

Asymptotic stability of rarefaction wave for a blood flow model

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Abstract

This paper is concerned with nonlinear stability of rarefaction wave to the Cauchy problem for a blood flow model, which describes the motion of blood through axi-symmetric compliant vessels. Inspired by the stability analysis of classical p -system, we show the solution of this typical model tends time-asymptotically toward the rarefaction wave under some suitably small conditions and there are more difficulties in the proof due to the appearance of strong nonlinear terms including second-order derivative of v with respect to the spatial variable x . The main result is proved by employing the elementary L^2 energy methods. This is the first result about nonlinear stability of some nontrivial profiles (i.e., non-constant function patterns) for the blood flow model.

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1 Introduction

1.1 The problem

In this paper, we consider the following one-dimensional blood flow model in a network of vessels with viscoelastic walls (see [5, 23]):

$$\begin{cases} A_t + m_x = 0, & x \in \mathbb{R}, t > 0, \\ m_t + \left(\frac{m^2}{A}\right)_x + \frac{A}{\rho} P_x = -k_f \frac{m}{A}. \end{cases} \quad (1.1)$$

Here $A(x, t)$ denotes the cross-sectional area of the vessel; $m(x, t) = A(x, t)u(x, t)$ represents the flow rate of the blood, where $u(x, t)$ denotes the averaged axial velocity $h_x(x, r, t)$ across

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the cross-section of the vessel of radius $R(x, t)$:

$$u(x, t) = \frac{1}{R^2(x, t)} \int_0^{R(x, t)} 2rh_x(x, r, t) dr.$$

The fluid density $\rho > 0$ is assumed to be constant. $k_f \geq 0$ is the friction coefficient per unit length. Moreover, $P(x, t)$ denotes the average internal pressure over a cross section. That is where the distensibility of the blood vessels comes into play. To close the system we need a constitutive law connecting the pressure P to the cross-sectional area A . Generally, the pressure law can be specified by:

$$P = G_0 \left(\left(\frac{A}{A_r} \right)^{\frac{\alpha_1}{2}} - 1 \right) + \alpha_2 \left(P_{\text{ext}} + \frac{\iota}{A_r} (\sqrt{A})_t \right). \quad (1.2)$$

Here the constants in (1.2) have some biological implications. For example, $G_0 > 0$ describes the stiffness of the vessel wall; $A_r > 0$ denotes the reference cross-sectional area; $P_{\text{ext}} > 0$ is the constant external pressure; $\iota > 0$ is the viscoelastic coefficient depending on the thickness of the vessel. Furthermore, the coefficient $\alpha_1 > 0$ reflects stress-strain response of the vessel radius and $\alpha_2 \geq 0$ represents the different weight on influence of $(P_{\text{ext}} + (\iota/A_r)(\sqrt{A})_t)$.

It is well known that the blood flow model can be used to describe many complex physiological phenomena related to human vascular system. Due to rich phenomena in actual physiological applications, the presence of strong nonlinearities in the mathematical model, a lot of physiological and mathematical researchers are attracted to study on this subject. In particular, when the coefficients α_1 and α_2 in (1.2) take different values, the system (1.1) occurs different forms. For example, in the Kelvin-Voigt blood flow model, the pressure is given by (see [23, 24])

$$P = \frac{\beta}{\sqrt{A_r}} \left(\left(\frac{A}{A_r} \right)^{\frac{1}{2}} - 1 \right) + P_{\text{ext}} + \frac{\iota}{A_r} (\sqrt{A})_t, \quad (1.3)$$

which is the case that $\alpha_1 = \alpha_2 = 1$ and $G_0 = \beta/\sqrt{A_r}$ in (1.2). Here β is a positive constant related to the vessel stiffness. In this case, the diffusive effect induced by the viscous term makes the system of hyperbolic/parabolic nature. In fact, as pointed out by the authors in [23], for the Kelvin-Voigt blood flow model, even if the hyperbolic nature of this system is dominant, because the viscous term is small compared to other terms, this additional viscous term plays an important role in numerical simulations [29], in estimation problems [6], and when data coming from numerical models are compared with in vivo data [2]. The authors in [1] also observed this phenomenon. As the blood pressure and vessel deformation are often overestimated by 1-D elastic models (see [32]), the incorporation of viscoelastic tube laws allows more physiological predictions than those obtained with elastic laws. We can also see that most of the mathematical researches for the model with pressure (1.3) focus on numerical simulation (see [7, 28]), but there are few rigorous mathematical analysis conclusions.

On the other hand, to include the fact that the vessel radius changes slower at higher pressures (non-linear response) taking $\alpha_1 > 1$ and $\alpha_2 = 0$ in (1.2), then the pressure is expressed

by the following formula (see [5]):

$$P = G_0 \left(\left(\frac{A}{A_r} \right)^{\frac{\alpha_1}{2}} - 1 \right), \quad (1.4)$$

where $\alpha_1 > 1$ describes non-linear stress-strain response. The model with pressure (1.4) can be derived from the 3-D Navier-Stokes equations which describe the motion of the viscous, incompressible, Newtonian fluid flow in a cylindrical tube (see [4]). And the rationality of this approximation was analysed by Čanić in [3]. Moreover, Čanić in [5] gave a new derivation of the blood flow model where the pressure term is given by (1.4). He also definitely pointed out that the viscous damping term on the right-hand side of the momentum equation (1.1)₂ is one order of magnitude smaller than the rest of the system. In other words, the damping term in the blood flow model has little effect in practical application. From this perspective, Čanić established a global existence theorem of the general 2×2 hyperbolic conservation law system and employed it to study the global existence of solution and shock formation for the blood flow model without viscous damping term. Finally, some numerical simulations were used to verify that the analysis of the first shock formation based on the zero viscous damping term provides a good estimate for the first shock formation in (1.1). The authors in [17] studied the Cauchy problem of the equations (1.1) and investigated the influence of the damping term $k_f m/A$ on the solution. Recently, Li-Zhao in [18] studied the initial-boundary value problem on bounded domains for the blood flow model with pressure (1.4) and showed that the L^∞ entropy weak solution exists globally in time when the initial value are large. Moreover, they also proved that as time goes to infinity, the entropy solution converges to a constant equilibrium state exponentially. Later Li-Zhao in [19] studied the same type of asymptotic states of smooth solutions with smooth enough initial data close to a constant equilibrium state.

Motivated by the papers [23] and [5], we will consider the model (1.1) with a more general pressure law than that in (1.3), namely, the pressure P is expressed by (1.2) where $\alpha_1 > 0$ and $\alpha_2 > 0$. On the other hand, the results in [5] obtained using numerical simulations are in accordance with the non-dimensional analysis which reveals that the viscous damping term is of one order of magnitude smaller than the remaining terms of the system. Based on Čanić's observation, we neglect the viscous damping term $k_f m/A$ and mainly investigate the influence of the viscoelastic term on the solution. To this aim, we reuse the variable $u = m/A$, and the model (1.1) can be written as follows:

$$\begin{cases} A_t + (Au)_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ (Au)_t + (Au^2)_x + p(A)_x = -\lambda A(\sqrt{A})_{xt}. \end{cases} \quad (1.5)$$

Here $p(A) = \kappa A^\gamma$ for $\kappa = (\alpha_1 G_0) / ((\alpha_1 + 2)\rho A_r^{\alpha_1/2})$ and $\gamma = (\alpha_1/2) + 1$; $\lambda = (\alpha_2 \iota) / (A_r \rho) > 0$. Without loss of generality, we assume $\kappa = 1$. We are interested in the large-time behavior of solution for the blood flow model (1.5) without vacuum of far field cross-sectional area. It is more convenient to use the Lagrangian coordinates to explore this problem. Therefore, we

introduce the Lagrangian coordinate transformation as follows:

$$(x, t) \rightarrow (y, \tau) : \quad y = \int_{\bar{x}(t)}^x A(z, t) dz, \quad \tau = t,$$

where $\bar{x}(t)$ satisfies the following integral curve:

$$\begin{cases} \frac{d\bar{x}(t)}{dt} = u(\bar{x}(t), t), \\ \bar{x}(0) = \bar{x}_0. \end{cases}$$

We still denote the Lagrangian coordinates (y, τ) by (x, t) for simplicity of notation and introduce a new variable $v = 1/A$. Then (1.5) can be transformed in Lagrangian coordinates as:

$$\begin{cases} v_t - u_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u_t + p(v)_x = \frac{\lambda}{2v} \left(v^{-\frac{1}{2}} \left(\frac{u}{v} \right)_x \right)_x, \end{cases} \quad (1.6)$$

where $p(v) = v^{-\gamma}$ for $\gamma > 1$. We will study the Cauchy problem of the blood flow model (1.6). The initial data is given by

$$(v, u)(x, 0) = (v_0, u_0)(x) \rightarrow (v_{\pm}, u_{\pm}) \quad \text{as } x \rightarrow \pm\infty, \quad (1.7)$$

where $\inf_{x \in \mathbb{R}} v_0(x) > 0$ and $v_{\pm} > 0$. To the best of our knowledge, there are few results about the large-time behavior of solutions towards some non-constant states, especially wave patterns for the blood flow model. In this paper, we only focus on the asymptotical stability of rarefaction wave to the Cauchy problem (1.6)–(1.7) and will give a explicit answer for this meaningful problem. The main idea is to generalize some known results of the Navier-Stokes equations, particularly about the global existence and large-time behavior of classical solutions near hyperbolic elementary waves.

