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?RESEARCH ARTICLE

Approximation of Dirichlet-to-Neumann operator for problems of diffraction by planar obstacles covered with thin dielectric multilayers

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Summary

We are interested in problems of diffraction of an acoustic or electromagnetic wave by a perfectly conducting planar obstacle coated with thin multilayers of dielectric materials. The aim is to obtain boundary condition that replaces the effect of dielectric thin layers. This condition is constructed from an approximation of the Dirichlet-to-Neumann operator. In this paper, we analyze the construction and the approximation of this operator.

KEYWORDS:
Acoustic and electromagnetic scattering, diffraction, thin dielectric layers, Dirichlet-to-Neumann operator

1 | INTRODUCTION

The diffraction of acoustic and electromagnetic waves by a perfectly conducting obstacle coated with thin dielectric layers, emerges in many applications in the industrial world such as electromagnetic compatibility problems in embedded systems, antennas, satellites, telecommunications, sonar, or also applications involving the detection of objects and radar stealth, see, for example[6,20,3,23,22,10] and the references therein.

In this paper, we are particularly interested in problems of diffraction by perfectly conducting planar obstacles, covered with homogenous thin dielectric multilayers. These problems are called transmission problems, which consist in solving a system of partial differential equations in an exterior domain with a radiation condition at infinity and in an interior domain relating to thin dielectric layers. The governing equations are coupled by connecting conditions set on the common interface between exterior and interior domains and between thin layers as well. Solving numerically these equations is challenging since it requires discretizing on the scale of the layers' thickness. The mesh then contains a very large number of elements, which makes the calculations long and sometimes imprecise[5,6,9,2,16]. For this reason, we try to replace our problem by another problem that does not bring in any more thin layers. The use of so-called Dirichlet-to-Neumann operator, relative to the equations set in thin layers allows to reduce the solving of our original problem to a problem that is posed only in the exterior domain with an appropriate boundary condition known as Dirichlet-to-Neumann condition[6,8,11,12,19] and abbreviated DtoN condition, which is also called Steklov-Poincaré condition or impedance condition as well[7,4]. The whole difficulty rests on the knowledge of this operator, which is generally non-explicit[6,13]. Fortunately, it is possible to explicit and approximate it in many cases as in planar obstacles[14,17,1], we are then able to construct the DtoN conditions in this case. We will exploit the fact that the thicknesses of thin layers tend to zero to derive the approximations of the DtoN operator.

This paper is organized as follows. In the next section we start with presenting briefly the physical problem and the mathematical governing equations. Then we reformulate our problem using DtoN operator, after that, we determine the exact formula of this operator. The third section is devoted to constructing approximations of the DtoN operator using two approaches: the first one consists of writing Taylor expansions iteratively in the thin layers and the second approach is to use asymptotic expansions. In the last section we apply the results obtained in the third section to a particular problem of scattering of electromagnetic waves.

2 | PROBLEM STATEMENT

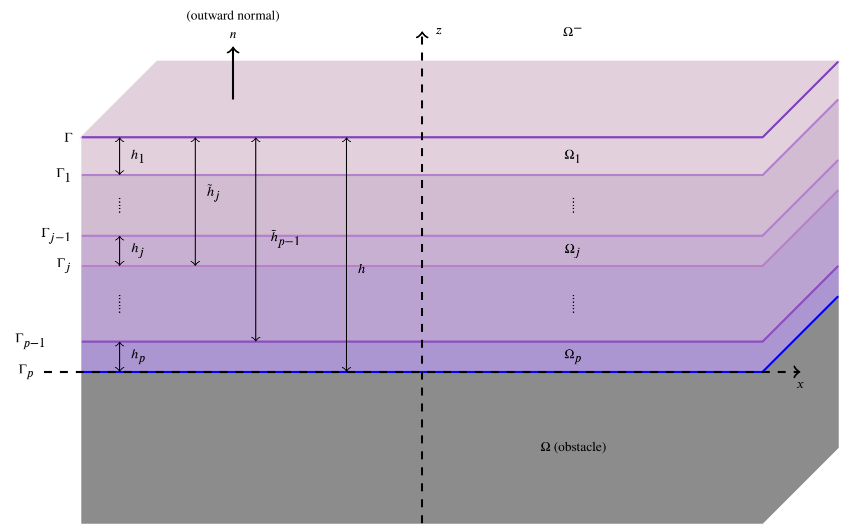
We consider the case of a perfectly conducting obstacle (made of metal) coated with p parallel thin dielectric layers of thicknesses h_j, $j = 1, .., p$. The dielectric of thickness h_j is characterized by a relative permittivity ε_j and a relative permeability $\mu_j, j = 1, .., p$. The metallic obstacle coated with thin dielectric layers is placed in a dielectric medium (propagation medium). This medium can be the vacuum and it is characterized by a permittivity ε_0 and permeability μ_0. This system is illuminated by an incident wave characterized by its number $k > 0$. When this wave encounters the obstacle, it generates a wave diffracted by this latter.

The metallic obstacle occupies a three-dimensional planar domain Ω; the thin layers are denoted by Ω_j with interior boundary $\partial_{int}\Omega_j, j = 1, .., p$.. The domain Ω adding to it the p thin layers is denoted by Ω^+ with boundary Γ and unit outward normal vector n. The exterior domain of Ω^+ is designated Ω^-. The thickness of the layers from the first till the j^{th} is $\tilde{h}_j = h_1 + ... + h_j$. We set $\tilde{h}_p = h$ and $h_j = \beta_j h$ with $\sum_{j=1}^{p} \beta_j = 1$, by convention $\tilde{h}_0 = h_0 = \beta_0 = 0$.

