Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

RESEARCH ARTICLE

The existence and averaging principle for stochastic fractional differential equations with impulses

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In this paper, a class of stochastic fractional differential equations (SFDEs) with impulses is considered. By virtue of the Mönch's fixed point theorem and Banach contraction principle, we explore the existence and uniqueness of solutions to the addressed system. Furthermore, with the aid of the Jensen's inequality, Hölder inequality, Burkholder-Davis-Gundy inequality, Grönwall-Bellman inequality and some novel assumptions, the averaging principle of our considered system is obtained. At the end of this paper, an example is provided to illustrate the theoretical results.

KEYWORDS:
fractional calculus, impulsive stochastic differential equations, existence, uniqueness, averaging principle

MSC CLASSIFICATION
26A33; 74H20; 74H25; 34C29; 60H10

1 | INTRODUCTION

The averaging principle is an effective method to simplify deterministic and stochastic systems [1]. To some extent, it is an approximation principle that can strike a balance between complex and simple systems. The basic idea of the averaging principle is to establish an approximation theorem to simplify stochastic differential equations, which replace the original system in a sense, and give the corresponding convergence theorem of optimal order. It is worthy to mention that the averaging method has received extensive attention since Krylov and Bogolyubov [2] proposed the non-periodic averaging principle for nonlinear system. Furthermore, the averaging principle of stochastic differential equations was addressed deeply by Khasminskii [3].

It is well known that stochastic processes are often seen as dynamic parts of probability theory. Stochastic processes have developed vigorously in theory and application due to the needs of practical problems and the efforts of mathematicians. Up to now, the stochastic averaging principle has been concerned by many scholars. For instance, using the averaging principle to explore stochastic differential equations [4]. In [5], the authors investigated the averaging principle of stochastic differential equations with jumps, and they proved that the speed of weak convergence is essentially twice that of strong convergence. Guo [6] studied the averaging principle of a class of generalized stochastic differential equations satisfying local Lipschitz condition and monotone condition. As an effective analysis tool, the stochastic averaging principle can help us approximate stochastic differential equations with various noises, like Poisson noise [7,8], multiplicative noise [9] and so on.

As the researching goes further, investigators begin to focus on the study of fractional stochastic differential equations. In recent decades, with the deepening influence of applied disciplines, the theory of fractional differential equations have gained special attentions by many scholars [10,11,12,13]. Since fractional differential equations have more advantages than integer differential equations in simulating some natural physical phenomena and dynamic system, they have been extensively expanded in physics, engineering, finance and other fields [14]. Xu et al. verified the averaging principle of fractional stochastic system by establishing some new assumptions [10,15,16]. Luo and Ahmed explored the averaging principle of stochastic fractional system with time delay [11,12], and their research focused on how to deal with time delay τ. In [17], Xu analysed the averaging principle of fractional neutral stochastic differential equations, and they proved that the mild solution before and after the average of two fractional systems are equivalent in the mean square sense. These theoretical results are of great significance to the study of perturbation theory and nonlinear dynamic system.

The impulsive effects are widely presented in many evolutionary processes in which state changes at a given moment. Therefore, impulsive fractional stochastic differential equations are generally used in many scientific branches, such as biology, economics, finance, telecommunications, electronics and medicine [18,19]. In particular, Liu and Xu developed an averaging result for impulsive fractional neutral stochastic differential equations [20]. In [21], Khalaf explored impulsive stochastic dynamical systems driven by fractional Brownian motion under non-Lipschitz conditions and they had proved that the solutions of these equations can be approximated to the solutions of the average stochastic differential equations without impulses in the sense of mean square and probability. To the best of our knowledge, the averaging principle of fractional stochastic differential equations with impulses is relatively less considered. So far, the research on the averaging principle of SFDEs with impulses is still immature and more meaningful results need to be found by scholars.

With the development of SFDEs with impulses, the existence and uniqueness have been correspondingly emphasized. In recent years, many scholars had studied the existence and uniqueness of solutions of the differential equations, see [22,23,24,25]. There are various methods to study the existence and uniqueness of solutions. In [26], the authors used Mönch fixed point theorem to obtain the existence of the solutions for impulsive neutral stochastic functional differential equations. In [27], the existence and uniqueness of the solutions for fractional stochastic delayed system with noninstantaneous impulses are obtained by using the Arzela-Ascoli theorem. In [28], the researchers discussed the existence and uniqueness of random fractional differential equation with impulses by Banach fixed point theorem and Schauder's fixed point theorem. In [29], the authors established a research framework related to the impulsive fractional sample path. A sequence of Picard functions was further constructed, then the successive approximation method was used to solve the impulsive fractional-order implicit differential equations with random effects. From the perspective of studied system, the system studied in reference [29] was more generalized than [28]. The mutation described by a noninstantaneous impulses [29] persists over a finite time interval.

Inspired by the previous discussions, we will consider the following SFDEs with impulses:

$$\begin{cases} dL(s) = f\,(s, L(s))\,ds + h\,(s, L(s))\,dW(s) + \sigma\,(s, L(s))\,(ds)^{\alpha}, s \in J, \\ \Delta L(s_i) = L({s_i}^{+}) - L({s_i}^{-}) = I_i(L(s_i)), s = s_i, i = 1, 2, \cdot\cdot\cdot, k, \\ L(0) = L_0, \end{cases} \tag{1}$$

where $J = [0, T]$, $\frac{1}{2} < \alpha < 1$, $f, \sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, and $h : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$, and f, h, σ are measurable continuous functions. $\{W(s) : s \in J\}$ is a m-dimensional standard Wiener process. $I_i : \mathbb{R}^d \to \mathbb{R}^d (i = 1, 2, \cdots, k)$ represent the impulsive disturbance of $L(s)$ at time s_i, s_i satisfy $0 = s_0 < s_1 < \cdots < s_i < \cdots < s_k < s_{k+1} = T$. $L\left(s_i^-\right)$ and $L\left(s_i^+\right)$ represent the left and right limits of $L(s)$ at time s_i, respectively. $\Delta L\left(s_i\right)$ represents the jump in the state L at time s_i. $\mathbb{R}^d$ is a Banach space with norm $\|\cdot\|$ and L_0 satisfies $\mathbb{E}\left\|L_0\right\|^2 < \infty$.

Compared with [12,13,26], the contributions of this paper are mainly reflected in the following two aspects:

- Due to the differences of systems, although the methods adopted in the study of existence are similar to [26], but there are discrepancies in the proof process. We apply the inequality to scale the fractional part of the system under consideration.

- In references [12,13], Luo and Guo investigate the averaging principle of fractional stochastic differential equations without impulsive effects, but the system (1) has impulsive term. Impulsive functions can more truly reflect the objective process of change and it is more widely model. Moreover, impulsive effects may bring new challenges in computation, and we are committed to overcoming hardships to achieve the desired results.

The arrangement for the rest article are outlined as belows: In Section 2, we present some necessary preliminaries to prepare for the later parts. In Section 3, we aim to prove the existence and uniqueness theorem of SFDEs with impulses. The averaging principle results are given in section 4. Finally, we give an example to verify the correctness of our conclusion in Section 5.

2 | PRELIMINARY

In this section, we introduce some preliminaries needed to establish our main results.

Definition 1. (Luo,Zhu and Luo[12]) Let's consider a function $f \in L^1\left[[0,T];\mathbb{R}^d\right]$, for any $\alpha \in (0,1)$, the Riemann-Liouville fractional integral operator of order α is defined for all $0 < t < T$ by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is defined as $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, which is a Gamma function.

