{p+2}} \left(\sum_{n=0}^{\infty} \widetilde{g}_n^2 \right)^{\frac{p}{p+2}} \le \left(\sum_{n=0}^{\infty} \left(\frac{\lambda_n}{L} \right)^p \left(\frac{L}{C_1} \right)^p f_n^2 \right)^{\frac{2}{p+2}} \left(\sum_{n=0}^{\infty} \widetilde{g}_n^2 \right)^{\frac{p}{p+2}} \\ &= \left(\frac{L}{C_1} \right)^{\frac{2p}{p+2}} \| f(x) \|_p^{\frac{4}{p+2}} \| \widetilde{g}(x) \|^{\frac{2p}{p+2}} \le \left(\frac{L}{C_1} \right)^{\frac{2p}{p+2}} E^{\frac{4}{p+2}} \| \widetilde{g}(x) \|^{\frac{2p}{p+2}} . \end{aligned}$$

4. Regularization methods

In this section, we apply four regularization methods to solve the source term identification. Moreover, we present the error estimates under the *a priori* and *a posteriori* regularization parameter choice rules, respectively.

4.1. Landweber iterative regularization method (method 1)

In this subsection, we give the Landweber iterative regularization method. Let $f^{m,\delta}(x)$ satisfy

$$f^{0,\delta}(x) := 0, f^{m,\delta}(x) = (I - \beta K^* K) f^{m-1,\delta}(x) + \beta K^* \widetilde{g}^\delta(x), m = 1, 2, \cdots ,$$

where $m>0$ is iteration step, β is the relaxation factor which satisfies $0<\beta<\frac{1}{\|K\|^2}$. By simple calculation, we obtain the regularized approximate solution of equation (1.1) as follows

$$f^{m,\delta}(x)=\sum_{n=0}^{\infty}\frac{1-(1-\beta\sigma_n^2)^m}{\sigma_n}\widetilde{g}_n^{\delta}\widetilde{X}_n(x). \tag{4.1}$$

4.1.1. A priori regularization parameter choice rule

In 4.1.1, we will give an error estimate under the *a priori* regularization parameter choice rule.

Theorem 4.1: Let $f(x)$ given by (3.1) be the exact solution. Let $f^{m,\delta}(x)$ given by (4.1) be the regularization solution. The conditions (1.3) and (3.2) hold. If we choose

$$m=[\left(\frac{E}{\delta}\right)^{\frac{4}{p+2}}],$$

where $[x]$ denotes the largest integer less than or equal to x, then we have the following error estimate

$$\| f^{m,\delta}(x)-f(x)\|\leq\left(\sqrt{\beta}+L^{\frac{p}{2}}\left(\frac{p}{\beta C_1^2}\right)^{\frac{p}{4}}\right)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.$$

Proof: Using triangle inequality, we have

$$\| f^{m,\delta}(x)-f(x)\|\leq\| f^{m,\delta}(x)-f^m(x)\|+\| f^m(x)-f(x)\|.$$

From Lemma 3.2 and (1.3), we have

$$\begin{aligned}\| f^{m,\delta}(x)-f^m(x)\| &=\|\sum_{n=0}^{\infty}\frac{1-(1-\beta\sigma_n^2)^m}{\sigma_n}(\widetilde{g}_n^{\delta}-\widetilde{g}_n)\widetilde{X}_n(x)\|\\ &\leq\sqrt{\beta m}\,\|\widetilde{g}^{\delta}(x)-\widetilde{g}(x)\|\leq\sqrt{\beta m}\,\delta,\end{aligned} \tag{4.2}$$

then

$$\begin{aligned}\| f^m(x)-f(x)\| &=\|\sum_{n=0}^{\infty}\frac{(1-\beta\sigma_n^2)^m}{\sigma_n}\widetilde{g}_n\widetilde{X}_n(x)\|\\ &=\|\sum_{n=0}^{\infty}(1-\beta\sigma_n^2)^m\left(\frac{\lambda_n}{L}\right)^{-\frac{p}{2}}\left(\frac{\lambda_n}{L}\right)^{\frac{p}{2}}f_n\widetilde{X}_n(x)\|\\ &\leq L^{\frac{p}{2}}\sup_{n\geq1}\left((1-\beta\sigma_n^2)^m\lambda_n^{-\frac{p}{2}}\right)E,\end{aligned}$$

from Lemma 3.3, we have

$$(1-\beta\sigma_n^2)^m\lambda_n^{-\frac{p}{2}}\leq\left(1-\beta\frac{C_1^2}{\lambda_n^2}\right)^m\lambda_n^{-\frac{p}{2}}\leq\left(\frac{p}{\beta C_1^2}\right)^{\frac{p}{4}}m^{-\frac{p}{4}},$$

thus,

$$\| f^m(x) - f(x) \| \leq L^{\frac{p}{2}} \left(\frac{p}{\beta C_1^2} \right)^{\frac{p}{4}} m^{-\frac{p}{4}} E. \tag{4.3}$$

Combining (4.2) and (4.3), and choosing the regularization parameter $m = [(\frac{E}{\delta})^{\frac{4}{p+2}}]$, we complete the proof.

4.1.2. A posteriori regularization parameter choice rule

In 4.1.1, the regularization parameter m is chosen by $m = [(\frac{E}{\delta})^{\frac{4}{p+2}}]$. It shows that the choice of m depends on the *a priori* bound E, which is hard to obtain in practical problems. So here we give *a posteriori* regularization parameter choice rule that does not depend on *a priori* information.

Using Morozov's discrepancy principle, we have

$$\| K f^{m,\delta}(x) - \widetilde{g}^\delta(x) \| \leq \omega\delta \leq \| K f^{m-1,\delta}(x) - \widetilde{g}^\delta(x) \|, \tag{4.4}$$

where $\omega > 1$ is a constant.

Theorem 4.2: Let $f(x)$ given by (3.1) be the exact solution. Let $f^{m,\delta}(x)$ given by (4.1) be the regularization solution. The conditions (1.3) and (3.2) hold. Regularization parameter m is chosen by (4.4), then we have the following error estimate

$$\| f^{m,\delta}(x) - f(x) \| \leq \left(L^{\frac{p}{p+2}} \left(\frac{p+2}{2C_1^2} \right)^{\frac{1}{2}} \left(\frac{C_2}{\omega - 1} \right)^{\frac{2}{p+2}} + \left(\frac{L(\omega+1)}{C_1} \right)^{\frac{p}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

Proof: Using triangle inequality, we have

$$\| f^{m,\delta}(x) - f(x) \| \leq \| f^{m,\delta}(x) - f^m(x) \| + \| f^m(x) - f(x) \| .$$

From Lemma 3.3 and (1.3), we have

$$\begin{aligned}
\omega\delta &\leq \| K f^{m-1,\delta}(x) - \widetilde{g}^\delta(x) \| = \| \sum_{n=0}^{\infty} (1 - \beta\sigma_n^2)^{m-1} \widetilde{g}_n^\delta \widetilde{X}_n(x) \| \\
&\leq \| \sum_{n=0}^{\infty} (1 - \beta\sigma_n^2)^{m-1} (\widetilde{g}_n^\delta - \widetilde{g}_n) \widetilde{X}_n(x) \| + \| \sum_{n=0}^{\infty} (1 - \beta\sigma_n^2)^{m-1} \widetilde{g}_n \widetilde{X}_n(x) \| \\
&\leq \delta + \| \sum_{n=0}^{\infty} (1 - \beta\sigma_n^2)^{m-1} \widetilde{g}_n \widetilde{X}_n(x) \| \\
&\leq \delta + \| \sum_{n=0}^{\infty} L^{\frac{p}{2}} C_2 (1 - \beta\sigma_n^2)^{m-1} f_n \left(\frac{\lambda_n}{L} \right)^{\frac{p}{2}} \lambda_n^{-1-\frac{p}{2}} \widetilde{X}_n(x) \| \\
&\leq \delta + L^{\frac{p}{2}} C_2 \sup_{n \geq 1} \left((1 - \beta\sigma_n^2)^{m-1} \lambda_n^{-1-\frac{p}{2}} \right) \| \sum_{n=0}^{\infty} f_n \left(\frac{\lambda_n}{L} \right)^{\frac{p}{2}} \widetilde{X}_n(x) \|
\end{aligned}$$

$$\leq \delta + L^{\frac{p}{2}} C_2 \left(\frac{p+2}{2\beta C_1^2}\right)^{\frac{p+2}{4}} m^{-\frac{p+2}{4}} E,$$

then, we deduce that

$$m \leq L^{\frac{2p}{p+2}} \left(\frac{p+2}{2\beta C_1^2}\right) \left(\frac{C_2 E}{(\omega-1)\delta}\right)^{\frac{4}{p+2}}, \tag{4.5}$$

similar to (4.2), it is easy to get that

$$\begin{aligned} \| f^{m,\delta}(x) - f^m(x) \| &\leq \sqrt{\beta m}\delta \\ &\leq L^{\frac{p}{p+2}} \left(\frac{p+2}{2C_1^2}\right)^{\frac{1}{2}} \left(\frac{C_2}{\omega-1}\right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \end{aligned} \tag{4.6}$$

On the other hand, we have

$$\begin{aligned} \| K(f^m(x) - f(x)) \| &= \| \sum_{n=0}^{\infty} (1-\beta\sigma_n^2)^m \widetilde{g}_n \widetilde{X}_n(x) \| \\ &\leq \| \sum_{n=0}^{\infty} (1-\beta\sigma_n^2)^m (\widetilde{g}_n^\delta - \widetilde{g}_n) \widetilde{X}_n(x) \| + \| \sum_{n=0}^{\infty} (1-\beta\sigma_n^2)^m \widetilde{g}_n^\delta \widetilde{X}_n(x) \| \\ &\leq \delta + \omega\delta, \end{aligned}$$

based on (3.2), we have

$$\begin{aligned} \| f^m(x) - f(x) \|_p &= \left(\sum_{n=0}^{\infty} \left(\frac{\lambda_n}{L}\right)^p \frac{(1-\beta\sigma_n^2)^{2m}}{\sigma_n^2} \widetilde{g}_n^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=0}^{\infty} \left(\frac{\lambda_n}{L}\right)^p (1-\beta\sigma_n^2)^{2m} f_n^2\right)^{\frac{1}{2}} \leq \| f(x) \|_p \leq E, \end{aligned}$$

using Therorem 3.1, we have

$$\| f^m(x) - f(x) \| \leq \left(\frac{L(\omega+1)}{C_1}\right)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{4.7}$$

Combining (4.6) and (4.7), we can easily obtain the desired result.