It is well known that the asymptotic behavior of solutions for the compressible Navier-Stokes equations are well characterized by the Riemann solutions for the corresponding hyperbolic part, i.e., the Euler system. And these basic Riemann solutions are dilation invariant solutions: shock wave, rarefaction wave, contact discontinuity and the linear combinations of above elementary waves (see [16, 25, 31]). Since the Euler system is an idealization when the dissipative effects are neglected, it is much more important to study the large-time asymptotic behavior of solutions for the corresponding viscous system (Navier-Stokes equations) towards the viscous versions of these elementary waves (see [9]). Indeed, there have been a lot of works on the asymptotic behaviors of solutions for the Navier-Stokes equations. For example, the stability results for the rarefaction wave can be found in [15, 22, 27, 26, 30]. The stability results for the shock wave can be found in [8, 20, 21]. And for the case of contact discontinuity, readers can see [10, 11, 13]. Moreover, we also refer to [9, 12, 14, 33] for the combination of two different kinds of wave patterns.

The asymptotic stability of elementary waves (rarefaction wave, shock wave and contact discontinuity) are especially important topics in the theory of PDEs in connection with fluid

dynamics, physiological flow, biology, chemistry and other natural sciences. Therefore it is meaningful and valuable to study the corresponding stability problems for the blood flow model. In the present paper, we are interested in the asymptotical stability of rarefaction wave to the Cauchy problem (1.6)–(1.7). Here, we briefly give some remarks on this problem and review some key analytical techniques. Before our comment, we firstly recall the classical p -system:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \lambda \left(\frac{u_x}{v} \right)_x, \end{cases}$$

where pressure p is a given smooth function of specific volume v satisfying $p'(v) < 0$ and $p''(v) > 0$. Compared with the result of [26] for the p -system, the nonlinear stability analysis of rarefaction wave for the blood flow model (1.6) is more complicated. The main difficulty lies in the appearance of the dissipative term $\lambda(v^{-\frac{1}{2}}(u/v)_x)_x/(2v)$ in (1.6)₂, which consists of the nonlinear terms including second-order derivative of v with respect to the spatial variable x .

The first trouble term we suffered in the zero-order estimate is $\int_0^t \int_{\mathbb{R}} \lambda v^{-\frac{7}{2}} \bar{u} \varphi_x \psi_x / 2 \, dx d\tau$. Indeed, when two spatial derivatives in $\lambda(v^{-\frac{1}{2}}(u/v)_x)_x/(2v)$ both act on the same v , it will appear the term $-\lambda v^{-\frac{7}{2}} u v_{xx} / 2$. Multiplying this term by ψ and calculating integration, one can obtain the term $\int_0^t \int_{\mathbb{R}} \lambda v^{-\frac{7}{2}} \bar{u} \varphi_x \psi_x / 2 \, dx d\tau$ (see (2.12)). In order to control this nonlinear bad term by the time-space integrable good term of ψ_x and φ_x , we require a technical condition that the upper bound of $|\bar{u}|$ (i.e., $\max\{|u_{\pm}|\}$) is suitably small. This is an important point in the zero-order estimate. One can see (2.12) and (2.19) for details simultaneity. So far it is unclear how to remove such restriction on the stability analysis of the rarefaction wave for the blood flow model.

Secondly, we obtain the higher estimates (2.2) of φ_x and (2.3) of ψ_x , which is similar to ones for the Cauchy problem of p -system but has more difficulties in the proof due to the appearance of the strong nonlinearity of v . For example, we will encounter some trouble terms like $\int_0^t \int_{\mathbb{R}} |\varphi_x|^3 \, dx d\tau$ in (2.24) and $\int_0^t \int_{\mathbb{R}} \varphi_x^2 |\psi_{xx}| \, dx d\tau$ in (2.33). To deal with these strong nonlinear terms we need the smallness of $\|\varphi_x\|_{L^\infty}$, which just requires the *a priori* assumption that $\|\varphi\|_{H^2}$ is small. Comparing with the *a priori* assumption that $\|\varphi\|_{H^1}$ is small for p -system (see [26]), in the present manuscript we need to control the space integration term $\|\varphi_{xx}\|^2$ (see Lemma 2.4 for more details). Here we would like to mention that it can not improve the regularity of ψ to the same Sobolev space $L^\infty((0, t); H_x^2(\mathbb{R}))$ where φ lies. In fact, when deriving the space integration term $\|\psi_{xx}\|^2$, someone will control some strong nonlinear terms by employing the smallness on $\|\varphi\|_{H^3}$ instead of $\|\varphi\|_{H^2}$. In this way, the regularity of $\|\varphi\|_{H^3}$ will be one order higher than that of $\|\psi\|_{H^2}$.

The rest of the paper is organized as follows. In Subsections 1.2 and 1.3, we construct the smooth rarefaction wave and state our main result respectively. In Section 2, we construct a perturbation system and make *a priori* estimates to prove the main result.

Notation: Throughout this paper, we denote positive constants generally large (respectively,

generally small) independent of x and t by C (respectively, by c). And the character ‘ C ’ and ‘ c ’ may vary from line to line. $\|\cdot\|_{L^q}$ stands the L^q -norm on the Lebesgue space $L^q(\mathbb{R})$ ($1 \leq q \leq \infty$). For the sake of convenience, we always denote $\|\cdot\| = \|\cdot\|_{L^2}$. What’s more, H^k will be used to denote the usual Sobolev space $W^{k,2}(\mathbb{R})$ ($k \in \mathbb{Z}_+$) with respect to variable x .

1.2 Rarefaction wave and smooth approximate profile

Our purpose is to show that the rarefaction wave solutions for (1.6)–(1.7) are nonlinearly stable. For rarefaction wave, the term with second-order derivative in (1.6) decays faster than the corresponding terms with first-order derivatives. Therefore system (1.6) with the far field constant states of initial data (1.7) may be replaced, time-asymptotically for rarefaction wave, by the corresponding hyperbolic system with following Riemann initial data:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0, \end{cases} \quad (1.8)$$

$$(v, u)(x, 0) = (v_0^r, u_0^r)(x) = \begin{cases} (v_-, u_-), & x < 0, \\ (v_+, u_+), & x > 0. \end{cases} \quad (1.9)$$

For any $(v_-, u_-) \in \mathbb{R}_+ \times \mathbb{R}$, the 1-rarefaction curve $R_1(v_-, u_-)$ corresponds to the integral curve of the first eigenvalue $\lambda_1 = -\sqrt{-p'(y)}$, and is defined by

$$R_1(v_-, u_-) = \left\{ (v, u) \in \mathbb{R}_+ \times \mathbb{R} \left| \begin{array}{l} u = u_- + \int_{v_-}^v \sqrt{-p'(y)} dy, \\ 0 < v_- < v, \quad u_- < u \end{array} \right. \right\}.$$

The 2-rarefaction curve $R_2(v_-, u_-)$ can be defined in the same way from the second eigenvalue $\lambda_2 = \sqrt{-p'(y)}$. One can see [31] for more details. In this paper, we only consider the 1-rarefaction wave solution, and the case for 2-rarefaction wave can be treated similarly. Hence the constant states (v_{\pm}, u_{\pm}) should satisfy the restriction condition

$$u_+ = u_- + \int_{v_-}^{v_+} \sqrt{-p'(y)} dy, \quad 0 < v_- < v_+. \quad (1.10)$$