Definition 2. (Luo,Zhu and Luo[12]) Assume $\alpha \in (0,1]$ and $\sigma(t)$ denotes a continuous function, then the integral of $\sigma(t)$ with respect to $(dt)^\alpha$ is defined by

$$\int_0^t \sigma(s)(ds)^\alpha = \alpha \int_0^t (t-s)^{\alpha-1}\sigma(s)ds.$$

In this paper, we define the space

$$U = \{L : \mathbb{R}^d \to \mathbb{R}^d : \text{ there exist } L(s_i^+) \text{ and } L(s_i^-) \text{ with } L(s_i) = L(s_i^-), i = 1,2,\cdots,k, L(0) = L_0\},$$

and $\|\cdot\|_U$ defined by

$$\|L\|_U = \sup_{t\in[0,T]} (\mathbb{E}\,\|L(t)\|^2)^{1/2}, L \in U.$$

Then $(U, \ \|\cdot\|_U)$ is a Banach space.

In order for our work to proceed smoothly, we make the following assumptions:

- (A_1) For f, h, σ in system (1), $\forall x, y \in U$, there exists a positive bounded function $\lambda(s)$ satisfying

$$\|f(s,x) - f(s,y)\|^2 \vee \|h(s,x) - h(s,y)\|^2 \vee \|\sigma(s,x) - \sigma(s,y)\|^2 \le \lambda(s)\,\|x-y\|^2,$$

  and $\sup_{0\le s\le T} \lambda(s) = \Lambda$.

- (A_2) Impulsive functions I_i satisfy Lipschitz condition and boundedness. Namely, $\forall x, y \in U$, there exists a positive constant m such that

$$\left\|I_i(x) - I_i(y)\right\|^2 \le m\,\|x-y\|^2, \left\|I_i(0)\right\| = 0.$$

Lemma 1. For any $r > 0$, suppose $U_r = \{L \in U : \|L\|^2 \le r\}$, then $U_r \subset U$ is a bounded,closed and convex subset.

Next, on each bounded subset B of Banach space X, the Hausdorff measure of noncompactness $\eta(\cdot)$ is given by

$$\eta(B) = \inf\{\varepsilon > 0; B \text{ has a finite } \varepsilon \text{ net in } X\}.$$

Then some basic properties of $\eta(\cdot)$ are given as follows:

Definition 3. (Deng,Shu and Mao[26]) Assume the map $\Phi : D \to Y$ is continuous and satisfies Mönch's condition, if $D \subset Y$ and D is a closed convex subset of Y such that $0 \in D$. Φ satisfies the following property: M is countable and M is a subset of D, $M \subset \overline{co}(\{0\} \bigcup \Phi(M)) \Rightarrow \bar{M}$ is compact. Therefore, Φ has a fixed point in D.

Lemma 2. (Deng,Shu and Mao[26]) Assume W is a standard Winer process, then

$$\eta\left(\int_0^s X(t)dW(t)\right) \le \sqrt{T}\eta(X(t)),$$

provided that $X \subset C\left([0,T];\mathbb{R}^{d\times m}\right)$, where

$$\int_0^s X(t)dW(t) = \left\{\int_0^s x(t)dW(t) : \text{ for all } x \in X, s \in [0,T]\right\}.$$

Lemma 3. (Deng,Shu and Mao[26]) Assume B, D are subsets of X, moreover, B, D are bounded, and X is a real Banach space, then they will satisfy the following properties
(1) Define that $\overline{B}$ is the closure of B, and conv B is the convex hull of B. Then there have $\eta(B) = \eta(\overline{B}) = \eta(convB)$;
(2) B is precompact when and only when $\eta(B) = 0$;
(3) $\eta(B \cup D) \leq \max\{\eta(B), \eta(D)\}$;
(4) For arbitrary $\lambda \in \mathbb{R}$, there exists $\eta(\lambda B) \leq |\lambda|\eta(B)$;
(5) When it comes to $B + D = \{x + y; x \in B, y \in D\}$, then $\eta(B + D) \leq \eta(B) + \eta(D)$;
(6) Assume $\{u_n\}_1^\infty$ be a sequence of Bochner integrable functions from J to X with $\|u_n(t)\| \leq \widehat{m}(t)$ for almost all $t \in J$ and every $n \geq 1$, where $\widehat{m}(t) \in L(J; R^+)$, then the function $\psi(t) = \eta(\{u_n\}_{n=1}^\infty) \in L(J; R^+)$ and satisfies

$$\eta\left(\left\{\int_0^t u_n(s)ds : n \geq 1\right\}\right) \leq 2\int_0^t \psi(s)ds;$$

(7) Let $W \subset C([0,T]; X)$ and W is bounded and equicontinuous, then $t \to \eta(W(t))$ is continuous on $[0,T]$, and

$$\eta\left(\int_0^t W(s)ds\right) \leq \int_0^t \eta(W(s))ds, \text{ for all } t \in [0,T],$$

where

$$\int_0^t W(s)ds = \left\{\int_0^t u(s)ds : u \in W\right\};$$

(8) Assume $W \subset C([0,T])$ and W is bounded, then

$$\eta(W(t)) \leq \eta(W), \text{ for all } t \in [0,T],$$

where $W(t) = \{u(t) : u \in W \subset X\}$. Furthermore, if W is equicontinuous on $[0,T]$, then $t \to W(t)$ is continuous on $[0,T]$, and

$$\eta(W) = \sup\{W(t) : t \in [0,T]\}.$$

3 | EXISTENCE AND UNIQUENESS

In this section, we shall study the existence and uniqueness of mild solution for system (1). For $\forall s \in [0,\ T]$, in order to facilitate our research, we will give the standard form of the system (1) as shown below.

Definition 4. A stochastic process $\{L(s), s \in [0,T]\}$ is referred to be a mild solution for system (1) if it satisfies the following conditions
(i)$L(0) = L_0$,
(ii) $L(s) \in U$ is $\mathcal{F}_t$-adapted, and has càdlàg path on $[0,T]$ a.s.,
(iii) here, when we give any s that belongs to $[0,T]$, $L(s)$ needs to satisfies the integral equation of the following form

$$\begin{aligned} L(s) =& L_0 + \int_0^s f(t, L(t))\,dt + \int_0^s h(t, L(t))\,dW(t) \\ &+ \alpha\int_0^s (s-t)^{\alpha-1}\sigma(t, L(t))\,dt + \sum_{i=1}^k I_i\left(L(s_i)\right). \end{aligned}$$

In this article, in order to make our work smoothly, we make the following hypotheses.

- (H_1) The function $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies
  (i) $f(s, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is continuous for all $s \in J$, $f(\cdot, x) : [0, T] \to \mathbb{R}^d$ is measurable for all $x \in \mathbb{R}^d$.
  (ii) Here, we give a continuous function $m_f : [0, T] \to \mathbb{R}^+$ and a continuous nondecreasing function $\varphi_f : \mathbb{R}^+ \to [0, \infty)$, then satisfy
  $$\|f(s, x)\|^2 \le m_f(s)\varphi_f\left(\|x\|^2\right).$$
  (iii) A positive function $b_f \in L^1(J, \mathbb{R}^+)$ given satisfies the following inequality for any bounded subsets $P_1 \subset \mathbb{R}^d$,
  $$\eta(f(s, x)) \le b_f(s) \sup_{a \in (-\infty, 0]} \eta\left(P_1(a)\right).$$
  (iv) $\lim\limits_{r\to\infty} \frac{\varphi_f(r)}{r} = 1$.