4.2. Fractional Landweber iterative regularization method (method 2)

In this subsection, we introduce the Fractional Landweber iterative regularization method, which overcomes the over-smoothness of solution compared to Landweber iterative regularization method. Let $f^{m,\delta}(x)$ satisfy

$$f^{0,\delta}(x) := 0, f^{m,\delta}(x) = (I - \beta(K^*K)^{\frac{\gamma+1}{2}}) f^{m-1,\delta}(x) + \beta(K^*K)^{\frac{\gamma-1}{2}} K^* \widetilde{g}^\delta(x), m = 1, 2, \cdots,$$

where $\frac{1}{2} < \gamma \leq 1$. By simple calculation, we obtain the regularized approximate solution of equation (1.1) as follows

$$f^{m,\delta}(x) = \sum_{n=0}^{\infty} \frac{[1-(1-\beta\sigma_n^2)^m]^\gamma}{\sigma_n} \widetilde{g}_n^\delta \widetilde{X}_n(x). \tag{4.8}$$

4.2.1. A priori regularization parameter choice rule

In 4.2.1, we will give an error estimate under the *a priori* regularization parameter choice rule.

Theorem 4.3: Let $f(x)$ given by (3.1) be the exact solution. Let $f^{m,\delta}(x)$ given by (4.8) be the regularization solution. The conditions (1.3) and (3.2) hold. If we choose

$$m = [\left(\frac{E}{\delta}\right)^{\frac{4}{p+2}}],$$

then we have the following error estimate

$$\| f^{m,\delta}(x) - f(x) \| \leq \left(\sqrt{\beta} + L^{\frac{p}{2}}\left(\frac{p}{\beta C_1^2}\right)^{\frac{p}{4}}\right) E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.$$

Proof: Using triangle inequality, we have

$$\| f^{m,\delta}(x) - f(x) \| \leq \| f^{m,\delta}(x) - f^m(x) \| + \| f^m(x) - f(x) \| .$$

From Lemma 3.2 and (1.3), we have

$$\begin{aligned} \| f^{m,\delta}(x) - f^m(x) \| &= \| \sum_{n=0}^{\infty} \frac{[1-(1-\beta\sigma_n^2)^m]^\gamma}{\sigma_n}(\widetilde{g}_n^\delta - \widetilde{g}_n)\widetilde{X}_n(x) \| \\ &\leq \sup_{0<\sigma_n\leq\sigma_1}\left(\frac{[1-(1-\beta\sigma_n^2)^m]^\gamma}{\sigma_n}\right) \| \widetilde{g}^\delta(x) - \widetilde{g}(x) \| \leq \sqrt{\beta m}\delta, \end{aligned} \tag{4.9}$$

then

$$\begin{aligned} \| f^m(x) - f(x) \| &= \| \sum_{n=0}^{\infty} \frac{[1-(1-(1-\beta\sigma_n^2)^m)^\gamma]}{\sigma_n}\widetilde{g}_n\widetilde{X}_n(x) \| \\ &\leq \| \sum_{n=0}^{\infty}(1-\beta\sigma_n^2)^m L^{\frac{p}{2}}\lambda_n^{-\frac{p}{2}}\left(\frac{\lambda_n}{L}\right)^{\frac{p}{2}} f_n\widetilde{X}_n(x) \| \\ &\leq L^{\frac{p}{2}}\sup_{n\geq 1}\left((1-\beta\sigma_n^2)^m\lambda_n^{-\frac{p}{2}}\right)E, \end{aligned}$$

similar to the proof of Theorem 4.1, we have

$$\| f^m(x) - f(x) \| \leq L^{\frac{p}{2}}\left(\frac{p}{\beta C_1^2}\right)^{\frac{p}{4}} m^{-\frac{p}{4}}E. \tag{4.10}$$

Combining (4.9) and (4.10), and choosing the regularization parameter $m = [\left(\frac{E}{\delta}\right)^{\frac{4}{p+2}}]$, we complete the proof.

4.2.2. A posteriori regularization parameter choice rule

In 4.2.2, we will give an error estimate under the *a posteriori* regularization parameter choice rule.

Theorem 4.4: Let $f(x)$ given by (3.1) be the exact solution. Let $f^{m,\delta}(x)$ given by (4.8) be the regularization solution. The conditions (1.3) and (3.2) hold. Regularization parameter m is chosen by (4.4), then we have the following error estimate:

$$\| f^{m,\delta}(x) - f(x) \| \leq \left(\left(\frac{L^{\frac{p}{2}} \theta_{\frac{p+2}{4}}}{C_1^{\frac{p}{2}}(\omega - 1)} \right)^{\frac{2}{p+2}} + \left(\frac{L(\omega+1)}{C_1} \right)^{\frac{p}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

Proof: Using triangle inequality, we have

$$\| f^{m,\delta}(x) - f(x) \| \leq \| f^{m,\delta}(x) - f^m(x) \| + \| f^m(x) - f(x) \| .$$

From Lemma 3.4 and (1.3), we have

$$\begin{aligned}
\omega\delta &\leq \| K f^{m-1,\delta}(x) - \widetilde{g}^{\delta}(x) \| = \| \sum_{n=0}^{\infty} [1 - (1 - (1 - \beta\sigma_n^2)^{m-1})^{\gamma}] \widetilde{g}_n^{\delta} \widetilde{X}_n(x) \| \\
&\leq \| \sum_{n=0}^{\infty} (1 - \beta\sigma_n^2)^{m-1} (\widetilde{g}_n^{\delta} - \widetilde{g}_n) \widetilde{X}_n(x) \| + \| \sum_{n=0}^{\infty} (1 - \beta\sigma_n^2)^{m-1} \widetilde{g}_n \widetilde{X}_n(x) \| \\
&\leq \delta + \| \sum_{n=0}^{\infty} (1 - \beta\sigma_n^2)^{m-1} \sigma_n f_n \widetilde{X}_n(x) \| \\
&\leq \delta + L^{\frac{p}{2}} \sup_{n \geq 1} \left((1 - \beta\sigma_n^2)^{m-1} \lambda_n^{-\frac{p}{2}} \sigma_n \right) \| \sum_{n=0}^{\infty} f_n \left(\frac{\lambda_n}{L} \right)^{\frac{p}{2}} \widetilde{X}_n(x) \| \\
&\leq \delta + L^{\frac{p}{2}} \sup_{n \geq 1} \left((1 - \beta\sigma_n^2)^{m-1} C_1^{-\frac{p}{2}} \sigma_n^{\frac{p+2}{2}} \right) E \\
&\leq \delta + \left(\frac{L}{C_1} \right)^{\frac{p}{2}} \theta_{\frac{p+2}{4}} (m\beta)^{-\frac{p+2}{4}} E,
\end{aligned}$$

where

$$\theta_{\frac{p+2}{4}} = \begin{cases} 1, & 0 \leq p \leq 2, \\ \left(\frac{p+2}{4} \right)^{\frac{p+2}{4}}, & p > 2, \end{cases}$$

then, we deduce that

$$m \leq \frac{1}{\beta} \left(\frac{L}{C_1} \right)^{\frac{2p}{p+2}} \left(\frac{\theta_{\frac{p+2}{4}} E}{(\omega - 1)\delta} \right)^{\frac{4}{p+2}},$$

similar to the proof of Theorem 4.2, it is easy to get that

$$\| f^{m,\delta}(x) - f^m(x) \| \leq \left(\frac{L^{\frac{p}{2}} \theta_{\frac{p+2}{4}}}{C_1^{\frac{p}{2}}(\omega - 1)} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \tag{4.11}$$

and

$$\| f^m(x) - f(x) \| \le \left(\frac{L(\omega+1)}{C_1} \right)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{4.12}$$

Combining (4.11) and (4.12), we can easily obtain the desired result.

