# REMARKS ON THE INFINITE DIMENSIONAL COUNTERPARTS OF THE DARBOUX THEOREM

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*We dedicate the paper to Professor Jean Mawhin with admiration and appreciation on the occasion of his 80th anniversary*

ABSTRACT. The Darboux theorem, one of the fundamental results in analysis, states that the derivative of a real (not necessarily continuously) differentiable function defined on a compact interval has the intermediate value property, i.e. attains each value between the derivatives at the endpoints. The Bolzano intermediate value theorem, which implies Darboux's theorem when the derivative is continuous, states that a continuous real-valued function $f$ defined on $[-1,1]$ satisfying $f(-1)<0$ and $f(1)>0$ 0, has a zero, i.e. $f(x)=0$ for at least one number $-1<x<1$. It has numerous counterparts in multivariate calculus as well as in the infinite-dimensional setting. The present paper is devoted to the discussion of some infinite-dimensional variants of the Darboux theorem, which does not seem to be sufficiently deeply discussed. The study relies on different notions of non-smooth differentiability of real functions and some appropriate compactness conditions. Problems involving functionals bounded below, monotone operators as well as some general questions concerning the existence of the so-called generalized equilibria are discussed.

## 1. INTRODUCTION

Suppose that $K$ is a closed subset of a Banach space $X$ and $f:K\to\mathbb{R}$ is a sufficiently regular functional. We are going to study the existence of *equilibria* and *generalized equilibria* of $f$, i.e. points $x\in K$ such that $0\in\partial f$ or $0\in\partial_f(x)+N(x;K)$, where $\partial f$ and $N(x;K)$ stand for an appropriate subdifferential of $f$ and a (corresponding) normal cone to $K$ at $x$.

If, for instance $K=[-1,1]$, and $f:[-1,1]\to\mathbb{R}$ is differentiable (one-sided derivatives are considered at the end-points), then, in view of the *Darboux theorem*, there is $x_0\in(-1,1)$ with $f'(x_0)=0$ provided $f'_+(-1)<0$ and $f'_-(1)>0$. Observe that, in this case the (Bouligand) normal cone $N_B(-1;K)=(-\infty,0]$, $N_B(1;K)=[0,+\infty)$ and $N_B(x;K)=\{0\}$ for any $x\in(-1,1)$. Hence $f'_+(-1)<0\Leftrightarrow f'_+(-1)\cap -N_B(-1;K)=\emptyset$, $f'_-(1)>0\Leftrightarrow \cap-N_B(1;K)=\emptyset$, and $f'(x_0)=0\Leftrightarrow 0\in f'(x_0)+N_B(x_0;K)$ for $x_0\in(-1,1)$. The Darboux theorem may therefore be stated as follows: *if $f'(x)\cap -N_B(x;K)=\emptyset$ for $x\in\partial K$, then there is $x_0\in\operatorname{int}K$ such that $0\in f'(x_0)+N_B(x;K)$.*

The Darboux theorem did not receive many different generalizations to multi-dimensional or infinite-dimensional setting as did the famous and fundamental intermediate value theorem of Bolzano, which – in fact – implies the Darboux theorem in case of $C^1$-function. The mean-value Bolzano theorem is perhaps one of the most important topological devices when studying equations of the form $f(x)=0$. It was extensively studied and generalized by numerous authors for almost 150 years (see, e.g., [32] and [33]) and various important results were established. One of the best known statements in this direction is the Poincaré–Miranda theorem, which is a direct $N$-dimensional version of the Bolzano theorem.

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Bolzano's and Darboux's theorems strongly rely on the compactness of the domain which, together with the Fermat rule and the Weierstrass theorems, allows to observe the existence of local extrema of studied functions. It is to be noticed that a different argument for the Darboux theorem has been provided in [36]. The intrinsic lack of compactness is problematic in infinite-dimensional spaces. Hence, the infinite-dimensional results require some additional assumptions. Such assumptions, having the character of the so-called inf-compactness hypotheses in case of functionals bounded below, belong to the rich family of the Palais-Smale type conditions and help to find equilibria $\big($zeros of generalized gradients$\big)$. Namely these conditions, combined with the Ekeland principle, make it possible to show that the argmin set of a studied functionals is nonempty, which leads to the existence of equilibria via Fermat's rule. A similar approach has been applied in case of infinite dimensional versions of another milestone result of analysis, i.e. the Rolle Theorem – see [3], where the approximate Rolle theorem was established and [1], where the Palais-Smale condition was employed.

In case of generalized equilibria, i.e. coincidences of abstract maps $F : K \to X^*$, the topological dual of $X$, with the normal bundle to $K$, different compactness assumptions are necessary. For instance one can require that a map $F$ is *guided* by an accretive operator having compact resolvent. This has proved to be useful e.g. in [29], where the truly infinite-dimensional counterparts of the Bolzano theorem results, i.e. going beyond compactness, were established.

A different approach to the existence of equilibria is available in case of monotone operators, which correspond to subdifferentials of convex functionals. The celebrated Minty-Browder methods permits to rely on reflexivity of an ambient space $X$ and weak compactness issues rather. This attitude, for instance, has been pursued in [34] in the context of equilibria.

In this paper, having its brevity in mind, we did not enter into the numerous possible applications. However, we have provided several natural examples showing the scope of potential applications is very wide. We also have not tried to study a different, but very interesting direction of Darboux's theorem extensions studied e.g. in [42] and [27], where the connectedness properties in $\mathbb{R}^n$ of the image of the (connected) domain under the gradient of a smooth map were studied.

After this introduction, the paper is organized as follows. In the second section, we provide a brief survey of concepts and results concerning the non-smooth differentiability of functionals defined on closed sets. The results are, with a few exceptions, folklore or well-known and scattered over the very abundant literature on the subject, though we mention only a couple of sources. We deal mainly with the so-called Dini-Hadamard differentiability, and Clarke-Rockafellar differentiability, with emphasis put on the Clarke generalized gradients for locally Lipschitz functionals. In the third section, we first study functionals bounded below. Our approach is rather standard and depends on the Ekeland variations principle with an aid of the above mentioned Palais-Smale type conditions. In particular, we provide many examples of functionals satisfying various conditions of this type along with corresponding results. In this way, we establish some new results or complement older ones. Next, we study the existence of generalized equilibria of some general set-valued upper hemicontinuous operators defined on closed sets in a Banach space with values in the dual $X^*$. The results stated here are new. The last part of the third section is devoted to monotone operators. Here we establish some new results that correspond well to results in [34]. The last secion is Appendix, where we collect some auxiliary facts concerning set-limits in the sense of Painlevé-Kuratowski and concepts of tangency in Banach spaces.

*Notation:* In the paper $X$ stands for a real Banach space $X$ equipped with a norm $\|\cdot\|$; $0_X$ is the zero vector in $X$. By $X^*$ and $\langle\cdot,\cdot\rangle$ we denote the (topological) dual of $X$ and the duality pairing in $X^* \times X$, respectively. The open (resp. closed) ball centered at $x \in X$ with radius $r > 0$ is denoted by $B(x,r)$ $\big($resp. $D(x,r)\big)$. For $A \subset X$, $\mathrm{int}A, \overline{A}$, $\partial K$, $\mathrm{conv}A$, and $\overline{\mathrm{conv}}A$ denote its interior, closure, boundary, convex hull, and closed convex hull, respectively; $B(A,r) := \{x \in X \mid d(x,A) := \inf_{a\in A} \|x-a\| < r\}$, $r > 0$.

Moreover $A^{-} := \{p \in X^* \mid \forall x \in A \ \ \langle p,x\rangle \leqslant 0\}$ is the (negative) *polar cone* of $K$ and $A^{\perp} := \{p \in X^* \mid \forall x \in A \ \ \langle p,x\rangle = 0\}$ is the *annihilator* of $A$. Throughout the work, we often use (conditional) set-limits in the sense of Kuratowski-Painlevé and some "tangent" cones: the reader may recall these concepts in the Appendix 4.1, 4.2. When speaking of a *set-valued map* $\varphi : X \multimap Y$, where $Y$ is a space, we mean a mapping assigning to each $x \in X$ a (possibly empty) set $\varphi(x) \subset Y$; $\mathrm{Dom}(\varphi) := \{x \in X \mid \varphi(x) \neq \emptyset\}$; always, if necessary, the properties of values are specified. For continuity concepts of set-valued maps – see [2].

## 2. Differentiablity

There is a variety of methods to study differentiability properties of functionals defined on closed domains: those of *ad hoc* character necessary in specific situations (i.e. in the divergence theorem) and those following a systematic approach – see [25], for instance – and taking into account a possible lack of any regularity of considered functionals. In the article, we are interested in generalizations of the Darboux theorem and, thus, we pay rather limited attention to calculus issues and/or deeper insight into the very notion of differentiability. However, to get a better understanding, we are going to treat several cases and compare the resulting approaches. Most of the results concerning different methods to introduce derivatives and their properties seem to be well-known, we include some of them for the sake of completeness and a reader's convenience. It is often difficult to point out who contributed to the theory and to whom particular results should be attributed. Hence, we will rather restrict ourselves to some general references as [2, 41, 9, 25, 6] and the extensive bibliography there.

Let $K \subset X$ be closed and consider a functional $f : K \to \mathbb{R}$. With a view to have some sort of clearance with variables, but with a slight abuse of notation, we consider the *extension* of $f$ given as

$$f : X \to \overline{\mathbb{R}} := (-\infty, +\infty], \ \text{ where } \ f(x) = +\infty \ \text{ for } \ x \notin K. \tag{2.1}$$

The *effective domain* $\mathrm{Dom}(f) := \{x \in X \mid f(x) < \infty\} = K$. In order not to lose some basic regularity exhibited by $f$ we assume that $f$ is at least *lower semicontinuous* on $K$, i.e., for any $x \in K$,

$$f(x) = \sup_{\varepsilon>0} \inf_{y\in B(x,\varepsilon)\cap K} f(y). \tag{2.2}$$

Then the extension of $f$ is lower semicontinuous, too. Moreover, the epigraphs of $f$ over $K$ and over $X$ coincide ([1]). If $f$ is convex, then so is its extension and the domain $K$.

### 2.1. Hadamard subdifferential.

For $x \in K$ and $u \in X$ the (lower right) *Dini-Hadamard directional derivative*

$$f'_H(x;u) := \liminf_{t\to 0^+,\, v\to u} \frac{f(x+tv) - f(x)}{t}. \tag{2.3}$$

It is clear that $f'_H(x;u) = d \in \overline{\mathbb{R}} \cup \{-\infty\}$.

**Lemma 2.1.** *If $u \notin T_B(x;K)$, where $T_B(x;K)$ stands for the Bouligand (contingent) cone to $K$, then $f'_H(x;u) = \infty$.*

*Proof.* Suppose that $d = f'_H(x;u) < +\infty$. Then there are sequences $(u_n) \subset X$ and $(t_n) \subset \mathbb{R}$ such that $u_n \to u$, $t_n \to 0^+$, $x + t_n u_n \in K$ for all $n \in \mathbb{N}$, and $d = \lim_{n\to\infty} t_n^{-1}\big(f(x+t_n u_n) - f(x)\big)$. This, in view of (4.2), means that $u \in T_B(x;K)$. □

[1]Recall that the *epigraph* $\mathrm{Epi}\,(f) := \big\{(x,\lambda) \in X \times \mathbb{R} \mid f(x) \leqslant \lambda\big\}$ of a lower semicontinuous functional is closed.

**Example 2.2.** Let $f:\mathbb{R}^2\to(-\infty,+\infty]$ be given by

$$f(x,y)=\begin{cases}x^3y^{-2} & \text{if } x\in(0,1],\ x^2\leqslant y\leqslant x,\\ 0 & \text{for } x=0=y,\\ +\infty & \text{otherwise.}\end{cases}$$

In this case $K=\{(x,y)\in\mathbb{R}^2\mid x\in[0,1],\ x^2\leqslant y\leqslant x\}$, $T_B\big((0,0);K\big)=\big\{u=[x,y]\in\mathbb{R}^2\mid x\geqslant y\geqslant 0\big\}$, $f$ is positive, convex, lower semicontinuous and $f'_H\big((0,0);[1,0]\big)=+\infty$. $\square$

The following proposition provides a convenient geometric description of the Hadamard directional derivative.

**Proposition 2.3.** *Let $x\in K$ and $u\in X$.*

(1) *Then $(u,\lambda)\in T_B\big((x,f(x));\mathrm{Epi}\,(f)\big)$ if and only if $u\in T_B(x;K)$ and $f'_H(x;u)\leqslant\lambda$. In particular $f'_H(x;0_X)\leqslant 0$. If $f'_H(x;0_X)<0$, then $f'(x;0_X)=-\infty$.*

(2) *More precisely*

$$f'_H(x;u)=\inf\big\{\lambda\in\mathbb{R}\mid(u,\lambda)\in T_B\big((x,f(x));\mathrm{Epi}\,(f)\big)\big\},\tag{2.4}$$

*where* $\inf\emptyset:=+\infty$, *and hence*

$$\mathrm{Epi}\,\big(f'_H(x;\cdot)\big)=T_B\big((x,f(x));\mathrm{Epi}\,(f)\big).\tag{2.5}$$

(3) *The functional $X\ni u\mapsto f'_H(x;\cdot)\in\overline{\mathbb{R}}\cup\{-\infty\}$ is positively homogeneous and lower semicontinuous;*

(4) *if $f$ is convex, then so is $f'_H(x;\cdot)$.*

*Proof.* Part (1) follows from the definition of the Bouligand cone and (2.3). The second assertion is immediate since $(0_X,0)\in T_B\big((x,f(x));\mathrm{Epi}\,(f)\big)$. If $f'_H(x;0_X)=d<0$, then $(0_X,d')\in T_B\big((x,f(x));\mathrm{Epi}\,(f)\big)$ for some $d\leqslant d'<0$, so $(0,\lambda d')\in T_B\big((x,f(x));\mathrm{Epi}\,(f)\big)$, implying that $f'_H(x;0_X)\leqslant\lambda d'$ for any $\lambda>0$, and hence $f'_H(x;0_X)=0$ or $-\infty$.

The equality (2.4) (equivalently (2.5)) was established by numerous authors and follows immediately from the definition (2.3) and the above part (1). Parts (3), (4) rely on (2.5) since $T_B\big((x,f(x));\mathrm{Epi}\,(f)\big)$ is a closed cone being convex whenever so is $f$. $\square$

We say that $f$ is *calm* (resp. *quiet*) on $K$ at $x$ if there are $\ell\in\mathbb{R}$ and $\varepsilon>0$ such that $f(y)\geqslant f(x)-\ell\|y-x\|$ $\big($resp. $f(y)\leqslant f(x)+\ell\|y-x\|\big)$, when $y\in K\cap B(x,\varepsilon)$. It is clear that if $f$ is calm with constant $\ell$ and $\ell'\geqslant\ell$, then $f$ is calm with constant $\ell'$. In particular if $f$ admits a local minimum at $x\in X$, then it is calm at $x$ on $K$ with constant $\ell=0$. Let us make the following observation.