We introduce the family $\Gamma(s)$ of parallel surfaces

$$\Gamma(s) = \{y; y = x - sn(x), x \in \Gamma\}, s \in (-\infty, h].$$

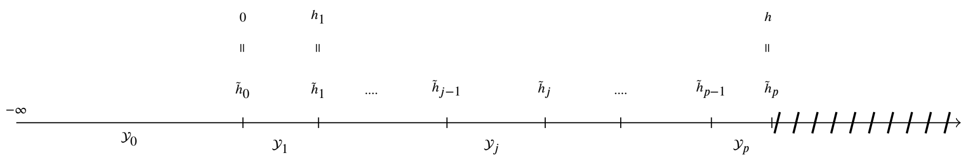
We notice that $\Gamma(0) = \Gamma$, $\partial_{int}\Omega_j = \Gamma\left(\tilde{h}_j\right)$ and Ω_j is the domain limited by $\Gamma_{\tilde{h}_j}$ and $\Gamma_{\tilde{h}_{j-1}}$. We set $\Gamma_j = \Gamma\left(\tilde{h}_j\right), j = 1, .., p$.



Many scattering problems of acoustic and electromagnetic waves by a perfectly conducting obstacle coated with thin dielectric layers can be represented in curvilinear coordinates [20] by the equations

$$\left.\begin{array}{ll}
\frac{\partial}{\partial s}\mathcal{Y}_p(s) = \mathcal{M}_p\mathcal{Y}_p(s) & \text{in } C\left(\left(\tilde{h}_{p-1}, \tilde{h}_p\right); X\right), \\
\mathcal{Y}_p\left(\tilde{h}_p\right) = \left[\varphi_1, \varphi_2\right]^{\mathrm{T}}, & \text{with } \sigma_1\varphi_1 + \sigma_2\varphi_2 = \varphi, \\
\frac{\partial}{\partial s}\mathcal{Y}_j(s) = \mathcal{M}_j\mathcal{Y}_j(s) & \text{in } C\left(\left(\tilde{h}_{j-1}, \tilde{h}_j\right); X\right), \; j = 1, ...p-1, \\
\mathcal{Y}_j\left(\tilde{h}_j\right) = \mathcal{Y}_{j+1}\left(\tilde{h}_j\right) & j = 1, ...p-1, \\
\frac{\partial}{\partial s}\mathcal{Y}_0(s) = \mathcal{M}_0\mathcal{Y}_0(s) & \text{in } C\left((-\infty, 0); X\right), \\
\mathcal{Y}_0(0) = \mathcal{Y}_1(0), & \\
+ \text{ condition for } s \to -\infty. &
\end{array}\right\} \tag{1}$$

Where $\mathcal{Y}_j = \left[U_j, V_j\right]^{\mathrm{T}}$ is in $C^1\left(\left(\tilde{h}_{p-1}, \tilde{h}_p\right); X\right)$, $\mathcal{M}_j = \begin{bmatrix} A_j & G_j \\ F_j & B_j \end{bmatrix}$ are linear differential operators with values in a Hilbert space X on $\mathbb{R}^n, n = 1, 2, 3$. The constants σ_1, σ_2 and φ are given.



Remark 1. For planar obstacles, the operators $\mathcal{M}_j$ are independent of s, however for arbitrary shaped obstacles it does dependent on s.

In order to ensure that the problem (1) is *well-posed* in the sense of[15, page 83], we assume that $\mathcal{M}_j,\ j = 1, ...p$ are generators of strongly continuous semigroups, see[21].

2.1 | Dirichlet-to-Neumann operator

As we mentioned in the introduction, solving numerically the problem (1) is challenging since it requires discretizing on the scale of the layers' thickness. The mesh then contains a very large number of elements, which makes the calculations long and sometimes imprecise. For this reason, we reformulate our problem (1) and replace it by another problem that does not bring in any more thin layers. The use of Dirichlet-to-Neumann operator, relative to the equations set in thin layers allows to reduce the solving of our original problem to a problem that is posed only in the exterior domain Ω^- corresponding to $s \in (-\infty, 0)$.
Our goal, therefore, is to rewrite the problem (1) as a problem in the exterior domain with an appropriate boundary condition on Γ corresponding to $s = 0$, which is known as Dirichlet-to-Neumann condition. To express this condition accurately, we introduce the Dirichlet-to-Neumann operator, abbreviated DtoN, which is also called Steklov-Poincaré operator and is known as impedance operator as well[7],[4].
We begin by defining this new operator. For $\phi = \left[\phi_1, \phi_2\right]^{\mathrm{T}}$ sufficiently smooth defined on $s = 0$, we consider $\mathcal{Y}^+ = \left(\mathcal{Y}_1, ..., \mathcal{Y}_p\right)$ the solution of the following problem:

$$\left.\begin{aligned}
&\tfrac{\partial}{\partial s}\mathcal{Y}_1(s) = \mathcal{M}_1\mathcal{Y}_1(s) \quad \text{in } C\left(\left(0, \tilde{h}_1\right); X\right), \\
&\mathcal{Y}_1(0) = \left[\phi_1, \phi_2\right]^{\mathrm{T}}, \\
&\tfrac{\partial}{\partial s}\mathcal{Y}_j(s) = \mathcal{M}_j\mathcal{Y}_j(s) \qquad \text{in } C\left(\left(\tilde{h}_{j-1}, \tilde{h}_j\right); X\right),\ j = 2, ...p, \\
&\mathcal{Y}_j\left(\tilde{h}_{j-1}\right) = \mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right) \qquad\qquad j = 2, ...p, \\
&\mathcal{Y}_p\left(\tilde{h}_p\right) = \left[\varphi_1, \varphi_2\right]^{\mathrm{T}} \quad \text{with } \sigma_1\varphi_1 + \sigma_2\varphi_2 = \varphi.
\end{aligned}\right\} \tag{2}$$

Definition 1. We define the DtoN operator by the mapping

$$S\ :\ \phi_1 \mapsto S\phi_1 = \phi_2. \tag{3}$$

The problem for $s \in (-\infty, 0)$, becomes then

$$\left.\begin{aligned}
&\tfrac{\partial}{\partial s}\mathcal{Y}_0(s) = \mathcal{M}_0\mathcal{Y}_0(s) \qquad \text{in } C\left((-\infty, 0); X\right), \\
&\left(\mathcal{Y}_0(0)\right)_2 = S\left(\mathcal{Y}_0(0)\right)_1, \\
&+ \text{condition for } s \to -\infty.
\end{aligned}\right\} \tag{4}$$

The inverse operator $S^{-1}\ :\ \phi_2 \mapsto S^{-1}\phi_2 = \phi_1$ is called Neumann-to-Dirichlet operator[18].