- (H_2) The function $h : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ satisfies
  (i) $h(s, \cdot) : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is continuous for all $s \in J$, $h(\cdot, x) : [0, T] \to \mathbb{R}^{d\times m}$ is measurable for all $x \in \mathbb{R}^d$.
  (ii) Here, we give a continuous function $m_h : [0, T] \to \mathbb{R}^+$ and a continuous nondecreasing function $\varphi_h : \mathbb{R}^+ \to [0, \infty)$, then satisfy
  $$\|h(s, x)\|^2 \le m_h(s)\varphi_h\left(\|x\|^2\right).$$
  (iii) A positive function $b_h \in L^2(J, \mathbb{R}^+)$ given satisfies the following inequality for any bounded subsets $P_2 \subset \mathbb{R}^d$,
  $$\eta(h(s, x)) \le b_h(s) \sup_{a \in (-\infty, 0]} \eta\left(P_2(a)\right).$$
  (iv) $\lim\limits_{r\to\infty} \frac{\varphi_h(r)}{r} = 1$.

- (H_3) The function $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies
  (i) $\sigma(s, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is continuous for all $s \in J$, $\sigma(\cdot, x) : [0, T] \to \mathbb{R}^d$ is measurable for all $x \in \mathbb{R}^d$.
  (ii) Here, we give a continuous function $m_\sigma : [0, T] \to \mathbb{R}^+$ and a continuous nondecreasing function $\varphi_\sigma : \mathbb{R}^+ \to [0, \infty)$, then satisfy
  $$\|\sigma(s, x)\|^2 \le m_\sigma(s)\varphi_\sigma\left(\|x\|^2\right).$$
  (iii) A positive function $b_\sigma \in L^1(J, \mathbb{R}^+)$ given satisfies the following inequality for any bounded subsets $P_3 \subset \mathbb{R}^d$,
  $$\eta(\sigma(s, x)) \le b_\sigma(s) \sup_{a \in (-\infty, 0]} \eta\left(P_3(a)\right).$$
  (iv) $\lim\limits_{r\to\infty} \frac{\varphi_\sigma(r)}{r} = 1$.

- (H_4) The continuous functions $I_i : \mathbb{R}^d \to \mathbb{R}^d$, and there exist $b_i > 0, i = 1, 2, \cdots, k$ given satisfy the following inequality for any bounded subsets $P_4 \subset U$,
  $$\eta\left(I_i\left(P_4\right)\right) \le b_i \sup_{a \in (-\infty, 0]} \eta\left(P_4(a)\right).$$

- (H_5) $5T \int_0^s m_f(t)dt + 20 \int_0^T m_h(t)dt + 5\frac{\alpha^2}{2\alpha-1} T^{2\alpha-1} \int_0^s m_\sigma(t)dt + 5k \sum\limits_{i=1}^{k} m < 1$.

Theorem 1. Assume that all of the above Hypotheses $(H_1) - (H_5), (A_1) - (A_2)$ are satisfied, then system (1) has at least one solution on U_r.

Proof. We define the operator $\Gamma : U \to U$ as following

$$\Gamma(L(s)) = L_0 + \int_0^s f(t, L(t))\,dt + \int_0^s h(t, L(t))\,dW(t) + \alpha \int_0^s (s-t)^{\alpha-1}\sigma(t, L(t))\,dt + \sum_{i=1}^{k} I_i\left(L(s_i)\right).$$

Our aim is to prove the existence of solutions of system (1), which can be equivalently translated into proving that the operator Γ has a fixed point. In order to prove this theorem, we will proceed in the following steps:

Step 1. In this part, we want to prove that there is a positive number r such that $\Gamma(U_r) \subset U_r$. In order to prove it, we assume $\Gamma(U_r) \not\subset U_r$. For the positive number r discussed, we can find a corresponding function L_r satisfies $L_r \in U_r$, but $\Gamma(U_r) \notin U_r$. In other words, for some $s = s(r) \in [0, T]$, $\mathbb{E}\left\|\left(\Gamma L_r\right)(s)\right\|^2 > r$. As a matter of fact, we have

$$\begin{aligned} r < \mathbb{E}\left\|\left(\Gamma L_r\right)(s)\right\|^2 \leq & 5\mathbb{E}\left\|L_0\right\|^2 + 5\mathbb{E}\left\|\int_0^s f\left(t, L_r(t)\right)dt\right\|^2 + 5\mathbb{E}\left\|\int_0^s h\left(t, L_r(t)\right)dW(t)\right\|^2 \\ & + 5\mathbb{E}\left\|\alpha\int_0^s (s-t)^{\alpha-1}\sigma\left(t, L_r(t)\right)dt\right\|^2 + 5\mathbb{E}\left\|\sum_{i=1}^{k} I_i\left(L_r(s_i)\right)\right\|^2 \\ := & I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

It's easy to get

$$I_1 = 5L_0^2. \tag{2}$$

According to the Hölder inequality, (H_1) and lemma 1, we can get

$$\begin{aligned} I_2 &\leq 5T\mathbb{E}\int_0^s \left\|f\left(t, L_r(t)\right)\right\|^2 dt \\ &\leq 5T\int_0^s m_f(t)\varphi_f\left(\|L_r(t)\|^2\right)dt \\ &\leq 5T\int_0^s m_f(t)\varphi_f(r)\,dt. \end{aligned} \tag{3}$$

By the Burkholder-Davis-Gundy inequality, (H_2) and lemma 1, we have

$$\begin{aligned} I_3 &\leq 20\mathbb{E}\int_0^T \left\|h\left(t, L_r(t)\right)\right\|^2 dt \\ &\leq 20\mathbb{E}\int_0^T m_h(t)\varphi_h\left(\|L_r(t)\|^2\right)dt \\ &\leq 20\int_0^T m_h(t)\varphi_h(r)\,dt. \end{aligned} \tag{4}$$

Owing to the Hölder inequality, (H_3) and lemma 1, we can get

$$\begin{aligned} I_4 &\leq 5\frac{\alpha^2}{2\alpha-1}s^{2\alpha-1}\mathbb{E}\int_0^s \left\|\sigma\left(t, L_r(t)\right)\right\|^2 dt \\ &\leq 5\frac{\alpha^2}{2\alpha-1}s^{2\alpha-1}\mathbb{E}\int_0^s m_\sigma(t)\varphi_\sigma\left(\|L_r(t)\|^2\right) dt \\ &\leq 5\frac{\alpha^2}{2\alpha-1}T^{2\alpha-1}\int_0^s m_\sigma(t)\varphi_\sigma(r)\, dt. \end{aligned} \tag{5}$$

By (A_2), we can obtain

$$\begin{aligned} I_5 &\leq 5k\sum_{i=1}^k \left\|I_i(L_r(s_i)\right\|^2 \\ &\leq 5k\sum_{i=1}^k mr. \end{aligned} \tag{6}$$

To sum up, from (2) - (6), we have

$$\begin{aligned} r <&\mathbb{E}\left\|\left(\Gamma L_r\right)(s)\right\|^2 \\ \leq& 5L_0^2 + 5T\int_0^s m_f(t)\varphi_f(r)\, dt + 20\int_0^T m_h(t)\varphi_h(r)\, dt \\ &+ 5\frac{\alpha^2}{2\alpha-1}T^{2\alpha-1}\int_0^s m_\sigma(t)\varphi_\sigma(r)\, dt + 5k\sum_{i=1}^k mr. \end{aligned} \tag{7}$$

Combining (iv) of (H_1), (iv) of (H_2), (iv) of (H_3), letting $r \to \infty$ and dividing both sides of (7) by r, we have

$$5T\int_0^s m_f(t)dt + 20\int_0^T m_h(t)dt + 5\frac{\alpha^2}{2\alpha-1}T^{2\alpha-1}\int_0^s m_\sigma(t)dt + 5k\sum_{i=1}^k m \geq 1.$$

The results obtained are contradicts with (H_5). Thus, there are some positive numbers r that satisfies $\Gamma(U_r) \subset U_r$.