**Proposition 2.4.** (1) *Suppose that $f$ is calm $\big($resp. quiet$\big)$ on $K$ at $x\in K$ with constant $\ell$. Then for any $u\in X$,*

$$f'_H(x;u)\geqslant-\ell\|u\|\ \ \big(\textit{resp. } f'_H(x;u)|\leqslant\ell\|u\|\big).\qquad\square$$

*In particular if $f$ has a local minimum at $x$, then $f'_H(x;u)\geqslant 0$ for any $u\in X$.*

(2) *In general $f'_H(x;0_X)=0$ if and only if $f$ is calm on $K$ at $x$.*

*Proof.* (1) is immediate from definition (2.3). In view of (1) only the ‘only if’ part of (2) should be shown. If $f$ is not calm at $x$, then there is a sequence $u_n\to 0_X$ such that $x+u_n\in K$, $\|u_n\|<1/n$ and $f(x+u_n)-f(x)<-n\|u_n\|$. If $h_n:=\sqrt{n}\|u_n\|$ and $v_n:=(\sqrt{n}\|u_n\|)^{-1}u_n$, then $h_n\to 0$, $v_n\to 0$ and for any $n\in\mathbb{N}$

$$\frac{f(x+h_nv_n)-f(x)}{h_n}<-\sqrt{n}.$$

This implies that $f'_H(x,0) = -\infty$. □

**Remark 2.5.** Let $x \in K$. For $h > 0$, $h^{-1}\Big(\mathrm{Epi}\,(f) - \big(x, f(x)\big)\Big) = \mathrm{Epi}\,(\varphi_h)$, where

$$\varphi_h(u) := h^{-1}\big(f(x+hu) - f(x)\big), \;\; u \in X,$$

and, therefore, in view of the definition of $T_B\big((x, f(x)); \mathrm{Epi}\,(f)\big)$ $\big($see (4.2)$\big)$ we have

$$\mathrm{Epi}\,\big(f'_H(x;\cdot)\big) = \operatorname*{Lim\,sup}_{h\to 0^+} \mathrm{Epi}\,(\varphi_h). \tag{2.6}$$

This, together with the notion of *epi-convergence* (see e.g. [14, Theorem 4.16]), implies that

$$f'_H(x;u) = \sup_{\varepsilon>0}\sup_{\eta>0}\inf_{0<h<\eta}\inf_{v\in B(u,\varepsilon)} \frac{f(x+hv)-f(x)}{h}; \tag{2.7}$$

and provides the expression coinciding with the usual limes inferior in (2.3) and does not bring anything new. □

Let $x \in K$. The *Hadamard subdifferential*

$$\partial_H f(x) := \big\{p \in X^* \mid \forall u \in X \;\; \langle p, u\rangle \leqslant f'_H(x;u)\big\}.$$

Note that

$$\partial_H f(x) = \big\{p \in X^* \mid \forall u \in T_B(x;K) \;\; \langle p, u\rangle \leqslant f'_H(x;u)\big\}. \tag{2.8}$$

It is immediate to see that $\partial_H f(x)$ is convex and weakly$^*$-closed. Observe that if $\partial_H f(x)$ is nonempty, then

$$\forall u \in X \;\; \varphi(u) := \sup_{p\in\partial_H f(x)} \langle p, u\rangle \leqslant f'_H(x;u).$$

The function $\varphi$ is lower semicontinuous, convex and positively homogeneous. This implies that

$$T_B\big((x, f(x)); \mathrm{Epi}\,(f)\big) \subset \mathrm{Epi}\,(\varphi),$$

i.e. $f$ is 'almost' convex at $x$ in the sense that the Bouligand cone to $\mathrm{Epi}\,(f)$ at $x$ is contained in a convex cone. Hence the normal cone $\big($see Remark 4.2$\big)$

$$N_B\big((x, f(x)), \mathrm{Epi}\,(f)\big) := \big[T_B\big((x, f(x)); \mathrm{Epi}\,(f)\big)\big]^- \neq \emptyset.$$

If $f$ has a local minimum at $x$, then the *Fermat rule* holds: $0 \in \partial_H f(x)$.

**Proposition 2.6.** *Let $x \in K$.*

(1) *For any $p \in X^*$*

$$p \in \partial_H f(x) \iff (p, -1) \in N_B\big((x, f(x)), \mathrm{Epi}\,(f)\big), \tag{2.9}$$

(2) *If $p \in \partial_H f(x)$, $q \in X^*$ and $q - p \in N_B(x;K)$, then $q \in \partial_H f(x)$; in other words*

$$\partial_H f(x) + N_B(x;K) = \partial_H f(x). \tag{2.10}$$

*Proof.* The first part is standard and directly follows from Proposition 2.3. To see the second statement take $u \in T_B(x;K)$. Then

$$\langle q, u\rangle = \langle q - p, u\rangle + \langle p, u\rangle \leqslant f'_H(x;u).$$

This completes the proof in view of (2.8). □

**Remark 2.7.** It is immediate to see that $(0, \lambda) \in T_B\big((x, f(x)); \mathrm{Epi}\,(f)\big)$ for any $x \in K$ and $\lambda \geqslant 0$. Therefore

$$N_B\big((x, f(x)); \mathrm{Epi}\,(f)\big) \subset X^* \times (-\infty, 0]$$

and $(p, 0) \in N_B\big((x, f(x)); \mathrm{Epi}\,(f)\big)$ if $p \in N_B(x;K)$. □

Let us compare the Hadamard derivative with the Dini derivative in case of a convex functional. Suppose that now that $f$ is convex (in addition to the above assumptions). Given $x \in K$ and $u \in X$, the Dini derivative

$$f'_D(x;u) := \lim_{h\to 0^+} \frac{f(x+hu)-f(x)}{h}.$$

It is clear that $\pm f_D(x;u) < \infty$ if and only if $\pm u \in A(x;K)$. This means that $f'(x;u)$ is finite for all $u \in X$ if and only if $K-x$ absorbs $X$.

Evidently for any $x \in K$ and $u \in X$, $f'_H(x;u) \leqslant f'_D(x;u)$; precisely one has the following proposition.

**Proposition 2.8.** (1) *If $f$ is convex, $x \in K$, then the Hadamard derivative $f'_H(x;\cdot)$ is the* lower closure *of $f'_D(x;\cdot)$, i.e. the greatest among all lower semicontinuous functionals $\leqslant f'_D(x;\cdot)$, and hence for any $u \in X$*

$$f'_H(x;u) = \liminf_{v\to u} f'_D(x;v). \tag{2.11}$$

*For $u \in X$, $f'_H(x;u) = f'_D(x;u)$ if and only if $f'_D(x;\cdot)$ is lower semicontinuous at $u$.*

(2) *For any $x \in K$, $\partial_H f(x) = \partial_D f(x)$, where*

$$\partial_D f(x) := \big\{p \in X^* \mid \forall u \in X\ f(x+u) \geqslant f(x) + \langle p,u\rangle\big\}$$

*is the subdifferential of $f$ at $x$ in the sense of convex analysis.*

*Proof.* The characterization similar to that from Proposition 2.3 holds true. Namely it is immediate to see that

$$f'_D(x,u) = \inf\big\{\lambda \in \mathbb{R} \mid (u,\lambda) \in A\big((x,f(x));\mathrm{Epi}\,(f)\big)\big\} \tag{2.12}$$

but

$$A\big((x,f(x));\mathrm{Epi}\,(f)\big) \subset \mathrm{Epi}\,\big(f'_D(x;\cdot)\big) \subset T_B\big((x,f(x));\mathrm{Epi}\,(f)\big) = \overline{A\big((x,f(x));\mathrm{Epi}\,(f)\big)},$$

the second inclusion follows since, in general, $f'_H(x;u) \leqslant f'_D(x;u)$. Hence, for each $x \in K$,

$$\overline{\mathrm{Epi}\,\big(f'_D(x;\cdot)\big)} = \mathrm{Epi}\,\big(f'_H(x;\cdot)\big)$$

This completes the proof of (2.11) in view of the characterization of the lower closure. Next statements are self evident. □

2.2. **Rockafellar-Clarke subdifferential.** Following the approach from Proposition 2.3, (2.5) and given $x \in K$ one defines the *Rockafellar-Clarke directional derivative* $X \ni u \mapsto f'_C(x;u)$ by

$$f'_C(x;u) := \inf\big\{\lambda \in \mathbb{R} \mid (u,\lambda) \in T_C\big((x,f(x));\mathrm{Epi}\,(f)\big)\big\}, \tag{2.13}$$

i.e.

$$\mathrm{Epi}\,\big(f'_C(x;\cdot)\big) = T_C\big((x,f(x));\mathrm{Epi}\,(f)\big), \tag{2.14}$$

where $T_C\big((x,f(x));\mathrm{Epi}\,(f)\big)$ is the Clarke cone to $\mathrm{Epi}\,(f)$ at $\big(x,f(x)\big)$ $\big($see (4.3)$\big)$.

For $y \in K$ and $h > 0$ let

$$\varphi_{y,h}(v) := \frac{f(y+hv)-f(y)}{h},\ \ v \in X.$$

Clearly $\varphi_{y,h}$ is lower semicontinuous, $\varphi_{y,h}(v) \in \overline{\mathbb{R}}$ and $\varphi_{y,h}(v) < \infty$ if and only if $y+hv \in K$. It is easy to see that

$$h^{-1}\big(\mathrm{Epi}\,(f) - \big(y,f(y)\big)\big) = \mathrm{Epi}\,(\varphi_{y,h}). \tag{2.15}$$

**Lemma 2.9.** *Let $x \in K$. Then*

$$T_C\big((x,f(x));\mathrm{Epi}\,(f)\big) = \operatorname*{Lim\,inf}_{\substack{h\to 0^+,\, y\xrightarrow{K}x\\ f(y)\to f(x)}} h^{-1}\big(\mathrm{Epi}\,(f) - \big(y,f(y)\big)\big).$$

*The condition determining the right-hand side set-limit is non-void: there are sequences $x_n \to x$ with $f(x_n) \to f(x)$ (see (2.2)).*

*Proof.* The set in the left-hand side is contained in the one in the right side. Suppose that

$$(u,\lambda) \in \operatorname*{Lim\,inf}_{\substack{h\to 0^+,\, y\xrightarrow{K}x\\ f(y)\to f(x)}} h^{-1}(K-y)$$

and take $(h_n) \subset \mathbb{R}$, $(x_n,\mu_n) \subset \operatorname{Epi}(f)$ with $h_n \to 0^+$, $x_n \to x$ and $\mu_n \to f(x)$. Hence $f(x_n) \to f(x)$ since

$$f(x) \leqslant \liminf_{n\to\infty} f(x_n) \leqslant \limsup_{n\to\infty} f(x_n) \leqslant \lim_{n\to\infty} \mu_n = f(x).$$

There are sequences $u_n \to u$ and $\lambda_n \to \lambda$ such that $\big(x_n + h_n u_n, f(x_n) + h_n\lambda_n\big) \in \operatorname{Epi}(f)$, i.e.

$$f(x_n + h_n u_n) \leqslant f(x_n) + h_n\lambda \leqslant \mu_n + h_n\lambda_n,$$

showing the assertion. $\square$

Lemma 2.9, (2.14) and (2.15) show that for $x \in K$,

$$\operatorname{Epi}\big(f'_C(x;\cdot)\big) = T_C\big((x,f(x));\operatorname{Epi}(f)\big) = \operatorname*{Lim\,inf}_{\substack{h\to 0^+,\, y\xrightarrow{K}x\\ f(y)\to f(x)}} \operatorname{Epi}(\varphi_{y,h}). \tag{2.16}$$

According to [14, Theorem 4.16], for $u \in X$,

$$f'_C(x;u) = \sup_{\varepsilon>0} \limsup_{\substack{h\to 0^+,\, y\xrightarrow{K}x\\ f(y)\to f(x)}} \inf_{v\in B(u,\varepsilon)} \frac{f(y+hv)-f(y)}{h}. \tag{2.17}$$

As we see the analytic expression for $f'_C(x;u)$ is not very convenient to deal with.

**Remark 2.10.** (1) In general for $x \in K$, the function $X \ni u \mapsto f'_C(x;u) \in \overline{\mathbb{R}} \cup \{-\infty\}$ is lower semicontinuous, convex and positively homogeneous. It is clear that for any $x \in K$ and $u \in X$

$$f'_H(x;u) \leqslant f'_C(x;u) \tag{2.18}$$

since $T_C\big((x,f(x));\operatorname{Epi}(f)\big) \subset T_B\big((x,f(x));\operatorname{Epi}(f)\big)$.

(2) If $f'_C(x;u) < \infty$, then $u \in T_C(x,K)$. Indeed in this case there is $\lambda \in \mathbb{R}$ such that

$$(u,\lambda) \in T_C\big((x,f(x));\operatorname{Epi}(f)\big)$$

and for any sequences $x_n \xrightarrow{K} x$ and $h_n \to 0^+$ there are sequences $u_n \to u$ and $\lambda_n \to \lambda$ such that $f(x_n + h_n u_n) \leqslant f(x_n) + h_n\lambda_n$. In particular this shows that $x_n + h_n u_n \in K$, i.e. $u \in T_C(x;K)$.

(3) If $f$ is calm on $K$ at $x \in K$ with constant $\ell \in \mathbb{R}$, then for any $u \in X$,

$$f'_C(x;u) \geqslant -\ell\|u\|$$

by (2.18) and Proposition 2.4 (1).

(4) If $f$ is locally Lipschitz on $K$ at $x \in K$, i.e. there is $\varepsilon > 0$ and $\ell > 0$ such that $|f(x') - f(x'')| \leqslant \ell\|x' - x''\|$ for $x', x'' \in B(x,\varepsilon)$, then $f'_C(x;u) \leqslant \ell\|u\|$ for $u \in T_C(x;K)$. To this aim take $u \in T_C(x;K)$. We shall show that $(u,\lambda) \in T_C\big((x,f(x));\operatorname{Epi}(f)\big)$, where $\lambda = \ell\|u\|$. Take sequence $x_n \xrightarrow{K} x$ (then $f(x_n) \to f(x)$) and $h_n \to 0^+$. Since $u \in T_C(x;K)$ there is a sequence $u_n \to u$ such that $x_n + h_n u_n \in K$ for all $n$. Thus

$$f(x_n + h_n u_n) \leqslant f(x_n) + h_n\lambda_n,$$

where $\lambda_n := \ell\|u_n\|$, showing the claim. Having this we infer that $f'_C(x;u) \leqslant \ell\|u\|$.