Remark 2. If we are interested in the values inside the thin layers, we define in a similar manner, the DtoN operator S posed in the exterior domain.

2.2 Determination of the exact Dirichlet-to-Neumann operator

Note that the calculation of the DtoN operator returns to express $\mathcal{Y}_p\left(\tilde{h}_p\right) = \left[\varphi_1, \varphi_2\right]^{\mathrm{T}}$ in terms of $\mathcal{Y}_0(0) = \left[\phi_1, \phi_2\right]^{\mathrm{T}}$.

Theorem 1. The exact Dirichlet-to-Neumann operator is given by

$$S\ :\ \phi_1 \mapsto S\phi_1 = \left(\sigma_1 Q_1 + \sigma_2 Q_2\right)^{-1}\left(\varphi - \left(\sigma_1 P_1 + \sigma_2 P_2\right)\phi_1\right). \tag{5}$$

where

$$\begin{bmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{bmatrix} = \exp\left(h_p \mathcal{M}_p\right) \ldots \exp\left(h_j \mathcal{M}_j\right) \ldots \exp\left(h_1 \mathcal{M}_1\right).$$

Proof. Existence and uniqueness of the DtoN operator S comes from solving successively Cauchy problems.
The unique solution of the Cauchy problem

$$\left.\begin{aligned} &\tfrac{\partial}{\partial s}\mathcal{Y}_j(s) = \mathcal{M}_j \mathcal{Y}_j(s) \qquad \text{in } C\left(\left(\tilde{h}_{j-1}, \tilde{h}_j\right); X\right), \\ &\mathcal{Y}_j\left(\tilde{h}_{j-1}\right) = \mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right), \end{aligned}\right\} \tag{6}$$

is given by

$$\mathcal{Y}_j(s) = \exp\left(\left(s - \tilde{h}_{j-1}\right)\mathcal{M}_j\right)\mathcal{Y}_j\left(\tilde{h}_{j-1}\right),\ s \in \left(\tilde{h}_{j-1}, \tilde{h}_j\right)$$

therefore for $s = \tilde{h}_j$ and replacing $\mathcal{Y}_j\left(\tilde{h}_{j-1}\right)$ by $\mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right)$ yields

$$\mathcal{Y}_j\left(\tilde{h}_j\right) = \exp\left(\left(\tilde{h}_j - \tilde{h}_{j-1}\right)\mathcal{M}_j\right)\mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right) = \exp\left(h_j \mathcal{M}_j\right)\mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right).$$

By induction it follows that

$$\left[\varphi_1, \varphi_2\right]^{\mathrm{T}} = \mathcal{Y}_p\left(\tilde{h}_p\right) = \tilde{\mathcal{M}}_p\left[\phi_1, \phi_2\right]^{\mathrm{T}}, \tag{7}$$

where

$$\tilde{\mathcal{M}}_p \equiv \begin{bmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{bmatrix} = \exp\left(h_p \mathcal{M}_p\right) \ldots \exp\left(h_j \mathcal{M}_j\right) \ldots \exp\left(h_1 \mathcal{M}_1\right). \tag{8}$$

The equation (7) is equivalent to

$$\left.\begin{aligned} P_1\phi_1 + Q_1\phi_2 = \varphi_1, \\ P_2\phi_1 + Q_2\phi_2 = \varphi_2. \end{aligned}\right\}$$

Since $\sigma_1\varphi_1 + \sigma_2\varphi_2 = \varphi$ then we get

$$\left(\sigma_1 P_1 + \sigma_2 P_2\right)\phi_1 + \left(\sigma_1 Q_1 + \sigma_2 Q_2\right)\phi_2 = \varphi.$$

Consequently

$$S\ :\ \phi_1 \mapsto S\phi_1 = \phi_2 = \left(\sigma_1 Q_1 + \sigma_2 Q_2\right)^{-1}\left(\varphi - \left(\sigma_1 P_1 + \sigma_2 P_2\right)\phi_1\right).$$

□

In most cases either σ_1 or σ_2 is equal to zero.
In the case where $\sigma_2 = 0$ the DtoN operator S_1 is

$$S_1\ :\ \phi_1 \mapsto S_1\phi_1 = Q_1^{-1}\left(\varphi_1 - P_1\phi_1\right). \tag{9}$$

Similarly if $\sigma_1 = 0$ the DtoN operator S_2 is

$$S_2\ :\ \phi_1 \mapsto S_2\phi_1 = Q_2^{-1}\left(\varphi_2 - P_2\phi_1\right). \tag{10}$$

Unfortunately, the formula of the exact DtoN operator is not practical for computation and it will be useful and interesting to approximate it. Our goal in this paper is to approximate this operator by an operator that is a rational fraction with respect to the thickness of thin layers.

3 APPROXIMATION OF DIRICHLET-TO-NEUMANN OPERATOR

We present two different approaches to approximate the DtoN operator. A first approach consists in using a Taylor expansions. A second approach concerns the asymptotic analysis of the problem with respect to the thickness of thin layers.
In order to simplify the formulas of the approximate DtoN operator, we introduce the multi-index notation.

3.1 | Multi-index notation

An p-dimensional multi-index is an p-tuple $\alpha = \left(\alpha_1, \alpha_2, ..., \alpha_p\right)$ of non-negative integers, which is in the set p-dimensional natural numbers, denoted $\mathbb{N}_0^p$.
For multi-indices α and β in $\mathbb{N}_0^p$ we define:
Componentwise sum and difference as $\alpha \pm \beta = \left(\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2, ..., \alpha_p \pm \beta_p\right)$.
Sum of components or absolute value as $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_p$.
Factorial as $\alpha! = \alpha_1!\alpha_2!...\alpha_p!$.
A vector $\mathbf{V} = \left(V_1, V_2, ..., V_p\right)$ to the power of multi-index α as $\mathbf{V}^\alpha = V_1^{\alpha_1} V_2^{\alpha_2} ... V_p^{\alpha_p}$.