Step 2. We need to prove the operator Γ is continuous in U_r. Give a sequence $\{L^n(s)\}_{n=1}^\infty$ in U_r, and when $n \to +\infty$, $L^n \to L$ in U_r. For a positive number r, any $n \in N^+$ and all $s \in J$ such that $\|L^n(s)\|^2 < r$. That implies $L^n \in U_r$ and $L \in U_r$. By $(H_1) - (H_3)$ and $(A_1) - (A_2)$, we can get

(i) $\lim_{n\to\infty} f(s, L^n(s)) = f(s, L(s))$;

(ii) $\lim_{n\to\infty} h(s, L^n(s)) = h(s, L(s))$;

(iii) $\lim_{n\to\infty} \sigma(s, L^n(s)) = \sigma(s, L(s))$;

(iv) $\lim_{n\to\infty} \mathbb{E}\left\|I_i\left(L^n(s_i)\right) - I_i\left(L(s_i)\right)\right\|^2 = 0, i = 1, 2, \ldots, k.$

For any $s \in J$, from the dominated convergence theorem, the following inequality can be derived

$$\begin{aligned}
&\mathbb{E}\left\|\Gamma L^n(s) - \Gamma L(s)\right\|^2 \\
\leq & 4\mathbb{E}\left\|\int_0^s [f(t, L^n(t)) - f(t, L(t))]dt\right\|^2 \\
&+ 4\mathbb{E}\left\|\int_0^s [h(t, L^n(t)) - h(t, L(t))]dW(t)\right\|^2 \\
&+ 4\alpha^2\mathbb{E}\left\|\int_0^s (s-t)^{\alpha-1}[\sigma(t, L^n(t)) - \sigma(t, L(t))]dt\right\|^2 \\
&+ 4k\sum_{i=1}^{k}\mathbb{E}\left\|[I_i(L^n(s_i)) - I_i(L(s_i))]\right\|^2.
\end{aligned}$$

When $n \to \infty$, we can get

$$\mathbb{E}\left\|\Gamma L^n(s) - \Gamma L(s)\right\|^2 \to 0.$$

That means Γ is a continuous operator in U_r.

Step 3. We want to verify the operator $\Gamma(U_r)$ is equicontinuous on J. Firstly, according to the definition of Γ, it can be seen that $\{\Gamma L : L \in U_r\}$ is equicontinuous at $s = 0$. Secondly, for $s_1, s_2 \in J, 0 < s_1 < s_2 \leqslant T$ and $L \in U_r$, under the above assumptions, we can get

$$\begin{aligned}
&\mathbb{E}\left\|\Gamma L(s_2) - \Gamma L(s_1)\right\|^2 \\
\leq & 4\mathbb{E}\left\|\int_0^{s_2} f(t, L(t))dt - \int_0^{s_1} f(t, L(t))dt\right\|^2 \\
&+ 4\mathbb{E}\left\|\int_0^{s_2} h(t, L(t))dW(t) - \int_0^{s_1} h(t, L(t))dW(t)\right\|^2 \\
&+ 4\mathbb{E}\left\|\alpha\int_0^{s_2} (s_2-t)^{\alpha-1}\sigma(t, L(t))dt - \alpha\int_0^{s_1} (s_1-t)^{\alpha-1}\sigma(t, L(t))dt\right\|^2 \\
&+ 4\mathbb{E}\left\|\sum_{i=1}^{k} I_i(L(s_2)) - \sum_{i=1}^{k} I_i(L(s_1))\right\|^2 \\
\leq & 4\mathbb{E}\left\|\int_0^{s_2} f(t, L(t))dt - \int_0^{s_1} f(t, L(t))dt\right\|^2 \\
&+ 4\mathbb{E}\left\|\int_0^{s_2} h(t, L(t))dW(t) - \int_0^{s_1} h(t, L(t))dW(t)\right\|^2 \\
&+ 8\alpha^2\mathbb{E}\left\|\int_0^{s_1} [(s_2-t)^{\alpha-1} - (s_1-t)^{\alpha-1}]\sigma(t, L(t))dt\right\|^2 \\
&+ 8\alpha^2\mathbb{E}\left\|\int_{s_1}^{s_2} (s_2-t)^{\alpha-1}\sigma(t, L(t))dt\right\|^2 \\
&+ 4\mathbb{E}\left\|\sum_{i=1}^{k} I_i(L(s_2)) - \sum_{i=1}^{k} I_i(L(s_1))\right\|^2.
\end{aligned}$$

When $s_2 \to s_1$, the right side of the equation goes to zero. It means $E\left\|\Gamma L(s_2)-\Gamma L(s_1)\right\|^2 \to 0$ as $s_2 \to s_1$. Thus, $\left\{\Gamma L : L \in U_r\right\}$ is equicontinuous on $\boldsymbol{J}$.

Step 4. We show that Mönch condition is true. Suppose $G \subset U_r$ be countable and $G \subset \overline{co}(\{0\} \cup \Gamma(G))$. Then we verify that $\eta(G) = 0$. As usual, we assume that $G = \{L^n\}_{n=1}^{\infty}$. $G \subset \overline{co}(\{0\} \cup \Gamma(G))$ is equicontinuous on $\boldsymbol{J}$ due to $\Gamma(G)$ is equicontinuous on $\boldsymbol{J}$.

In order to verify Mönch condition, we first give the following

$$
\begin{aligned}
\Gamma L(s) &= \int_0^s f(t, L(t))dt + \int_0^s h(t, L(t))dW(t) \\
&\quad + \alpha \int_0^s (s-t)^{\alpha-1}\sigma(t, L(t))dt + \sum_{i=1}^{k} I_i\left(L(s_i)\right) \\
&=\left[\int_0^s f(t, L(t))dt + \alpha \int_0^s (s-t)^{\alpha-1}\sigma(t, L(t))dt\right] \\
&\quad + \int_0^s h(t, L(t))dW(t) + \sum_{i=1}^{k} I_i\left(L(s_i)\right) \\
&=\Gamma_1 + \Gamma_2 + \Gamma_3.
\end{aligned}
$$

From (iii) of (H_2), (iii) of (H_4) and lemma 3, we can obtain

$$
\begin{aligned}
\eta\left(\{\Gamma_1 L^n(s)\}_{n=1}^{\infty}\right) &=\eta\left(\int_0^s f\left(t, L^n(t)\right)dt + \alpha\int_0^s (s-t)^{\alpha-1}\sigma\left(t, L^n(t)\right)dt\right) \\
&\le \int_0^s b_f(t) \sup_{a\in(-\infty,0]} \eta\left(\{L^n(a)\}_{n=1}^{\infty}\right)dt \\
&\quad + 2\alpha\int_0^s (s-t)^{\alpha-1} b_\sigma(t) \sup_{a\in(-\infty,0]} \eta\left(\{L^n(a)\}_{n=1}^{\infty}\right)dt \\
&\le T\left\|b_f\right\|_{L^1(J,\mathbb{R}^+)} \sup_{0\le s\le T} \eta\left(\{L^n(s)\}_{n=1}^{\infty}\right) \\
&\quad + 2T^{\alpha}\left\|b_\sigma\right\|_{L^1(J,\mathbb{R}^+)} \sup_{0\le s\le T} \eta\left(\{L^n(s)\}_{n=1}^{\infty}\right).
\end{aligned} \tag{8}
$$

