# ALGEBRAIC ANDÔ DILATION

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**Abstract**: We solve the Andô dilation problem for linear maps on vector space asked by Krishna and Johnson in *[Oper. Matrices, 2022]*. More precisely, we show that any commuting linear maps on vector space can be dilated to commuting injective linear maps.
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## 1. Introduction

After a decade of work of Sz.-Nagy [13, 14], Andô [1] made a breakthrough result in the dilation theory of contractions on Hilbert space which states as follows.

**Theorem 1.1.** *[1] (**Andô Dilation**) Let $\mathcal{H}$ be a Hilbert space and $T, S : \mathcal{H} \to \mathcal{H}$ be commuting contractions. Then there exists a Hilbert space $\mathcal{K}$ which contains $\mathcal{H}$ isometrically and a pair of commuting unitaries $U, V : \mathcal{K} \to \mathcal{K}$ such that*

$$T^nS^m = P_{\mathcal{H}}U^nV^m|_{\mathcal{H}}, \quad \forall n, m \in \mathbb{Z}_+ \coloneqq \mathbb{N} \cup \{0\},$$

*where $P_{\mathcal{H}} : \mathcal{K} \to \mathcal{H}$ is the orthogonal projection onto $\mathcal{H}$.*

After the work of Andô, Parrott [10] showed that it is not possible to improve Theorem 1.1 for more than two commuting contractions. Later, Andô dilation is derived for commuting contractions on Banach spaces [12]. In 2021 (arXiv version), in the paper [7], while continuing the work of Bhat, De and Rakshit [2] on dilations of linear maps on vector spaces, Krishna and Johnson [7] asked following problem.

**Question 1.2.** *[7] **Whether there is an Andô dilation for linear maps on vector spaces? More precisely, whether commuting linear maps on vector space can be dilated to commuting bijective linear maps?***

In this paper, we solve Question 1.2 partially by showing that we can go upto injective linear maps.

## 2. Algebraic Andô Dilation

We first give a different proof Theorem 2.1 than given in [2] which helps us to give a proof of algebraic version of Andô dilation.

**Theorem 2.1.** *[2] (Algebraic Sz.-Nagy Dilation or Bhat-De-Rakshit Dilation) Let $\mathcal{V}$ be a vector space and $T : \mathcal{V} \to \mathcal{V}$ be a linear map. Then there is a vector space $\mathcal{W}$ containing $\mathcal{V}$ through a natural coordinate* *injective map and an injective linear map $U : \mathcal{W} \to \mathcal{W}$ such that*

$$\text{(Dilation equation)} \quad T^n = P_{\mathcal{V}} U^n|_{\mathcal{V}}, \quad \forall n \in \mathbb{Z}_+,$$

*where $P_{\mathcal{V}} : \mathcal{W} \to \mathcal{V}$ is a coordinate projection (idempotent) onto $\mathcal{V}$.*

*Proof.* Our construction is motivated from the construction of Sz.-Nagy dilation of a contraction on a Hilbert space given in Chapter 1 of [14]. Given a vector space $\mathcal{V}$, let $I_{\mathcal{V}}$ be the identity operator on $\mathcal{V}$ and $\oplus_{n=0}^{\infty}\mathcal{V}$ be the vector space defined by

$$\oplus_{n=0}^{\infty}\mathcal{V} \coloneqq \{(x_n)_{n=0}^{\infty}, x_n \in \mathcal{V}, \forall n \in \mathbb{Z}_+, x_n \neq 0 \text{ only for finitely many } n's\}.$$

Let $T : \mathcal{V} \to \mathcal{V}$ be a linear map. Define $\mathcal{W} \coloneqq \oplus_{n=0}^{\infty}\mathcal{V}$ and

$$\begin{aligned}
&I : \mathcal{V} \ni x \mapsto (x, 0, \dots) \in \mathcal{W},\\
&U : \mathcal{W} \ni (x_n)_{n=0}^{\infty} \mapsto (Tx_0, (I_{\mathcal{V}} - T)x_0, x_1, x_2, \dots) \in \mathcal{W},\\
&P : \mathcal{W} \ni (x_n)_{n=0}^{\infty} \mapsto x_0 \in \mathcal{V}.
\end{aligned}$$

Then clearly the dilation equation is satisfied. The proof is complete if we show that $U$ is injective. Let $(x_n)_{n=0}^{\infty} \in \mathcal{W}$ be such that $U(x_n)_{n=0}^{\infty} = 0$. then

$$(Tx_0, (I_{\mathcal{V}} - T)x_0, x_1, x_2, \dots) = (0, 0, 0, \dots).$$

We then have $x_1 = x_2 = \cdots = 0$ and $Tx_0 = (I_{\mathcal{V}} - T)x_0 = 0$. Rewriting

$$x_0 = Tx_0 = 0.$$

Thus $(\mathcal{W}, U)$ is an injective linear dilation of $T$. ∎

Following is the most important result of this paper which we call algebraic Andô dilation.