3.2 | Approximation of the DtoN operator by Taylor expansions

Recall that the calculation of the DtoN operator returns to express $\mathcal{Y}_p\left(\tilde{h}_p\right)$ in terms of $\mathcal{Y}_0(0)$.
We start from the condition of electrical conductor at $s = \tilde{h}_p$ which is $\mathcal{Y}_p\left(\tilde{h}_p\right) = \left[\varphi_1, \varphi_2\right]^{\mathrm{T}}$ and we write a Taylor expansion at the points $s = \tilde{h}_{j-1}, j = p, ..., 1$. Using the fact that $\mathcal{Y}_j(s)$ satisfies the equation $\frac{\partial}{\partial s}\mathcal{Y}_j(s) = \mathcal{M}_j \mathcal{Y}_j(s)$ in $\left(\tilde{h}_{j-1}, \tilde{h}_j\right)$ and taking into account of the transmission conditions $\mathcal{Y}_j\left(\tilde{h}_{j-1}\right) = \mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right)$, we obtain a formula that connects $\mathcal{Y}_p\left(\tilde{h}_p\right)$ and $\mathcal{Y}_0(0)$. Then we can easily derive an approximation of the DtoN operator; the order of Taylor expansion will give the order of the approximation.

Theorem 2. An approximation of order n for DtoN operator (5) is given by

$$S : \phi_1 \mapsto S\phi_1 = \left(\sigma_1 Q_{1,n} + \sigma_2 Q_{2,n}\right)^{-1} \left(\varphi - \left(\sigma_1 P_{1,n} + \sigma_2 P_{2,n}\right)\phi_1\right), \tag{11}$$

where

$$P_{1,n} = \sum_{l=0}^{n} \mathbf{A}_l h^l, \ \ Q_{1,n} = \sum_{l=0}^{n} \mathbf{G}_l h^l, \ \ P_{2,n} = \sum_{l=0}^{n} \mathbf{F}_l h^l, \ \ Q_{2,n} = \sum_{l=0}^{n} \mathbf{B}_l h^l, \tag{12}$$

with

$$N_l \equiv \begin{bmatrix} \mathbf{A}_l & \mathbf{G}_l \\ \mathbf{F}_l & \mathbf{B}_l \end{bmatrix} = \sum_{|\alpha|=l} \frac{\mathbf{M}^\alpha}{\alpha!}, \ \ \mathbf{M} = \left(\beta_p \mathcal{M}_p, ..., \beta_1 \mathcal{M}_1\right).$$

Proof. By Taylor expansions

$$\mathcal{Y}_j\left(\tilde{h}_j\right) = \sum_{l=0}^{n} \mathcal{Y}_j^{(l)}\left(\tilde{h}_{j-1}\right) \frac{\left(h_j\right)^l}{l!}, \tag{13}$$

where $\mathcal{Y}_j^{(l)}$ is the derivative of order l of $\mathcal{Y}_j$ with respect to s with the convention $\mathcal{Y}_j^{(0)} \equiv \mathcal{Y}_j$. For simplicity in writing we omitted the term $o\left(\left(h_j\right)^n\right)$.
Since the matrix operator $\mathcal{M}_j$ is independent of s, we can easily see that

$$\mathcal{Y}_j^{(l)}(s) = \left(\mathcal{M}_j\right)^l \mathcal{Y}_j(s). \tag{14}$$

Replacing $\mathcal{Y}_j^{(l)}\left(\tilde{h}_{j-1}\right)$ by its value of (14) in (13), then substituting $\mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right)$ for $\mathcal{Y}_j\left(\tilde{h}_{j-1}\right)$, we obtain

$$\mathcal{Y}_j\left(\tilde{h}_j\right) = M_{j,n}\mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right), \tag{15}$$

where

$$M_{j,n} = \sum_{l=0}^{n} \left(\mathcal{M}_j\right)^l \frac{\left(h_j\right)^l}{l!} = \sum_{l=0}^{n} \left(\beta_j \mathcal{M}_j\right)^l \frac{h^l}{l!}, \tag{16}$$

with $\left(\mathcal{M}_j\right)^0$ is the 2×2 identity matrix. By induction we obtain

$$\mathcal{Y}_p\left(\tilde{h}_p\right) = \tilde{M}_{p,n}\mathcal{Y}_0(0), \tag{17}$$

with

$$\tilde{M}_{p,n} = M_{p,n} M_{p-1,n} ... M_{2,n} M_{1,n}. \tag{18}$$

According to the formula of exact DtoN operator (5), its approximation of order n can be expressed as

$$S : \phi_1 \mapsto S\phi_1 = \left(\sigma_1 Q_{1,n} + \sigma_2 Q_{2,n}\right)^{-1} \left(\varphi - \left(\sigma_1 P_{1,n} + \sigma_2 P_{2,n}\right)\phi_1\right), \tag{19}$$

where

$$\begin{bmatrix} P_{1,n} & Q_{1,n} \\ P_{2,n} & Q_{2,n} \end{bmatrix} = \tilde{M}_{p,n}. \tag{20}$$

To get approximation of any order with respect to h, we need to express $\tilde{M}_{p,n}$ as a polynomial ordered by increasing powers of h. The matrix $\tilde{M}_{p,n}$ can be written as

$$\tilde{M}_{p,n} = M_{p,n}...M_{2,n}M_{1,n} = \left(\sum_{l_p=0}^{n} \left(\beta_p \mathcal{M}_p\right)^{l_p} \frac{h^{l_p}}{l_p!}\right) .... \left(\sum_{l_1=0}^{n} \left(\beta_1 \mathcal{M}_1\right)^{l_1} \frac{h^{l_1}}{l_1!}\right), \tag{21}$$

which can be rearranged to the conventional form

$$\tilde{M}_{p,n} = \sum_{l=0}^{n} N_l h^l, \tag{22}$$

where

$$N_l \equiv \begin{bmatrix} \mathbf{A}_l & \mathbf{G}_l \\ \mathbf{F}_l & \mathbf{B}_l \end{bmatrix} = \sum_{l_1+l_2+...+l_p=l} \left(\frac{\left(\beta_p \mathcal{M}_p\right)^{l_p}}{l_p!} ... \frac{\left(\beta_2 \mathcal{M}_2\right)^{l_2}}{l_2!} \frac{\left(\beta_1 \mathcal{M}_1\right)^{l_1}}{l_1!} \right), \tag{23}$$

or alternatively it can be written as

$$N_l = \sum_{0 \le l_1 \le l_2 \le ... \le l_{p-1} \le l} \left(\frac{\left(\beta_p \mathcal{M}_p\right)^{l-l_{p-1}}}{\left(l - l_{p-1}\right)!} ... \frac{\left(\beta_2 \mathcal{M}_2\right)^{l_2 - l_1}}{\left(l_2 - l_1\right)!} \frac{\left(\beta_1 \mathcal{M}_1\right)^{l_1}}{l_1!} \right).$$