(5) Summing up: if $f$ is Lipschitz on $K$ at $x \in K$, then for any $u \in X$, $f'_C(x;u) \in \mathbb{R} \Leftrightarrow u \in T_C(x;K)$ and in this situation

$$-\ell\|u\| \leqslant f'_C(x;u) \leqslant \ell\|u\|. \qquad \square$$

**Proposition 2.11.** (comp. [9, Proposition 2.2.1]) *Let $u \in T_C(x;K)$, $f$ be locally Lipschitz on $K$ at $x$ with constant $\ell > 0$. Then*

$$f'_C(x;u) = \limsup_{\substack{h\to 0^+, y\to x, v\to u,\\ y, y+hv\in K}} h^{-1}\big(f(y+hv) - f(y)\big).$$

*If $u$ is a hypertangent, then*

$$f'_C(x;u) = \limsup_{h\to 0^+, y\to x, y\in K} h^{-1}\big(f(y+hu) - f(y)\big).$$

*In particular if $x \in \operatorname{int} K$, then for any $u \in X$*

$$f'_C(x;u) = \limsup_{h\to 0^+, y\to x} h^{-1}\big(f(y+hu) - f(y)\big) \tag{2.19}$$

*is the Clarke directional derivative, usually denoted by $f^\circ(x;u)$. Hence the function $\operatorname{int} K \times X \ni (x,u) \mapsto f'_C(x;u)$ is upper semicontinuous.*

*Proof.* It is immediate to see that if $(u,\lambda) \in T_C\big((x, f(x)); Epi(f)\big)$, then for any sequences $h_n \to 0^+$, $x_n \to x$, $x_n \in K$, there is a sequence $u_n \to u$ such that $x_n + h_n u_n \in K$ for all $n$ and

$$\limsup_{n\to\infty} h_n^{-1}\big(f(x_n + h_n u_n) - f(x_n)\big) \leqslant \lambda.$$

Since $f$ is locally Lipschitz on $K$ at $x$ and Clarke's tangent cone is closed, from (2.13) and Remark 2.10 (4), $\big(u, f'_C(x;u)\big) \in T_C\big((x,f(x)); \operatorname{Epi}(f)\big)$. Fix $\varepsilon > 0$. For sufficiently large $n$ we have $\|u_n - u\| < \varepsilon$. Hence

$$\inf_{v\in B(u,\varepsilon)} h_n^{-1}\big(f(x_n + h_n v) - f(x_n)\big) \leqslant h_n^{-1}\big(f(x_n + h_n u_n) - f(x_n)\big).$$

The sequences $(x_n), (h_n)$ are arbitrary, therefore

$$\limsup_{\substack{h\to 0^+, y\to x,\\ y\in K}} \inf_{v\in B(u,\varepsilon)} h^{-1}\big(f(y+hv) - f(y)\big) \leqslant \limsup_{\substack{h\to 0^+, y\to x, v\to u,\\ y, y+hv\in K}} h^{-1}\big(f(y+hv) - f(y)\big),$$

with the right hand side being less or equal to $f'_C(x;u)$. Passing to the least upper bound on the left hand side over $\varepsilon > 0$ gives the equality

$$f'_C(x;u) = \limsup_{\substack{h\to 0^+, y\to x, v\to u,\\ y, y+hv\in K}} h^{-1}\big(f(y+hv) - f(y)\big).$$

In order to complete the proof, observe that, from the definition of a hypertangent, for any sequences $x_n \to x, x_n \in K$, $h_n \to 0^+$, for sufficiently large $n$ we have $x_n + h_n u, x_n + h_n u_n \in K$ and

$$\Big| h_n^{-1}\big(f(x_n + h_n u_n) - f(x_n)\big) - h_n^{-1}\big(f(x_n + h_n u) - f(x_n)\big)\Big| \leqslant \ell \|u_n - u\|.$$

The sequences $x_n, h_n, u_n$ were arbitrary, therefore

$$\limsup_{\substack{h\to 0^+, y\to x, v\to u,\\ y, y+hv\in K}} h^{-1}\big(f(y+hv) - f(y)\big) = \limsup_{h\to 0^+, y\to x, y\in K} h^{-1}\big(f(y+hu) - f(y)\big).$$

□

**Proposition 2.12.** *Let $x \in K$. Then*

$$\forall u \in X \;\; f'_H(x;u) = f'_C(x;u) \;\Leftrightarrow\; T_C\big((x,f(x)); \operatorname{Epi}(f)\big) = T_B\big((x,f(x)); \operatorname{Epi}(f)\big).$$

*If, for some $u \in T_B(x;K)$, the function $f'_H(\cdot;u)$ is upper semicontinuous on $K$ at $x$, then $f'_H(x;u) = f'_C(x;u)$.*

*Proof.* The 'if' part follows readily from (2.13) and (2.4). If $(u,\lambda)\in T_B\big((x,f(x));\mathrm{Epi}\,(f)\big)$, then $f'_C(x;u)=f'_H(x;u)\leqslant\lambda$. For any $\varepsilon>0$ there is $\mu\in\mathbb{R}$ with $f'_C(x;u)\leqslant\mu<\lambda+\varepsilon$ and $(u,\mu)\in T_C\big((x,f(x));\mathrm{Epi}\,(f)\big)$. This implies that $(u,\lambda)\in T_C\big((x,f(x));\mathrm{Epi}\,(f)\big)$ and that $T_B\big((x,f(x));\mathrm{Epi}\,(f)\big)\subset T_C\big((x,f(x));\mathrm{Epi}\,(f)\big)$.

To see the second part we shall first show that the multivalued map

$$K\ni y\mapsto G(y):=\big\{\lambda\in\mathbb{R}\mid(u,\lambda)\in T_B\big((y,f(y));\mathrm{Epi}\,(f)\big)\big\}$$

is lower semicontinuous at $x$, i.e.

$$G(x)\subset\operatorname*{Lim\,inf}_{y\xrightarrow{K}x}G(y).\tag{2.20}$$

It is clear that the set $G(y)$ is closed. To this aim take a sequence $x_n\xrightarrow{K}x$ and $\lambda\in G(x)$. Then $f'_H(x;u)\leqslant\lambda$. The function $f'_H(\cdot;u)$ is upper semicontinuous, i.e.

$$\limsup_{n\to\infty}f'_H(x_n;u)\leqslant d:=f'_H(x;u).$$

Hence for any $\varepsilon>0$ and for almost all $n$, $f'_H(x_n,u)<d+\varepsilon$. Thus $(u,d+\varepsilon)\in G(x_n)$ for almost all $n$. This implies that $\lambda\in G(x_n)$ for large $n$ and shows (2.20).

Recall that, in view of [47, Theorem 3.1],

$$\operatorname*{Liminf}_{(y,\lambda)\xrightarrow{\mathrm{Epi}\,(f)}(x,f(x))}T_B\big(z;\mathrm{Epi}\,(f)\big)\subset T_C\big((x,f(x));\mathrm{Epi}\,(f)\big).\tag{2.21}$$

We shall now prove that

$$\operatorname*{Lim\,inf}_{y\xrightarrow{K}x}G(y)\subset\operatorname*{Liminf}_{(y,\lambda)\xrightarrow{\mathrm{Epi}\,(f)}(x,f(x))}T_B\big((y,\lambda);\mathrm{Epi}\,(f)\big).$$

Note that if $(x,\lambda)\in\mathrm{Epi}\,(f)$, then, directly from the definition of the Bouligand cone,

$$T_B\big((x,f(x));\mathrm{Epi}\,(f)\big)\subset T_B\big((x,\lambda);\mathrm{Epi}\,(f)\big).$$

Let $\lambda\in\operatorname*{Liminf}_{y\xrightarrow{K}x}G(y)$ and $(y_n,\lambda_n)\xrightarrow{\mathrm{Epi}\,(f)}(x,f(x))$. Then, in particular, $y_n\xrightarrow{K}x$; therefore, there is $\lambda_n\to\lambda$ such that $\lambda_n\in G(y_n)$. But then $(u,\lambda_n)\in T_B\big((y_n,f(y_n));\mathrm{Epi}\,(f)\big)$ and $(u,\lambda_n)\to(u,\lambda)$, so $(u,\lambda_n)\in T_B\big((y_n,k_n);\mathrm{Epi}\,(f)\big)$. Since $(y_n,\lambda_n)$ was arbitrary, we finally get that

$$(u,\lambda)\in\operatorname*{Liminf}_{(y,k)\xrightarrow{\mathrm{Epi}\,(f)}(x,f(x))}T_B\big((y,k);\mathrm{Epi}\,(f)\big)$$

as required. This, together with (2.20) and (2.21), shows that

$$G(x)\subset\operatorname*{Lim\,inf}_{y\xrightarrow{K}x}G(y)\subset\big\{\lambda\mid(u,\lambda)\in T_C\big((x,f(x));\mathrm{Epi}\,(f)\big)\big\}\subset G(x).$$

This clearly implies the second assertion. □

The *Rockafellar-Clarke* subdifferential

$$\partial_C f(x):=\big\{p\in X^*\mid\forall u\in X\ \ \langle p,u\rangle\leqslant f'_C(x;u)\big\}.\tag{2.22}$$

In view of Remark 2.10 (2)

$$\partial_C f(x)=\big\{p\in X\mid\forall T_C(x;K)\ \ \langle p,u\rangle\leqslant f'_C(x;u)\big\}.$$

Hence, in analogy with (2.10)

$$\partial_C f(x)=\partial_C f(x)+N_C(x;K),\tag{2.23}$$

where $N_C(x;K):=[T_C(x;K)]^-$ is the (Clarke) normal cone – see Remark 4.2.

**Remark 2.13.** (1) Clearly $\partial_C f(x) \neq \emptyset$ if and only if $f'_C(x;u) > -\infty$ for all $u \in X$ for then $f'_C(x;\cdot)$, as a convex, lower semicontinuous and positively homogeneous function, is the support function of the nonempty set in the right-hand side of (2.22), i.e.

$$f'_C(x;u) = \sup_{p\in\partial_C f(x)} \langle p,u\rangle,\ u \in X.$$

In particular if $f$ is calm on $K$ at $x$ (with constant $\ell \geqslant 0$), then by Remark 2.10 (2), $f'_C(x;u) \geqslant -\ell\|u\|$ for all $u \in X$. Thus $\partial_C f(x) \neq \emptyset$ and, by the use of an appropriate version of the separation theorem, we may separate Epi $\big(f'_C(x;\cdot)\big)$ from the cone $\{(u,\lambda) \in X\times\mathbb{R} \mid \lambda \leqslant -\ell\|u\|\}$. So there is $p \in X^*$ such that

$$-\ell\|u\| \leqslant \langle p,u\rangle \leqslant f'_C(x;U).$$

Thus $p \in \partial_C f(x)$ and $\|p\| \leqslant \ell$. By the min-max inequality

$$\sup_{u\in X,\,\|u\|\leqslant 1}\ \inf_{p\in\partial_C f(x)} \langle p,u\rangle \leqslant \inf_{p\in\partial_C f(x)}\ \sup_{u\in X,\,\|u\|\leqslant 1} \langle p,u\rangle = \inf_{p\in\partial_C f(x)} \|p\| \leqslant \ell. \tag{2.24}$$

(2) If $x \in K$ and $f$ locally Lipschitz on $K$ at $x$, then $\partial_C f(x)$ is a convex and weakly$^*$-compact subset of $X^*$. If $f$ is Lipschitz on a ball $B(x,\varepsilon)$, $\varepsilon > 0$, (i.e., in particular, $x \in \operatorname{int} K$), then the set-valued map $B(x,\varepsilon) \ni y \mapsto \partial_C f(y) \subset X^*$ is *upper hemicontinuous*, i.e. for any $u \in X$ the function $B(x,\varepsilon) \ni y \mapsto \sup_{p\in\partial_C f(y)}\langle p,u\rangle$ is upper semicontinuous (as a real extended function) – see [2, Sec. 2.6], having nonempty convex, closed and bounded values.

(3) Exactly as before one can establish the following fact:

$$p \in \partial_C f(x) \iff (p,-1) \in N_C\big((x,f(x));\mathrm{Epi}\,(f)\big) := \big[T_C\big((x,f(x));\mathrm{Epi}\,(f)\big)\big]^-.$$

It is often very convenient to have that the *graph* of $\partial_C f$ $\big($i.e. the set $\{(x,p) \in K\times X^* \mid p \in \partial_C f(x)\}\big)$ is closed with respect to the product of the strong topology in $K$ and the weak$^*$-topology in $X^*$. It is always the case locally over $x \in \operatorname{int} K$ provided $f$ is Lipschitz around $x$ in view of Remark 2.13 (2). In general we may define a 'subdifferential' $\overline{\partial}_C f$ as the (sequential) closure of the graph of $\partial_C$ in the strong $\times$ weak$^*$-topology. Therefore

$$p \in \overline{\partial}_C f(x) \Leftrightarrow \exists x_n \to x,\, f(x_n) \to f(x)\ \exists p_n \in \partial_C f(x_n)\ \ p_n \overset{*}{\rightharpoonup} p,$$

i.e. $\overline{\partial}_C f(x)$ is the weak$^*$-upper limit of $\partial_C f(y)$, where $y \to x$ along with $f(y) \to f(x)$.

### 2.3. Hadamard and Fréchet differentiablity.

Let us conclude this section by considering variants of Hadamard's and Fréchet's differentiablity of $f$ $\big(^2\big)$. We say that $f$ is *H-differentiable at $x$* if there is $p \in X^*$ such that for any $u \in T_B(x;K)$ and for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\big|f(x+hv) - f(x) - h\langle p,v\rangle\big| \leqslant \varepsilon|h|$$

if $h \in \mathbb{R}$, $v \in X$ are such that $|h| < \delta$, $\|v-u\| < \delta$ and $x+hv \in K$. We say that $p \in X^*$ satisfying the above condition is a *H-gradient* of $f$ at $x$.

**Example 2.14.** *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ with $K = \mathrm{Dom}(f) = [a,b]$. Then $f$ is H-differentiable at $x \in (a,b)$ $\big($resp. $x = a$ or $x = b\big)$ if and only if the ordinary derivative $f'(x)$ exists $\big($resp. there exist one-sided derivatives $f'_+(a)$ or $f'_-(b)\big)$.* □

It is easy to see that $f$ is $H$-differentiable at $x$ and $p \in X^*$ is an $H$-gradient of $f$ at $x$ if and only if for any $u \in T_B(x;K)$, sequences $h_n \to 0^+$ and $u_n \to u$ such that $x + h_n u_n \in K$ for all $n$ one has

$$\langle p,u\rangle = \lim_{n\to\infty} \frac{f(x+h_n u_n) - f(x)}{h_n}. \tag{2.25}$$

[2] $H$-differentiablity is discussed at length in a somewhat forgotten but very useful book [20, Chapter 4.2].

Hence in this case

$$\forall\, u \in T_B(x;K) \;\; f'_H(x;u) = \langle p,u\rangle \;\text{ and }\; p \in \partial_H f(x). \tag{2.26}$$

In view of (2.10) we have the following

**Proposition 2.15.** *If $f$ is $H$-differentiable at $x \in K$, then*

$$\partial_H f(x) = p + N_B(x;K),$$

*where $p$ is an $H$-gradient of $f$ at $x$.* □

**Remark 2.16.** (1) Observe that not all elements in $\partial_H f(x)$ are $H$-gradients. Precisely, in view of (2.25), $q \in X^*$ is an $H$-gradient of $f$ at $x \in K$ if and only if $q - p \in T_B(x;K)^{\perp}$, where $p$ is an $H$-gradient of $f$ at $x$.