And the Riemann problem (1.8)–(1.9) admits a weak solution of the form $(v^r, u^r)(x/t)$ as

$$\begin{cases} u^r\left(\frac{x}{t}\right) = u_- + \int_{v_-}^{v^r\left(\frac{x}{t}\right)} \sqrt{-p'(y)} dy, \\ \lambda_1\left(v^r\left(\frac{x}{t}\right), u^r\left(\frac{x}{t}\right)\right) = \begin{cases} \lambda_1(v_-, u_-), & x < \lambda_1(v_-, u_-)t, \\ \frac{x}{t}, & \lambda_1(v_-, u_-)t \leq x \leq \lambda_1(v_+, u_+)t, \\ \lambda_1(v_+, u_+), & \lambda_1(v_+, u_+)t < x. \end{cases} \end{cases}$$

Since the rarefaction wave $(v^r, u^r)(x/t)$ is not smooth enough, it is convenient to construct its smooth approximation $(\bar{v}, \bar{u})(x, t)$ called the smooth rarefaction wave as follows (see [26]):

$$\begin{cases} \lambda_1(\bar{v}, \bar{u}) = \omega(x, 1+t), & \lambda_1(v_\pm, u_\pm) = \omega_\pm, \\ \bar{u} = u_- + \int_{v_-}^{\bar{v}} \sqrt{-p'(y)} dy, \end{cases} \quad (1.11)$$

where $\omega(x, t)$ is the solution of the following Cauchy problem for the Burgers equation

$$\begin{cases} \omega_t + \omega\omega_x = 0, \\ \omega(x, 0) = \frac{\omega_+ + \omega_-}{2} + \frac{\omega_+ - \omega_-}{2} \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}}. \end{cases} \quad (1.12)$$

And $\omega(x, t)$ have the following properties (see [26]):

Lemma 1.1. *Set $\delta_r = \omega_+ - \omega_-$ for $\omega_- < \omega_+$. Then the Cauchy problem (1.12) has a unique smooth global solution $\omega(x, t)$ satisfying*

- (1) $\omega_x > 0$, $\omega_- < \omega(x, t) < \omega_+$ for $x \in \mathbb{R}$ and $t \geq 0$.
- (2) For any $1 \leq q \leq +\infty$, there exists a constant C depending only on q such that for any $t > 0$,

$$\|\omega_x\|_{L^q} \leq C \min\{\delta_r, \delta_r^{\frac{1}{q}} t^{-1+\frac{1}{q}}\},$$

$$\|(\omega_{xx}, \omega_{xxx})\|_{L^q} \leq C \min\{\delta_r, t^{-1}\}.$$

- (3) $\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |w(x, t) - w^r(x/t)| = 0$, where $w^r(x/t)$ is the solution of the Burgers equation with Riemann initial data $w(x, 0) = w_-$, if $x < 0$ and $w(x, 0) = w_+$, if $x > 0$.

It is easy to check that (\bar{v}, \bar{u}) satisfies the system (1.8). Hence by the Lemma 1.1 and (1.11), we can obtain that (\bar{v}, \bar{u}) satisfies the following Lemma (cf. [26]).

Lemma 1.2. *Let $\delta = |v_+ - v_-| + |u_+ - u_-|$ be the wave strength. Then the smooth approximate profile $(\bar{v}, \bar{u})(x, t)$ which is defined by (1.11) satisfies the following properties:*

- (1) $0 < v_- < \bar{v}(x, t) < v_+$, $u_- < \bar{u}(x, t) < u_+$ for any $x \in \mathbb{R}$ and $t > 0$. And there exists a constant C such that

$$\bar{u}_x > 0 \quad \text{and} \quad |\bar{v}_x| \leq C \bar{u}_x.$$

- (2) For any $1 \leq q \leq +\infty$, there exists a constant C which only depends on q such that for any $t > 0$ and $0 \leq \alpha \leq 1$:

$$\|(\bar{v}_x, \bar{u}_x)\|_{L^q} \leq C \min\{\delta, \delta^{\frac{1}{q}} (1+t)^{-1+\frac{1}{q}}\},$$

$$\|(\bar{v}_{xx}, \bar{u}_{xx})\|_{L^q} \leq C \min\{\delta, (1+t)^{-1}\} \leq C \delta^\alpha (1+t)^{-(1-\alpha)},$$

$$\|(\bar{v}_{xxx}, \bar{u}_{xxx})\|_{L^q} \leq C \min\{\delta, (1+t)^{-1}\} \leq C \delta^\alpha (1+t)^{-(1-\alpha)}.$$

- (3)

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |(\bar{v}, \bar{u})(x, t) - (v^r, u^r)(x/t)| = 0. \quad (1.13)$$

1.3 Main result

The main purpose of this paper is to show that the solution (v, u) of the Cauchy problem (1.6)–(1.7) tends toward the rarefaction wave (v^r, u^r) constructed in Subsection 1.2, provided the initial data $(v_0, u_0)(x)$ is suitably close to $(v_0^r, u_0^r)(x)$. The main result is stated in the following theorem.

Theorem 1.1. *Suppose the initial data and the far-field data satisfy (1.7) and (1.10). There exist sufficiently small positive constants δ_1 , \bar{C} and ε which are independent of T , such that if $0 < \delta < \delta_1$, $0 < \max\{|u_\pm|\} < \bar{C}$ and the initial data satisfies*

$$\|v_0(x) - \bar{v}(x, 0)\|_{H^2} + \|u_0(x) - \bar{u}(x, 0)\|_{H^1} \leq \varepsilon,$$

then the Cauchy problem (1.6)–(1.7) exists a unique time-global solution $(v, u)(x, t)$. Moreover, the solution $(v, u)(x, t)$ tends time-asymptotically to the rarefaction wave in the sense that

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |(v, u)(x, t) - (v^r, u^r)(x, t)| = 0. \quad (1.14)$$

2 Uniform a priori estimates

We next use the elementary energy method to prove the Theorem 1.1. Define the perturbation as

$$\varphi = v - \bar{v}, \quad \psi = u - \bar{u}.$$

Then we can easily verify that (φ, ψ) satisfies

$$\begin{cases} \varphi_t - \psi_x = 0, \\ \psi_t + (p(v) - p(\bar{v}))_x = \frac{\lambda}{2v} \left(v^{-\frac{1}{2}} \left(\frac{u}{v} \right)_x \right)_x, \\ (\varphi_0, \psi_0)(x) := (\varphi, \psi)(x, 0) = (v_0(x) - \bar{v}(x, 0), u_0(x) - \bar{u}(x, 0)). \end{cases} \quad (2.1)$$

For $0 < T < +\infty$, define the function space $X(T)$ as

$$X(T) = \left\{ (\varphi, \psi) \left| \begin{array}{l} \varphi \in L^\infty((0, T); H^2(\mathbb{R})), \psi \in L^\infty((0, T); H^1(\mathbb{R})) \\ (\varphi_x, \psi_x) \in L^2((0, T); H^1(\mathbb{R})) \end{array} \right. \right\}.$$

The global existence of solutions to the Cauchy problem (2.1) can be obtained by the classical continuation argument based on the local existence of solutions and *a priori* estimates. And the local existence can be established by the standard iteration argument. In order to prove Theorem 1.1 for brevity, we only devote ourselves to establishing the global-in-time *a priori* estimates as follows.

Proposition 2.1. *Suppose all the conditions in Theorem 1.1 hold. Let $(\varphi, \psi) \in X(T)$ be a solution to the Cauchy problem (2.1) on $0 < t < T$ for $T > 0$. There exist some small positive constants \bar{C} , δ_0 and ε_0 such that if $\max\{|u_\pm|\} < \bar{C}$, $\delta < \delta_0$ and*

$$\sup_{0 \leq t \leq T} (\|\varphi\|_{H^2} + \|\psi\|_{H^1}) \leq \varepsilon_0, \quad (2.2)$$

then $(\varphi, \psi)(x, t)$ satisfies

$$\sup_{0 \leq t \leq T} (\|\varphi\|_{H^2}^2 + \|\psi\|_{H^1}^2) + \int_0^T \|(\varphi_x, \psi_x)\|_{H^1}^2 d\tau \leq C (\|\varphi_0\|_{H^2}^2 + \|\psi_0\|_{H^1}^2) + C\delta^{\frac{1}{6}}. \quad (2.3)$$

By using the *a priori* assumption (2.2) and the following Sobolev inequality

$$\|f\|_{L^\infty} \leq \sqrt{2} \|f\|^{\frac{1}{2}} \|f_x\|^{\frac{1}{2}}, \quad \text{for } f(x) \in H^1(\mathbb{R}), \quad (2.4)$$

we can directly get

$$\|(\varphi, \varphi_x, \psi)\|_{L^\infty} \leq \sqrt{2}\varepsilon_0, \quad (2.5)$$

which will be frequently used in the sequel.