With the multi-indices notations introduced above, the term N_l can simply be written as

$$N_l = \sum_{|\alpha|=l} \frac{\mathbf{M}^{\alpha}}{\alpha!}, \tag{24}$$

where $\mathbf{M} = \left(\beta_p \mathcal{M}_p, ..., \beta_1 \mathcal{M}_1\right)$. The calculation of $N_l, l = 1, ...n$ determines the approximation of order n of DtoN operator, which is given by the formula (19) with

$$P_{1,n} = \sum_{l=0}^{n} \mathbf{A}_l h^l, \;\; Q_{1,n} = \sum_{l=0}^{n} \mathbf{G}_l h^l, \;\; P_{2,n} = \sum_{l=0}^{n} \mathbf{F}_l h^l, \;\; Q_{2,n} = \sum_{l=0}^{n} \mathbf{B}_l h^l.$$

□

3.2.1 | Approximation of order 0

We begin the calculations with something that is more simple, i.e. an approximation of order 0. In this case the corresponding matrix N_0 is a 2×2 identity matrix. The approximation of order 0 is therefore given by

$$S\phi_1 = \frac{1}{\sigma_2} \left(\varphi - \sigma_1 \phi_1\right) \;\text{ if }\; \sigma_2 \neq 0 \;\text{ and }\; S^{-1}\phi_2 = \varphi_1 \;\text{ if }\; \sigma_2 = 0.$$

The associated DtoN conditions are

$$\left(\mathcal{Y}_0(0)\right)_2 = \frac{1}{\sigma_2} \left(\varphi - \sigma_1 \left(\mathcal{Y}_0(0)\right)_1\right) \;\text{ if }\; \sigma_2 \neq 0 \;\text{ and }\; \left(\mathcal{Y}_0(0)\right)_1 = \varphi_1 \;\text{ if }\; \sigma_2 = 0.$$

These conditions are in fact quite reasonable, they simply consist of completely removing the thin layers. However, they are uninteresting because they do not take into account the effect of thin layers. They are not satisfactory only when the thicknesses of the layers become almost zero. We should therefore go further in our Taylor expansion to lead to conditions of higher order that are more useful. These conditions of order 0 must be recovered in all higher order approximations by letting the thickness h tend to zero.

3.2.2 | Approximation of order 1

Let us now examine the approximation of order 1. The matrix N_1 is given by

$$N_1 \equiv \begin{bmatrix} \mathbf{A}_1 & \mathbf{G}_1 \\ \mathbf{F}_1 & \mathbf{B}_1 \end{bmatrix} = \sum_{j=1}^{p} \beta_j \mathcal{M}_j = \sum_{j=1}^{p} \beta_j \begin{bmatrix} A_j & G_j \\ F_j & B_j \end{bmatrix}.$$

Therefore, the approximate DtoN operator of order 1 is

$$S\phi_1 = \frac{\varphi - \left(\sigma_1 + \left(\sigma_1 \mathbf{A}_1 + \sigma_2 \mathbf{F}_1\right) h\right) \phi_1}{\sigma_2 + \left(\sigma_1 \mathbf{G}_1 + \sigma_2 \mathbf{B}_1\right) h},$$

where

$$\mathbf{A}_1 = \sum_{j=1}^{p} \beta_j A_j, \;\; \mathbf{G}_1 = \sum_{j=1}^{p} \beta_j G_j, \;\; \mathbf{F}_1 = \sum_{j=1}^{p} \beta_j F_j, \;\; \mathbf{B}_1 = \sum_{j=1}^{p} \beta_j B_j.$$

3.2.3 | Approximation of order 2

The matrix N_2 can be written as

$$N_2 = \sum_{i,j,i>j}^{p} \beta_i \beta_j \mathcal{M}_i \mathcal{M}_j + \frac{1}{2} \sum_{j=1}^{p} \beta_j^2 \mathcal{M}_j^2 \equiv \begin{bmatrix} \mathbf{A}_2 & \mathbf{G}_2 \\ \mathbf{F}_2 & \mathbf{B}_2 \end{bmatrix}, \tag{25}$$

and thus the approximation of DtoN operator in this case is

$$S\phi_1 = \frac{\varphi - \left(\sigma_1 + \left(\sigma_1 \mathbf{A}_1 + \sigma_2 \mathbf{F}_1\right) h + \left(\sigma_1 \mathbf{A}_2 + \sigma_2 \mathbf{F}_2\right) h^2\right) \phi_1}{\sigma_2 + \left(\sigma_1 \mathbf{G}_1 + \sigma_2 \mathbf{B}_1\right) h + \left(\sigma_1 \mathbf{G}_2 + \sigma_2 \mathbf{B}_2\right) h^2} \tag{26}$$

with

$$\mathbf{A}_2 = \sum_{i,j,i>j}^{p} \beta_i \beta_j \left(A_i A_j + G_i F_j\right) + \frac{1}{2} \sum_{j=1}^{p} \beta_j^2 \left(A_j^2 + G_j F_j\right), \tag{27}$$

$$\mathbf{G}_2 = \sum_{i,j,i>j}^{p} \beta_i \beta_j \left(A_i G_j + G_i B_j\right) + \frac{1}{2} \sum_{j=1}^{p} \beta_j^2 \left(A_j G_j + G_j B_j\right), \tag{28}$$

$$\mathbf{F}_2 = \sum_{i,j,i>j}^{p} \beta_i \beta_j \left(F_i A_j + B_i F_j\right) + \frac{1}{2} \sum_{j=1}^{p} \beta_j^2 \left(F_j A_j + B_j F_j\right), \tag{29}$$

$$\mathbf{B}_2 = \sum_{i,j,i>j}^{p} \beta_i \beta_j \left(F_i G_j + B_i B_j\right) + \frac{1}{2} \sum_{j=1}^{p} \beta_j^2 \left(F_j G_j + B_j^2\right). \tag{30}$$

The expression of S in terms of A_j, G_j, F_j and B_j for the approximations of higher order can be derived easily from the formula (24). However, they are too long formulas and we would rather not give them here.