That comes from the lemma 2 and (iii) of (H_2), we have

$$
\begin{aligned}
\eta\left(\{\Gamma_2 L^n(s)\}_{n=1}^{\infty}\right) &=\eta\int_0^s h\left(t, L^n(t)\right)dW(t) \\
&\le\sqrt{T}\eta h\left(s, L^n(s)\right) \\
&\le\sqrt{T} b_h(s) \sup_{a\in(-\infty,0]} \eta\left(\{L^n(a)\}_{n=1}^{\infty}\right) \\
&\le\sqrt{T}\left\|b_h\right\|_{L^2(J,\mathbb{R}^+)} \sup_{0\le s\le T} \eta\left(\{L^n(s)\}_{n=1}^{\infty}\right).
\end{aligned} \tag{9}
$$

By (ii) of (H_4), we can obtain

$$\begin{aligned}\eta\left(\{\Gamma_3 L^n(s)\}_{n=1}^{\infty}\right) =&\eta \sum_{i=1}^{k} I_i\left(L^n(s_i)\right)\\ \leq& \sum_{i=1}^{k} b_i \sup_{a\in(-\infty,0]} \eta\left(\{L^n(a)\}_{n=1}^{\infty}\right)\\ \leq& \sum_{i=1}^{k} b_i \sup_{0\leq s\leq T} \eta\left(\{L^n(s)\}_{n=1}^{\infty}\right).\end{aligned} \tag{10}$$

Therefore, it follows from (8) - (10) that

$$\begin{aligned}\eta\left(\{\Gamma L^n(s)\}_{n=1}^{\infty}\right) \leq&[T\left\|b_f\right\|_{L^1(J,\mathbb{R}^+)} + 2T^{\alpha}\left\|b_\sigma\right\|_{L^1(J,\mathbb{R}^+)} + \sqrt{T}\left\|b_h\right\|_{L^2(J,\mathbb{R}^+)} + \sum_{i=1}^{k} b_i]\cdot\eta\left(\{L^n(s)\}_{n=1}^{\infty}\right)\\ =&Q\eta\left(\{L^n(s)\}_{n=1}^{\infty}\right),\end{aligned}$$

where $Q = T\left\|b_f\right\|_{L^1(J,\mathbb{R}^+)} + 2T^{\alpha}\left\|b_\sigma\right\|_{L^1(J,\mathbb{R}^+)} + \sqrt{T}\left\|b_h\right\|_{L^2(J,\mathbb{R}^+)} + \sum_{i=1}^{k} b_i < 1$.

Thus, through M*ö*nch condition, we can obtain

$$\eta(G) \leq \eta(\overline{co}(\{0\}\cup\Gamma(G))) = \eta(\Gamma(G)) \leq Q\eta(G),$$

which implies $\eta(G) = 0$, and it follows that G is a relatively compact set. According to definition 3, we can draw a conclusion that Γ has a fixed point in G. The proof of this theorem is completed. □

Theorem 2. Assume that $(A_1)-(A_2)$ are fulfilled, if $[4T\Lambda + 16\Lambda + 4\frac{\alpha^2}{2\alpha-1}T^{2\alpha-1} + 4k\sum_{i=1}^{k} m] < 1$, then system (1) has a unique solution on U_r.

Proof. $L(s)$ is a solution of system (1), if

$$\begin{aligned}L(s) =&L(0) + \int_0^s f(t, L(t))\,dt + \int_0^s h(t, L(t))\,dW(t)\\ &+ \alpha\int_0^s (s-t)^{\alpha-1}\sigma(t, L(t))\,dt + \sum_{i=1}^{k} I_i\left(L(s_i)\right).\end{aligned}$$

If $L(s)$ is a fixed point of Γ, then it follows that $L(s)$ is a solution of the system (1). Let $L_1(s), L_2(s) \in U$ and $L_1(0) = L_2(0) = L_0$. For $\forall s \in J$, we can obtain the following

$$
\begin{aligned}
\mathbb{E}\left\|\Gamma L_1(s)-\Gamma L_2(s)\right\|^2 \leq & 4\mathbb{E}\left\|\int_0^s [f(t,L_1(t))-f(t,L_2(t))]dt\right\|^2 \\
& +4\mathbb{E}\left\|\int_0^s [h(t,L_1(t))-h(t,L_2(t))]dW(t)\right\|^2 \\
& +4\mathbb{E}\left\|\alpha\int_0^s (s-t)^{\alpha-1}[\sigma(t,L_1(t))-\sigma(t,L_2(t))]dt\right\|^2 \\
& +4\mathbb{E}\left\|\sum_{i=1}^k I_i\left(L_1(s_i)\right)-\sum_{i=1}^k I_i\left(L_2(s_i)\right)\right\|^2 \\
\leq & 4T\mathbb{E}\int_0^s \left\|f(t,L_1(t))-f(t,L_2(t))\right\|^2 dt \\
& +16\mathbb{E}\int_0^T \left\|h(t,L_1(t))-h(t,L_2(t))\right\|^2 dt \\
& +4\frac{\alpha^2}{2\alpha-1}s^{2\alpha-1}\mathbb{E}\int_0^s \left\|\sigma(t,L_1(t))-\sigma(t,L_2(t))\right\|^2 dt \\
& +4k\mathbb{E}\sum_{i=1}^k \left\|I_i(L_1(s_i))-I_i(L_2(s_i))\right\|^2 \\
\leq & 4T\Lambda\left\|L_1(s)-L_2(s)\right\|^2+16\Lambda\left\|L_1(s)-L_2(s)\right\|^2 \\
& +4\frac{\alpha^2}{2\alpha-1}T^{2\alpha-1}\Lambda\left\|L_1(s)-L_2(s)\right\|^2+4k\sum_{i=1}^k m\left\|L_1(s)-L_2(s)\right\|^2 \\
= & (4T\Lambda+16\Lambda+4\frac{\alpha^2}{2\alpha-1}T^{2\alpha-1}\Lambda+4k\sum_{i=1}^k m)\left\|L_1(s)-L_2(s)\right\|^2.
\end{aligned}
$$

Due to $[4T\Lambda+16\Lambda+4\frac{\alpha^2}{2\alpha-1}T^{2\alpha-1}+4k\sum_{i=1}^k m]<1$ and Banach contraction principle, we can obtain that system (1) has a unique solution. As a consequence, the proof of this theorem is completed. □

4 | AVERAGING PRINCIPLE

For this section, in order to establish an averaging principle for the SFDEs with impulses, we first give the following averaging conditions. For $\forall s\in[0,\ T]$, the standard form for system (1) can be expressed as follows

$$
\begin{aligned}
L_\varepsilon(s) = & L_0+\varepsilon\int_0^s f\left(t,L_\varepsilon(t)\right)dt+\sqrt{\varepsilon}\int_0^s h\left(t,L_\varepsilon(t)\right)dW(t) \\
& +\alpha\varepsilon^\alpha\int_0^s (s-t)^{\alpha-1}\sigma\left(t,L_\varepsilon(t)\right)dt+\varepsilon\sum_{i=1}^k I_i\left(L_\varepsilon(s_i)\right),
\end{aligned}
\tag{11}
$$

where ε_0 is a fixed point with $\varepsilon\in(0,\varepsilon_0]$ is a sufficiently small and positive parameter.