(2) If $f$ is $H$-differentiable at $x$, then it is continuous and locally stable at $x$. The linearity implies that $f'_H(x;0_X) = 0$. Hence, in view of Proposition 2.4 (2), $f$ is calm. The $H$-differentiability of $-f$ implies that $f$ is quiet; the continuity follows indirectly. □

We say that $f$ is *$F$-differentiable at* $x \in K$ and that $p \in X^*$ is an *$F$-gradient* of $f$ at $x$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $u \in X$ such that $x+u \in K$ and $\|u\| < \delta$

$$\big| f(x+u) - f(x) - \langle p,u\rangle \big| \leqslant \varepsilon \|u\|.$$

It is clear that if $f$ is $F$-differentiable at $x$, then it is $H$-differentiable and any $F$-gradient is an $H$-gradient.

2.4. **Lipschitz continuous case.** Here we follow closely [24] with some minor changes. Suppose now that $f$ is Lipschitz on $K$, i.e. there is $\ell > 0$ such that $|f(x) - f(y)| \leqslant \ell \|x-y\|$ for any $x,y \in K$. In this situation the restriction $f|_K$ $\big($i.e. the 'original' $f\big)$ admits the so-called infimal convolution extension $\widetilde{f} : X \to \mathbb{R}$ defined by

$$\widetilde{f}(y) := \inf_{x\in K} \big( f(x) + L\|y-x\| \big), \;\; y \in X,$$

where $L > \ell$ (see [45] and [24]). Observe that for $y \in K$ and $x \in K$, $f(x) + L\|y-x\| \geqslant f(y)$ because $f(y) - f(x) \leqslant \ell\|y-x\| \leqslant L\|y-x\|$, and

$$f(y) \geqslant \widetilde{f}(y) = \inf_{x\in K} \big( f(x) + L\|y-x\| \big) \geqslant f(y).$$

Therefore $\widetilde{f} = f$ on $K$. For any $y,z \in X$ and $x \in K$ we have

$$f(x) + L\|y-x\| \leqslant f(x) + L\|y-z\| + L\|z-x\|$$

and passing to g.l.b with $x \in K$ we get that

$$\widetilde{f}(y) \leqslant \widetilde{f}(z) + L\|y-z\|.$$

This shows that $\widetilde{f}$ is Lipschitz continuous.

According to Proposition 2.11 and (2.19) for any $y \in X$ and $u \in X$

$$\widetilde{f}'_C(y;u) = \widetilde{f}^{\circ}(y;u) = \limsup_{z\to y,\, h\to 0^+} \frac{\widetilde{f}(z+hu) - \widetilde{f}(z)}{h} = \sup_{p\in\partial\widetilde{f}(y)} \langle p,u\rangle,$$

where

$$\partial \widetilde{f}_C(y) := \big\{ p \in X^* \mid \forall\, u \in X \;\; \langle p,u\rangle \leqslant \widetilde{f}^{\circ}(y;u) \big\}$$

and

$$\forall\, p \in \partial_C \widetilde{f}(x) \;\; \|p\| \leqslant L. \tag{2.27}$$

**Proposition 2.17.** *Let $x \in K$. Then*

*(1) for any $u \in X$, $\widetilde{f}^{\circ}(x;u) \leqslant f'_C(x;u)$;*

(2) $\partial_C \widetilde{f}(x) \subset \{p \in \partial_C f(x) \mid \|p\| \leqslant L\}$;

(3) $\partial_C f(x) = \partial_C \widetilde{f}(x) + N_C(x;K)$.

*Proof.* It is obvious that $\mathrm{Epi}\,(f) \subset \mathrm{Epi}\,(\widetilde{f}) \cap K \times [0,+\infty$. Hence

$$T_B\big((x,f(x));\mathrm{Epi}\,(f)\big) \subset T_B\big((x,f(x));\mathrm{Epi}\,(\widetilde{f})\big)$$

and $\widetilde{f}'_H(x;u) \leqslant f'_H(x;u)$ for any $u \in X$. In general a similar inclusion is not true. However, following a delicate argument from [24, Theorem 2], we get that

$$T_C\big((x,f(x));\mathrm{Epi}\,(f)\big) \subset T_C\big((x,f(x));\mathrm{Epi}\,(\widetilde{f})\big).$$

This implies that (1) holds true and then $\partial_C \widetilde{f}(x) \subset \partial_C f(x)$. Therefore $\partial_C \widetilde{f}(x) + N_C(x;K) \subset \partial_C f(x) + N_C(x;K) = \partial_C f(x)$ by (2.23). In view of (2.27) property (2) follows. It is clear that $f = \widetilde{f} + I_K$, where $I_K$ denotes the indicator of $K$. Therefore

$$\partial_C f(x) \subset \partial_C \widetilde{f}(x) + N_C(x;K)$$

in view of [40, Theorem 2] and (4.5). □

## 3. EQUILIBRIA AND GENERALIZED EQUILIBRIA OF NONSMOOTH FUNCTIONALS

3.1. **Functionals bounded below.** In this section, we are going to study some minimization techniques for nonsmooth functionals. The celebrated Ekeland variational principle will constitute the main tool to deal with these problems. We shall survey some known results and complement them with some new examples and facts relying on a systematic study of various inf-compactness conditions of the Palais-Smale type. The proofs are very simple thanks to the preliminary facts provided above.

Let us recall the following version of the Ekeland principle (stated in terms of our functional $f$, i.e. $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $\mathrm{Dom}(f) = K$ is closed) – see [18] and [38]. We suppose that $f$ is bounded from below and let

$$m := \inf_{x \in K} f(x) = \inf_{x \in X} f(x). \tag{3.1}$$

**Theorem 3.1.** *For any $\varepsilon > 0$, $\lambda > 0$ and $x_0 \in K$ such that $f(x_0) \leqslant m + \varepsilon\lambda$ there is $x_\varepsilon \in K$ such that*

(1) $\|x_\varepsilon - x_0\| \leqslant \lambda$;

(2) $f(x_\varepsilon) + \varepsilon\|x_\varepsilon - x_0\| \leqslant f(x_0)$;

(3) $f(x_\varepsilon) < f(x) + \varepsilon\|x - x_\varepsilon\|$ *for any* $x \in X$, $x \neq x_\varepsilon$. □

Putting it briefly: *if $f$ is lower semicontinuous and bounded from below, then for any $\varepsilon > 0$ there is $x_\varepsilon$ such that $f(\cdot) + \varepsilon\|x_\varepsilon - \cdot\|$ attains its minimum at $x_\varepsilon$.*

As de Figueiredo in [19] writes - ‘*...this principle discovered in 1972 has found a multitude of applications in different fields of analysis. It has also served to provide simple and elegant proofs of known results (...) it is a tool that unifies many results where the underlying idea is some sort of approximation.*’ To see this idea take a sequence $(y_n) \subset K$ such that $f(y_n) \leqslant m + n^{-2}$, i.e. a minimizing sequence. In view of Ekeland's principle one is in a position to ‘improve’ it by choosing a sequence $(x_n) \subset K$ such that

$$\begin{aligned} &\text{(a)} \qquad f(x_n) \leqslant f(y_n),\\ &\text{(b)} \qquad \|x_n - y_n\| \leqslant n^{-1},\\ &\text{(c)} \qquad \forall x \in X, x \neq x_n \;\; f(x_n) < f(x) + n^{-1}\|x_n - x\|. \end{aligned} \tag{3.2}$$

Recall that, having the lower semicontinuity of $f$, the so-called *inf-compactness* of $f$, i.e. the hypothesis concerning compactness of sublevel sets $f^\lambda := \{x \in K \mid f(x) \leqslant \lambda\}$, $m < \lambda$, guarantees the existence a minimizer $x_0$, $f(x_0) = m$. If $X$ is reflexive and $f$ is additionally convex, then level sets are weakly compact provided they are bounded.

In order to deal with a general problem of minimization some relaxed forms of inf-compactness are needed. For instance one can consider a functional $f$ such that

$$\lim_{\lambda \to m^+} \alpha_f(\lambda) = 0,$$

where $\alpha_f(\lambda) := \chi(f^\lambda)$ is the ball (or Hausdorff) measure of noncompactness of $f^\lambda$. This can be achieved if (a generalized) pseudo-gradient field of sorts associated with $f$ is condensing with respect to $\chi$. This topic however, as quite distant, will not be pursued here. Instead we will discuss some forms of the compactness assumptions generally known as Palais-Smale type conditions. There is a variety of such conditions considered for smooth and non-smooth functionals – see [31] for an excellent survey of many results concerning the Palais-Smale conditions, and e.g [49] for a survey of such conditions in case of nonsmooth situation and rather an extensive discussion and bibliography.

Let us recall some of these conditions (in a slightly different form).

**Hypothesis 3.2.** The functional $f$ is said to satisfy the compactness condition :

- $(C_1)$, the *Palais-Smale condition in the sense of Costa and Goncalves*, if given sequences $(x_n) \subset K$ and $(\varepsilon_n)$, $(\delta_n)$ of positive numbers such that $\varepsilon_n, \delta_n \to 0^+$,
$$f(x_n) \to m \;\text{ and }\; f(x_n) \leqslant f(x) + \varepsilon_n \|x_n - x\| \;\text{ if }\; x \in K,\; \|x - x_n\| \leqslant \delta_n,$$
then $(x_n)$ has an accumulation point;
- $(C_2)$ if for any sequence $(x_n) \subset K$ such that
$$f(x_n) \to m \;\text{ and }\; \liminf_{n\to\infty} \inf_{\|u\|\leqslant 1} f'_H(x_n, u) \geqslant 0$$
has a convergent subsequence.
- $(C_3)$, the *Palais-Smale condition in the sense of Chang*, if for any sequence such that
$$f(x_n) \to m \;\text{ and }\; \inf_{p \in \partial_C f(x_n)} \|p\| \to 0$$
possesses a convergent subsequence.

**Remark 3.3.** (1) $(C_2)$ implies $(C_1)$. Given sequences $(x_n) \subset K$ and $(\varepsilon_n)$ and $(\delta_n)$ of positive numbers such that $\varepsilon_n, \delta_n \to 0$, $f(x_n) \to m$ and $f(x_n) \leqslant f(x) + \varepsilon_n \|x_n - x\|$ if $x \in K$ when $\|x - x_n\| \leqslant \delta_n$, we actually get that, for any $n$, $f$ is calm (with constant $\ell = \varepsilon_n$) on $K \cap B(x_n, \delta_n)$. Therefore in view of Proposition for 2.4

$$\inf_{\|u\|\leqslant 1} f'_H(x_n; u) \leqslant -\varepsilon_n.$$

Thus $(x_n)$ has an accumulation point.

In a similar manner we prove that $(C_3)$ implies $(C_1)$. Given a sequence as above we may refer to Remark 2.13 (1) in order to obtain for any $n \in \mathbb{N}$ an element $p_n \in \partial_C f(x_n)$ with $\|p_n\| \leqslant \varepsilon_n$. Having this condition $(C_3)$ gives a convergent subsequence of $(x_n)$.

(2) Condition $(C_1)$ implies the standard Palais-Smale condition for $F$-differentiable functionals (see subsection 2.3): if $(x_n) \subset K$ is a sequence such that $f(x_n) \to m$, $p_n$ is an $F$-gradient of $f$ at $x_n$ and $\|p_n\| \to 0$, then $x_n$ has a convergent subsequence. If we have such a sequence, then for a sequence $\varepsilon_n \to 0$ such that $\varepsilon_n > 2\|p_n\|$ there is $\delta_n > 0$ such that for any $x \in K$

$$\left| f(x) - f(x_n) - \langle p_n, x - x_n \rangle \right| \leqslant 2^{-1} \varepsilon_n \|x - x_n\|$$

if $\|x - x_n\| < \delta_n$. Hence

$$f(x) \geqslant f(x_n) + \langle p, x - x_n \rangle - 2^{-1}\varepsilon_n \|x - x_n\| \geqslant -\varepsilon_n \|x - x_n\| \;\text{ i }\; \|x - x_n\| \leqslant \delta_n.$$

In view of $(C_1)$, $(x_n)$ possesses a convergent subsequence.

(3) In view of Theorem [49, Theorem 1] if $(C_1)$ is satisfied, then $(C_3)$ holds true if $f$ is convex.

We are in a position to get the following theorem that corresponds to numerous minimization results obtained by different authors. Let us mention only a few references: [16], where the set $K$ of constraints was assumed to be convex and where results of Struwe [46] have been generalized and [13].

**Theorem 3.4.** *Let $f : K \to \mathbb{R}$ be lower semicontinuous and bounded from below, let $m = \inf_{x\in K} f(x)$. Suppose that it $f$ satisfies one of conditions $(C_1)$, $(C_2)$ or $(C_3)$. Then there is $x_0 \in K$ such that $0 \in \partial_H f(x_0) \subset \partial_C(x_0)$ and $f(x_0) = 0$. Moreover $N_B(x_0;K) \subset \partial_H f(x_0)$ and $N_C(x_0;K) \subset \partial_C f(x_0)$.*

*Proof.* By the Ekeland principle one gets a sequences $(x_n)$ and $(y_n)$ satisfying (3.2). Clearly $m \leqslant \lim_{n\to\infty} f(x_n) \leqslant \lim_{n\to\infty} f(y_n) = m$. By condition $(C_1)$ (in view of Remark 3.3 $(C_2)$ (resp. $(C_3)$) implies $(C_1)$), up to a subsequence, $x_n \to x_0$. Thus, by lower semicontinuity,

$$m \leqslant f(x_0) \leqslant \liminf_{n\to\infty} f(x_n) = m.$$

Since $x_0$ is a minimizer of $f$ we gather that $f$ is calm (with constant $\ell = 0$). Therefore $0 \leqslant f'_h(x_0;u) \leqslant f'_C(x_0;u)$ for any $u \in X$ and, hence, $0 \in \partial_H f(x_0) \subset \partial_C f(x_0)$. The last statement follows from (2.10) and (2.23), respectively. □

Let us now provide some examples of functionals satisfying Palais-Smale type conditions. We start with the following lemma (after [17]) – for the sake of completeness we provide a proof.

**Lemma 3.5.** *Suppose that $Y$ is a Banach space such that $j : X \hookrightarrow\hookrightarrow Y$ (i.e. the embedding $j$ is continuous and compact). Let $G : Y \to \mathbb{R}$ be locally Lipschitz. Then the functional $g := G \circ j : X \to \mathbb{R}$ is locally Lipschitz and the set-valued map*

$$X \ni x \mapsto \varphi(x) := \partial_C g(x) \subset X^*$$

*is completely continuous with nonempty compact convex values, i.e. it is upper semicontinuous, has compact values and the image $\varphi(B)$ is relatively compact nonempty values in $X^*$, provided $B \subset X$ is bounded.*

*Proof.* It is clear that $g$ is locally Lipschitz, the values of $\varphi$ are convex weakly$^*$-compact and $\varphi$ is upper hemicontinuous. In order to prove the upper semicontinuity and the compactness of values of $\varphi$ it is sufficient to show that given sequences $x_n \to x$ in $X$ and $p_n \in \varphi(x_n)$, there is a subsequence $(p_{n_k})$ of $(p_n)$ such that $\lim_{k\to\infty} p_{n_k} = p \in \varphi(x)$. To this aim observe that, by the upper hemicontinuity of $\partial_C G$ ([3]), the image $\partial_C G\big(j(Z)\big)$, where $Z := \overline{\{x_n \mid n \in \mathbb{N}\}}$, is bounded as $j(Z)$ is relatively compact. By the Schauder Theorem, the adjoint $j^* : Y^* \to X^*$ is compact. Therefore the set $j^* \circ \partial_C G\big(j(Z)\big)$ is relatively compact. In view of [10, Theorem 2.3.11, Remark 2.3.11] for each $n \in \mathbb{N}$,

$$p_n \in \varphi(x_n) \subset j^* \circ \partial_C G\big(j(x_n)\big) \subset j^* \circ \partial_C G\big(j(Z)\big).$$

Passing to a subsequence if necessary, we may suppose that $p_n \to p \in X^*$. In particular $p_n \overset{w^*}{\rightharpoonup} p$ (weakly$^*$). The closeness of the graph of $\varphi$ (in $X \times X^*$ with $X^*$ considered with the weak$^*$-topology) implies that $p \in \varphi(x)$ as required.