Once Proposition 2.1 is proved, someone can close the *a priori* assumption (2.2). Moreover, for $0 < \max\{|u_\pm|\} < \bar{C}$, the estimate (2.3) and the equations (2.1) imply that

$$\int_0^{+\infty} \left[\|(\varphi_x, \psi_x)(t)\|^2 + \left| \frac{d}{dt} \|(\varphi_x, \psi_x)(t)\|^2 \right| \right] dt < +\infty,$$

which easily leads to

$$\lim_{t \rightarrow +\infty} \|(\varphi_x, \psi_x)(t)\| = 0.$$

Then by using the Sobolev inequality (2.4) and the estimate (2.3), together with (1.13), we can state the asymptotic behavior (1.14) of the solution to the problem (1.6)–(1.7).

Proposition 2.1 can be proved by the subsequent four lemmas. Here we first give the zero-order energy estimates.

Lemma 2.1. *Suppose all the conditions in Proposition 2.1 are true and denote $\xi := \max\{|u_\pm|\}$. Then for all $0 < t < T$, there exists a constant \bar{C} depending only on v_\pm , λ and γ such that if $0 < \xi < \bar{C}$, the following energy estimate holds:*

$$\begin{aligned} & \|(\varphi, \psi)\|^2 + \int_0^t \int_{\mathbb{R}} \bar{u}_x \varphi^2 dx d\tau + \int_0^t \|\psi_x\|^2 d\tau \\ & \leq C\delta^{\frac{1}{6}} + C\|(\varphi_0, \psi_0)\|^2 + C(\varepsilon_0^{\frac{1}{2}} + \xi) \int_0^t \|\varphi_x\|^2 d\tau. \end{aligned} \quad (2.6)$$

Proof. Inspired by the work of the p -system in [26], we define the relative entropy function:

$$\eta(x, t) = \frac{1}{2} \psi^2 - \int_{\bar{v}}^v p(s) ds + p(\bar{v}) \varphi.$$

Then taking the derivative of $\eta(x, t)$ with respect to t , and integrating the resulting equality with respect to x on \mathbb{R} gives

$$\frac{d}{dt} \int_{\mathbb{R}} \eta(x, t) dx + \int_{\mathbb{R}} \bar{u}_x [p(v) - p(\bar{v}) - p'(\bar{v}) \varphi] dx = \int_{\mathbb{R}} \frac{\lambda}{2v} \left(v^{-\frac{1}{2}} \left(\frac{u}{v} \right)_x \right)_x \psi dx. \quad (2.7)$$

In order to get the time-space integrable good term of ψ_x , we expand the last term of (2.7) as:

$$\int_{\mathbb{R}} \frac{\lambda}{2v} \left(v^{-\frac{1}{2}} \left(\frac{u}{v} \right)_x \right)_x \psi dx = \int_{\mathbb{R}} \frac{\lambda}{2v} \left(v^{-\frac{1}{2}} \left(\frac{u_x}{v} - \frac{uv_x}{v^2} \right) \right)_x \psi dx$$

$$\begin{aligned}
&= \int_{\mathbb{R}} \left[\frac{\lambda}{2} v^{-\frac{3}{2}} \left(\frac{u_x}{v} \right)_x - \frac{3}{4} \lambda v^{-\frac{7}{2}} u_x v_x - \frac{\lambda}{2} v^{-\frac{7}{2}} u v_{xx} + \frac{5}{4} \lambda v^{-\frac{9}{2}} u v_x^2 \right] \psi \, dx \\
&= \int_{\mathbb{R}} \frac{\lambda}{2} v^{-\frac{3}{2}} \left(\frac{\psi_x}{v} \right)_x \psi \, dx + \int_{\mathbb{R}} \left[\frac{\lambda}{2} v^{-\frac{3}{2}} \left(\frac{\bar{u}_x}{v} \right)_x - \frac{3}{4} \lambda v^{-\frac{7}{2}} u_x v_x - \frac{\lambda}{2} v^{-\frac{7}{2}} u v_{xx} + \frac{5}{4} \lambda v^{-\frac{9}{2}} u v_x^2 \right] \psi \, dx \\
&= -\frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \psi_x^2 \, dx + \frac{3}{4} \lambda \int_{\mathbb{R}} v^{-\frac{7}{2}} \psi \psi_x v_x \, dx + \int_{\mathbb{R}} \frac{\lambda}{2} v^{-\frac{3}{2}} \left(\frac{\bar{u}_x}{v} \right)_x \psi \, dx \\
&\quad - \int_{\mathbb{R}} \frac{3}{4} \lambda v^{-\frac{7}{2}} u_x v_x \psi \, dx - \int_{\mathbb{R}} \frac{\lambda}{2} v^{-\frac{7}{2}} u v_{xx} \psi \, dx + \int_{\mathbb{R}} \frac{5}{4} \lambda v^{-\frac{9}{2}} u v_x^2 \psi \, dx \\
&=: -\frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \psi_x^2 \, dx + \sum_{i=1}^5 I_i. \tag{2.8}
\end{aligned}$$

By putting (2.8) into (2.7), one can get

$$\frac{d}{dt} \int_{\mathbb{R}} \eta \, dx + \int_{\mathbb{R}} \bar{u}_x [p(v) - p(\bar{v}) - p'(\bar{v})\varphi] \, dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \psi_x^2 \, dx = \sum_{i=1}^5 I_i. \tag{2.9}$$

Next we estimate the terms on the right-hand side of (2.9) one by one. By applying the Sobolev inequality (2.4), the *a priori* assumption (2.2) and the decay property of (\bar{v}_x, \bar{u}_x) in Lemma 1.2, together with the Hölder and the Cauchy inequalities, one can deduce

$$\begin{aligned}
I_1 + I_3 &= \frac{3}{4} \lambda \int_{\mathbb{R}} v^{-\frac{7}{2}} \psi \psi_x v_x \, dx - \frac{3}{4} \lambda \int_{\mathbb{R}} v^{-\frac{7}{2}} u_x v_x \psi \, dx = -\frac{3}{4} \lambda \int_{\mathbb{R}} v^{-\frac{7}{2}} \psi v_x \bar{u}_x \, dx \\
&\leq C \int_{\mathbb{R}} (|\psi \varphi_x \bar{u}_x| + |\psi \bar{v}_x \bar{u}_x|) \, dx \\
&\leq C \|\psi\|^{\frac{1}{2}} \|\psi_x\|^{\frac{1}{2}} \|\varphi_x\| \|\bar{u}_x\| + C \|\psi\|^{\frac{1}{2}} \|\psi_x\|^{\frac{1}{2}} \|\bar{v}_x\| \|\bar{u}_x\| \\
&\leq C \varepsilon_0^{\frac{1}{2}} (\|\psi_x\|^2 + \|\varphi_x\|^2) + C \delta^{\frac{4}{3}} (1+t)^{-\frac{4}{3}}, \tag{2.10}
\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{3}{2}} \left(\frac{\bar{u}_x}{v} \right)_x \psi \, dx = \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{3}{2}} (v^{-1} \bar{u}_{xx} - v^{-2} \bar{u}_x v_x) \psi \, dx \\
&= \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \bar{u}_{xx} \psi \, dx - \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} \bar{u}_x (\varphi_x + \bar{v}_x) \psi \, dx \\
&\leq C \int_{\mathbb{R}} (|\bar{u}_{xx} \psi| + |\psi \bar{u}_x \varphi_x| + |\psi \bar{u}_x \bar{v}_x|) \, dx \\
&\leq C \|\psi\|^{\frac{1}{2}} \|\psi_x\|^{\frac{1}{2}} (\|\bar{u}_{xx}\|_{L^1} + \|\varphi_x\| \|\bar{u}_x\| + \|\bar{u}_x\| \|\bar{v}_x\|) \\
&\leq C \varepsilon_0^{\frac{1}{2}} (\|\psi_x\|^2 + \|\varphi_x\|^2) + C \delta^{\frac{1}{6}} (1+t)^{-\frac{7}{6}}, \tag{2.11}
\end{aligned}$$