The next step is to verify the original solution $L_\varepsilon(s)$ converges to the solution $L_\varepsilon^*(s)$ of the average system when ε approaches zero, and $L_\varepsilon^*(s)$ is defined as follows

$$\begin{aligned} L_\varepsilon^*(s) =& L_0 + \varepsilon \int_0^s \overline{f}\left(L_\varepsilon^*(t)\right) dt + \sqrt{\varepsilon} \int_0^s \overline{h}\left(L_\varepsilon^*(t)\right) dW(t) \\ &+ \alpha\varepsilon^\alpha \int_0^s (s-t)^{\alpha-1} \overline{\sigma}\left(L_\varepsilon^*(t)\right) dt + \varepsilon \int_0^s \overline{I_i}\left(L_\varepsilon^*(t)\right) dt, \end{aligned} \tag{12}$$

where $\overline{f}, \overline{\sigma} : \mathbb{R}^d \to \mathbb{R}^d$, $\overline{h} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ and $\overline{I_i} : \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions satisfying the following hypotheses:

- (H_6) For $\forall x \in U, T_1 \in [0, T]$, the corresponding positive bounded functions $\gamma_1, \gamma_2, \gamma_3$ such that
  (i)
  $$\frac{1}{T_1} \int_0^{T_1} \left\| f(s,\ x) - \overline{f}(x) \right\| ds \le \gamma_1(T_1)(\|x\|),$$
  (ii)
  $$\frac{1}{T_1} \int_0^{T_1} \left\| h(s,\ x) - \overline{h}(x) \right\|^2 ds \le \gamma_2(T_1)(\|x\|^2),$$
  (iii)
  $$\frac{1}{T_1} \int_0^{T_1} \left\| \sigma(s,\ x) - \overline{\sigma}(x) \right\|^2 ds \le \gamma_3(T_1)(\|x\|^2),$$
  (iv)
  $$\overline{I}(x) \le \frac{1}{T_1} \sum_{i=1}^{k} I_i(x),$$
  where $\lim_{T_1\to\infty} \gamma_i(T_1) = 0$, and $i = 1, 2, 3$.
- (H_7) For every I_i, there exists a positive constant $\overline{m}$ such that for $\forall x \in U$
  $$\left\| I_i(x) \right\|^2 \le \overline{m}.$$

Theorem 3. Suppose that $(A_1), (H_6)$ and (H_7) are true. For an arbitrarily small number $\delta_1 > 0$, we can find the corresponding $P > 0$, $\varepsilon_1 \in \left(0, \varepsilon_0\right]$ and $\beta \in (0,1)$ satisfying for all $\varepsilon \in \left(0, \varepsilon_1\right]$

$$\mathbb{E}\left(\sup_{s\in\left[0, P\varepsilon^{-\beta}\right]} \left\| L_\varepsilon(s) - L_\varepsilon^*(s) \right\|^2 \right) \le \delta_1.$$

Proof. For $\forall u \in [0, T]$, it follows from (11)-(12) and owing to the Jensen's inequality, it follows that

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\le s\le u} \left\| L_\varepsilon(s) - L_\varepsilon^*(s) \right\|^2 \right) \\ \le& 4\varepsilon^2 \cdot \mathbb{E}\left(\sup_{0\le s\le u} \left\| \int_0^s [f\left(t, L_\varepsilon(t)\right) - \overline{f}\left(L_\varepsilon^*(t)\right)] dt \right\|^2 \right) \\ &+ 4\varepsilon \cdot \mathbb{E}\left(\sup_{0\le s\le u} \left\| \int_0^s [h\left(t, L_\varepsilon(t)\right) - \overline{h}\left(L_\varepsilon^*(t)\right)] dW(t) \right\|^2 \right) \\ &+ 4\alpha^2\varepsilon^{2\alpha} \cdot \mathbb{E}\left(\sup_{0\le s\le u} \left\| \int_0^s (s-t)^{\alpha-1} [\sigma\left(t, L_\varepsilon(t)\right) - \overline{\sigma}\left(L_\varepsilon^*(t)\right)] dt \right\|^2 \right) \end{aligned}$$

$$
\begin{aligned}
&+4\varepsilon^2\cdot\mathbb{E}\left(\sup_{0\le s\le u}\left\|\sum_{i=1}^{k} I_i\left(L_\varepsilon(s_i)\right)-\int_0^s \overline{I_i}\left(L_\varepsilon^*(t)\right)dt\right\|^2\right)\\
:=&J_1+J_2+J_3+J_4.
\end{aligned}
$$

Specifically, in order to get the desired results by the Jensen's inequality again, we can get

$$
\begin{aligned}
J_1\le&8\varepsilon^2\cdot\mathbb{E}\left(\sup_{0\le s\le u}\left\|\int_0^s[f\left(t,L_\varepsilon(t)\right)-f\left(t,L_\varepsilon^*(t)\right)]dt\right\|^2\right)\\
&+8\varepsilon^2\cdot\mathbb{E}\left(\sup_{0\le s\le u}\left\|\int_0^s[f\left(t,L_\varepsilon^*(t)\right)-\overline{f}\left(L_\varepsilon^*(t)\right)]dt\right\|^2\right)\\
:=&J_{11}+J_{12}.
\end{aligned}
$$

By the Hölder inequality and (A_1), we can get

$$
\begin{aligned}
J_{11}\le&8\varepsilon^2u\cdot\mathbb{E}\left(\sup_{0\le s\le u}\int_0^s[\lambda(t)\left\|L_\varepsilon(t)-L_\varepsilon^*(t)\right\|^2]dt\right)\\
\le&8\varepsilon^2\Lambda u\cdot\int_0^u\mathbb{E}(\sup_{0\le t_1\le t}\left\|L_\varepsilon(t_1)-L_\varepsilon^*(t_1)\right\|^2)dt.
\end{aligned}
\tag{13}
$$

According to Hypothesis (H_6), it is easy to get

$$
\begin{aligned}
J_{12}\le&8\varepsilon^2u^2\gamma_1^2(u)\cdot\mathbb{E}\left(\sup_{0\le s\le u}\left\|L_\varepsilon^*(s)\right\|^2\right)\\
:=&\Theta_{12}\varepsilon^2u^2,
\end{aligned}
\tag{14}
$$

where $\Theta_{12}=8\gamma_1^2(u)\cdot\mathbb{E}\left(\sup_{0\le s\le u}\left\|L_\varepsilon^*(s)\right\|^2\right)$.

In dealing with the second part, with the aid of the Burkholder-Davis-Gundy inequality and Jensen's inequality, we can obtain

$$
\begin{aligned}
J_2\le&16\varepsilon\cdot\mathbb{E}\int_0^u\left\|h\left(t,L_\varepsilon(t)\right)-\overline{h}\left(L_\varepsilon^*(t)\right)\right\|^2dt\\
\le&32\varepsilon\cdot\mathbb{E}\int_0^u\left\|h\left(t,L_\varepsilon(t)\right)-h\left(t,L_\varepsilon^*(t)\right)\right\|^2dt\\
&+32\varepsilon\cdot\mathbb{E}\int_0^u\left\|h\left(t,L_\varepsilon^*(t)\right)-\overline{h}\left(L_\varepsilon^*(t)\right)\right\|^2dt\\
:=&J_{21}+J_{22}.
\end{aligned}
$$

By (A_1), we can obtain

$$
\begin{aligned}
J_{21}\le&32\varepsilon\cdot\mathbb{E}\int_0^u[\lambda(t)\left\|L_\varepsilon(t)-L_\varepsilon^*(t)\right\|^2]dt\\
\le&32\varepsilon\Lambda\cdot\int_0^u\mathbb{E}\left(\sup_{0\le t_1\le t}\left\|L_\varepsilon(t_1)-L_\varepsilon^*(t_1)\right\|^2\right)dt.
\end{aligned}
\tag{15}
$$

According to (H_6), we can get

$$\begin{aligned} J_{22} \leq & 32\varepsilon u\gamma_2(u) \cdot \mathbb{E}\left(\sup_{0\leq s\leq u}\left\|L_\varepsilon^*(s)\right\|^2\right) \\ := & \Theta_{22}\varepsilon u, \end{aligned} \tag{16}$$

where $\Theta_{22} = 32\gamma_2(u) \cdot \mathbb{E}\left(\sup_{0\leq s\leq u}\left\|L_\varepsilon^*(s)\right\|^2\right)$.