4.3. TSVD method (method 3)

In this subsection, we introduce TSVD method, which is simple in structure but effective. Its advantage is to filter large and small singular values separately to ensure the stability of the solution. We give the regularized approximate solution as follows

$$f^{m,\delta}(x) = \sum_{\sigma_n^2 \ge \frac{1}{m}} \frac{1}{\sigma_n} \widetilde{g}_n^\delta \widetilde{X}_n(x). \tag{4.13}$$

4.3.1. A priori regularization parameter choice rule

In 4.3.1, we will give an error estimate under the *a priori* regularization parameter choice rule.

Theorem 4.5: Let $f(x)$ given by (3.1) be the exact solution. Let $f^{m,\delta}(x)$ given by (4.13) be the regularization solution. The conditions (1.3) and (3.2) hold. If we choose

$$m = [\left(\frac{E}{\delta} \right)^{\frac{4}{p+2}}],$$

then we have the following error estimate

$$\| f^{m,\delta}(x) - f(x) \| \le \left(1 + \left(\frac{L}{C_1} \right)^{\frac{p}{2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

Proof: Using triangle inequality, we have

$$\| f^{m,\delta}(x) - f(x) \| \le \| f^{m,\delta}(x) - f^m(x) \| + \| f^m(x) - f(x) \| .$$

Then

$$\| f^{m,\delta}(x) - f^m(x) \| = \| \sum_{\sigma_n^2 \ge \frac{1}{m}} \frac{1}{\sigma_n} (\widetilde{g}_n^\delta - \widetilde{g}_n) \widetilde{X}_n(x) \| \le \sqrt{m}\delta, \tag{4.14}$$

from (3.2), we have

$$\begin{aligned} \| f^m(x) - f(x) \| &= \| \sum_{\sigma_n^2 < \frac{1}{m}} \frac{1}{\sigma_n} \widetilde{g}_n \widetilde{X}_n(x) \| = \left(\sum_{\sigma_n^2 < \frac{1}{m}} \left(\frac{\lambda_n}{L} \right)^{-p} \left(\frac{\lambda_n}{L} \right)^{p} f_n^2 \right)^{\frac{1}{2}} \\ &\le \left(\sum_{\sigma_n^2 < \frac{1}{m}} \left(\frac{L}{C_1} \right)^{p} \sigma_n^p \left(\frac{\lambda_n}{L} \right)^{p} f_n^2 \right)^{\frac{1}{2}} \le \left(\frac{L}{C_1} \right)^{\frac{p}{2}} m^{-\frac{p}{4}} E. \end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15), and choosing the regularization parameter $m = [\left(\frac{E}{\delta}\right)^{\frac{4}{p+2}}]$, we complete the proof.

4.3.2. A posteriori regularization parameter choice rule

In 4.3.2, we will give an error estimate under the *a posteriori* regularization parameter choice rule.

Theorem 4.6: Let $f(x)$ given by (3.1) be the exact solution. Let $f^{m,\delta}(x)$ given by (4.13) be the regularization solution. The conditions (1.3) and (3.2) hold. Regularization parameter m is chosen by (4.4), then we have the following error estimate

$$\| f^{m,\delta}(x) - f(x) \| \leq \left(\left(\frac{\sqrt{2}L^{\frac{p}{2}}}{C_1^{\frac{p}{2}}(\omega - 1)} \right)^{\frac{2}{p+2}} + \left(\frac{L(\omega+1)}{C_1} \right)^{\frac{p}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}$$

Proof: Using triangle inequality, we have

$$\| f^{m,\delta}(x) - f(x) \| \leq \| f^{m,\delta}(x) - f^m(x) \| + \| f^m(x) - f(x) \| .$$

Then

$$\begin{aligned}
\omega\delta &\leq \| Kf^{m-1,\delta}(x) - \widetilde{g}^{\delta}(x) \| = \| \sum_{\sigma_n^2 < \frac{1}{m-1}} \widetilde{g}_n^{\delta} \widetilde{X}_n(x) \| \\
&\leq \| \sum_{\sigma_n^2 < \frac{1}{m-1}} (\widetilde{g}_n^{\delta} - \widetilde{g}_n) \widetilde{X}_n(x) \| + \| \sum_{\sigma_n^2 < \frac{1}{m-1}} \widetilde{g}_n \widetilde{X}_n(x) \| \\
&\leq \delta + \| \sum_{\sigma_n^2 < \frac{2}{m}} \sigma_n f_n \widetilde{X}_n(x) \| \\
&\leq \delta + \left(\sum_{\sigma_n^2 < \frac{2}{m}} \left(\frac{L}{C_1} \right)^p \sigma_n^{p+2} \left(\frac{\lambda_n}{L} \right)^p f_n^2 \right)^{\frac{1}{2}} \\
&\leq \delta + \left(\frac{L}{C_1} \right)^{\frac{p}{2}} \left(\frac{2}{m} \right)^{\frac{p+2}{4}} E,
\end{aligned}$$

then, we deduce that

$$m \leq 2 \left(\frac{L}{C_1} \right)^{\frac{2p}{p+2}} \left(\frac{E}{(\omega-1)\delta} \right)^{\frac{4}{p+2}},$$

similar to the proof of Theorem 4.2, it is easy to get that

$$\| f^{m,\delta}(x) - f^m(x) \| \leq \left(\frac{\sqrt{2}L^{\frac{p}{2}}}{C_1^{\frac{p}{2}}(\omega - 1)} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \tag{4.16}$$

and

$$\| f^m(x) - f(x) \| \leq \left(\frac{L(\omega+1)}{C_1} \right)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{4.17}$$

Combining (4.16) and (4.17), we can easily obtain the desired result.

4.4. Combining TSVD method and Fractional Landweber iterative regularization method (method 4)

In this subsection, we combine two methods, its advantage is that not only retains the characteristics of Fractional Landweber iterative regularization method, but also increases the advantages of TSVD method for filtering large and small singular values separately. We give the regularized approximate solution as follows

$$f^{m,\delta}(x) = \sum_{\sigma_n^2 \geq \frac{1}{m}} \frac{1}{\sigma_n} \widetilde{g}_n^\delta \widetilde{X}_n(x) + \sum_{\sigma_n^2 < \frac{1}{m}} \frac{[1-(1-\beta\sigma_n^2)^m]^\gamma}{\sigma_n} \widetilde{g}_n^\delta \widetilde{X}_n(x). \tag{4.18}$$

4.4.1. A priori regularization parameter choice rule

In 4.4.1, we will give an error estimate under the *a priori* regularization parameter choice rule.

Theorem 4.7: Let $f(x)$ given by (3.1) be the exact solution. Let $f^{m,\delta}(x)$ given by (4.18) be the regularization solution. The conditions (1.3) and (3.2) hold. If we choose

$$m = [\left(\frac{E}{\delta}\right)^{\frac{4}{p+2}}],$$

then we have the following error estimate

$$\| f^{m,\delta}(x) - f(x) \| \leq \left(\sqrt{\beta} + 1 + \left(\frac{L}{C_1}\right)^{\frac{p}{2}}\right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

Proof: Using triangle inequality, we have

$$\| f^{m,\delta}(x) - f(x) \| \leq \| f^{m,\delta}(x) - f^m(x) \| + \| f^m(x) - f(x) \| .$$

From Lemma 3.2, we have

$$\begin{aligned} \| f^{m,\delta}(x) - f^m(x) \| &= \| \sum_{\sigma_n^2 \geq \frac{1}{m}} \frac{1}{\sigma_n} (\widetilde{g}_n^\delta - \widetilde{g}_n) \widetilde{X}_n(x) + \sum_{\sigma_n^2 < \frac{1}{m}} \frac{[1-(1-\beta\sigma_n^2)^m]^\gamma}{\sigma_n} (\widetilde{g}_n^\delta - \widetilde{g}_n) \widetilde{X}_n(x) \| \\ &\leq (\sqrt{m} + \sqrt{\beta m}) \| \widetilde{g}^\delta(x) - \widetilde{g}(x) \| \leq (\sqrt{m} + \sqrt{\beta m})\delta, \end{aligned} \tag{4.19}$$

from (3.2), we have

$$\begin{aligned} \| f^m(x) - f(x) \| &= \| \sum_{\sigma_n^2 < \frac{1}{m}} \frac{[(1-(1-\beta\sigma_n^2)^m)^\gamma - 1]}{\sigma_n} \widetilde{g}_n \widetilde{X}_n(x) \| \\ &= \| \sum_{\sigma_n^2 < \frac{1}{m}} \frac{1}{\sigma_n} \widetilde{g}_n \widetilde{X}_n(x) \| = \left(\sum_{\sigma_n^2 < \frac{1}{m}} \left(\frac{\lambda_n}{L}\right)^{-p} \left(\frac{\lambda_n}{L}\right)^{p} f_n^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq \left(\sum_{\sigma_n^2<\frac{1}{m}} \left(\frac{L}{C_1}\right)^p \sigma_n^p \left(\frac{\lambda_n}{L}\right)^p f_n^2 \right)^{\frac{1}{2}} \leq \left(\frac{L}{C_1}\right)^{\frac{p}{2}} m^{-\frac{p}{4}} E. \tag{4.20}$$

Combining (4.19) and (4.20), and choosing the regularization parameter $m = [\left(\frac{E}{\delta}\right)^{\frac{4}{p+2}}]$, we complete the proof.

4.4.2. A posteriori regularization parameter choice rule

In 4.4.2, we will give an error estimate under the *a posteriori* regularization parameter choice rule.