$$\begin{aligned}
I_4 &= -\frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} u v_{xx} \psi \, dx = -\frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} u \bar{v}_{xx} \psi \, dx - \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} u \varphi_{xx} \psi \, dx \\
&= -\frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} u \bar{v}_{xx} \psi \, dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} u \varphi_x \psi_x \, dx
\end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} u_x \varphi_x \psi \, dx - \frac{7}{4} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} u \varphi_x \psi v_x \, dx \\
& = -\frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} (\psi + \bar{u}) \bar{v}_{xx} \psi \, dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} (\psi + \bar{u}) \varphi_x \psi_x \, dx \\
& \quad + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} (\psi_x + \bar{u}_x) \varphi_x \psi \, dx - \frac{7}{4} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} (\psi + \bar{u}) \varphi_x \psi (\varphi_x + \bar{v}_x) \, dx \\
& \leq C \int_{\mathbb{R}} (|\psi^2 \bar{v}_{xx}| + |\bar{u} \psi \bar{v}_{xx}| + |\psi \varphi_x \psi_x| + |\bar{u} \varphi_x \psi_x| + |\bar{u}_x \varphi_x \psi| \\
& \quad + |\psi^2 \bar{v}_x \varphi_x| + |\psi^2 \varphi_x^2| + |\bar{u} \bar{v}_x \varphi_x \psi| + |\bar{u} \varphi_x^2 \psi|) \, dx \\
& \leq C(\varepsilon_0^{\frac{1}{2}} + \xi) (\|\varphi_x\|^2 + \|\psi_x\|^2) + C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
I_5 & = \frac{5}{4} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} u v_x^2 \psi \, dx = \frac{5}{4} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} (\psi + \bar{u}) (\varphi_x + \bar{v}_x)^2 \psi \, dx \\
& \leq \int_{\mathbb{R}} (|\psi^2 \varphi_x^2| + |\psi^2 \bar{v}_x^2| + |\bar{u} \varphi_x^2 \psi| + |\bar{u} \bar{v}_x^2 \psi|) \, dx \\
& \leq C\varepsilon_0^{\frac{1}{2}} (\|\psi_x\|^2 + \|\varphi_x\|^2) + C\delta^{\frac{4}{3}}(1+t)^{-\frac{4}{3}},
\end{aligned} \tag{2.13}$$

where $\xi = \max\{|u_{\pm}|\}$ is the upper bound of $|\bar{u}|$. By substituting the estimates (2.10)–(2.13) into (2.9) and first taking ξ then ε_0 , δ suitably small, one can get

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}} \eta \, dx + \int_{\mathbb{R}} \bar{u}_x [p(v) - p(\bar{v}) - p'(\bar{v})\varphi] \, dx + c\|\psi_x\|^2 \\
& \leq C(\varepsilon_0^{\frac{1}{2}} + \xi) \|\varphi_x\|^2 + C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}.
\end{aligned} \tag{2.14}$$

In addition, the Taylor expansion implies the following equivalence relation:

$$\eta(x, t) \sim (\varphi^2 + \psi^2), \quad p(v) - p(\bar{v}) - p'(\bar{v})\varphi \sim \varphi^2. \tag{2.15}$$

Thus after integrating the inequality (2.14) with respect to t and employing (2.15), one can arrive at (2.6). This completes the proof of Lemma 2.1. \square

Lemma 2.2. *Suppose all the conditions in Proposition 2.1 are true. Then for all $0 < t < T$, there exists a constant \bar{C} depending only on v_{\pm} , λ and γ such that if $0 < \xi < \bar{C}$, the following energy estimate holds:*

$$\|\varphi_x\|^2 + \int_0^t \|\varphi_x\|^2 \, d\tau + \int_0^t \int_{\mathbb{R}} \bar{u}_x \varphi_x^2 \, dx \, d\tau \leq C\delta^{\frac{1}{6}} + C(\|\varphi_0\|_{H^1}^2 + \|\psi_0\|^2). \tag{2.16}$$

Proof. Motivated by the work of the p -system in [26], we firstly rewrite the form of the equation (2.1)₂. Due to

$$\left(\frac{u_x}{v}\right)_x = \left(\frac{v_t}{v}\right)_x = (\ln v)_{tx} = \left(\frac{v_x}{v}\right)_t = \left(\frac{\varphi_x}{v}\right)_t + \left(\frac{\bar{v}_x}{v}\right)_t,$$

and recalling (2.8), one has

$$\begin{aligned} \frac{\lambda}{2v} \left(v^{-\frac{1}{2}} \left(\frac{u}{v} \right)_x \right)_x &= \frac{\lambda}{2} v^{-\frac{3}{2}} \left(\frac{u_x}{v} \right)_x + \underbrace{\left(-\frac{3}{4} \lambda v^{-\frac{7}{2}} u_x v_x - \frac{\lambda}{2} v^{-\frac{7}{2}} u v_{xx} + \frac{5}{4} \lambda v^{-\frac{9}{2}} u v_x^2 \right)}_J \\ &= \frac{\lambda}{2} v^{-\frac{3}{2}} \left(\frac{\varphi_x}{v} \right)_t + \frac{\lambda}{2} v^{-\frac{3}{2}} \left(\frac{\bar{v}_x}{v} \right)_t + J. \end{aligned}$$

Thus the equation (2.1)₂ can be rewritten as

$$\frac{\lambda}{2} v^{-\frac{3}{2}} \left(\frac{\varphi_x}{v} \right)_t - p'(v) \varphi_x = \psi_t + (p'(v) - p'(\bar{v})) \bar{v}_x - \frac{\lambda}{2} v^{-\frac{3}{2}} \left(\frac{\bar{v}_x}{v} \right)_t - J. \quad (2.17)$$

Multiplying the equation (2.17) by φ_x/v and integrating it with respect to x on \mathbb{R} , one can get

$$\begin{aligned} &\frac{\lambda}{4} \frac{d}{dt} \int_{\mathbb{R}} v^{-\frac{7}{2}} \varphi_x^2 dx - \int_{\mathbb{R}} p'(v) v^{-1} \varphi_x^2 dx \\ &= \int_{\mathbb{R}} v^{-1} \psi_t \varphi_x dx + \int_{\mathbb{R}} (p'(v) - p'(\bar{v})) \bar{v}_x v^{-1} \varphi_x dx - \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{3}{2}} \left(\frac{\bar{v}_x}{v} \right)_t \frac{\varphi_x}{v} dx \\ &\quad + \int_{\mathbb{R}} \frac{3}{4} \lambda v^{-\frac{9}{2}} u_x v_x \varphi_x dx + \int_{\mathbb{R}} \frac{\lambda}{2} v^{-\frac{9}{2}} u v_{xx} \varphi_x dx - \int_{\mathbb{R}} \frac{5}{4} \lambda v^{-\frac{11}{2}} u v_x^2 \varphi_x dx \\ &\quad - \frac{3}{8} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} \varphi_x^2 u_x dx =: I_6 + I_7 + \dots + I_{12}. \end{aligned} \quad (2.18)$$