As for the third term, we have the following

$$\begin{aligned} J_3 \leq & 8\alpha^2\varepsilon^{2\alpha} \cdot \mathbb{E}\left(\sup_{0\leq s\leq u}\left\|\int_0^s (s-t)^{\alpha-1}[\sigma\left(t, L_\varepsilon(t)\right) - \sigma\left(t, L_\varepsilon^*(t)\right)]dt\right\|^2\right) \\ & + 8\alpha^2\varepsilon^{2\alpha} \cdot \mathbb{E}\left(\sup_{0\leq s\leq u}\left\|\int_0^s (s-t)^{\alpha-1}[\sigma\left(t, L_\varepsilon^*(t)\right) - \overline{\sigma}\left(L_\varepsilon^*(t)\right)]dt\right\|^2\right) \\ := & J_{31} + J_{32}. \end{aligned}$$

The H$\ddot{o}$lder inequality and (A_1) can export

$$\begin{aligned} J_{31} \leq & 8\frac{\alpha^2}{2\alpha-1}\varepsilon^{2\alpha}u^{2\alpha-1} \cdot \mathbb{E}(\sup_{0\leq s\leq u}\int_0^s \left\|\sigma\left(t, L_\varepsilon(t)\right) - \sigma\left(t, L_\varepsilon^*(t)\right)\right\|^2 dt \\ \leq & 8\frac{\alpha^2}{2\alpha-1}\varepsilon^{2\alpha}u^{2\alpha-1} \cdot \mathbb{E}(\sup_{0\leq s\leq u}\int_0^s \lambda(t)\left\|L_\varepsilon(t) - L_\varepsilon^*(t)\right\|^2 dt) \\ \leq & 8\frac{\alpha^2}{2\alpha-1}\varepsilon^{2\alpha}\Lambda u^{2\alpha-1} \cdot \int_0^u \mathbb{E}(\sup_{0\leq t_1\leq t}\left\|L_\varepsilon(t_1) - L_\varepsilon^*(t_1)\right\|^2)dt. \end{aligned} \tag{17}$$

By (H_6), we can obtain

$$\begin{aligned} J_{32} \leq & 8\frac{\alpha^2}{2\alpha-1}\varepsilon^{2\alpha}u^{2\alpha}\gamma_3(u) \cdot \mathbb{E}\left(\sup_{0\leq t\leq u}\left\|L_\varepsilon^*(t)\right\|^2\right) \\ := & \Theta_{32}\varepsilon^{2\alpha}u^{2\alpha}, \end{aligned} \tag{18}$$

where $\Theta_{32} = 8\frac{\alpha^2}{2\alpha-1}\gamma_3(u) \cdot \mathbb{E}\left(\sup_{0\leq t\leq u}\left\|L_\varepsilon^*(t)\right\|^2\right)$.

With the aid of (H_6) and (H_7), we can obtain

$$\begin{aligned} J_4 \leq & 8\varepsilon^2 \cdot \mathbb{E}(\sup_{0\leq s\leq u}\left\|\sum_{i=1}^k I_i(L_\varepsilon(s_i))\right\|^2) \\ & + 8\varepsilon^2 \cdot \mathbb{E}(\sup_{0\leq s\leq u}\left\|\int_0^s \overline{I_i}(L_\varepsilon^*(t))dt\right\|^2) \\ \leq & 8\varepsilon^2 k \cdot \mathbb{E}(\sup_{0\leq s\leq u}\sum_{i=1}^k \left\|I_i(L_\varepsilon(s_i))\right\|^2) \\ & + 8\varepsilon^2\frac{k}{T_1^2}u \cdot \mathbb{E}(\sup_{0\leq s\leq u}\int_0^s \sum_{i=1}^k \left\|I_i(L_\varepsilon^*(t)\right\|^2 dt) \\ \leq & (8k^2\overline{m} + 8k^2u^2\overline{m}\frac{1}{T_1^2})\varepsilon^2 \\ := & \Theta_4\varepsilon^2, \end{aligned} \tag{19}$$

where $\Theta_4 = 8k^2\overline{m} + 8k^2u^2\overline{m}\frac{1}{T_1^2}$.

It follows from (13)–(19) that

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le s\le u}\left\|L_\varepsilon(s)-L_\varepsilon^*(s)\right\|^2\right) \le & 8\varepsilon^2\Lambda u\cdot\int_0^u \mathbb{E}(\sup_{0\le t_1\le t}\left\|L_\varepsilon(t_1)-L_\varepsilon^*(t_1)\right\|^2)dt+\Theta_{12}\varepsilon^2u^2\\
&+32\varepsilon\Lambda\cdot\int_0^u \mathbb{E}\left(\sup_{0\le t_1\le t}\left\|L_\varepsilon(t_1)-L_\varepsilon^*(t_1)\right\|^2\right)dt+\Theta_{22}\varepsilon u\\
&+8\frac{\alpha^2}{2\alpha-1}\varepsilon^{2\alpha}\Lambda u^{2\alpha-1}\cdot\int_0^u \mathbb{E}(\sup_{0\le t_1\le t}\left\|L_\varepsilon(t_1)-L_\varepsilon^*(t_1)\right\|^2)dt+\Theta_{32}\varepsilon^{2\alpha}u^{2\alpha}+\Theta_4\varepsilon^2\\
\le & \Theta_{12}\varepsilon^2u^2+\Theta_{22}\varepsilon u+\Theta_{32}\varepsilon^{2\alpha}u^{2\alpha}+\Theta_4\varepsilon^2+(8\varepsilon^2\Lambda u+32\varepsilon\Lambda+8\frac{\alpha^2}{2\alpha-1}\varepsilon^{2\alpha}\Lambda u^{2\alpha-1})\\
&\cdot\int_0^u \mathbb{E}\left(\sup_{0\le t_1\le t}\left\|L_\varepsilon(t_1)-L_\varepsilon^*(t_1)\right\|^2\right)dt.
\end{aligned}$$

By means of the Grönwall-Bellman inequality, we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le s\le u}\left\|L_\varepsilon(s)-L_\varepsilon^*(s)\right\|^2\right) \le & (\Theta_{12}\varepsilon^2u^2+\Theta_{22}\varepsilon u+\Theta_{32}\varepsilon^{2\alpha}u^{2\alpha}+\Theta_4\varepsilon^2)\\
&\cdot exp(\int_0^u 8\varepsilon^2\Lambda u+32\varepsilon\Lambda+8\frac{\alpha^2}{2\alpha-1}\varepsilon^{2\alpha}\Lambda u^{2\alpha-1}dt)\\
=&(\Theta_{12}\varepsilon^2u^2+\Theta_{22}\varepsilon u+\Theta_{32}\varepsilon^{2\alpha}u^{2\alpha}+\Theta_4\varepsilon^2)\\
&\cdot exp[(8\varepsilon^2\Lambda u+32\varepsilon\Lambda+8\frac{\alpha^2}{2\alpha-1}\varepsilon^{2\alpha}\Lambda u^{2\alpha-1})u],
\end{aligned}$$