Theorem 4.8: Let $f(x)$ given by (3.1) be the exact solution. Let $f^{m,\delta}(x)$ given by (4.18) be the regularization solution. The conditions (1.3) and (3.2) hold. Regularization parameter m is chosen by (4.4), then we have the following error estimate

$$\| f^{m,\delta}(x) - f(x) \| \leq \left((\sqrt{\beta}+1) \frac{2^{\frac{p}{2p+4}}}{\beta^{\frac{1}{p+2}}} \left(\frac{L}{C_1}\right)^{\frac{p}{p+2}} \frac{1}{(\omega-1)^{\frac{2}{p+2}}} + \left(\frac{L(\omega+1)}{C_1}\right)^{\frac{p}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

Proof: Using triangle inequality, we have

$$\| f^{m,\delta}(x) - f(x) \| \leq \| f^{m,\delta}(x) - f^m(x) \| + \| f^m(x) - f(x) \| .$$

Then

$$\begin{aligned}
\omega\delta &\leq \| Kf^{m-1,\delta}(x) - \widetilde{g}^\delta(x) \| = \| \sum_{\sigma_n^2<\frac{1}{m-1}} [1-(1-(1-\beta\sigma_n^2)^{m-1})^\gamma] \widetilde{g}_n^\delta \widetilde{X}_n(x) \| \\
&\leq \| \sum_{\sigma_n^2<\frac{1}{m-1}} (1-\beta\sigma_n^2)^{m-1} (\widetilde{g}_n^\delta - \widetilde{g}_n) \widetilde{X}_n(x) \| + \| \sum_{\sigma_n^2<\frac{1}{m-1}} (1-\beta\sigma_n^2)^{m-1} \widetilde{g}_n \widetilde{X}_n(x) \| \\
&\leq \delta + \| \sum_{\sigma_n^2<\frac{2}{m}} (1-\beta\sigma_n^2)^{m-1} \sigma_n f_n \widetilde{X}_n(x) \| \\
&\leq \delta + \frac{1}{\sqrt{\beta m}} \| \sum_{\sigma_n^2<\frac{2}{m}} f_n \widetilde{X}_n(x) \| \\
&= \delta + \frac{1}{\sqrt{\beta m}} \left(\sum_{\sigma_n^2<\frac{2}{m}} \left(\frac{\lambda_n}{L}\right)^{-p} \left(\frac{\lambda_n}{L}\right)^p f_n^2 \right)^{\frac{1}{2}} \\
&\leq \delta + \frac{1}{\sqrt{\beta m}} \left(\frac{L}{C_1}\right)^{\frac{p}{2}} \left(\frac{2}{m}\right)^{\frac{p}{4}} E,
\end{aligned}$$

then, we deduce that

$$m \leq \frac{2^{\frac{p}{p+2}}}{\beta^{\frac{2}{p+2}}} \left(\frac{L}{C_1}\right)^{\frac{2p}{p+2}} \left(\frac{E}{(\omega-1)\delta}\right)^{\frac{4}{p+2}},$$

similar to the proof of Theorem 4.2, it is easy to get that

$$\| f^{m,\delta}(x) - f^m(x) \| \le (\sqrt{\beta}+1)\frac{2^{\frac{p}{2p+4}}}{\beta^{\frac{1}{p+2}}}\left(\frac{L}{C_1}\right)^{\frac{p}{p+2}}\frac{1}{(\omega-1)^{\frac{2}{p+2}}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{4.21}$$

and

$$\| f^m(x) - f(x) \| \le \left(\frac{L(\omega+1)}{C_1}\right)^{\frac{p}{p+2}} E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{4.22}$$

Combining (4.21) and (4.22), we can easily obtain the desired result.

5. Numerical experiments

In this section, numerical experiments are presented to illustrate the effectiveness of our methods. We take Case 1 as an example, that is, let $\alpha_1 = \beta_1 = 1$. Assume that $F(x,t) = r(t)f(x) + Z(x,t)$, then equation (1.1) can be rewritten as

$$\begin{cases} D_t^\alpha u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} + F(x,t), & 0 < x < L, t > 0, \\ u(x,0) = \phi(x), & 0 \le x \le L, \\ u(0,t) + u_x(0,t) = \mu_1(t), & 0 \le t \le T, \\ u(L,t) + u_x(L,t) = \mu_2(t), & 0 \le t \le T. \end{cases} \tag{5.1}$$

Due to the difficulty in getting the exact solution of the direct problem, we use the finite difference method (FDM) to solve the direct problem (assumed that fuctions $F(x,t)$, $\phi(x)$, $\mu_1(t)$ and $\mu_2(t)$ are known). Then the final value data $g(x)$ is easily obtained.

Denote the discrete points in the space interval $[0,L]$ as $x_i = ih (i = 0, 1, \cdots, M)$ with the space step size $h = \frac{L}{M}$, the discrete points in the time interval $[0,T]$ as $t_n = n\tau (n = 0, 1, \cdots, N)$ with the time step size $\tau = \frac{T}{N}$. Let the value at each gird point is $u_i^n = u(x_i, t_n)$.

Take the average of two adjacent time layers. Adopting the finite difference scheme, we discrete the equation $D_t^\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + F(x,t)$ as follows

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[u_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u_i^k - a_{n-1}^{(\alpha)}\phi(x_i)] = \frac{1}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + F_i^n, \tag{5.2}$$

where $a_k^{(\alpha)} = (k+1)^{1-\alpha} - k^{1-\alpha} (k > 0)$, $F_i^n = F(x_i, t_n)$, denote $\lambda = \frac{h^2}{\tau^\alpha \Gamma(2-\alpha)}$ and (5.2) can be simplified as

$$\begin{cases} -2u_1^n+(\lambda+2-2h)u_0^n=-2h\mu_1(t)+\lambda\sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})u_0^k+\lambda a_{n-1}^{(\alpha)}\phi(x_0)+h^2F_0^n, \\ -u_{i+1}^n+(\lambda+2)u_i^n-u_{i-1}^n=\lambda\sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})u_i^k+\lambda a_{n-1}^{(\alpha)}\phi(x_i)+h^2F_i^n, \qquad 1\le i\le M-1, \\ (\lambda+2+2h)u_M^n-2u_{M-1}^n=2h\mu_2(t)+\lambda\sum\limits_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})u_M^k+\lambda a_{n-1}^{(\alpha)}\phi(x_M)+h^2F_M^n. \end{cases}$$

Then we get the following matrix equation:

$$AU_n=A_1+\lambda A_2+\lambda a_{n-1}^{(\alpha)}A_3+h^2A_4,$$

where A is a tridiagonal matrix:

$$A_{(M+1)\times(M+1)}=\begin{pmatrix} \lambda+2-2h & -2 & 0 & 0 & \cdots \\ -1 & \lambda+2 & -1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & -1 & \lambda+2 & -1 \\ \cdots & 0 & 0 & -2 & \lambda+2+2h \end{pmatrix},$$

$U^n=(u_0,\cdots,u_M)^T$, $A_1=(-2h\mu_1(t),0,\cdots,0,2h\mu_2(t))^T$, $A_2=(\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})u_0^k,\cdots,\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)})u_M^k)^T$, $A_3=(\phi(x_0),\cdots,\phi(x_M))^T$, $A_4=(F_0^n,\cdots,F_M^n)^T$.

Noisy data is generated by adding a random disturbance, i.e.

$$g^\delta(x)=g(x)+\epsilon g(x)\cdot(2rand(size(g(x)))-1),$$

where $\epsilon>0$ reflects the noise level and $\delta=\|g^\delta(x)-g(x)\|$.

In order to facilitate the subsequent description, we call the Landweber iterative regularization method as method 1, Fractional Landweber iterative regularization method as method 2, TSVD method as method 3, combining TSVD method and Fractional Landweber iterative regularization method as method 4. The absolute error $e(f,\epsilon)_{1\sim4}$ and relative error $e_r(f,\epsilon)_{1\sim4}$ between the regularized approximate solution and the exact solution of method $1\sim4$ are

$$e(f,\epsilon)_i=\|f^{m,\delta}(x)-f(x)\|, e_r(f,\epsilon)_i=\frac{\|f^{m,\delta}(x)-f(x)\|}{\|f(x)\|}, i=1,\cdots,4,$$

respectively. In our following numerical experiments, we take $L=1,T=1,M=50,N=100,\beta=\frac{1}{\sigma_n^2+1},\gamma=0.75$. The regularized approximate solutions of four methods are given by (4.1), (4.8), (4.13) and (4.18), respectively. The regularization parameter m under the *a priori* choice rule is given by $m = [(\frac{E}{\delta})^{\frac{4}{p+2}}]$, where $E = \| f(x) \|_p$. The regularization parameter m under the *a posteriori* choice rule is given by (4.4), with $\omega = 1.1$.