Next we estimate I_i ($6 \leq i \leq 12$) in the equation (2.18) term by term. Similar to the estimation of the right-hand side terms in (2.9), by applying the decay properties in Lemma 1.2, the Sobolev inequality and (2.5), one can obtain that:

$$\begin{aligned} I_6 &= \int_{\mathbb{R}} v^{-1} \psi_t \varphi_x dx = \int_{\mathbb{R}} (v^{-1} \psi \varphi_x)_t dx - \int_{\mathbb{R}} v^{-1} \psi \varphi_{xt} dx + \int_{\mathbb{R}} v^{-2} \psi \varphi_x v_t dx \\ &= \int_{\mathbb{R}} (v^{-1} \psi \varphi_x)_t dx - \int_{\mathbb{R}} v^{-1} \psi \psi_{xx} dx + \int_{\mathbb{R}} v^{-2} \psi \varphi_x u_x dx \\ &= \int_{\mathbb{R}} (v^{-1} \psi \varphi_x)_t dx + \int_{\mathbb{R}} v^{-1} \psi_x^2 dx - \int_{\mathbb{R}} v^{-2} v_x \psi \psi_x dx + \int_{\mathbb{R}} v^{-2} \psi \varphi_x u_x dx \\ &\leq \int_{\mathbb{R}} (v^{-1} \psi \varphi_x)_t dx + C \|\psi_x\|^2 + C \varepsilon_0^{\frac{1}{2}} \|\varphi_x\|^2 + C \delta^2 (1+t)^{-2}, \end{aligned} \quad (2.19)$$

$$\begin{aligned} I_7 &= \int_{\mathbb{R}} (p'(v) - p'(\bar{v})) \bar{v}_x v^{-1} \varphi_x dx \leq C \int_{\mathbb{R}} |\varphi \bar{v}_x \varphi_x| dx \\ &\leq C \|\varphi\|^{\frac{1}{2}} \|\varphi_x\|^{\frac{3}{2}} \|\bar{v}_x\| \leq C \varepsilon_0^{\frac{1}{2}} \|\varphi_x\|^2 + C \delta^2 (1+t)^{-2}, \end{aligned} \quad (2.20)$$

$$\begin{aligned} I_8 &= -\frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{3}{2}} \left(\frac{\bar{v}_x}{v} \right)_t \frac{\varphi_x}{v} dx = -\frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \varphi_x \left(\frac{\bar{v}_{xt}}{v} - v^{-2} \bar{v}_x v_t \right) dx \\ &= -\frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \varphi_x (v^{-1} \bar{u}_{xx} - v^{-2} \bar{v}_x u_x) dx \\ &\leq C \delta^{\frac{1}{4}} (\|\varphi_x\|^2 + \|\psi_x\|^2) + C \delta^{\frac{1}{4}} (1+t)^{-\frac{3}{2}}, \end{aligned} \quad (2.21)$$

$$\begin{aligned}
I_9 &= \frac{3}{4}\lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} u_x v_x \varphi_x \, dx = \frac{3}{4}\lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} (\psi_x + \bar{u}_x)(\varphi_x + \bar{v}_x) \varphi_x \, dx \\
&\leq C(\varepsilon_0 + \delta)(\|\psi_x\|^2 + \|\varphi_x\|^2) + C\delta^{\frac{1}{2}}(\|\varphi_x\|^2 + (1+t)^{-3}),
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
I_{10} &= \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{9}{2}} u v_{xx} \varphi_x \, dx = \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{9}{2}} u \varphi_{xx} \varphi_x \, dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{9}{2}} u \bar{v}_{xx} \varphi_x \, dx \\
&= -\frac{\lambda}{4} \int_{\mathbb{R}} v^{-\frac{9}{2}} u_x \varphi_x^2 \, dx + \frac{9}{8}\lambda \int_{\mathbb{R}} v^{-\frac{11}{2}} v_x u \varphi_x^2 \, dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{9}{2}} u \bar{v}_{xx} \varphi_x \, dx \\
&\leq C(\varepsilon_0 + \delta)(\|\psi_x\|^2 + \|\varphi_x\|^2) + C\delta^{\frac{1}{4}}(\|\varphi_x\|^2 + (1+t)^{-\frac{3}{2}}),
\end{aligned} \tag{2.23}$$

$$I_{11} = -\frac{5}{4}\lambda \int_{\mathbb{R}} v^{-\frac{11}{2}} u v_x^2 \varphi_x \, dx \leq C(\varepsilon_0 + \delta^{\frac{1}{2}})\|\varphi_x\|^2 + C\delta^{\frac{1}{2}}(1+t)^{-3}, \tag{2.24}$$

and

$$\begin{aligned}
I_{12} &= -\frac{3}{8}\lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} \varphi_x^2 u_x \, dx = -\frac{3}{8}\lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} \varphi_x^2 \psi_x \, dx - \frac{3}{8}\lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} \varphi_x^2 \bar{u}_x \, dx \\
&\leq C\varepsilon_0 (\|\psi_x\|^2 + \|\varphi_x\|^2) - \frac{3}{8}\lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} \varphi_x^2 \bar{u}_x \, dx.
\end{aligned} \tag{2.25}$$

By plugging (2.19)–(2.25) into (2.18), then integrating the resulting inequality with respect to t and employing (2.6), finally choosing ξ , ε_0 and δ small enough, one can obtain (2.16).

The proof of Lemma 2.2 is finished. \square

Lemma 2.3. *Suppose all the conditions in Proposition 2.1 are true. Then for all $0 < t < T$, there exists a constant \bar{C} depending only on v_{\pm} , λ and γ such that if $0 < \xi < \bar{C}$, the following energy estimate holds:*

$$\|\psi_x\|^2 + \int_0^t \|\psi_{xx}\|^2 \, d\tau \leq C\delta^{\frac{1}{6}} + C\|(\varphi_0, \psi_0)\|_{H^1}^2 + C(\varepsilon_0 + \xi) \int_0^t \|\varphi_{xx}\|^2 \, d\tau. \tag{2.26}$$

Proof. Multiplying (2.1)₂ by $-\psi_{xx}$ and integrating the resulting equality with respect to x leads to

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \psi_x^2 \, dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \psi_{xx}^2 \, dx \\
&= \int_{\mathbb{R}} (p(v) - p(\bar{v}))_x \psi_{xx} \, dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} v_x \psi_x \psi_{xx} \, dx - \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{3}{2}} \left(\frac{\bar{u}_x}{v} \right)_x \psi_{xx} \, dx \\
&\quad + \frac{3}{4}\lambda \int_{\mathbb{R}} v^{-\frac{7}{2}} u_x v_x \psi_{xx} \, dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} u v_{xx} \psi_{xx} \, dx - \frac{5}{4}\lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} u v_x^2 \psi_{xx} \, dx \\
&=: I_{13} + I_{14} + \cdots + I_{18}.
\end{aligned} \tag{2.27}$$

Applying (2.5), the Sobolev inequality, the Hölder inequality, the Cauchy inequality with small parameter σ and integration by parts, together with Lemma 1.2 yields that

$$I_{13} = \int_{\mathbb{R}} (p(v) - p(\bar{v}))_x \psi_{xx} \, dx = \int_{\mathbb{R}} [p'(v) \varphi_x \psi_{xx} + (p'(v) - p'(\bar{v})) \bar{v}_x \psi_{xx}] \, dx$$

$$\begin{aligned}
&\leq C \int_{\mathbb{R}} (|\varphi_x \psi_{xx}| + |\varphi \bar{v}_x \psi_{xx}|) \, dx \\
&\leq C \|\varphi_x\| \|\psi_{xx}\| + C \|\varphi\|^{\frac{1}{2}} \|\varphi_x\|^{\frac{1}{2}} \|\bar{v}_x\| \|\psi_{xx}\| \\
&\leq C(\sigma + \varepsilon_0^{\frac{1}{2}}) \|\psi_{xx}\|^2 + C_\sigma \|\varphi_x\|^2 + C\delta^2(1+t)^{-2},
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
I_{14} &= \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} v_x \psi_x \psi_{xx} \, dx = \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} (\varphi_x + \bar{v}_x) \psi_x \psi_{xx} \, dx \\
&\leq C(\varepsilon_0 + \delta) (\|\psi_x\|^2 + \|\psi_{xx}\|^2),
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
I_{15} &= -\frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{3}{2}} \left(\frac{\bar{u}_x}{v} \right)_x \psi_{xx} \, dx \\
&= -\frac{\lambda}{2} \int_{\mathbb{R}} \left(v^{-\frac{5}{2}} \bar{u}_{xx} \psi_{xx} - v^{-\frac{7}{2}} \bar{u}_x v_x \psi_x \right) \, dx \\
&\leq C \int_{\mathbb{R}} (|\bar{u}_{xx} \psi_{xx}| + |\bar{u}_x \varphi_x \psi_{xx}| + |\bar{u}_x \bar{v}_x \psi_{xx}|) \, dx \\
&\leq C \|\bar{u}_{xx}\| \|\psi_{xx}\| + C \|\bar{u}_x\|_{L^\infty} \|\varphi_x\| \|\psi_{xx}\| + C \|\psi_{xx}\| \|\bar{u}_x\|_{L^4}^2 \\
&\leq C\delta^{\frac{1}{4}} (\|\varphi_x\|^2 + \|\psi_{xx}\|^2) + C\delta^{\frac{1}{4}} (1+t)^{-\frac{3}{2}},
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
I_{16} &= \frac{3}{4} \lambda \int_{\mathbb{R}} v^{-\frac{7}{2}} u_x v_x \psi_{xx} \, dx \\
&\leq C(\varepsilon_0 + \delta^{\frac{1}{2}}) (\|\varphi_x\|^2 + \|\psi_x\|^2 + \|\psi_{xx}\|^2) + C\delta^{\frac{1}{2}} (1+t)^{-3},
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
I_{17} &= \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{7}{2}} u v_{xx} \psi_{xx} \, dx \\
&\leq C(\varepsilon_0 + \xi) (\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2) + C\delta^{\frac{1}{4}} (\|\psi_{xx}\|^2 + (1+t)^{-3}),
\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}
I_{18} &= -\frac{5}{4} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} u v_x^2 \psi_{xx} \, dx \\
&\leq C\varepsilon_0 (\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2 + \|\varphi_x\|^2) + C\delta^{\frac{1}{2}} (\|\psi_{xx}\|^2 + (1+t)^{-3}),
\end{aligned} \tag{2.33}$$