In order to complete the proof it is enough to show that given a *bounded* sequence $(x_n)$ in $X$, if $p_n \in \varphi(x_n)$, then $(p_n)$ has a convergent subsequence. Indeed: in view of the compactness of $j$, the set $C := \overline{\big\{j(x_n) \mid n \in \mathbb{N}\big\}} \subset X$ is compact; hence $\partial_C G(C)$ is bounded in $X^*$. The same proof as above shows that, for each $n \geqslant 1$, $p_n \in j^* \circ \partial_C G(C)$. The relative compactness of $j^* \circ \partial_C G(C))$ ends the proof. □

[3]For the definition and properties of upper hemicontinuous set-valued maps – see [2, Section 2.6].

Suppose that the space $X$ is reflexive and let $a : X \times X \to \mathbb{R}$ be a symmetric continuous bilinear form. Consider a linear operator $A : X \to X^*$ given by

$$\forall y \in X \ \ \langle Ax, y \rangle = a(x,y), \ \ x \in X. \tag{3.3}$$

It is immediate to see that $A$ is continuous and self-adjoint, i.e. $A = A^* \circ J$, where $J : X \to X^{**}$ is the canonical evaluation isomorphism. By $N(A)$ and $R(A)$ we denote the null-space and the range of $A$. Let us suppose that the dimension $\dim N(A) < \infty$ and that $R(A)$ is closed. Then, by the closed range theorem $\operatorname{codim} R(A) = \dim {}^{\perp}R(A) = \dim N(A)$. This shows that $A$ is a Fredholm operator of index 0.

**Example 3.6.** In the context of Lemma 3.5 assume that $Y$ is reflexive and there is a constant $c > 0$ such that for any $x \in X$

$$c\|x\| \leqslant \|Ax\| + \|j(x)\|.$$

Then, by Peetre's lemma (see e.g. [30, Chapter 2, Section 5.2.]) $\dim N(A) < \infty$ and $R(A)$ is closed. □

**Proposition 3.7.** *Let a bilinear, symmetric and continuous form $a(\cdot,\cdot)$ be nonnegative, i.e. $a(x,x) \geqslant 0$ for any $x \in X$, and let $g$ is given in Lemma 3.5 be bounded from below. Then $f : X \to \mathbb{R}$ given by*

$$f(x) = \frac{1}{2} a(x,x) + g(x) \ \ \text{for} \ \ x \in X,$$

*satisfies condition $(C_3)$ on any bounded subset of $X$.*

*Proof.* Since the function $X \ni x \mapsto a(x,x)$ is continuously Fréchet differentiable, it is immediate to see that $f$ is locally Lipschitz and bounded below; moreover, in view of [10, Proposition 2.3.3, Proposition 2.2.1] for any $x, u \in X$,

$$f'_C(x;u) = f^\circ(x;u) \leqslant \langle Ax, u \rangle + g^\circ(x;u),$$

where $A : X \to X^*$ given by (3.3). Hence, for all $x \in X$,

$$\partial_C f(x) \subset Ax + \partial_C g(x).$$

Let $m = \inf f$ and take a bounded sequence $(x_n)$ such that $f(x_n) \to m$ and $\inf_{q \in \partial_C f(x_n)} \|q\| \to 0$. This implies that that there are sequences $(q_n)$ and $(p_n)$ such that $q_n = Ax_n + p_n$, where $p_n \in \partial_C g(x_n)$ and $\|q_n\| \to 0$ as $n \to \infty$. In view of Proposition 3.5, there is a subsequence $\lim_{k \to \infty} p_{n_k} = p \in X^*$. Hence $Ax_{n_k} \to p$ as $k \to \infty$.

Since $A$ is a Fredholm operator, $X$ and $X^*$ admit direct sum decompositions of the form

$$X = N(A) \oplus X_1, \ \ X^* = R(A) \oplus Z,$$

where $X_1$ and $Z$ are closed linear subspaces of $X$ and $X^*$, respectively. Consider the continuous projection $P : X \to N(A)$ onto $N(A)$ and $Z$. In view of the Banach inverse mapping theorem $A|_{X_1} : X_1 \to R(A)$ is a linear homeomorphism. Let $B := [A|_{X_1}]^{-1} : R(A) \to X_1$ be the (continuous) inverse of $A|_{X_1}$. Let $x_{n_k} = y_{n_k} + z_{n_k}$, where $y_{n_k} = B\big(Ax_{n_k}\big) \in X_1$ and $z_n \in N(A)$. Since $N(A)$ is finite-dimensional we may assume without loss of generality that $z_{n_k}$ is convergent. □

**Proposition 3.8.** *Suppose that $X$ is a reflexive space endowed with a norm $\|\cdot\|$ (the so-called the* Troyanski *norm) under which $X$ and $X^*$ are locally uniformly convex and let $f : X \to \mathbb{R}$ be given by*

$$f(x) = \frac{1}{2}\|x\|^2 + g(x), \ \ x \in X,$$

*where $g$ is as in Lemma 3.5 and bounded from below. Then $f$ satisfies condition $(C_3)$ on bounded sets.*

*Proof.* Again $f$ is locally Lipschitz, bounded below and

$$\partial_C f(x) \subset J(x) + \partial_C g(x) \ \ \text{for} \ \ x \in X,$$

where $J$ is the (normalized) duality mapping, i.e., $J(x) := \{p \in X^* \mid \langle p, x\rangle = \|x\|^2 = \|p\|^2\}$, $x \in X$. Since $X$ and $X^*$ are locally uniformly convex $J$ is a single-valued homoeomorphism (see e.g. [4, Theorem 1.2]. Take a bounded sequence $(x_n)$ such that $f(x_n) \to m = \inf f$ and $J(x_n) + p_n \to 0$ for some $p_n \in \partial_C g(x_n)$. Again, passing to a subsequence if necessary we may assume that $p_n \to p \in X^*$. This implies that $J(x_n) \to p$ and, therefore, $x_n \to J^{-1}(p)$. $\square$

Palais-Smale compactness conditions are related to the so-called $(S)$-type conditions.

**Hypothesis 3.9.** Let $A : X \multimap X^*$ be a multivalued operator. We say $A$ satisfies condition $(S)$ provided, for a sequence $(x_n) \subset \mathrm{Dom}(A)$ such that $x_n \rightharpoonup x$, if $\inf_{p\in A(x_n)} \langle p, x_n - x\rangle \to 0$, then $x_n \to x$.

Condition $(S)$, introduced by F. Browder, seems to be a very weak monotonicity property – see [22] for details.

**Lemma 3.10.** *Suppose that $X$ is uniformly convex. If a map $A : X \multimap X^*$ is* $d$-monotone*, i.e. there is a* gauge*, i.e. an increasing function $\rho : [0, +\infty) \to \mathbb{R}$ such that for all $x, y \in X$, $p \in A(x)$ and $q \in A(y)$*

$$\langle p - q, x - y\rangle \geqslant \big(\rho(\|x\|) - \rho(\|y\|)\big)\big(\|x\| - \|y\|\big),$$

*then $A$ satisfies condition $(S)$.*

*Proof.* Indeed if $x_n \rightharpoonup x$, and $\inf_{p\in A(x_n)}\langle p, x_n - x\rangle \to 0$, the there is a sequence $(p_n)$ such that $p_n \in A(x_n)$ and $\langle p_n, x_n - x\rangle \to 0$. Take $p \in A(x)$, then $\langle p, x_n - x\rangle \to 0$; hence $\langle p_n - p, x_n - x\rangle \to 0$. Then

$$\big(\rho(\|x_n\|) - \rho(\|x\|)\big)\big(\|x_n\| - \|x\|\big) \to 0.$$

This clearly implies that $\|x_n\| \to \|x\|$. The uniform convexity of $X$ entails that $x_n \to x$. $\square$

**Corollary 3.11.** *Let $X$ be reflexive with uniformly convex $X^*$ and let $g : X \to \mathbb{R}$ be convex and bounded below. If $f : X \to \mathbb{R}$ is given by*

$$f(x) := \frac{1}{s}\|x\|^s + g(x), \ \ x \in X,$$

*where $s > 1$, then $f$ satisfies condition $(C_3)$ on bounded sets.*

*Proof.* Evidently $f$ is convex (and locally Lipschitz) and bounded below. Moreover the function $h : X \ni x \mapsto \frac{1}{s}\|x\|^s$ is continuously Fréchet differentiable and, for any $x \in X$,

$$h'(x) = J_s(x) \in X^*,$$

where $\langle J_s(x), x\rangle = \|x\|^s$, $\|J_s(x)\| = \|x\|^{s-1}$, i.e. $J_s$ is the so-called *duality mapping with the gauge* $\rho(t) = t^{s-1}$, $t \geqslant 0$. Observe that $J_s$ is $d$-monotone $\big($with the same gauge$\big)$ since for any $x, y \in X$,

$$\begin{aligned}\big\langle J_s(x) - J_s(y), x - y\big\rangle &= \|x\|^s - \big\langle J_s(x), y\big\rangle - \big\langle J_s(y), x\big\rangle + \|y\|^s \geqslant \\ &\geqslant \|x\|^s - \|y\|^s - \|x\|\|y\|^{s-1} + \|y\|\|x\|^{s-1} = \\ &= \big(\|x\| - \|y\|\big)\big(\|x\|^{s-1} - \|y\|^{s-1}\big).\end{aligned}$$

On the other hand $\partial_C g$ is (maximal) monotone in view of [12]. Hence

$$\partial_C f(x) \subset J_s(x) + \partial g(x), \ \ x \in X,$$

is $d$-monotone and satisfies condition $(S)$.

Take a bounded sequence $(x_n) \subset X$ such that $f(x_n) \to \inf f$ and $q_n = J_s(x_n) + p_n \to 0$, where $p_n \in \partial f(x_n)$. In view reflexivity, passing to a subsequence, $x_n \rightharpoonup x \in X$. Then $\langle q_n, x_n - x\rangle \to 0$. Hence $x_n \to x$. $\square$

As we have seen all the above testing of the Palais Smale condition included an additional assumption concerning the boundedness of ‘Palais-Smale’ sequences necessary to handle compact embedding issues to prove strong convergence (or to get a weak accumulation point in a reflexive space). This shows that often problems in verifying the Palais-Smale condition are related to the missing boundedness. Therefore, in order, to get respective minimization results for functionals studied above we need to assume that the constraining set $K$ is bounded or additional assumptions concerning a functional are considered.

**Proposition 3.12.** *Let us consider the situation of Proposition 3.7 and, additionally assume that the form a is positive definite, i.e.* $a(x,x) \geqslant \alpha\|x\|^2$*, where* $\alpha > 0$*, and there is* $\gamma > 2$ *and* $R > 0$ *such that*

$$\gamma g(x) \geqslant \sup_{p\in\partial_C(x)} \langle p,x\rangle \ \text{for } \|x\| \geqslant R. \tag{AR}$$

*Then* $f$ *satisfies the Palais-Smale type condition* $(C_3)$.

**Remark 3.13.** Assumption $(AR)$ concerns the ‘subquadratic’ growth of $g$ and corresponds to the celebrated Ambrosetti-Rabinowitz condition considered e.g. in a semilinear Dirichlet problem illustration the Mountain Pass Theorem – see [39].

*Proof.* It is sufficient to show that a sequence $(x_n)$ such that $f(x_n) \to m$, $\inf_{q\in\partial_C f(x_n)} \|p_n\| \to 0$ is bounded.

Suppose it is not the case and for almost all $n \in \mathbb{N}$, $\|x_n\| \geqslant R$. Take $p_n \in \partial_C g(x_n)$ such that $\|Ax_n + p_n\| \to 0$. For sufficiently large $n$

$$\begin{aligned} m+1+\|x_n\| &\geqslant f(x_n) - \gamma^{-1}\langle Ax_n + p_n, x_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{\gamma}\right) a(x_n,x_n) + \left(g(x_n) - \gamma^{-1}\langle p_n,x_n\rangle\right) \\ &\geqslant \left(\frac{1}{2}-\frac{1}{\gamma}\right)\alpha\|x_n\|^2, \end{aligned}$$

in view of $(AR)$. This implies that $\|x_n\|$ is bounded. □

Another remedy of sorts for difficulties with boundedness of Palais-Smale sequences is provided by weaker comapctness conditions, e.g. the so-called *Cerami-type* conditions.

**Hypothesis 3.14.** Let $\alpha : [0,+\infty) \to (0,+\infty)$ be a continuous nonincreasing function such that $\int^{\infty} \alpha(t)\,\mathrm{d}t +$ $+\infty$. We say the $f$ satisfies condition

- $(C_4)$, the *Cerami condtion* if given sequences $(x_n) \subset K$, $\varepsilon_n, \delta_n \to 0^+$ such that $f(x_n) \to m := \inf_{x\in K} f(x)$ and

  $$f(x_n) \leqslant f(x) + \varepsilon_n \frac{\|x-x_n\|}{1+\|x_n\|} \ \text{ for } x \in K,\ \|x-x_n\| \leqslant \delta_n,$$

  then $(x_n)$ has a convergent subsequence;
- $(C_5)$ if for any sequence $(x_n) \subset K$ such that

  $$f(x_n) \to m \ \text{ and } \left(1+\|x_n\|\right) \inf_{p\in\partial_C f(x_n)} \|p\| \to 0$$

  has a convergent subsequence.

Exactly as above we observe that condition $(C_5)$ implies $(C_4)$. It is also clear that $(C_1)$ implies $(C_4)$ while $(C_3)$ implies $(C_5)$.