Example 1: Take functions

$$f(x) = \sin(\frac{\pi}{2}x), \qquad r(t) = 2(t^2+1), \qquad Z(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}x,$$
$$\phi(x) = \sin(\frac{\pi}{2}x), \qquad \mu_1(x) = -(t^2+1), \qquad \mu_2(x) = -2(t^2+1).$$

ϵ	$e_r(f,\epsilon)_1$	$e_r(f,\epsilon)_2$	$e_r(f,\epsilon)_3$	$e_r(f,\epsilon)_4$	$e(f,\epsilon)_1$	$e(f,\epsilon)_2$	$e(f,\epsilon)_3$	$e(f,\epsilon)_4$
0.001	0.0391	0.0361	0.0398	0.0386	0.1974	0.1823	0.2010	0.1949
0.005	0.0572	0.0513	0.0783	0.0560	0.2888	0.2591	0.3954	0.2828
0.01	0.0661	0.0659	0.0861	0.0767	0.3338	0.3328	0.4349	0.3873

Table 1: Errors by using *a priori* parameter choice rule for Example 1

ϵ	$e_r(f,\epsilon)_1$	$e_r(f,\epsilon)_2$	$e_r(f,\epsilon)_3$	$e_r(f,\epsilon)_4$	$e(f,\epsilon)_1$	$e(f,\epsilon)_2$	$e(f,\epsilon)_3$	$e(f,\epsilon)_4$
0.001	0.0378	0.0312	0.0375	0.0359	0.1909	0.1576	0.1894	0.1813
0.005	0.0488	0.0478	0.0771	0.0546	0.2464	0.2414	0.3893	0.2757
0.01	0.0634	0.0626	0.0843	0.0725	0.3202	0.3161	0.4257	0.3661

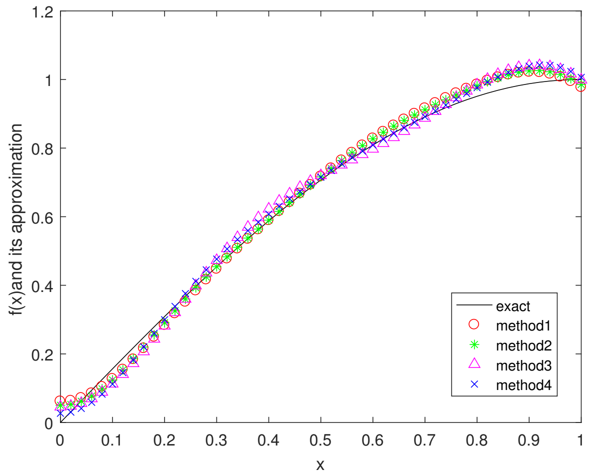
Table 2: Errors by using *a posteriori* parameter choice rule for Example 1

Example 2: Take functions

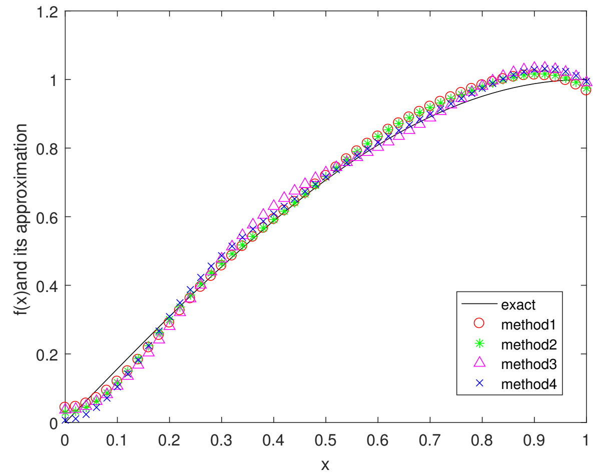
$$f(x) = \begin{cases} 2x, & 0 \le x < 0.5, \\ 2-2x, & 0.5 \le x \le 1, \end{cases} \qquad r(t) = t^2+1, \qquad Z(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}x,$$
$$\phi(x) = 2x, \qquad \mu_1(x) = -(t^2+1), \qquad \mu_2(x) = -2(t^2+1).$$

ϵ	$e_r(f,\epsilon)_1$	$e_r(f,\epsilon)_2$	$e_r(f,\epsilon)_3$	$e_r(f,\epsilon)_4$	$e(f,\epsilon)_1$	$e(f,\epsilon)_2$	$e(f,\epsilon)_3$	$e(f,\epsilon)_4$
0.001	0.0780	0.0779	0.0749	0.0772	0.3186	0.3182	0.3059	0.3153
0.005	0.0852	0.0846	0.0871	0.0897	0.3480	0.3455	0.3557	0.3663
0.01	0.1140	0.1088	0.1034	0.1101	0.4656	0.4444	0.4223	0.4497

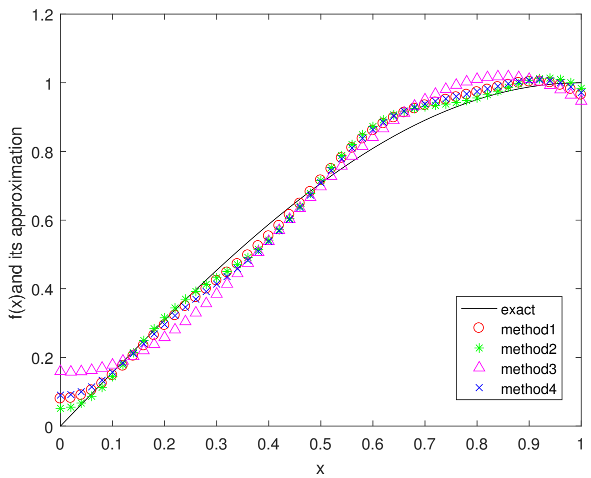
Table 3: Errors by using *a priori* parameter choice rule for Example 2



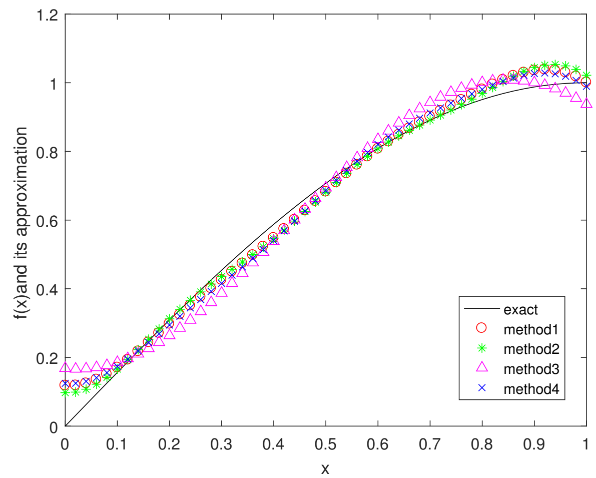
(a) ***A priori parameter choice rule*** $(\epsilon = 0.001)$



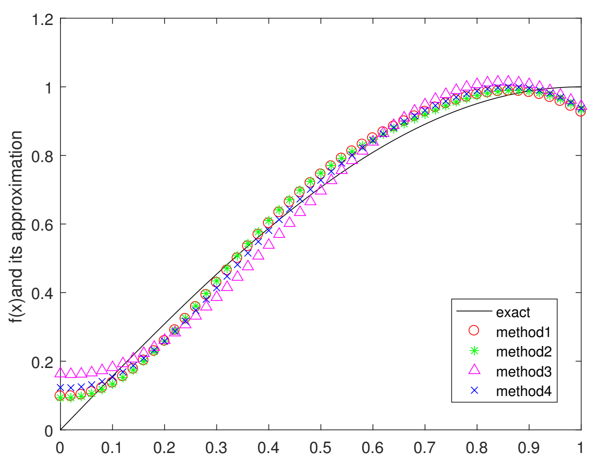
(b) ***A posteriori parameter choice rule*** $(\epsilon = 0.001)$



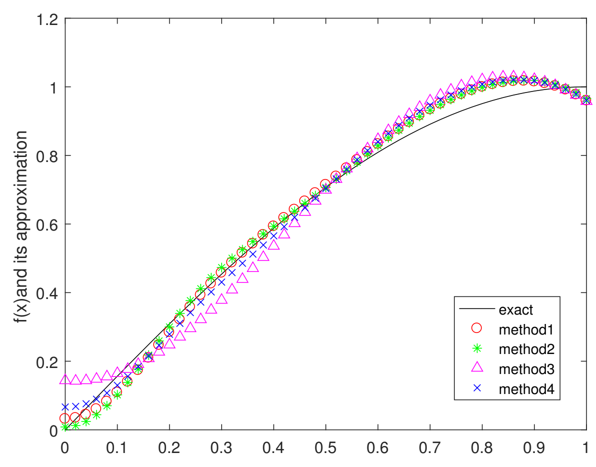
(c) ***A priori parameter choice rule*** $(\epsilon = 0.005)$