3.3 | Asymptotic analysis

We will now present another approach of constructing approximations of the DtoN operator, based on the construction of an asymptotic expansion with respect to the thickness of thin layers.

3.3.1 | Problem reformulation

The determination of the approximated DtoN operator by asymptotic expansions, based primarily on reformulating the problem (2), which helps eliminate the dependence of the problem geometry on the small parameter h. This can be done by the following change of variable:

$$t = \frac{s - \tilde{h}_j}{h_j} + j, \;\; \tilde{h}_{j-1} \le s \le \tilde{h}_j, \;\; j = 1, ..., p. \tag{31}$$

We then set

$$\mathcal{E}_j(t) = \mathcal{Y}_j(s), \;\; j-1 \le t \le j, \quad j = 1, ..., p. \tag{32}$$

We are now able to write the equations of the problem (2) verified by the new unknowns $\mathcal{E}_j(t)$, $j = 1, ..., p$.
The derivative of $\mathcal{E}$ with respect to the new variable t is written as

$$\tfrac{\partial}{\partial t} \mathcal{E}_j(t) = h_j \tfrac{\partial}{\partial s} \mathcal{Y}_j(s), \;\; j-1 \le t \le j, \quad j = 1, ..., p. \tag{33}$$

By inserting these formulas in the problem (2), we obtain

$$\left.\begin{array}{ll} \frac{\partial}{\partial t}\mathcal{E}_1(t) = h_1\mathcal{M}_1\mathcal{E}_1(t) & \text{in } C((0,1);X), \\ \mathcal{E}_1(0) = [\phi_1,\phi_2]^{\mathrm{T}}, & \\ \frac{\partial}{\partial t}\mathcal{E}_j(t) = h_j\mathcal{M}_j\mathcal{E}_j(t) & \text{in } C((j-1,j);X),\ j=2,...p, \\ \mathcal{E}_j(j-1) = \mathcal{E}_{j-1}(j-1) & j=2,...p, \\ \mathcal{E}_p(p) = [\varphi_1,\varphi_2]^{\mathrm{T}}. & \end{array}\right\} \tag{34}$$

3.3.2 | Asymptotic expansion

The thickness h of the thin layers is assumed to be small enough. This allows us to postulate the existence of an asymptotic expansion for the solution of the problem (34) in the following form:

$$\mathcal{E}_j(t) = \sum_{l=0}^{\infty}\mathcal{E}_{j,l}(t)\,h^l, \quad j-1 \le t \le j, \quad j=1,...,p, \tag{35}$$

where the functions $\mathcal{E}_{j,l}$ are independent of h.
By inserting these expressions in our problem (34) and formally identifying the same powers in h^l, it will lead to systems of equations that are independent of h. They allow to determine iteratively the terms of our asymptotic expansion.
We will start by writing the auxiliary problems arising from this formal identification in the equations of the problem (34).

$$\left.\begin{array}{ll} \frac{\partial}{\partial t}\mathcal{E}_{1,0} = 0;\ \ \frac{\partial}{\partial t}\mathcal{E}_{1,l} = \beta_1\mathcal{M}_1\mathcal{E}_{1,l-1}, l \ge 1 & \text{in } C((0,1);X), \\ \mathcal{E}_{1,0}(0) = [\phi_1,\phi_2]^{\mathrm{T}};\quad \mathcal{E}_{1,l}(0) = 0, l \ge 1 & \\ \frac{\partial}{\partial t}\mathcal{E}_{j,0} = 0;\ \ \frac{\partial}{\partial t}\mathcal{E}_{j,l} = \beta_j\mathcal{M}_j\mathcal{E}_{j,l-1}, l \ge 1 & \text{in } C((j-1,j);X),\ j=2,...p, \\ \mathcal{E}_{j,l}(j-1) = \mathcal{E}_{j-1,l}(j-1), l \ge 0, & j=2,...p. \end{array}\right\} \tag{36}$$

Solving these equations allows us to proceed to the determination of DtoN operator approximations. We immediately observe that

$$\mathcal{E}_{1,l}(t) = \frac{1}{l!}\left(t\beta_1\mathcal{M}_1\right)^l[\phi_1,\phi_2]^{\mathrm{T}},\ l \ge 0\ \text{ in } C((0,1);X).$$

Then solving iteratively for $\mathcal{E}_{j,l}$ we obtain

$$\mathcal{E}_{p,l}(t) = \left(\sum_{|\alpha|=l}\frac{(\mathbf{M}(t))^{\alpha}}{\alpha!}\right)[\phi_1,\phi_2]^{\mathrm{T}},\ l \ge 0,\ \text{in } C((p-1,p);X). \tag{37}$$

where $\mathbf{M}(t) = \left((t-p+1)\beta_p\mathcal{M}_p, \beta_{p-1}\mathcal{M}_{p-1}, ..., \beta_2\mathcal{M}_2, \beta_1\mathcal{M}_1\right)$.
Recall that the asymptotic expansion of the solution $\mathcal{E}_p$ is given by

$$\mathcal{E}_p(t) = \sum_{l=0}^{\infty}\mathcal{E}_{p,l}(t)\,h^l.$$

Substituting p for t in $\mathcal{E}_p(t)$, we obtain

$$\mathcal{E}_p(p) = \sum_{l=0}^{\infty}\mathcal{E}_{p,l}(p)\,h^l = [\varphi_1,\varphi_2]^{\mathrm{T}}.$$