**Theorem 3.15.** *Under assumptions of Theorem 3.4 if $f$ satsfies condtion $(C_4)$ or $(C_5)$, then there is $x_0 \in K$ such that $f(x_0) = m$ and $0 \in \partial_H f(x_0) \subset \partial_C f(x_0)$. Moreover $N_B(x_0;K) \subset \partial_H f(x_0)$ and $N_C(x_0;K) \subset \partial_C f(x_0)$.*

*Proof.* Evidently $\int_0^\infty (1+t)^{-1}\,dt = \infty$. In view of the generalized version of the Ekeland Principle due to C. K. Zhong [50] for any $n \in \mathbb{N}$ and $y_n \in K$ such that $f(y_n) \leqslant m + n^{-2}$, there is $x_n \in K$ such that

$$f(x_n) \leqslant f(y_n),\ \|x_n - y_n\| \leqslant r_n \ \text{ and } \ f(x_n) \leqslant f(x) + \frac{\|x - x_n\|}{n\big(1+\|x_n\|\big)} \ \text{ for all } \ x \in X,$$

where $r_n > 0$ is such that $n\int_0^{r_n}(1+t)^{-1}\,dt \geqslant 1$ for any $n$. This, together with condition $(C_4)$, completes the proof. □

We say that $g$ is *asymptotically quadratic* (in the sense of Fréchet) if there is $p \in X^*$ such that

$$\limsup_{\|x\|\to\infty} \frac{\sup_{q\in\partial_C g(x)} \|q-p\|}{\|x\|} = 0. \tag{3.4}$$

This condition says actually that the Clark subdifferential is asymptotically linear.

Again let us consider the situation of Proposition 3.12.

**Lemma 3.16.** *If $a$ is positive definite and $g$ is asymptotically quadratic, then any sequence such that $(1+\|x_n\|)\inf_{p\in\partial_C f(x_n)} \|p_n\| \to 0$ is bounded.*

*Proof.* Take a sequence $(x_n)$ such that $f(x_n) \to m$ and a sequence $p_n \in \partial_C g(x_n)$ such that $\big(1+\|x_n\|\big)\|p_n\| \to 0$. Since

$$\big(1+\|x_n\|\big)\|p_n\| \geqslant \big|\langle p_n, x_n\rangle\big| = \big|\langle Ax_n + q_n, x_n\rangle\big|$$

for some $q_n \in \partial_C g(x_n)$, we get that that the sequence $(a(x_n,x_n) + \langle q_n, x_n\rangle)$ is bounded. Take $0 < \varepsilon < \alpha$. Due to (3.4) there is $R > 0$ such that for $\|x\| \geqslant R$ and $q \in \partial_C g(x)$, $\|q - p\| \leqslant \varepsilon\|x\|$. If $\|x_n\| \geqslant R$ for large $n$, then

$$\begin{aligned}\langle Ax_n + q_n, x_n\rangle &= a(x_n,x_n) + \langle q_n - p, x_n\rangle + \langle p, x_n\rangle \\ &\geqslant (\alpha-\varepsilon)\|x_n\|^2 - \|p\|\|x_n\|.\end{aligned}$$

showing that $(x_n)$ is bounded. □

It actually appears that conditions $(C_1)$ and $(C_4)$ are equivalent (as do conditions $(C_3)$ and $(C_5)$). To see this we can argue as in [13].

**Lemma 3.17.** *If $f$ satisfies $(C_4)$ condition, then the sublevel $f^\lambda$ is bounded for some $\lambda > m$. This implies that $(C_4)$ implies $(C_1)$.*

*Proof.* If not then, for all large $n$ the sublevel $f^{m+n^{-2}}$ is not bounded and there are sequences $(x_n)$ and $(y_n)$ as in (3.2), where additionally $\|y_n\| > n$. But this prevents $(x_n)$ from having a partial limit. □

The observation made in the above Lemma corresponds to the question concerning relations of different Palais-Smale conditions (recall that $f$ is *coercive* if for every $\lambda \in \mathbb{R}$ (equivalently for all large $\lambda$), the sublevel $f^\lambda$ is bounded). For other results in this direction – see e.g. [7] where the differentiable case was studied, [48] for the Lipschitz functionals and [28] for some other issues.

It is possible to get a piece of slightly more precise information in the spirit of the Karush-Kuhn-Tucker theory $\big($not being a trivial consequence of (2.10) or (2.23)$\big)$.

**Corollary 3.18.** *Suppose that $f : K \to \mathbb{R}$ is locally Lipschitz, bounded below and satisfy one of conditions $(C_i)$, $i = 1,2,\ldots,5$. Then there is $x_0 \in K$ such that $0 \in \partial_C f(x_0) + N_C(x_0;K)$.*

*Proof.* In view of Theorems 3.4 or 3.15 there is $x_0 \in K$ such that $f(x_0) = m = \inf_{x\in K} f(x)$. Take $\varepsilon > 0$ such that $f$ is Lipschitz with constant $\ell > 0$ continuous on the closed set $S = \{x \in K \mid \|x - x_0\| \leqslant \varepsilon\}$ and let $\widetilde{f} : X \to \mathbb{R}$ be a Lipschitz continuous extension of the restriction $f|_S$ given in Subsection 2.4. In view of Proposition 2.17 (3) and (2) there is $p \in \partial_C \widetilde{f}(x_0) \cap \big[-N_C(x_0;K)\big]$ and $p \in \partial_C f(x_0)$. □

The above result does not seem to be very interesting as long as we are not in a position to establish the existence $x \in K$ with a nontrivial $p \in \partial_C f(x) \cap [-N_C(x;K)]$. This is what we are going to get in the next subsection.

3.2. **Generalized equilibria.** Now we are going to study a different situation being a general counterpart of the problem outlined above. Assume $K \subset X$ is closed, $F : K \multimap X^*$ is a set-valued map, for any $x \in K$, $F(x) \subset X^*$ is weakly$^*$-compact and nonempty. We will look for *generalized equilibria* of $F$, i.e., $x \in K$ such that

$$F(x) \cap \big[-N_C(x;K)\big] \neq \emptyset.$$

The origin of this problem is clear: in a typical situation $F(x)$ is a subdifferential (of some kind) of a functional $f : K \to \mathbb{R}$ – see Corollary 3.25.

Observe that $x \in K$ is a generalized equilibrium if and only if

$$\sup_{u\in -T_C(x;K),\, \|u\|\leqslant 1} \ \inf_{p\in F(x)} \langle p,u\rangle = 0. \tag{3.5}$$

Indeed, the necessity of (3.5) is evident. Observe that, due to the Sion min-max equality – see [43],

$$\inf_{p\in F(x)} \ \sup_{u\in -T_C(x;K),\, \|u\|\leqslant 1} \langle p,u\rangle = \sup_{u\in -T_C(x;K),\, \|u\|\leqslant 1} \ \inf_{p\in F(x)} \langle p,u\rangle.$$

Hence if (3.5) holds, then there is $p_0 \in F(x)$ such that $\langle p,u\rangle \leqslant 0$ for any $u \in -T_C(x;K)$ since the function $F(x) \ni p \mapsto \sup_{u\in -T_C(x;K),\, \|u\|\leqslant 1} \langle p,u\rangle$, as the upper envelope of the family $\{\langle\cdot,u\rangle\}_{u\in T_C(x;K),\, \|u\|\leqslant 1}$ of weakly$^*$-continuous functions, is weakly$^*$-lower semicontinuous and $F(x)$ is a weakly$^*$-compact set.

Let us now study the following situation.

**Hypothesis 3.19.** *We assume that*

- *$A : D(A) \subset X \to X$ is a densely defined closed linear operator generating a strongly continuous semigroup $\{e^{-tA}\}_{t\geqslant 0}$ of linear contractions, i.e. for each $T \geqslant 0$, $e^{-tA} : X \to X$ is a linear (bounded) operator with $\|e^{-tA}\| \leqslant 1$;*
- *$A$ is* resolvent compact, *i.e. for any $h > 0$, the resolvent $J_h := (I + hA)^{-1} : X \to X$ is compact;*
- *a closed $K \subset X$ is* resolvent invariant, *i.e. for all $h > 0$, $J_h(K) \subset K$.*

**Remark 3.20.** (1) In view of the Lumer-Phillips theorem the above hypotheses are satisfied if and only if $A$ is a densely defined $m$-accretive operator, i.e. for any $x \in D(A)$ and $\lambda > 0$, $\|x\| \leqslant \|x + \lambda Ax\|$ and $R(I + A) = X$ – see e.g. [44], [37]. Moreover $(0, +\infty) \subset \rho(A)$, where $\rho(A)$ stands for the resolvent set of $-A$. Hence, for any $h > 0$, the resolvent is well-defined and for $x, y \in X$

$$x = J_\lambda y \Leftrightarrow x \in D(A) \text{ and } x + \lambda Ax = y.$$

For $x \in X$, $t \geqslant 0$,

$$e^{-tA}x := \lim_{n\to\infty} J^n_{t/n}.$$

The operator $A$ is resolvent compact if and only if the embedding $D(A) \subset X$, where $D(A)$ is endowed with the graph norm, is compact.

(2) The most natural example of the situation studied is given by the following example – see e.g. [23, Section 7.3.2, Corollary 7.3.5]. Let $X$ be a Hilbert space and let a Banach $Y$ densely embedded in

$X$. Assume that $a : Y \times Y \to \mathbb{R}$ is a symmetric continuous and positive definite bilinear form. Similarly as in (3.3), let $\mathscr{A} : Y \to Y^*$ be given by $\langle \mathscr{A}x, y\rangle := a(x,y)$ for $x, y \in Y$. The part $A$ of $\mathscr{A}$ in $X = X^*$, i.e. given by $Au = \mathscr{A}u$ for $u \in D(A) = \{u \in Y \mid \mathscr{A}u \in X^*\}$ is a densely defined $m$-accretive operator. Moreover, $A$ is resolvent compact if $j$ is compact.

(3) In view of [29, Propositions 4.5, 4.6] if $K$ is convex, then the following conditions are equivalent:

(i) $K$ is resolvent invariant;

(ii) $K$ is semigroup invariant, i.e. $e^{-tA}(K) \subset K$;

(iii) $K \cap D(A)$ is dense in $K$ and $-Ax \in T_C(x;K)$ for any $x \in K \cap D(A)$;

(iv) if $A$ is defined via procedure described in (2) above then $\pi(Y) \subset Y$ and $a\big(\pi(x), x - \pi(x)\big) \geqslant 0$ for any $x \in Y$, where $\pi : X \to K$ is the metric projection of $X$ onto $K$, i.e. $\big\|x - \pi(x)\big\| = d_K(x)$ for $x \in X$.

**Proposition 3.21.** *Under Hypothesis 3.19 suppose that the set $K$ is convex and bounded, the map $F$ is upper hemicontinuous and bounded and* guided by *$A$, i.e.*

$$\forall\, x \in D(A) \cap K \quad \inf_{p \in F(x)} \langle p, Ax\rangle \leqslant 0. \tag{3.6}$$

*Then, for any $\varepsilon > 0$ there is $\bar{x} \in K$ such that*

$$\inf_{p \in F(\bar{x})} \; \sup_{u \in -T_C(\bar{x};K),\, \|u\| \leqslant 1} \langle p, u\rangle \leqslant \varepsilon. \tag{3.7}$$

*Proof.* We shall prove that for any $\varepsilon > 0$ there is $\bar{x} \in K$ and $\delta > 0$ such that for any $u \in X$, $\|u\| \leqslant 1$

$$d_K^\circ(\bar{x};u) < \delta \implies \inf_{p \in F(\bar{x})} \langle p, u\rangle \leqslant \varepsilon. \tag{3.8}$$

Note that, in view of (4.4), condition (3.8) implies (3.7).

Suppose to the contrary that there $\varepsilon > 0$ such that for all $x \in K$, and all $\delta > 0$ there is $u \in X$ with $\|u\| \leqslant 1$ such that $d_K^\circ(x;u) < \delta$ but $\inf_{p \in F(x)} \langle p, u\rangle > \varepsilon$.

Now choose $0 < \delta < \varepsilon/2M$, where $\sup\big\{\|p\| \mid p \in F(x), x \in K\big\} \leqslant M$. For each $x \in K$ let

$$\varphi(x) := \big\{u \in X \mid \|u\| \leqslant 1,\, d_K^\circ(x;u) < \delta \;\text{ and }\; \inf_{p \in F(x)} \langle p, u\rangle > \varepsilon\big\}.$$

Observe that $\varphi(x)$ is a nonempty convex set. Let $u \in X$, $\|u\| \leqslant 1$, and consider the set

$$\varphi^{-1}(u) = \big\{x \in K \mid u \in \varphi(x)\big\}.$$

The function $d_K^\circ(;u)$ is upper semicontinuous and the function $K \ni x \mapsto \inf_{p \in F(x)} \langle p, u\rangle$ is lower semicontinuous. This shows that the set $\varphi^{-1}(u)$ is open. Hence the family $\big\{\varphi^{-1}(u)\big\}_{u \in X,\, \|u\| \leqslant 1}$ is an open covering of $K$. Let $\big\{\lambda_i\big\}_{i \in I}$ be a partition of unity subordinated to this covering, i.e. for each $i \in I$, the function $\lambda_i : K \to [0,1]$ is continuous, there is $u_i \in X$, $\|u_i\| \leqslant 1$ such that the support $\operatorname{supp} \lambda_i \subset \varphi^{-1}(u_i)$, the family $\{\operatorname{supp} \lambda_i\}_{i \in I}$ is locally finite and $\sum_{i \in I} \lambda_i(x) = 1$ for any $x \in X$. Having this define

$$g(x) = \sum_{i \in I} \lambda_i(x) u_i, \;\; K.$$

Then $g$ is a well-defined continuous function, $\|g(x)\| \leqslant 1$ and $g(x) \in \varphi(x)$ for any $x \in K$ since if $x \in \operatorname{supp} \lambda_i$, then $x \in \varphi^{-1}(u_i)$, i.e. $u_i \in \varphi(x)$.

Now let $r : X \to K$ be a retraction. It is well-know that we may take $r$ such that

$$\|x - r(x)\| \leqslant 2d_K(x),\; x \in K.$$

Now let, for $h > 0$, a map $f_h : K \to X$ be given by

$$f_h(x) = J_h \circ r\big(x + hg(x)\big), \;\; x \in K. \tag{3.9}$$

By assumption 3.19 $K$ is resolvent invariant. Hence $f_h(K) \subset K$. On the other hand $J_h$ is compact, $K$ is bounded and so is $g$. Thus, by the Schauder fixed point theorem, for any $h > 0$, there is $x_h \in K \cap D(A)$ such that $f_h(x_h) = x_h$. This means that

$$x_h + hAx_h = r\big(x_h + hg(x_h)\big),$$

so consequently

$$h\big\|g(x_h) - Ax_h\big\| = \big\|x_h + hg(x_h) - (x_h + hAx_h)\big\| =$$
$$\big\|x_h + hg(x_h) - r\big(x + h + hg(x_h)\big)\big\| \leqslant 2d_K\big(x_h + hg(x_h)\big)$$

and hence for all $h > 0$

$$\big\|g(x_h) - Ax_h\big\| \leqslant 2\|g(x_h)\| \leqslant 2.$$

This implies that the set $\{Ax_h\}_{h>0}$ is bounded. Fix a $\lambda > 0$ and observe that for any $h > 0$

$$x_h = J_\lambda\big(x_h + \lambda Ax_h\big).$$

The compactness of $J_\lambda$ implies that, passing to a subsequence, if necessary we can assume that $x_h \to x_0$. Next

$$\big\|g(x_h) - Ax_h\big\| \leqslant 2\left(\frac{d_K\big(x_h + hg(x_0)\big)}{h} + \big\|g(x_h) - g(x_0)\big\|\right).$$

Hence

$$\limsup_{h\to 0^+}\big\|g(x_h) - Ax_h\big\| \leqslant \limsup_{h\to 0^+}\frac{d_K\big(x_h + hg(x_0)\big)}{h} = d_K^\circ\big(x_0; g(x_0)\big) < 2\delta < \varepsilon/M$$

since $g(x_0) \in \varphi(x_0)$. There is $\eta > 0$ such that for all $0 < h < \eta$, $\big\|g(x_h) - Ax_h\big\| < \varepsilon/M$. Take $0 < h < \eta$. Then, for any $p \in F(x_h)$, $\big\langle p, g(x_h)\big\rangle > \varepsilon$ since $g(x_h) \in \varphi(x_h)$. On the other hand, in view (3.6), there is $\bar p \in F(x_h)$ such that $\langle \bar p, Ax_h\rangle \leqslant 0$. Therefore

$$\varepsilon < \big\langle \bar p, g(x_h) - Ax_h\big\rangle \leqslant \|\bar p\|\big\|g(x_h) - Ax_h\big\| \leqslant \varepsilon.$$

A contradiction completes the proof of (3.8) and (3.7). □

Proposition 3.21 asserts the existence of ‘approximate’ generalized equilibria. In oredr to establish the existence of generalized equilibria we need to impose the next compactness condition corresponding to Palais-Smale type conditions studied above.