(d) ***A posteriori parameter choice rule*** $(\epsilon = 0.005)$

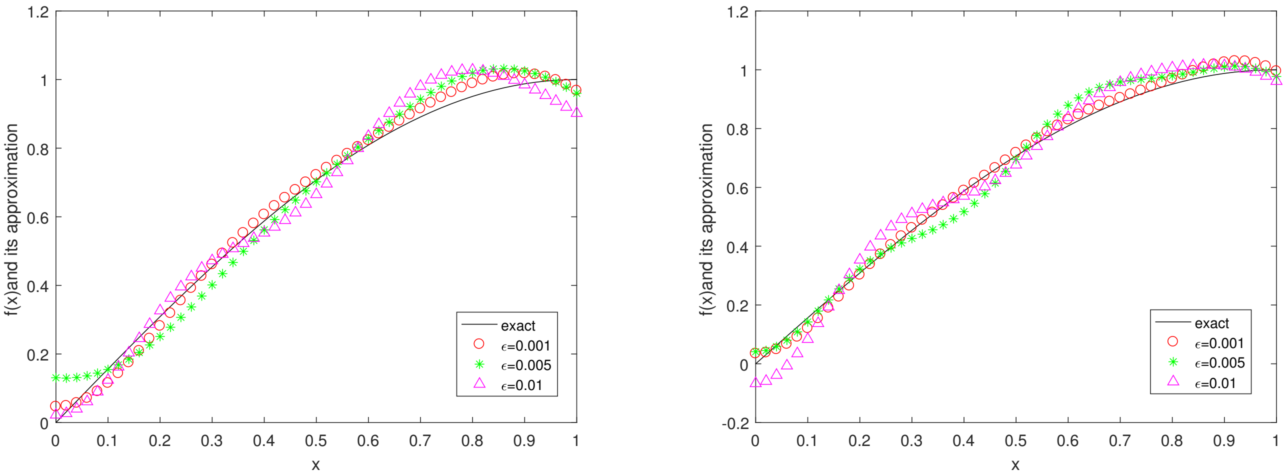


(e) ***A priori parameter choice rule*** $(\epsilon = 0.01)$



(f) ***A posteriori parameter choice rule*** $(\epsilon = 0.01)$

Fig 1: The exact solution $f(x)$ and its approximation with different methods for Example 1



(a) *A priori parameter choice rule (method 4)* **(b) *A posteriori parameter choice rule (method 4)***

Fig 2: The exact solution $f(x)$ and its approximation by using method 4 with different noise levels for Example 1

ϵ	$e_r(f,\epsilon)_1$	$e_r(f,\epsilon)_2$	$e_r(f,\epsilon)_3$	$e_r(f,\epsilon)_4$	$e(f,\epsilon)_1$	$e(f,\epsilon)_2$	$e(f,\epsilon)_3$	$e(f,\epsilon)_4$
0.001	0.0757	0.0737	0.0716	0.0765	0.3092	0.3010	0.2924	0.3124
0.005	0.0840	0.0810	0.0841	0.0862	0.3431	0.3308	0.3435	0.3520
0.01	0.1108	0.1033	0.0998	0.0996	0.4525	0.4219	0.4076	0.4068

Table 4: Errors by using *a posteriori* parameter choice rule for Example 2

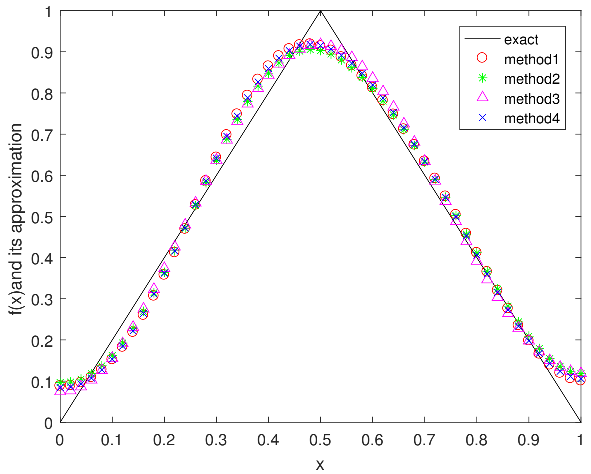
Example 3: Take functions

$$f(x) = \begin{cases} 0, & 0 \le x < 0.25, \\ 2, & 0.25 \le x < 0.75, \\ 0, & 0.75 \le x \le 1, \end{cases} \qquad r(t) = t^2 + 1, \qquad Z(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} x,$$

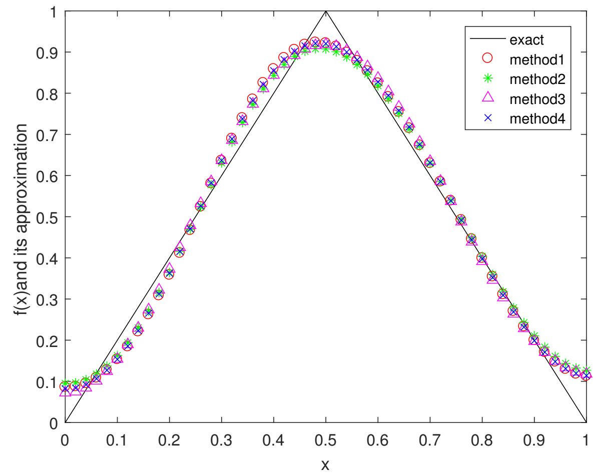
$$\phi(x) = 2x, \qquad \mu_1(x) = -(t^2+1), \qquad \mu_2(x) = -2(t^2+1).$$

ϵ	$e_r(f,\epsilon)_1$	$e_r(f,\epsilon)_2$	$e_r(f,\epsilon)_3$	$e_r(f,\epsilon)_4$	$e(f,\epsilon)_1$	$e(f,\epsilon)_2$	$e(f,\epsilon)_3$	$e(f,\epsilon)_4$
0.001	0.2281	0.2274	0.2280	0.2292	2.2810	2.2740	2.2800	2.2920
0.005	0.2833	0.2804	0.3055	0.2758	2.8330	2.8040	3.0550	2.7580
0.01	0.3106	0.3059	0.3273	0.3128	3.1060	3.0590	3.2730	3.1280

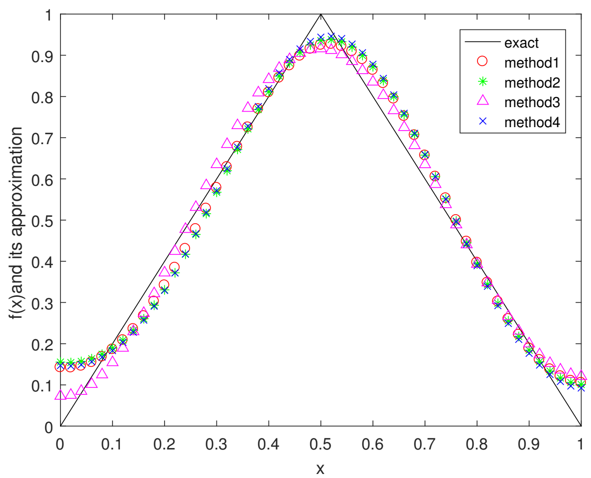
Table 5: Errors by using *a priori* parameter choice rule for Example 3



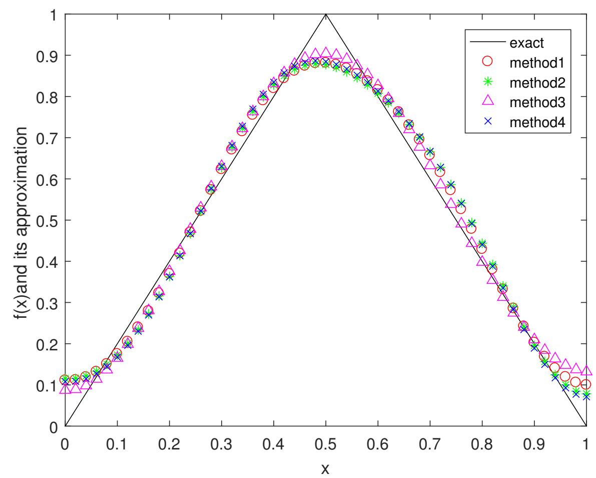
(a) *A priori parameter choice rule* ($\epsilon = 0.001$)



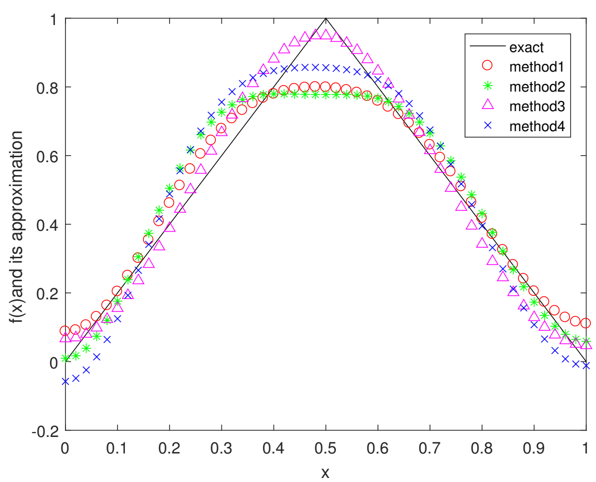
(b) *A posteriori parameter choice rule* ($\epsilon = 0.001$)



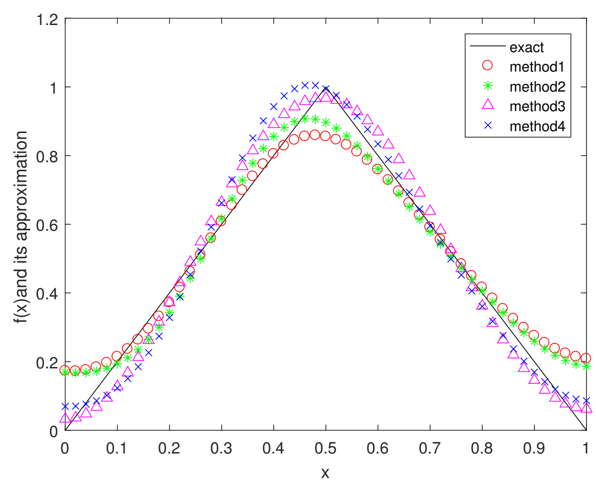
(c) *A priori parameter choice rule* ($\epsilon = 0.005$)