where σ in (2.28) is a suitably small positive constant which is arising from the Cauchy inequality.

By substituting (2.28)–(2.33) into (2.27), then integrating the resulting inequality with respect to t and choosing first σ then ξ , ε_0 , δ suitably small; together with (2.6) and (2.16), one can reach (2.26). This completes the proof of this Lemma 2.3.

□

Lemma 2.4. *Suppose all the conditions in Proposition 2.1 are true. Then for all $0 < t < T$, there exists a constant \bar{C} depending only on v_{\pm} , λ and γ such that if $0 < \xi < \bar{C}$, the following energy estimate holds:*

$$\|\varphi_{xx}\|^2 + \int_0^t \|\varphi_{xx}\|^2 d\tau \leq C\delta^{\frac{1}{6}} + C(\|\varphi_0\|_{H^2}^2 + \|\psi_0\|_{H^1}^2). \quad (2.34)$$

Proof. Taking the derivative of the equation (2.17) with respect to x and multiplying it by φ_{xx}/v , then integrating the result with respect to x on \mathbb{R} , one can get

$$\begin{aligned} & \frac{\lambda}{4} \frac{d}{dt} \int_{\mathbb{R}} v^{-\frac{7}{2}} \varphi_{xx}^2 dx - \int_{\mathbb{R}} p'(v) v^{-1} \varphi_{xx}^2 dx \\ &= -\frac{3}{8} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} v_t \varphi_{xx}^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \left(\frac{\varphi_x v_x}{v^2} \right)_t \varphi_{xx} dx - \frac{3}{4} \lambda \int_{\mathbb{R}} v^{-\frac{7}{2}} v_x \left(\frac{\varphi_x}{v} \right)_t \varphi_{xx} dx \\ & - \frac{\lambda}{2} \int_{\mathbb{R}} \left[v^{-\frac{3}{2}} \left(\frac{\bar{v}_x}{v} \right)_t \right]_x \frac{\varphi_{xx}}{v} dx + \int_{\mathbb{R}} \left(\frac{3}{4} \lambda v^{-\frac{7}{2}} u_x v_x + \frac{\lambda}{2} v^{-\frac{7}{2}} u v_{xx} - \frac{5}{4} \lambda v^{-\frac{9}{2}} u v_x^2 \right)_x \frac{\varphi_{xx}}{v} dx \\ & + \int_{\mathbb{R}} v^{-1} \varphi_{xx} \psi_{xt} dx + \int_{\mathbb{R}} p'(v)_x \varphi_x \frac{\varphi_{xx}}{v} dx + \int_{\mathbb{R}} [(p'(v) - p'(\bar{v})) \bar{v}_x]_x \frac{\varphi_{xx}}{v} dx \\ & =: I_{19} + I_{20} + \cdots + I_{26}. \end{aligned} \quad (2.35)$$

Similar to the proof of previous lemmas, we will estimate I_i ($19 \leq i \leq 26$) term by term. Firstly, we have the following estimates:

$$\begin{aligned} I_{19} &= -\frac{3}{8} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} v_t \varphi_{xx}^2 dx = -\frac{3}{8} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} u_x \varphi_{xx}^2 dx \\ &\leq C\varepsilon_0 (\|\psi_{xx}\|^2 + \|\varphi_{xx}\|^2 + \|\psi_x\|^2) + C\delta \|\varphi_{xx}\|^2, \end{aligned} \quad (2.36)$$

$$\begin{aligned} I_{20} &= \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \left(\frac{\varphi_x v_x}{v^2} \right)_t \varphi_{xx} dx = \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \left(\frac{\varphi_{xt} v_x}{v^2} + \frac{\varphi_x v_{xt}}{v^2} - \frac{\varphi_x v_x v_t}{v^3} \right) \varphi_{xx} dx \\ &= \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{5}{2}} \left(\frac{\psi_{xx} v_x}{v^2} + \frac{\varphi_x u_{xx}}{v^2} - \frac{\varphi_x v_x u_x}{v^3} \right) \varphi_{xx} dx \\ &\leq C(\varepsilon_0 + \delta) (\|\psi_{xx}\|^2 + \|\varphi_{xx}\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2), \end{aligned} \quad (2.37)$$

$$\begin{aligned} I_{21} &= -\frac{3}{4} \lambda \int_{\mathbb{R}} v^{-\frac{7}{2}} v_x \left(\frac{\varphi_x}{v} \right)_t \varphi_{xx} dx = -\frac{3}{4} \lambda \int_{\mathbb{R}} v^{-\frac{7}{2}} v_x \left(\frac{\varphi_{xt}}{v} - \frac{\varphi_x v_t}{v^2} \right) \varphi_{xx} dx \\ &= -\frac{3}{4} \lambda \int_{\mathbb{R}} v^{-\frac{7}{2}} v_x \left(\frac{\psi_{xx}}{v} - \frac{\varphi_x u_x}{v^2} \right) \varphi_{xx} dx \\ &\leq C(\varepsilon_0 + \delta) (\|\psi_{xx}\|^2 + \|\varphi_{xx}\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2), \end{aligned} \quad (2.38)$$

and

$$I_{22} = - \int_{\mathbb{R}} \left[v^{-\frac{3}{2}} \left(\frac{\bar{v}_x}{v} \right)_t \right]_x \frac{\varphi_{xx}}{v} dx = -\frac{\lambda}{2} \int_{\mathbb{R}} \left[v^{-\frac{3}{2}} \left(\frac{\bar{v}_{xt}}{v} - \frac{\bar{v}_x v_t}{v^2} \right) \right]_x \frac{\varphi_{xx}}{v} dx$$

$$\begin{aligned}
&= -\frac{\lambda}{2} \int_{\mathbb{R}} \left(v^{-\frac{5}{2}} \bar{v}_{xxt} - \frac{5}{2} v^{-\frac{7}{2}} v_x \bar{v}_{xt} - v^{-\frac{7}{2}} \bar{v}_{xx} v_t - v^{-\frac{7}{2}} \bar{v}_x v_{xt} + \frac{7}{2} v^{-\frac{9}{2}} v_x \bar{v}_x v_t \right) \frac{\varphi_{xx}}{v} dx \\
&= -\frac{\lambda}{2} \int_{\mathbb{R}} \left(v^{-\frac{5}{2}} \bar{u}_{xxx} - \frac{5}{2} v^{-\frac{7}{2}} v_x \bar{u}_{xx} - v^{-\frac{7}{2}} \bar{v}_{xx} u_x - v^{-\frac{7}{2}} \bar{v}_x u_{xx} + \frac{7}{2} v^{-\frac{9}{2}} v_x \bar{v}_x u_x \right) \frac{\varphi_{xx}}{v} dx \\
&\leq C\delta^{\frac{1}{4}}(1+t)^{-\frac{3}{2}} + C\delta^{\frac{1}{4}} (\|\psi_{xx}\|^2 + \|\varphi_{xx}\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2). \tag{2.39}
\end{aligned}$$