with

$$\mathcal{E}_{p,l}(p) = \left(\sum_{|\alpha|=l}\frac{\mathbf{M}^{\alpha}}{\alpha!}\right)[\phi_1,\phi_2]^{\mathrm{T}},\ l \ge 0,$$

where $\mathbf{M} = \mathbf{M}(p) = \left(\beta_p\mathcal{M}_p, \beta_{p-1}\mathcal{M}_{p-1}, ..., \beta_2\mathcal{M}_2, \beta_1\mathcal{M}_1\right)$.
Finally we obtain the following formula:

$$\left(\sum_{l=0}^{\infty}\left(\sum_{|\alpha|=l}\frac{\mathbf{M}^{\alpha}}{\alpha!}\right)h^l\right)[\phi_1,\phi_2]^{\mathrm{T}} = [\varphi_1,\varphi_2]^{\mathrm{T}},$$

which allows to determine the asymptotic expansion of DtoN operator that is given by

$$S\phi_1 = \left(\sigma_1 Q_1 + \sigma_2 Q_2\right)^{-1}\left(\varphi - \left(\sigma_1 P_1 + \sigma_2 P_2\right)\phi_1\right),$$

where

$$\begin{bmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{bmatrix} = \sum_{l=0}^{\infty}\left(\sum_{|\alpha|=l}\frac{\mathbf{M}^{\alpha}}{\alpha!}\right)h^{l}.$$

We restrict the sum over l from 0 to n to get an approximation of order n. We observe that is the same formula as obtained in Taylor expansions.

4 | APPLICATIONS

In this section we apply the results obtained in approximating the DtoN operator, to a problem of scattering of a transverse electric (TE) electromagnetic wave by perfectly conducting planar obstacles, covered with thin homogenous dielectric multilayers. In TE electromagnetic waves, there will be simplifications in the components of electric and magnetic fields, the total wave can be represented only by its non zero magnetic component, which is a scalar two variables x and y function denoted $u(x, y)$. Therefore, the scattering problem can be reduced to the following scalar problem in dimension two.

$$\left.\begin{array}{ll}
\triangle u_p + \kappa_p u_p = 0 \quad \text{in } \Omega_p, & \\
\dfrac{\partial u_p}{\partial y}\left(x, -\tilde{h}_p\right) = 0, \quad x \in \mathbb{R}, & \\
& \\
\triangle u_j + \kappa_j u_j = 0 & \text{in } \Omega_j,\ j = 1, ...p-1, \\
\dfrac{1}{\varepsilon_j}\dfrac{\partial u_j}{\partial y}\left(x, -\tilde{h}_j\right) = \dfrac{1}{\varepsilon_{j+1}}\dfrac{\partial u_{j+1}}{\partial y}\left(x, -\tilde{h}_j\right); & x \in \mathbb{R},\ j = 1, ...p-1, \\
u_j\left(x, -\tilde{h}_j\right) = u_{j+1}\left(x, -\tilde{h}_j\right) & x \in \mathbb{R},\ j = 1, ...p-1, \\
& \\
\triangle u_0 + \kappa_0 u_0 = 0 & \text{in } \Omega^-, \\
\dfrac{1}{\varepsilon_0}\dfrac{\partial u_0}{\partial y}(x, 0) = \dfrac{1}{\varepsilon_1}\dfrac{\partial u_1}{\partial y}(x, 0);\ u_0(x, 0) = u_1(x, 0), & x \in \mathbb{R}, \\
+ \text{ a condition called of radiation,} &
\end{array}\right\}$$

with $\kappa_j = k^2\varepsilon_j\mu_j, j = 0, ..., p$.

If we set $\mathcal{Y}_j(s) = \left[U_j(s), V_j(s)\right]^{\mathrm{T}} = \left[u_j(x, -s), \dfrac{1}{\varepsilon_j}\dfrac{\partial u_j}{\partial y}(x, -s)\right]^{\mathrm{T}}$, then $\sigma_1 = \varphi = 0$ and $\mathcal{M}_j = \begin{bmatrix} A_j & G_j \\ F_j & B_j \end{bmatrix} = \begin{bmatrix} 0 & -\varepsilon_j \\ \dfrac{1}{\varepsilon_j}\left(\kappa_j + \dfrac{\partial^2}{\partial x^2}\right) & 0 \end{bmatrix}$ in the corresponding problem (2). In this case, the approximated DtoN operator of order n is

$$S : \phi \mapsto S\phi = \frac{-P_{2,n}}{Q_{2,n}}\phi, \tag{38}$$

where $\left[P_{2,n}, Q_{2,n}\right]$ is the second row of the 2×2 matrix $\tilde{M}_{p,n} = \sum_{l=0}^{n} N_l h^l$, $N_l = \sum_{|\alpha|=l}\dfrac{\mathbf{M}^{\alpha}}{\alpha!}$ with $\mathbf{M} = \left(\beta_p\mathcal{M}_p, ..., \beta_1\mathcal{M}_1\right)$.

We can prove easily that N_l is a diagonal matrix if l is even and it is with zeros in its diagonal if l is odd number. Consequently, $P_{2,n}$ has only odd powers of h and $Q_{2,n}$ has only even ones.

Remark 3. If the scattered wave is transverse magnetic (TM), it will be reduced to the same problem with the condition $u_p\left(x, -\tilde{h}_p\right) = 0$ instead of $\dfrac{\partial u_p}{\partial y}\left(x, -\tilde{h}_p\right) = 0$ and substituting μ for ε.