(d) *A posteriori parameter choice rule* ($\epsilon = 0.005$)

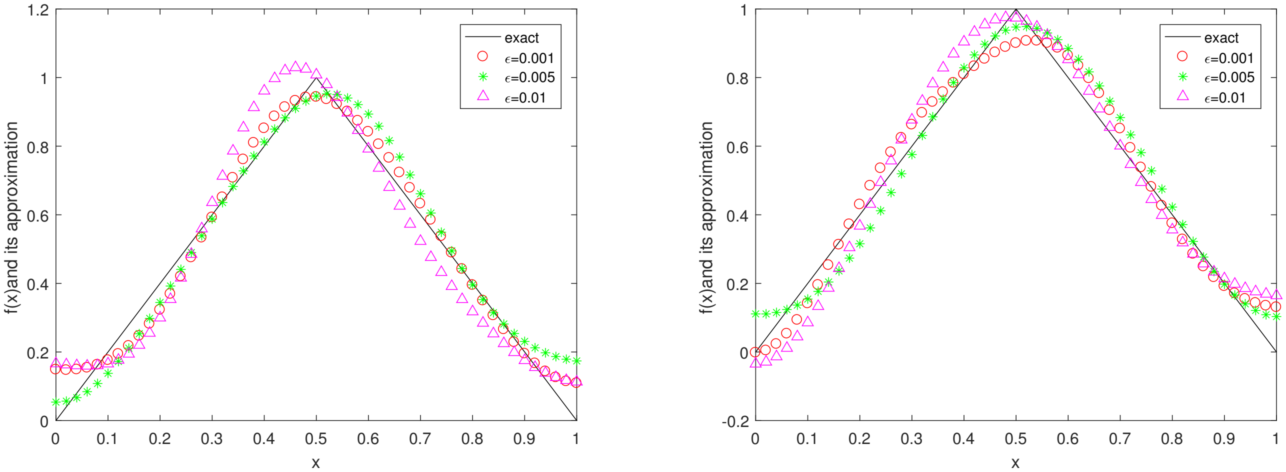


(e) *A priori parameter choice rule* ($\epsilon = 0.01$)



(f) *A posteriori parameter choice rule* ($\epsilon = 0.01$)

Fig 3: The exact solution $f(x)$ and its approximation with different methods for Example 2



(a) *A priori parameter choice rule (method 4)* **(b) *A posteriori parameter choice rule (method 4)***

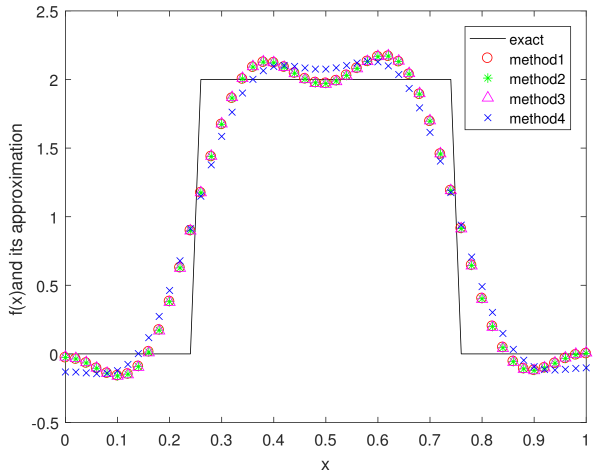
Fig 4: The exact solution $f(x)$ and its approximation by using method 4 with different noise levels for Example 2

ϵ	$e_r(f,\epsilon)_1$	$e_r(f,\epsilon)_2$	$e_r(f,\epsilon)_3$	$e_r(f,\epsilon)_4$	$e(f,\epsilon)_1$	$e(f,\epsilon)_2$	$e(f,\epsilon)_3$	$e(f,\epsilon)_4$
0.001	0.2259	0.2252	0.2263	0.2253	2.2590	2.2520	2.2630	2.2530
0.005	0.2797	0.2777	0.3026	0.2727	2.7970	2.7770	3.0260	2.7270
0.01	0.3002	0.2964	0.3233	0.3095	3.0020	2.9640	3.2330	3.0950

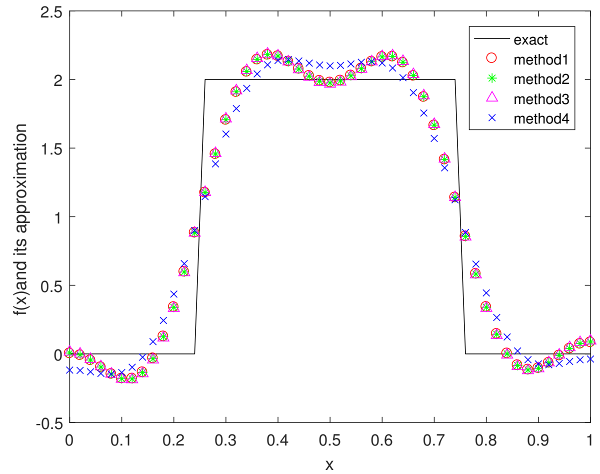
Table 6: Errors by using *a posteriori* parameter choice rule for Example 3

In Figs 1, 3 and 5, we provide the comparisons between the exact solution $f(x)$ and its regularized approximate solution by using *a priori* and *a posteriori* parameter choice rules with different methods. Since the *a prior* bound E is difficult to obtain in practical problems, we give an accurate *a prior* bound $E = \| f(x) \|_p$, then both *a priori* and *a posteriori* parameter choice rules achieve desired results. In Figs 2, 4 and 6, we take combining TSVD method and Fractional Landweber iterative regularization method as an example, then we provide the comparisons between the exact solution $f(x)$ and its regularized approximate solution by using *a priori* and *a posteriori* parameter choice rules with different noise levels. Tables 1-6 show the absolute errors and relative errors with different methods and different noise levels.

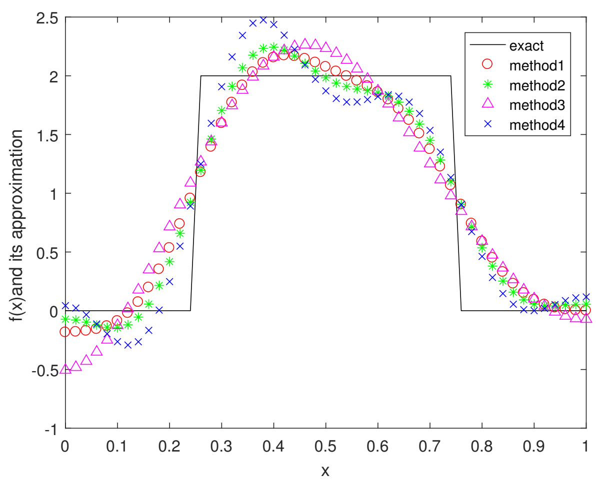
From these examples, it can be seen from Figs 1-6 that the numerical results of four regularization methods are similar, and when the smoothness of the solution gets better, the numerical results get better. These are consistent with our theoretical results. It can be seen from Table 1-6 that the smaller ϵ, the better the approximation effect. Moreover, we can see that the *a posteriori* parameter choice rule is even comparable to the *a*



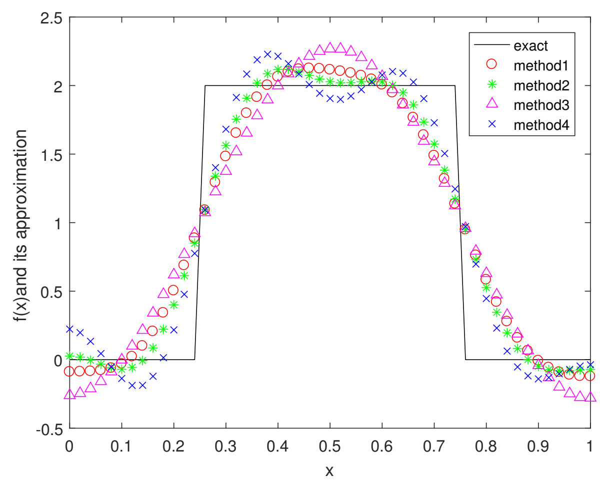
(a) ***A priori parameter choice rule*** $(\epsilon = 0.001)$



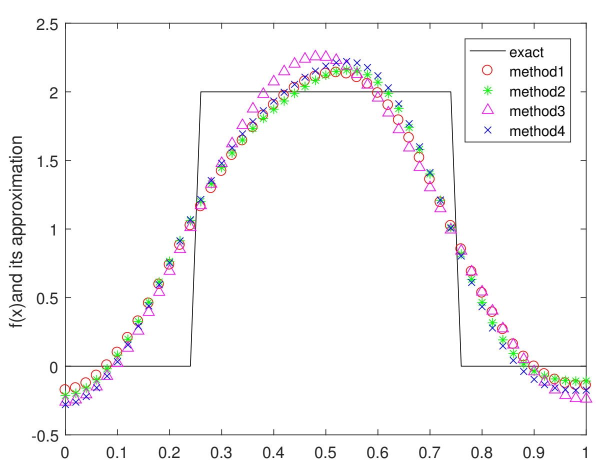
(b) ***A posteriori parameter choice rule*** $(\epsilon = 0.001)$