**Hypothesis 3.22.** Suppose that $K$ is convex. Condition $(SC)$, the *Struwe-Chang compactness condition* is satisfied if a sequence $(x_n) \subset K$ such that

$$\inf_{p\in F(x_n)} \sup_{y\in K,\, \|y-x_n\|\leqslant 1} \langle p, x_n - y\rangle \to 0, \tag{3.10}$$

has a convergent subsequence.

Condition $(SC)$ in case $F = f'$, where $f$ is a $C_1$-functional, has been introduced by Struwe (see [46]) in the ontext of the Plateau problem and, independently, by Chang (see [8]). A similar condition has been studied in [16].

Recall that if $x \in K$, then

$$T_C(x;K) = \overline{\bigcup_{h>0} h^{-1}(K - x)}.$$

Hence the coefficient

$$\gamma_F(x) := \inf_{p\in F(x)} \sup_{y\in K,\, \|y-x\|\leqslant 1} \langle p, x - y\rangle,\ \ x \in K$$

and condtion $(SC)$ well-corresponds to (3.7). Namely if (3.7) holds, them $\gamma_F(\bar x) \leqslant \varepsilon$ and if $\gamma_F(x_0) = 0$ for some $x_0 \in K$, then there is $p_0 \in F(x_0)$ such that $\langle p, x_0 - y\rangle = 0$ for any $y \in K$, i.e. $p_0 \in -N_C(x_0;K)$.

**Example 3.23.** It is easy to see that operators satisfying condition $(S)$ (see definition in 3.9) satisfy $(SC)$, too. Let us consider a different situation. Let $X$ be uniformly convex, $F(x)=F_1(x)+F_2(x)$, $x\in K$, where $F_1 : X \multimap X^*$ is $d$-monotone (see definition in Lemma 3.10), $K\subset \mathrm{Dom}(F_1)$, and $F_2 : X\to X^*$, $K\subset \mathrm{Dom}(F_2)$, satisfies condition $(P)$: if $x_n \rightharpoonup x$ in $K$, then $\limsup_{n\to\infty}\inf_{q\in F_2(x_n)}\langle q, x_n - x\rangle \geqslant 0$ ([4]). Then $F$ has the $(SC)$ property.

Indeed: take a sequence $(x_n)$ such that

$$\inf_{p\in F(x_n)} \sup_{y\in K,\, \|y-x_n\|\leqslant 1} \langle p, x_n - y\rangle \to 0.$$

Since $K$ is bounded and weakly closed, we may assume without loss of generality that $x_n \rightharpoonup x\in K$. For any $n\in\mathbb{N}$ there are $r_n\in F_1(x_n)$ and $q_n\in F_2(x_n)$ such that

$$\langle r_n, x_n - x\rangle + \langle q_n, x_n - x\rangle \to 0.$$

In view of condition $(P)$, $\limsup_{n\to\infty}\langle q_n, x_n - x\rangle \geqslant 0$. Therefore $\liminf_{n\to\infty}\langle r_n, x_n - x\rangle \leqslant 0$. Hence $\liminf_{n\to\infty}\langle r_n - r, x_n - x\rangle \leqslant 0$, where $r\in F_1(x)$. This, together with $d$-monotonicity of $F_1$, implies that $(x_n)$ has a convergent subsequence.

**Theorem 3.24.** *Assume that in addition to hypotheses of Proposition 3.21 condition $(SC)$ is satisfied. Then $F$ has a generalized equilibrium.*

*Proof.* We are going to show that $\gamma_F : K\to\mathbb{R}$ is lower semicontinuous.

*Claim:* The function

$$K\times X \ni (y,u) \to \sigma(y,u) := \sup_{p\in F(y)} \langle p, u\rangle$$

is upper semicontinuous. To this aim take $\lambda\in\mathbb{R}$ and sequences $(y_m)\subset K$, $(u_m)\subset X$ such that $y_m\to y\in K$, $u_m\to u$ and for any $m\in\mathbb{N}$, $\sigma(y_m,u_m)\geqslant\lambda$. We are to show that $\sigma(y,u)\geqslant\lambda$. Fix $\varepsilon>0$ and, for each $m\in N$, take $p_m\in F(y_m)$ such that $\langle p_m, u_m\rangle > \lambda-\varepsilon/2$. Since $F$ is upper hemicontinuous and $y_m\to y$, the image $\bigcup_{m=1}^\infty F(y_m)$ is bounded, i.e. there is $L>0$ such that for any $m$ and $p\in F(y_m)$, $\|p\|\leqslant L$. For large $m$, $L\|u_m-u\|\leqslant\varepsilon/2$. Hence for such $m$, we have that $\langle p_m, u\rangle > \lambda-\varepsilon$. Again by upper semicontinuity of $F$

$$\sigma(y,u) \geqslant \lim_{m\to\infty}\sigma(y_m,u) \geqslant \limsup_{m\to\infty}\langle p_m, u\rangle \geqslant \lambda-\varepsilon.$$

Since $\varepsilon$ was arbitrary, the proof of the upper semicontinuity of $\sigma$ is completed.

Now observe that, in view of e.g. [2, Theorem 4.2.2], the map $K\ni x \multimap T_C(x;K)$ is lower semicontinuous. Therefore the map $K\ni x\multimap \big\{u\in T_C(x;K) \mid \|u\|\leqslant 1\big\}$ is lower semicontinuous, too.

Finally

$$\gamma_F(x) = \sup_{u\in T_C(x;k),\, \|u\|\leqslant 1} \big(-\sigma(x,u)\big),\ \ x\in K.$$

By the standard properties of the so-called marginal functions – see [2, Theorem 1.4.16], we gather that $\gamma_F$ is lower semicontinuous.

In view of Proposition 3.21 there is a sequence $(x_n)$ such that $\gamma_F(x_n)\to 0$. The $(SC)$ condition implies that (passing to a subsequence), $x_n\to x_0\in K$. Therefore

$$0\leqslant \gamma_F(x_0) \leqslant \liminf_{n\to\infty}\gamma_F(x_n) = 0. \qquad \square$$

Let us now derive a couple of corollaries.

**Corollary 3.25.** *Assume that $K$ and $A$ are as above in Proposition 3.21, let $f : K\to\mathbb{R}$ be Lipschitz continuous, $f_C'(x,Ax)\leqslant 0$ for $x\in D(A)\cap K$ and such that*

[4]Condition $(P)$ was introduced by P. Hess (see also [22]). It is again a mild monotonicity condition satisfied by monotone maps, pseudomonotone maps and many others.

- *any sequence* $(x_n) \subset K$ *with*
$$\inf_{y \in K, \|y-x_n\| \leqslant 1} f_C'(x_n, y-x_n) \to 0,$$
*has a convergent subsequence.*

*Then there is* $x_0 \in K$ *such that* $0 \in \partial_C f(x_0) + N_C(x_0;K)$.

As we shall see the compactness condition considered above is a version of $(SC)$ expressed in terms of the Clarke subdifferential. Actually this is the condition considered in [8] in case $f$ is smooth.

*Proof.* Let $\widetilde{f} : X \to \mathbb{R}$ be a Lipschitz continuous extension of $f$ as given in subsection 2.4 and let $F : K \to X^*$ be given by
$$F(x) := \partial_C \widetilde{f}(x), \ \ x \in K.$$
Then $F$ is upper hemicontinuous, bounded ($\sup_{p \in F(K)} \|p\| \leqslant L$, where $L$ is the Lipschitz constant of $\widetilde{f}$) and for $x \in D(A) \cap K$ and $p \in F(x)$
$$\langle p, Ax\rangle \leqslant \widetilde{f}^\circ(x;Ax) \leqslant f_C'(x;Ax) \leqslant 0$$
We now show that condition $(SC)$ is satisfied. Take a sequence $(x_n) \subset K$ such that $\gamma_F(x_n) \to 0$. Then
$$\begin{aligned}\gamma_F(x_n) &= \sup_{y \in K, \|y-x_n\| \leqslant 1} \big( - \sup_{p \in F(x_n)} \langle p, y-x_n\rangle \big) = \sup_{y \in K, \|y-x_n\| \leqslant 1} \big( - \widetilde{f}^\circ(x_n; y-x_n)\big)\\ &\geqslant \sup_{y \in K, \|y-x_n\| \leqslant 1} \big( - f_C'(x_n, y-x_n)\big) \geqslant 0.\end{aligned}$$
This implies that $\inf_{y \in K, \|y-x_n\| \leqslant 1} f_C'(x_n, y-x_n) \to 0$. Hence $(x_n)$ has a convergent subsequence. In view of Theorem 3.24, there are $x_0 \in K$ and $p_0 \in F(x_0) \subset \partial_C f(x_0)$ such that $p_0 \in -N_C(x_0;K)$. $\square$

**Corollary 3.26.** *Let* $K \subset \mathbb{R}^n$ *be compact and convex.*

(1) *If* $F : K \to \mathbb{R}^n$ *is an upper semicontinuous set-valued mapping with nonempty compact convex values, then there is* $x \in K$ *such that* $0 \in F(x) + N_C(x;K)$. *If the interior* $\operatorname{int} K \neq \emptyset$ *and* $F(x) \cap -N_C(x;K) = \emptyset$ *for* $x \in \partial K$, *then there is* $x \in \operatorname{int} K$ *such that* $0 \in F(x_0)$.

(2) *If* $\operatorname{int} K \neq \emptyset$, $f : K \to \mathbb{R}$ *is locally Lipschitz and* $\partial_C f(x) \cap -N_C(x;K) = \emptyset$ *for* $x \in \partial K$, *then there is* $x_0 \in \operatorname{int} K$ *with* $0 \in \partial_C f(x_0)$.

(3) *If* $f : K \to \mathbb{R}$, *where* $K = \{x \in \mathbb{R}^n \mid \|x\| \leqslant R\}$, *is locally Lipschitz and* $f_C'(x;-x) < 0$ *for* $\|x\| = R$, *then there is* $x_0 \in K$, $\|x_0\| < 1$, *such that* $0 \in \partial_C f(x_0)$.

*Proof.* It is sufficient to show (1). It is easy to see that all assumptions of Theorem 3.24 are satisfied for $A \equiv 0$. $\square$

3.3. **Monotone mappings.** Now we shall establish the existence of equilibria of some operators of the monotone type via approach related to the Minty-Browder method.

We say that $G : X \multimap X^*$ is *radially upper hemicontinuous* at $x \in \mathrm{Dom}(G)$ if for any $z \in X$ such that $x + tz \in \mathrm{Dom}(G)$ for $t \in [0,1]$, the function
$$[0,1] \ni t \mapsto \sup_{p \in G(x+tz)} \langle p, z\rangle \in \overline{\mathbb{R}}$$
is upper semicontinuous at $t = 0$.

Note that if $G$ is upper demicontinuous (i.e. upper semicontinuous from $X$ to $X^*$ endowed with the weak$^*$-topology), then it is upper hemicontinuous and it is upper hemicontinuous along straight lines in $\mathrm{Dom}(G)$ and, hence radially upper hemicontinuous. $G$ is said to be radially upper hemicontinuous if it is so at any $x \in \mathrm{Dom}(G)$.

Let us start with the following ‘maximality’ property of monotone operators, being actually a variant of the well-know criterion for maximal monotonicity.

**Lemma 3.27.** *Suppose that $G : X \multimap X^*$ is a radially upper hemicontinuous monotone operator with closed convex values. For any $y \in \operatorname{int}\operatorname{Dom}(G)$ and $q \in X^*$, if*

$$\forall x \in \operatorname{Dom}(G),\, p \in G(x) \qquad \langle p-q, x-y\rangle \geqslant 0, \tag{3.11}$$

*then $q \in G(y)$.*

*Proof.* Assume that there is a pair $(y,q) \in \operatorname{int}\operatorname{Dom}(G) \times X^*$ satisfying condition (3.11), but $q \notin G(y)$. Then, by the separation theorem, there is $z \in X$ such that

$$\langle q, z\rangle > \sup_{p \in G(y)} \langle p, z\rangle.$$

We may assume that $y + tz \in \operatorname{int}\operatorname{Dom}(G)$ for $t \in [0,1]$ since, otherwise, we may take $\lambda z$ with $\lambda > 0$ small enough guarantee that $y + t\lambda z \in \operatorname{int}\operatorname{Dom}(G)$ for all $t \in [0,1]$. Let $y_t := y + tz$, $t \in [0,1]$. Then for any $t \in (0,1]$ and $q \in G(y_t)$,

$$0 \leqslant \langle p-q, y-y_t\rangle = t\langle q-p, z\rangle.$$

Hence for $t \in (0,1]$

$$\sup_{q \in G(y_t)} \langle q, z\rangle \geqslant \langle p, z\rangle.$$

The radial upper hemicontinuity implies that the function $[0,1] \ni t \mapsto \sup_{q \in G(y_t)}\langle q, z\rangle$ is upper semicontinuous (as a real function). Thus

$$\langle p, z\rangle > \sup_{q \in G(y)} \langle q, z\rangle \geqslant \limsup_{t \to 0^+} \sup_{q \in G(y_t)} \langle q, z\rangle \geqslant \langle p, z\rangle..$$

The contradiction completes the proof. □

The next result is a noncompact, infinite dimensional version of the Debrunner-Flor extension inequality [15].

**Lemma 3.28.** *Assume that the space $X$ is reflexive, $K$ is closed and convex, $G : X \multimap X^*$ is monotone $0 \in Dom(G) \subset K$, $g : X \to X^*$ single valued monotone,* demicontinuous $\big($*i.e. continuous with respect to the weak$^*$-topology in $X^*$*$\big)$ *and* coercive $\big($*in the sense that* $\lim_{\|x\| \to \infty} \|x\|^{-1}\big\langle g(x), x\big\rangle = \infty\big)$. *Then there is $x_0 \in K$ such that*

$$\forall x \in \operatorname{Dom}(G),\, p \in G(x) \quad \big\langle p + g(x_0), x - x_0\big\rangle \geqslant 0. \qquad \square$$

For the proof – see [4, Theorem 2.1].