**Theorem 2.2.** ***(Algebraic Andô Dilation)*** *Let $\mathcal{V}$ be a vector space and $T, S : \mathcal{V} \to \mathcal{V}$ be commuting linear maps. Then there is a vector space $\mathcal{W}$ containing $\mathcal{V}$ through a natural coordinate injective map and injective linear maps $U, V : \mathcal{W} \to \mathcal{W}$ such that*

$$\text{(Bivariate Dilation equation)} \quad T^n S^m = P_{\mathcal{V}} U^n V^m|_{\mathcal{V}}, \quad \forall n, m \in \mathbb{Z}_+ \coloneqq \mathbb{N} \cup \{0\},$$

*where $P_{\mathcal{V}} : \mathcal{W} \to \mathcal{V}$ is a coordinate projection (idempotent) onto $\mathcal{V}$.*

*Proof.* Our arguments are motivated from original argument for Andô dilation for commuting contractions on Hilbert spaces by Andô [1]. Define $\mathcal{W} \coloneqq \oplus_{n=0}^{\infty}\mathcal{V}$ and

$$\begin{aligned}
&W_1 : \mathcal{W} \ni (x_n)_{n=0}^{\infty} \mapsto (Tx_0, (I_{\mathcal{V}} - T)x_0, 0, x_1, x_2, \dots) \in \mathcal{W},\\
&W_2 : \mathcal{W} \ni (x_n)_{n=0}^{\infty} \mapsto (Sx_0, (I_{\mathcal{V}} - S)x_0, 0, x_1, x_2, \dots) \in \mathcal{W},\\
&P : \mathcal{W} \ni (x_n)_{n=0}^{\infty} \mapsto x_0 \in \mathcal{V}.
\end{aligned}$$

Let $x \in \mathcal{V}$ be such that $(I_{\mathcal{V}} - T)Sx = 0 = (I_{\mathcal{V}} - S)x$. Then

$$(I_{\mathcal{V}} - S)Tx = Tx - STx = Tx - TSx = T(I_{\mathcal{V}} - S)x = T0 = 0$$

and

$$(I_{\mathcal{V}} - T)x = x - Tx = x - T(Sx) = Sx - TSx = (I_{\mathcal{V}} - T)Sx = 0.$$

This obervation says that the map

$$v : \{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\} \to \{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}$$

defined by

$$v(I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) := ((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0)$$

is a well-defined injective linear map. Clearly $v$ is surjective. We now claim that $v$ can be extended as a bijective linear map (which we again denote by $v$) from $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ to $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. We get two cases.

Case (i): $\dim(\mathcal{V}) < \infty$.

Let $\mathcal{Y}$ be any vector space complement of $\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ and $\mathcal{Z}$ be any vector space complement of $\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. From the dimension formula for vector spaces, we then get

$$\begin{aligned}\dim(\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}) &= \dim(\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}) + \dim(\mathcal{Y})\\ &= \dim(\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}) + \dim(\mathcal{Z})\end{aligned}$$

Since $\dim(\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}) = \dim(\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\})$,

$$\dim(\mathcal{Y}) = \dim(\mathcal{Z}).$$

Thus $v$ can be extended bijectively and linearly from $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ to $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$.

Case (i): $\dim(\mathcal{V}) = \infty$.

Let $\mathcal{Y}$ be any vector space complement of $\{((I_{\mathcal{V}} - T)Sx, 0, (I_{\mathcal{V}} - S)x, 0) : x \in \mathcal{V}\}$ containing the space $\{(0, x, 0, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ and $\mathcal{Z}$ be any vector space complement of $\{((I_{\mathcal{V}} - S)Tx, 0, (I_{\mathcal{V}} - T)x, 0) : x \in \mathcal{V}\}$ containing the space $\{(0, x, 0, 0) : x \in \mathcal{V}\}$ in $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. Then

$$\dim(\mathcal{V}) = \dim(\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}) \geq \dim(\mathcal{Y}) \geq \dim(\mathcal{V})$$

and

$$\dim(\mathcal{V}) = \dim(\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}) \geq \dim(\mathcal{Z}) \geq \dim(\mathcal{V}).$$

Therefore $\dim(\mathcal{Y}) = \dim(\mathcal{Z})$ and hence $v$ can be extended bijectively and linearly from $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ to $\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$.