which implies there exist a constant $P > 0$ and $\beta \in (0,\ 1)$, for $\forall s \in [0,\ P\varepsilon^{-\beta}] \subseteq J$ satisfying

$$\mathbb{E}\left(\sup_{0\le s\le P\varepsilon^{-\beta}}\left\|L_\varepsilon(s)-L_\varepsilon^*(s)\right\|^2\right)\le\xi\varepsilon^{1-\beta},$$

where $\xi = [\Theta_{12}P^2\varepsilon^{1-\beta}+\Theta_{22}P+\Theta_{32}P^{2\alpha}\varepsilon^{2\alpha+\beta-2\alpha\beta-1}+\Theta_4\varepsilon^2]\cdot exp[8\Lambda P^2\varepsilon^{2-2\beta}+32\Lambda P\varepsilon^{1-\beta}+8\frac{\alpha^2}{2\alpha-1}\Lambda P^{2\alpha}\varepsilon^{2\alpha-2\alpha\beta}]$ is a constant. Hence, for arbitrarily given $\delta_1 > 0$, there exists $\varepsilon_1 \in (0,\ \varepsilon_0]$, for each $\varepsilon \in (0,\ \varepsilon_1]$ and $s \in [0,\ P\varepsilon^{-\beta}]$, we have

$$\mathbb{E}\left(\sup_{s\in[0,P\varepsilon^{-\beta}]}\left\|L_\varepsilon(s)-L_\varepsilon^*(s)\right\|^2\right)\le\delta_1.$$

□

Corollary 1. Suppose that Assumptions $(A_1)-(A_2)$ and $(H_6)-(H_7)$ set up. Then for an arbitrarily number $\delta_2 > 0$ such that for $P > 0, \beta \in (0, 1)$ and $\varepsilon_1 \in (0, \varepsilon_0)$ satisfying for all $\varepsilon \in (0, \varepsilon_1]$,

$$\lim_{\varepsilon\to 0}\mathbb{P}(\sup_{s\in[0,p\varepsilon^{-\beta}]}\left\|L_\varepsilon(s)-L_\varepsilon^*(s)\right\|>\delta_2)=0.$$

Proof. By Theorem 3 and utilizing the Chebyshev-Markov inequality, for any number $\delta_2 > 0$, we can obtain

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\in[0,P\varepsilon^{-\beta}]}\left\|L_\varepsilon(s)-L_\varepsilon^*(s)\right\|>\delta_2\right) &\le \frac{1}{\delta_2^2}\mathbb{E}\left(\sup_{s\in[0,P\varepsilon^{-\beta}]}\left\|L_\varepsilon(s)-L_\varepsilon^*(s)\right\|^2\right)\\
&\le\frac{\xi\varepsilon^{1-\beta}}{\delta_2^2}\\
&\le 0\ as\ \varepsilon\to\ 0,
\end{aligned}$$

where $\xi = [\Theta_{12}P^2\varepsilon^{1-\beta}+\Theta_{22}P+\Theta_{32}P^{2\alpha}\varepsilon^{2\alpha+\beta-2\alpha\beta-1}+\Theta_4\varepsilon^2]\cdot exp[8\Lambda P^2\varepsilon^{2-2\beta}+32\Lambda P\varepsilon^{1-\beta}+8\frac{\alpha^2}{2\alpha-1}\Lambda P^{2\alpha}\varepsilon^{2\alpha-2\alpha\beta}]$. This completes the proof. □

5 | EXAMPLE

In this section, we aim to give an example to illustrate the results of the averaging principle. Consider the following SFDEs with impulses

$$\begin{cases} dL_\varepsilon(s) = \varepsilon sin^2(s)L_\varepsilon(s)ds + \sqrt{\varepsilon}cos^2(s)L_\varepsilon(s)dW(s) + \frac{1}{2}\varepsilon^{0.6}sin(L_\varepsilon(s))(ds)^{0.6}, s \in J, \\ \Delta L_\varepsilon(s) = \varepsilon i^2 L_\varepsilon(s_i^-), s = s_i, i = 1, 2, \cdots, k, \\ L(0) = L_0, \end{cases} \tag{20}$$

where $\alpha = 0.6$. The coefficients $f(s, L_\varepsilon(s)) = sin^2(s)L_\varepsilon(s), h(s, L_\varepsilon(s)) = cos^2(s)L_\varepsilon(s), \sigma(s, L_\varepsilon(s)) = \frac{1}{2}sin(L_\varepsilon(s))$ and $I_i(s) = i^2 L_\varepsilon(s)$ satisfy the conditions (A_1) and (H_7). Let $T_1 = 1$. And we define

$$\overline{f}(L_\varepsilon(s)) = \frac{1}{T_1}\int_0^{T_1} f(s, L_\varepsilon)ds = \int_0^1 sin^2(t)L_\varepsilon dt = \frac{2 - sin2}{4}L_\varepsilon,$$

$$\overline{h}(L_\varepsilon(s)) = \frac{1}{T_1}\int_0^{T_1} h(s, L_\varepsilon)ds = \int_0^1 cos^2(t)L_\varepsilon dt = \frac{2 + sin2}{4}L_\varepsilon,$$

$$\overline{\sigma}(L_\varepsilon(s)) = \frac{1}{T_1}\int_0^{T_1} \sigma(s, L_\varepsilon)ds = \int_0^1 \frac{1}{2}sin(L_\varepsilon)dt = \frac{1}{2}sinL_\varepsilon,$$

$$\overline{I}(L_\varepsilon(s)) = \frac{1}{T_1}\sum_{i=1}^{k} I_i(s) = \sum_{i=1}^{k} i^2 L_\varepsilon = \frac{k(k+1)(2k+1)}{6}L_\varepsilon.$$

According to the above discussions, (H_6) is established. Then, we simplify SFDEs with impulses can be defined as

$$\begin{aligned} L_\varepsilon^*(s) =& L_0 + \varepsilon\int_0^s \frac{2 - sin2}{4}L_\varepsilon(t)dt + \sqrt{\varepsilon}\int_0^s \frac{2 + sin2}{4}L_\varepsilon(t)dW(t) \\ &+ 0.6\varepsilon^{0.6}\int_0^s (s-t)^{-0.4}\frac{1}{2}sin(L_\varepsilon(t))dt + \varepsilon\int_0^s \frac{k(k+1)(2k+1)}{6}L_\varepsilon(t)dt. \end{aligned}$$

Through verification, we can get that the conditions of Theorem 3 and Corollary 1 are satisfied. Hence, as $\varepsilon \to 0$, the average solution $L_\varepsilon^*(s)$ is equivalent to the original solution $L_\varepsilon(s)$ in the sense of mean square and in probability.

6 | CONCLUSIONS

In this paper, the existence, uniqueness and averaging principle of solutions for SFDEs with impulses are studied. Using M*ö*nch fixed point theorem, the existence theorem is derived. In the next step, the uniqueness theorem can be gotten by using Banach contraction principle. In order to obtain the averaging principle of SFDEs with impulses, we give the averaging conditions, and use various inequality scaling techniques, such as Jensen's inequality, H*ö*lder inequality, Burkholder-Davis-Gundy inequality and Gr*ö*nwall-Bellman inequality to derive the desired results. Finally, an example is given to illustrate the validity of theoretical results.

After finishing the work of this paper, we will be devoted to studying the averaging principle of fuzzy fractional stochastic differential equations. In addition, how to deal with random terms in fuzzy sense is a difficult but interesting subject, which will be discussed carefully in our following work.

ACKNOWLEDGMENTS

We would like to thank the editor and the anonymous referees sincerely for their useful comments and constructive suggestions on our paper, which have considerably improved the quality.

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