Approximation of order 2

The matrix $\tilde{M}_{p,2}$ for the approximation of order 2 is $\tilde{M}_{p,2} = N_0 + N_1 h + N_2 h^2$. As we have seen before, the matrices N_0 and N_1 are

$$N_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \; N_1 = \sum_{j=1}^{p} \beta_j \begin{bmatrix} A_j & G_j \\ F_j & B_j \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{G}_1 \\ \mathbf{F}_1 & 0 \end{bmatrix}, \tag{39}$$

where

$$\mathbf{G}_1 = -\sum_{j=1}^{p} \beta_j \varepsilon_j, \;\; \mathbf{F}_1 = \sum_{j=1}^{p} \frac{\beta_j}{\varepsilon_j} L_j \text{ with } L_j = \kappa_j + \frac{\partial^2}{\partial x^2}. \tag{40}$$

Using the results obtained in the formulas (25)-(30) we see that the matrix N_2 can be written as

$$N_2 = \begin{bmatrix} \mathbf{A}_2 & 0 \\ 0 & \mathbf{B}_2 \end{bmatrix},$$

where

$$\mathbf{A}_2 = -\sum_{j=1}^{p} \left(\tfrac{1}{2} \beta_j \varepsilon_j + \sum_{k=j+1}^{p} \beta_k \varepsilon_k \right) \frac{\beta_j}{\varepsilon_j} L_j, \;\; \mathbf{B}_2 = -\sum_{j=1}^{p} \left(\tfrac{1}{2} \beta_j \varepsilon_j + \sum_{k=1}^{j-1} \beta_k \varepsilon_k \right) \frac{\beta_j}{\varepsilon_j} L_j.$$

Thus, the approximate DtoN operator of order 2 is

$$S : \phi \mapsto S\phi = \frac{-\mathbf{F}_1 h}{1 + \mathbf{B}_2 h^2} \phi. \tag{41}$$

Approximation of order 4

As we mentioned above the matrices N_3 and N_4 are in the form

$$N_3 = \begin{bmatrix} 0 & \mathbf{G}_3 \\ \mathbf{F}_3 & 0 \end{bmatrix}, \; N_4 = \begin{bmatrix} \mathbf{A}_4 & 0 \\ 0 & \mathbf{B}_4 \end{bmatrix}.$$

Since the approximated DtoN operator of order 4 is

$$S : \phi \mapsto S\phi = \frac{-\mathbf{F}_1 h - \mathbf{F}_3 h^3}{1 + \mathbf{B}_2 h^2 + \mathbf{B}_4 h^4} \phi, \tag{42}$$

we need to calculate only $\mathbf{F}_3$ and $\mathbf{B}_4$.
Using the general formula (24) for calculating N_l we see that

$$\mathbf{F}_3 = -\sum_{i,j,i>j}^{p} \omega_{ij} \tfrac{\beta_i \beta_j}{\varepsilon_i \varepsilon_j} L_j L_i - \tfrac{1}{6} \sum_{j=1}^{p} \tfrac{\beta_j^3}{\varepsilon_j} L_j^2,$$

$$\mathbf{B}_4 = \sum_{i,j,i>j}^{p} \gamma_{ij} \tfrac{\beta_i \beta_j}{\varepsilon_i \varepsilon_j} L_j L_i + \tfrac{1}{6} \sum_{j=1}^{p} b_j \tfrac{\beta_j^3}{\varepsilon_j} L_j^2,$$

where

$$\omega_{ij} = \tfrac{1}{2} \left(\beta_i \varepsilon_i + \beta_j \varepsilon_j \right) + \sum_{k=j+1}^{i-1} \beta_k \varepsilon_k, \;\; \gamma_{ij} = \omega_{ij} a_j - \tfrac{1}{12} \beta_j^2 \varepsilon_j^2,$$

$$a_j = \tfrac{1}{2} \beta_j \varepsilon_j + \sum_{k=1}^{j-1} \beta_k \varepsilon_k, \;\; b_j = \tfrac{1}{4} \beta_j \varepsilon_j + \sum_{k=1}^{j-1} \beta_k \varepsilon_k.$$

An approximation of order 4 that is a polynomial with respect to the thickness of thin layers is

$$S : \phi \mapsto S\phi = \left(-\mathbf{F}_1 h - \left(\mathbf{F}_3 - \mathbf{F}_1 \mathbf{B}_2 \right) h^3 \right) \phi,$$

where

$$\mathbf{F}_3 - \mathbf{F}_1 \mathbf{B}_2 = 2 \sum_{i,j,i>j}^{p} a_j \tfrac{\beta_i \beta_j}{\varepsilon_i \varepsilon_j} L_j L_i + \sum_{j=1}^{p} c_j \tfrac{\beta_j^2}{\varepsilon_j^2} L_j^2,$$

with

$$a_j = \tfrac{1}{2} \beta_j \varepsilon_j + \sum_{k=1}^{j-1} \beta_k \varepsilon_k, \;\; c_j = \tfrac{1}{3} \beta_j \varepsilon_j + \sum_{k=1}^{j-1} \beta_k \varepsilon_k.$$

In the case where the scattered wave is transverse magnetic (TM), the approximated DtoN operator of order 4 is

$$S : \phi \mapsto S\phi = \frac{-1 - \mathbf{A}_2 h^2 - \mathbf{A}_4 h^4}{\mathbf{G}_1 h + \mathbf{G}_3 h^3}\phi, \tag{43}$$

where

$$\mathbf{G}_1 = -\sum_{i=1}^{p} \beta_i \mu_i,\ \mathbf{A}_2 = -\sum_{i=1}^{p} a_i \frac{\beta_i}{\mu_i} L_i,\ \mathbf{G}_3 = \sum_{i=1}^{p} c_i \frac{\beta_i}{\mu_i} L_i,$$
$$\mathbf{A}_4 = \sum_{i,j,i>j}^{p} \gamma_{ij} \frac{\beta_i \beta_j}{\mu_i \mu_j} L_j L_i + \frac{1}{6}\sum_{i=1}^{p} b_i \frac{\beta_i^3}{\mu_i} L_i^2,$$

with

$$\omega_{ij} = \frac{1}{2}\left(\beta_i \mu_i + \beta_j \mu_j\right) + \sum_{k=j+1}^{i-1} \beta_k \mu_k,\ \ \gamma_{ij} = \omega_{ij} a_i - \frac{1}{12}\beta_i^2 \mu_i^2,$$
$$a_i = \frac{1}{2}\beta_i \mu_i + \sum_{k=i+1}^{p} \beta_k \mu_k,\ b_i = \frac{1}{4}\beta_i \mu_i + \sum_{k=i+1}^{p} \beta_k \mu_k,$$
$$c_i = \left(\frac{1}{2}\beta_i \mu_i + \sum_{k=1}^{i-1} \beta_k \mu_k\right) a_i - \frac{1}{12}\beta_i^2 \mu_i^2.$$

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