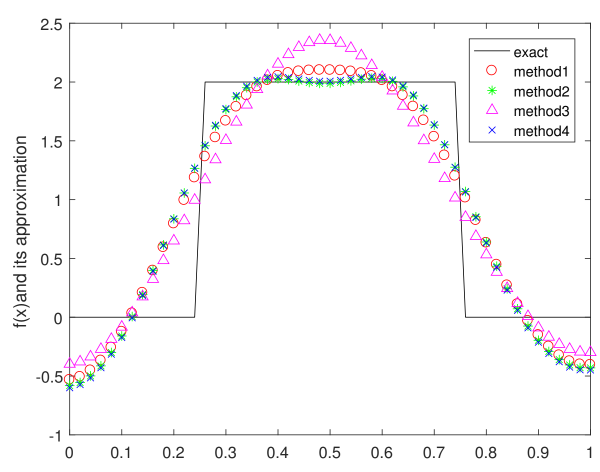
(c) ***A priori parameter choice rule*** $(\epsilon = 0.005)$



(d) ***A posteriori parameter choice rule*** $(\epsilon = 0.005)$

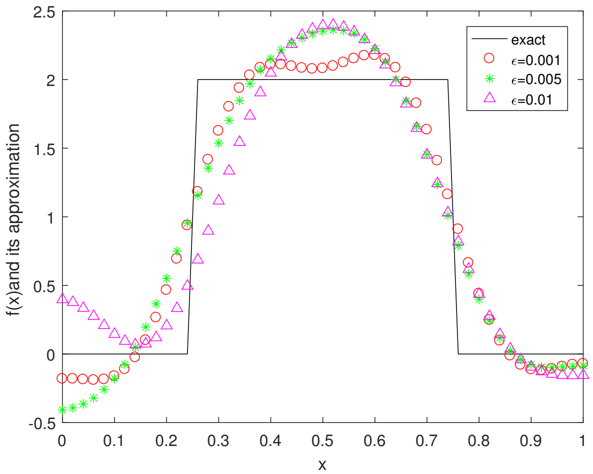


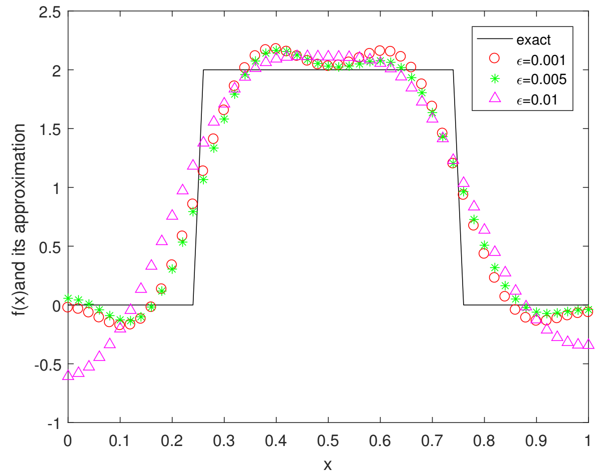
(e) ***A priori parameter choice rule*** $(\epsilon = 0.01)$



(f) ***A posteriori parameter choice rule*** $(\epsilon = 0.01)$

Fig 5: The exact solution $f(x)$ and its approximation with different methods for Example 3





(a) ***A priori parameter choice rule (method 4)*** **(b)** ***A posteriori parameter choice rule (method 4)***

Fig 6: The exact solution $f(x)$ and its approximation by using method 4 with different noise levels for Example 3

priori parameter choice rule. These numerical examples verify the validity of our iterative regularization method.

6. Conclusion

In this paper, we consider the source term identification of the time-fractional diffusion equation under Robin boundary condition. Landweber iterative regularization method, Fractional Landweber iterative regularization method, TSVD method, combining TSVD method and Fractional Landweber iterative regularization method are used to solve the equation (1.1). Error estimations between the regularized approximate solution and exact solution under two parameter choice rules are given. The numerical simulation results show that: (1) Fractional Landweber iterative regularization method outperform Landweber iterative regularization method, (2) numerical results of four methods are similar, which is consistent with our theoretical results. Combining TSVD method and Fractional Landweber iterative regularization method does not show advantages in numerical examples of this paper, probably because the equation (1.1) is linear. On other types of inverse problem, such as nonlinear equations, combining TSVD method and Fractional Landweber iterative regularization method will achieve better numerical results, which awaits our further study.

References

[1] Y. Luchko, Maximum principle and its application for the time-fractional diffusion equations, Fractional Calculus & Applied Analysis, 14(1), 110-124(2011).

[2] M. Borikhanov, M. Kirane, B.K. Torebek, Maximum principle and its application for the nonlinear time-fractional diffusion equations with Cauchy-Dirichlet conditions, Applied Mathematics Letters, 81, 14-20(2018).

[3] M. Al-Refai, Y. Luchko, Maximum principle for the multi-term time-fractional diffusion equations with the Riemann–Liouville fractional derivatives, Applied Mathematics & Computation, 257, 40-51(2015).

[4] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations, Journal of Computational & Applied Mathematics, 172(1), 65-77(2004).

[5] S.B. Yuste, J. Quintana-Murillo, A finite difference method with non-uniform timesteps for fractional diffusion equations, European Physical Journal Special Topics, 183(12), 2594-2600(2012).

[6] M. Dehghan, M. Abbaszadeh, A finite difference/finite element technique with error estimate for space fractional tempered diffusion-wave equation, Computers & Mathematics with Applications, 75(8), 2903-2914(2018).

[7] T.A.M. Langlands, B.I. Henry, The accuracy and stability of an implicit solution method for the fractional diffusion equation, Journal of Computational Physics, 205(2), 719-736(2005).

[8] L.M. Li, D. Xu, M. Luo, Alternating direction implicit Galerkin finite element method for the two-dimensional fractional diffusion-wave equation, Journal of Computational Physics, 255, 471-485(2013).

[9] J. Manimaran, L. Shangerganesh, A. Debbouche, Finite element error analysis of a time-fractional nonlocal diffusion equation with the Dirichlet energy, Journal of Computational and Applied Mathematics, 382(2), 113066(2021).

[10] M.R.S. Ammi, I. Jamiai, D.F.M. Torres, A finite element approximation for a class of Caputo time-fractional diffusion equations, Computers & mathematics with applications, 78(5), 1334-1344(2019).

[11] A.J. Hussein, A weak Galerkin finite element method for solving time-fractional coupled Burgers' equations in two dimensions, Applied Numerical Mathematics, 156(2020), 265-275(2020).

[12] T. Wei, J.G. Wang, A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation, Applied Numerical Mathematics, 78, 95-111(2014).

[13] Y. Zhang, X. Xu, Inverse source problem for a fractional diffusion equation, Inverse Problems, 27(3), 035010(2011).

[14] F. Yang, X. Liu, X.X. Li, et al, Landweber iterative regularization method for identifying the unknown source of the time-fractional diffusion equation, Advances in Difference Equations, 2017(1), 1-15(2017).

[15] X.T. Xiong, Q. Zhou, Y.C. Hon, An inverse problem for fractional diffusion equation in 2-dimensional case: Stability analysis and regularization, Journal of Mathematical Analysis and Applications, 393(1), 185–199(2012).

[16] N.H. Tuan, L.D. Long, N.V. Thinh, Regularized solution of an inverse source problem for a time fractional diffusion equation, Applied Mathematical Modelling, 40, 8244–8264(2016).

[17] M. Kirane, S.A. Malik, M.A. Al-Gwaiz, An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions, Mathematical Methods in the Applied Sciences, 36(9), 1056-1069(2013).

[18] A. Kirsch, An Introduction to the Mathematical Theory of Inverse Problems, Springer, New York(2011).

[19] X.B. Yan, Z.Q. Zhang, T Wei, Simultaneous inversion of a time-dependent potential coefficient and a time source term in a time fractional diffusion-wave equation, Chaos, Solitons & Fractals, 157, 111901(2022).

[20] F. Yang, Y.P. Ren, X.X. Li, Landweber iteration regularization method for identifying unknown source on a columnar symmetric domain, Inverse Problems in Science and Engineering, 26(5-8), 1109-1129(2018).

[21] E. Klann, R. Ramlau, Regularization by fractional filter methods and data smoothing, Inverse Problems, 24(2), 025018(2018).

[22] C.S. Wu, Special Topics in Mathematical Physical Methods, Beijing University Press, Beijing(2013).

[23] Q. Gu, Mathematical methods for physics, Science Press, Beijing(2012).

[24] X.X. Geng, H. Cheng, W.P. Fan, A note on "Analytical solution for the time-fractional telegraph equation by the method of separating variables", Journal of Mathematical Analysis and Applications, 512(2), 126144(2022).

[25] I. Podlubny, Fractional Differential Equations, Academic Press, New York(1999).

[26] Y.K. Ma, P. Prakash, A. Deiveegan, Generalized Tikhonov methods for an inverse source problem of the time-fractional diffusion equation, Chaos, Solitons & Fractals, 108, 39–48(2018).

[27] Y.J.Deng, Z.H. Liu, Iteration methods on sideways parabolic equations, Inverse Problem, 25(9), 095004(2009).