Define $\mathcal{V}^{(4)} := \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$. We identify $\mathcal{W}$ and $\mathcal{V} \oplus (\oplus_{n=1}^{\infty} \mathcal{V}^{(4)})$ by the map

$$(x_n)_{n=0}^{\infty} \mapsto (x_0, (x_1, x_2, x_3, x_4), (x_5, x_6, x_7, x_8), \dots)$$

Now we define $W : \mathcal{W} \to \mathcal{W}$ by

$$W(x_n)_{n=0}^{\infty} := (x_0, v(x_1, x_2, x_3, x_4), v(x_5, x_6, x_7, x_8), \dots)$$

which becomes bijective linear map with inverse

$$W^{-1}(x_n)_{n=0}^{\infty} := (x_0, v^{-1}(x_1, x_2, x_3, x_4), v^{-1}(x_5, x_6, x_7, x_8), \dots)$$

We finally define $U := WW_1$, $V := W_2W^{-1}$ and show that $(\mathcal{W}, (U, V))$ is the required injective linear dilation of $(T, S)$. Clearly $U$ and $V$ are injective. By induction, we also have the multivariate dilation equation

$$T^n S^m x = P_{\mathcal{V}} U^n V^m x, \quad \forall n, m \in \mathbb{Z}_+, \forall x \in \mathcal{V}.$$

Now we are left only with proving that $U$ and $V$ commute. Let $(x_n)_{n=0}^\infty \in \mathcal{W}$. Then

$$\begin{aligned}
UV(x_n)_{n=0}^\infty &= WW_1W_2W^{-1}(x_n)_{n=0}^\infty\\
&= WW_1W_2(x_0, v^{-1}(x_1,x_2,x_3,x_4), v^{-1}(x_5,x_6,x_7,x_8),\dots)\\
&= WW_1(Sx_0,(I_\mathcal{V}-S)x_0,0,v^{-1}(x_1,x_2,x_3,x_4), v^{-1}(x_5,x_6,x_7,x_8),\dots)\\
&= W(TSx_0,(I_\mathcal{V}-T)Sx_0,0,(I_\mathcal{V}-S)x_0,0,v^{-1}(x_1,x_2,x_3,x_4), v^{-1}(x_5,x_6,x_7,x_8),\dots)\\
&= (TSx_0, v((I_\mathcal{V}-T)Sx_0,0,(I_\mathcal{V}-S)x_0,0),(x_1,x_2,x_3,x_4),(x_5,x_6,x_7,x_8),\dots)\\
&= (STx_0,(I_\mathcal{V}-S)Tx_0,0,(I_\mathcal{V}-T)x_0,0),(x_1,x_2,x_3,x_4),(x_5,x_6,x_7,x_8),\dots)
\end{aligned}$$

and

$$\begin{aligned}
VU(x_n)_{n=0}^\infty &= W_2W^{-1}WW_1(x_n)_{n=0}^\infty = W_2W_1(x_n)_{n=0}^\infty\\
&= W_2(Tx_0,(I_\mathcal{V}-T)x_0,0,x_1,x_2,\dots)\\
&= (STx_0,(I_\mathcal{V}-S)Tx_0,0,(I_\mathcal{V}-T)x_0,0),x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,\dots).
\end{aligned}$$

Therefore $VU = UV$. □

Theorem 2.2 and the works presented in [3, 4, 8, 9, 11] gives the following problem.

**Question 2.3.** (i) *Whether there is an explicit (matrix) construction of algebraic Andô dilation?*
(ii) *Whether there is a Halmos dilation for commuting linear maps on vector spaces?*
(iii) *Whether there is an Egerváry N-dilation for commuting linear maps on vector spaces?*
(iv) *Does Theorem 2.2 holds for more than two commuting linear maps?*
(v) *Can the dilated injective linear maps $U, V$ in Theorem 2.2 be improved to bijective linear maps?*

**Remark 2.4.** *Andô dilation for p-adic magic contractions and self-adjoint morphisms on indefinite inner product modules over *-rings of characteristic 2 are still open [5, 6].*

## 3. Conclusions

(1) In 1950, Halmos showed that every contraction on a Hilbert space can be lifted to a unitary [4].
(2) In 1953, Sz.-Nagy derived his dilation theorem [13].
(3) In 1955, Schaffer gave simple proof of Sz.-Nagy dilation result [11].
(4) In 1963, Andô showed that Sz.-Nagy dilation holds for two commuting contractions [1].
(5) In 1973, Stroescu derived Andô dilation for contractions on Banach spaces [12].
(6) In 2021, Bhat, De and Rakshit introduced set theoretic and vector space approach to dilation theory [2]. Later, Krishna and Johnson continued this study in 2022 [7].
(7) In this paper, we derived Andô dilation for linear maps on vector spaces.

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