**Theorem 3.29.** *Let $X$ be reflexive, let $U \subset X$ be open and $0 \in U$. We assume that*

- *$G : X \multimap X^*$ is a radially upper hemicontinuous monotone set-valued map with closed convex values and let $\overline{U} \subset \operatorname{Dom}(G)$;*
- *$g : X \to X^*$ is a montone demicontinuous and coercive single-valued mapping;*

*If for any $x \in \partial U$*

$$\inf_{p \in G(x)} \big\langle p + g(x), x\big\rangle > 0, \tag{3.12}$$

*then there is $x_0 \in U$ such that $0 \in G(x_0) + g(x_0)$.*

*Proof.* Let $K := \overline{\mathrm{conv}}U$. Then all assumptions of Debrunner-Flor lemma 3.28 are satisfied $\big($if necessary we may redefine $G$ by putting $G(x) = \emptyset$ for $x \notin K$ to guarantee that $0 \in U \subset \mathrm{Dom}(G) \subset K\big)$. Hence there is $x_0 \in K$ such that

$$\big\langle p + g(x_0), x - x_0\big\rangle \geqslant 0 \tag{3.13}$$

for all $x \in \mathrm{Dom}(G)$ and $p \in G(x)$.

*Claim 1:* $x_0 \in \overline{U}$. If not, then there is $x := \lambda x_0 \in \partial U$, $0 < \lambda < 1$ ($x$ lies in the intersection of the segment joining 0 to $x_0$ with the boundary $\partial U$). Take $p \in G(x)$. Then in view of (3.13) and (3.12)

$$0 \leqslant \big\langle p + g(x) + g(x_0) - g(x), x - x_0\big\rangle = \big\langle p + g(x), (1 - \lambda^{-1})x\big\rangle - \big\langle g(x_0) - g(x), x_0 - x\big\rangle < 0$$

since $g$ is monotone. This contradiction justifies Claim 1.

*Claim 2:* $x_0 \in U$. Suppose that $x_0 \in \partial U$. We may suppose that $sx_0 \in U$ for all $s \in [0,1)$. In view of (3.12)

$$\inf_{p \in G(x_0)} \big\langle p + g(x_0), x_0\big\rangle = \inf_{p \in G(x_0)} \langle p, x_0\rangle + \big\langle g(x_0), x_0\big\rangle > 0.$$

The radial upper semicontinuity implies that the function

$$\sigma(t) := [0,1] \ni t \mapsto \inf_{p \in G\big((1-t)x_0\big)} \langle p, x_0\rangle + \big\langle g((1-t)x_0), x_0\big\rangle$$

is lower semicontinuous. Since $\sigma(0) > 0$, we find $0 < \delta < 1$ such that $\sigma(t) > 0$ for $t \in [0,\delta]$. Take $p \in G(\delta x_0)$; then by (3.13)

$$0 \leqslant \big\langle p + g(\delta x_0) + g(x_0) - g(\delta x_0), \delta x_0 - x_0\big\rangle = (\delta - 1)\big\langle p + g(\delta x_0), x_0\big\rangle - \big\langle g(x_0) - g(\delta x_0), x_0 - \delta x_0\big\rangle < 0.$$

This is again a contradiction proving Claim 2.

By Lemma 3.27 $G$ is 'maximal' over $U$. Therefore, in view of (3.13), we infer that $-g(x_0) \in G(x_0)$. This completes the proof. □

As consequence we get the following variant of the Minty-Browder theorem – see [4, Theorem 2.2]. Recall that a reflexive Banach space $X$ is usually considered equivalently renormed as a strictly convex space with $X^*$ strictly convex, too.

**Corollary 3.30.** *Let $X$ be a reflexive space, $G : X \multimap X^*$ be a radially upper hemicontinuous set-valued map with closed convex values and let $\overline{U} \subset \mathrm{Dom}(G)$, where $U \subset X$ is open and $0 \in U$. If for any $x \in \partial U$*

$$\inf_{p \in G(x)} \langle p, x\rangle \geqslant 0, \tag{3.14}$$

*then for any $\lambda > 0$, there is $x_\lambda \in U$ such that $0 \in Jx_\lambda + G(x_\lambda)$, where $J$ stands for the $\big($normalized$\big)$ duality operator $J : X \to X^*$.*

*If $U$ is bounded or $G$ is coercive, $\overline{U} \subset \mathrm{int}\,\mathrm{Dom}(G)$ or condition (3.14) holds with the sharp inequality, i.e. $\inf_{p \in G(x)} \langle p, x\rangle > 0$ for $x \in \partial U$, then there exists $x_0 \in \overline{U}$ such that $0 \in G(x_0)$.*

*Proof.* The duality operator is monotone and, since $X$ is reflexive with $X^*$ strictly convex. It is also coercive since for an $x \in X$, $\langle Jx, x\rangle = \|x\|^2$. This also show that condition (3.12) is satisfied when $g = J$. Thus the first part follows immediately from Theorem 3.29.

To see the second part take $x_n \in U$ such that $0 \in n^{-1}Jx_n + G(x_n)$, i.e. $0 = n^{-1}Jx_n + p_n$, for some $p_n \in G(x_n)$. The exisitence of $(x_n)$ follows from the first part.

For any $n \in \mathbb{N}$

$$0 = \langle p_n + n^{-1}Jx_n, x_n\rangle = \langle p_n, x_n\rangle + n^{-1}\|x_n\|^2.$$

If $U$ is not bounded, then in view of the coercivity we infer that $(x_n)$ is bounded. Hence, passing to a subsequence if necessary, we may assume that $x_n \rightharpoonup x_0 \in K = \overline{\mathrm{conv}}U$. Moreover $p_n \to 0$, since $(Jx_n)$ is bounded.

As in the proof of Theorem 3.29 we show that $x_0 \in \overline{U}$. Now, if (3.14) holds with sharp inequality, then one shows that $x_0 \in U$. Otherwise $x_0 \in \operatorname{int}\operatorname{Dom}(G)$. Hence in both cases $x_0 \in \operatorname{int}\operatorname{Dom}(G)$.

We have that $x_n \rightharpoonup x_0$, $p_n \to p$. Take any $y \in D(G)$ and $q \in G(y)$. The monotonicity of $G$ implies that

$$0 \leqslant \langle p_n - q, x_n - y\rangle \to \langle -q, x_0 - y\rangle.$$

Lemma 3.27 implies that $0 \in G(x_0)$. □

**Remark 3.31.** (1) Theorem 3.29 and Corollary 3.30 has been stated in the language of monotone operators. We leave it to the interested reader to state it in terms of functionals and their subdifferentials, as it is very well-know that the subdifferential of a convex function is monotone (and maximal monotone if $X$ is rotund – see [5]); moreover, in view of [12], the Clarke subdifferential $\partial_C f$ of a lower semicontinuous function $f$ on a reflexive space is monotone if and only if $f$ is convex.

(2) The above results depend on boundary conditions (3.12) or (3.14). It would be very interesting to replace them by geometrical conditions stated in terms of normal (or tangent) cones. For instance if $X$ is a Hilbert space and $U = \{x \in X \mid \|x\| < 1\}$, then (3.14) means $\big($after identification of $X$ with $X^*\big)$ that $G(x) \subset T_C(x, \overline{U})$.

## 4. Appendix

4.1. **Kuratowski-Painlevé conditional limits.** Let $E$ be a set, let $\varphi : E \multimap X$, where $X$ is a Banach space, be a set-valued map, i.e. for any $y \in E$, $\varphi(y)$ is a nonempty subset of $X$. Additionally consider a function $\eta : E \to \mathbb{R}$ and a condition (predicate) $w(y)$, $y \in E$. Assume that

$$\forall\, \delta > 0 \quad A_\delta := \big\{y \in E \mid |\eta(y)| < \delta,\ w(y) \text{ is satisfied}\big\} \neq \emptyset.$$

Having this it is very convenient to introduce the following concepts of the *conditional* set-limits being simple variants of the original notions due to Kuratowski and Painlevé. Namely the *Kuratowski-Painlevé upper* (resp. *lower*) *set-limit* of $\varphi(y)$ as $\eta(y) \to 0$ under condition $w(y)$ are defined by

$$\begin{aligned} \operatorname*{Lim\,inf}_{\eta(y)\to 0,\, w(y)} \varphi(y) &:= \big\{u \in X \mid \inf_{\delta>0} \sup_{y \in A_\delta} d\big(u, \varphi(y)\big) = 0\big\}, \\ \operatorname*{Lim\,sup}_{\eta(y)\to 0,\, w(y)} \varphi(y) &:= \big\{u \in X \mid \sup_{\delta>0} \inf_{y \in A_\delta} d\big(u, \varphi(y)\big) = 0\big\}. \end{aligned} \tag{4.1}$$

For instance if $E$ is a subset of another metric space $\big($with the distance denoted by $\rho\big)$, $y_0$ is an accumulation point of $E$ and $\eta : E \to \mathbb{R}$ is given as $\eta(y) = \rho(y, y_0)$ for $y \in E$, then one writes

$$\operatorname*{Lim\,inf}_{y \to y_0,\, w(y)} \varphi(y) := \operatorname*{Lim\,inf}_{\eta(y)\to 0,\, w(y)} \varphi(y);$$

similarly for $\operatorname{Lim\,sup}_{y\to y_0,\, w(y)} \varphi(y)$.

It is easy to show that

$$\begin{aligned} & u \in \operatorname*{Lim\,sup}_{\eta(y)\to 0,\, w(y)} \varphi(y) \iff \\ & \exists\, (y_n)_{n=1}^\infty \subset E,\ \ \eta(y_n) \to 0,\ w(y_n)\ \ \exists (u_n)_{n=1}^\infty\ \ u_n \in \varphi(y_n)\ \text{ for all } n \in \mathbb{N}, \text{ and }\ u_n \to u, \end{aligned}$$

$$\begin{aligned} & u \in \operatorname*{Lim\,inf}_{\eta(y)\to 0,\, w(y)} \varphi(y) \iff \\ & \forall\, (y_n)_{n=1}^\infty \subset E,\ \ \eta(y_n) \to 0,\ w(y_n)\ \ \exists (u_n)_{n=1}^\infty\ \ u_n \in \varphi(y_n)\ \text{ for all } n \in \mathbb{N}, \text{ and }\ u_n \to u. \end{aligned}$$

Both limits are closed sets in $X$, the lower limit is contained in the upper limit and, in general, the inclusion is strict.

4.2. **Approximate cones.** Let $K$ be a closed subset of $X$ and $x \in K$. The *contingent* (or *Bouligand*) cone $T_B(x;K)$ is defined by

$$T_B(x;K) := \operatorname*{Lim\,sup}_{h\to 0^+} h^{-1}(K-x); \tag{4.2}$$

here $E := (0,\infty)$ and $\eta(h) := h$ for $h > 0$, the condition $w(h)$ is absent and $\varphi(h) := h^{-1}(K-x)$. Hence $u \in T_K(x)$ if and only if there are sequences $h_n \to 0$ and $u_n \to u$ such that $x + h_n u_n \in K$ for all $n \in \mathbb{N}$. It is immediate to see that the $T_B(x;K)$ is a closed cone, i.e. if $u \in T_K(x)$, $\lambda \geqslant 0$, then $\lambda u \in T_K(x)$.

Apart from the contingent cone we shall consider the *tangent* or the *Clarke cone*

$$T_C(x;K) := \operatorname*{Lim\,inf}_{h\to 0^+,\, y\xrightarrow{K}x} h^{-1}(K-y),\ x \in K, \tag{4.3}$$

where $y \xrightarrow{K} x$ means that $y \to x$ and $y \in K$. Here $E := X \times (0,+\infty)$, $\eta(y,h) := \|y-x\| \vee h$ for $y \in X$, $h > 0$, $w(y) \equiv [y \in K]$, and $\varphi(y,h) := h^{-1}(K-y)$. Therefore $u \in T_C(x;K)$ if and only if for any sequences $y_n \to x$ such that $y_n \in K$ for all $n \in \mathbb{N}$ and $h_n \to 0^+$ there is a sequence $u_n \to u$ such that $y_n + h_n u_n \in K$ for all $n \in \mathbb{N}$. A remarkable property is that the cone $T_C(x;K)$ is *always* convex.

A vector $u \in X$ is an *admissible* (with respect to $K$ at $x \in K$) *direction* if there is a sequence $(h_n)_{n=1}^{\infty}$ such that $h_n \to 0^+$ and $x + h_n u \in K$ for all $n \geqslant 1$. The set of admissible directions will be denoted by by $A(x;K)$. In general

$$A(x;K) \subset \bigcup_{h>0} h^{-1}(K-x)$$

and in case of a convex closed $K$

$$A(x;K) = \bigcup_{t>0} t^{-1}(K-x) = \bigcup_{t>0} \bigcap_{0<h\leqslant t} h^{-1}(K-x).$$

**Lemma 4.1.** *For any closed $K \subset X$ and $x \in K$*

$$T_C(x;K) \subset T_B(x;K) \subset \overline{A(x;K)}.$$

*If $K \subset X$ is convex, then*

$$T_C(x;K) = T_B(x;K) = \overline{A(x;K)}. \qquad \square$$

**Remark 4.2.** (1) Observe that for any $x \in K$ and $r > 0$

$$T_B(x;K) = T_B\big(x; K \cap D(x,r)\big),$$

where $D(x,r)$ is the closed ball around $x$ of radius $r > 0$. The similar statements are true with regard to other cones $T_C(x;K)$ and $A(x;K)$.

(2) It is immediate to see that

$$\begin{aligned} T_C(x;k) &= \Big\{u \in X \mid \limsup_{y\to x,\, h\to 0^+} h^{-1} d_K(y+hu) = 0\Big\} \\ &= \Big\{u \in X \mid d_K^{\circ}(x;u) \leqslant 0\Big\} = \partial_C d_K(x)^-, \end{aligned} \tag{4.4}$$

where $d_K(y) := \inf_{k\in K} \|y-k\|$ and

$$\begin{aligned} T_B(x;K) &= \Big\{u \in X \mid \liminf_{h\to 0^+} h^{-1} d_K(x+hu) = 0\Big\} \\ &\subset \Big\{u \in X \mid (d_K)'_H(x;u) = 0\Big\} \subset \partial_H d_K(x)^-. \end{aligned}$$

In a similar manner

$$N_C(x;K) = T_C(x;K)^- = \partial_C I_K(x) \ \text{ and } \ N_B(x;K) = \partial_H I_K(x), \tag{4.5}$$

where $I_K$ is the indicator of $K$.

(3) Observe that if $M \subset X$ is closed and $x \in K \subset M$, then $T_B(x;K) \subset T_B(x;M)$. This property does not hold true for the Clarke cones.

We say that a vector $u \in X$ is *hypertangent* (comp. [2, Def. 4.5.8]) to $K$ at $x \in K$ if

$$\exists \varepsilon > 0 \, \forall y \in B(x,\varepsilon) \cap K \;\; \forall h \in (0,\varepsilon) \qquad y + hu \in K. \tag{4.6}$$

Hypertangent vectors belong to $T_C(x;K)$.

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