Next, we estimate the term I_{23} . Since

$$\begin{aligned}
I_{23} &= \int_{\mathbb{R}} \left(\frac{3}{4} \lambda v^{-\frac{7}{2}} u_x v_x + \frac{\lambda}{2} v^{-\frac{7}{2}} u v_{xx} - \frac{5}{4} \lambda v^{-\frac{9}{2}} u v_x^2 \right) \frac{\varphi_{xx}}{v} dx \\
&= - \int_{\mathbb{R}} \frac{31}{8} \lambda v^{-\frac{9}{2}} u_x v_x^2 \frac{\varphi_{xx}}{v} dx + \int_{\mathbb{R}} \frac{3}{4} \lambda v^{-\frac{7}{2}} u_{xx} v_x \frac{\varphi_{xx}}{v} dx + \int_{\mathbb{R}} \frac{5}{4} \lambda v^{-\frac{7}{2}} u_x v_{xx} \frac{\varphi_{xx}}{v} dx \\
&\quad - \int_{\mathbb{R}} \frac{17}{4} \lambda v^{-\frac{9}{2}} u v_x v_{xx} \frac{\varphi_{xx}}{v} dx + \int_{\mathbb{R}} \frac{\lambda}{2} v^{-\frac{7}{2}} u v_{xxx} \frac{\varphi_{xx}}{v} dx + \int_{\mathbb{R}} \frac{45}{8} \lambda v^{-\frac{11}{2}} u v_x^3 \frac{\varphi_{xx}}{v} dx \\
&=: J_1 + J_2 + \cdots + J_6, \tag{2.40}
\end{aligned}$$

I_{23} can be estimated by the following terms:

$$J_1 = -\frac{31}{8} \lambda \int_{\mathbb{R}} v^{-\frac{11}{2}} u_x v_x^2 \varphi_{xx} dx \leq C(\varepsilon_0 + \delta)(\|\varphi_{xx}\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2) + C\delta(1+t)^{-3},$$

$$J_2 = \frac{3}{4} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} u_{xx} v_x \varphi_{xx} dx \leq C(\varepsilon_0 + \delta)(\|\varphi_{xx}\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2) + C\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}},$$

$$J_3 = \frac{5}{4} \lambda \int_{\mathbb{R}} v^{-\frac{9}{2}} u_x v_{xx} \varphi_{xx} dx \leq C(\varepsilon_0 + \delta)(\|\varphi_{xx}\|^2 + \|\psi_x\|^2 + \|\psi_{xx}\|^2) + C\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}},$$

$$J_4 = -\frac{17}{4} \lambda \int_{\mathbb{R}} v^{-\frac{11}{2}} u v_x v_{xx} \varphi_{xx} dx \leq C(\varepsilon_0 + \delta)(\|\varphi_{xx}\|^2 + \|\varphi_x\|^2) + C\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}},$$

$$\begin{aligned}
J_5 &= \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{9}{2}} u v_{xxx} \varphi_{xx} dx = \frac{\lambda}{4} \int_{\mathbb{R}} v^{-\frac{9}{2}} u (\varphi_{xx}^2)_x dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{9}{2}} u v \bar{v}_{xxx} \varphi_{xx} dx \\
&= -\frac{\lambda}{4} \int_{\mathbb{R}} v^{-\frac{9}{2}} u_x \varphi_{xx}^2 dx + \frac{9}{8} \lambda \int_{\mathbb{R}} v^{-\frac{11}{2}} u v_x \varphi_{xx}^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}} v^{-\frac{9}{2}} u \bar{v}_{xxx} \varphi_{xx} dx \\
&\leq C(\varepsilon_0 + \delta^{\frac{1}{4}})(\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2 + \|\psi_x\|^2) + C\delta^{\frac{1}{4}}(1+t)^{-\frac{3}{2}},
\end{aligned}$$

and

$$\begin{aligned}
J_6 &= \frac{45}{8} \lambda \int_{\mathbb{R}} v^{-\frac{13}{2}} u v_x^3 \varphi_{xx} dx = \frac{45}{8} \lambda \int_{\mathbb{R}} v^{-\frac{13}{2}} u (\varphi_x + \bar{v}_x)^3 \varphi_{xx} dx \\
&\leq C(\varepsilon_0 + \delta)(\|\varphi_{xx}\|^2 + \|\varphi_x\|^2) + C\delta(1+t)^{-3}.
\end{aligned}$$

Substituting above estimates into (2.40) gives

$$I_{23} \leq C(\varepsilon_0 + \delta^{\frac{1}{4}}) (\|\varphi_{xx}\|^2 + \|\psi_{xx}\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2) + C\delta^{\frac{1}{4}}(1+t)^{-\frac{3}{2}}. \tag{2.41}$$

In addition, we have the following estimates:

$$\begin{aligned}
I_{24} &= - \int_{\mathbb{R}} v^{-1} \varphi_{xx} \psi_{xt} \, dx = \int_{\mathbb{R}} (v^{-1} \varphi_{xx} \psi_x)_t \, dx - \int_{\mathbb{R}} v^{-1} \varphi_{xxt} \psi_x \, dx + \int_{\mathbb{R}} v^{-2} \varphi_{xx} \psi_x v_t \, dx \\
&= \int_{\mathbb{R}} (v^{-1} \varphi_{xx} \psi_x)_t \, dx - \int_{\mathbb{R}} v^{-1} \psi_{xxx} \psi_x \, dx + \int_{\mathbb{R}} v^{-2} \varphi_{xx} \psi_x u_x \, dx \\
&= \int_{\mathbb{R}} (v^{-1} \varphi_{xx} \psi_x)_t \, dx + \int_{\mathbb{R}} v^{-1} \psi_{xx}^2 \, dx - \int_{\mathbb{R}} v^{-2} \psi_{xx} \psi_x v_x \, dx + \int_{\mathbb{R}} v^{-2} \varphi_{xx} \psi_x u_x \, dx \\
&\leq \int_{\mathbb{R}} (v^{-1} \varphi_{xx} \psi_x)_t \, dx + C \|\psi_{xx}\|^2 + C(\varepsilon_0 + \delta)(\|\psi_x\|^2 + \|\varphi_{xx}\|^2), \tag{2.42}
\end{aligned}$$

$$I_{25} = \int_{\mathbb{R}} p'(v)_x \varphi_x \frac{\varphi_{xx}}{v} \, dx \leq C(\varepsilon_0 + \delta)(\|\varphi_x\|^2 + \|\varphi_{xx}\|^2), \tag{2.43}$$

and

$$\begin{aligned}
I_{26} &= \int_{\mathbb{R}} [(p'(v) - p'(\bar{v})) \bar{v}_x]_x \frac{\varphi_{xx}}{v} \, dx \\
&= \int_{\mathbb{R}} v^{-1} (p'(v) - p'(\bar{v})) \bar{v}_{xx} \varphi_{xx} \, dx + \int_{\mathbb{R}} v^{-1} [p'(v) v_x - p'(\bar{v}) \bar{v}_x] \bar{v}_x \varphi_{xx} \, dx \\
&= \int_{\mathbb{R}} v^{-1} (p'(v) - p'(\bar{v})) \bar{v}_{xx} \varphi_{xx} \, dx + \int_{\mathbb{R}} v^{-1} p'(v) \varphi_x \bar{v}_x \varphi_{xx} \, dx \\
&\quad + \int_{\mathbb{R}} v^{-1} (p'(v) - p'(\bar{v})) \bar{v}_x^2 \varphi_{xx} \, dx \\
&\leq C \int_{\mathbb{R}} (|\varphi \bar{v}_{xx} \varphi_{xx}| + |\varphi_x \bar{v}_x \varphi_{xx}| + |\varphi \bar{v}_x^2 \varphi_{xx}|) \, dx \\
&\leq C(\varepsilon_0 + \delta) (\|\varphi_x\|^2 + \|\varphi_{xx}\|^2) + C\delta^{\frac{1}{2}} (1+t)^{-\frac{3}{2}}. \tag{2.44}
\end{aligned}$$

By substituting the estimates (2.36)–(2.39) and (2.41)–(2.44) into (2.35), then integrating the resulting inequality with respect to t and choosing ε_0, δ small enough; together with (2.6), (2.16), (2.26) and the smallness of ξ , one can arrive at (2.34). The proof of Lemma 2.4 is finished. \square

Proof of Proposition 2.1: We combine Lemmas 2.1–2.4, then choose ξ, ε_0 and δ small enough to establish the *a priori* estimates (2.3). Thus the proof of Proposition 2.1 is completed. \square

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