Received ; | Revised ; | Accepted

DOI: xxx/xxxx

ARTICLE TYPE

Stability Analysis of a Fractional-Order SEIR Epidemic Model with General Incidence Rate and Time Delay †

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Summary

In the present paper we investigate the qualitative behaviour of a fractional SEIR model with general incidence rate function and time delay where the fractional derivative is defined in the Caputo sense. The basic reproduction number $\mathcal{R}_0$ is derived using the method of next generation matrix and we give a complete study of local stability of both free and endemic steady state. Using Liapunov method we prove the global stability of free and endemic steady state under some hypotheses on the parameters of the system. Finally to illustrate our results, we use the model to predict the first peak of the COVID-19 epidemic in Algeria.

KEYWORDS:
SEIR model, Fractional derivative, Steady state, Time-delay, Stability, COVID-19

1 | INTRODUCTION

The role of epidemiology is the study of the spread of infectious diseases in a population and the factors that are responsible to contribute of their occurrence. Many authors used different types of mathematical models of infectious diseases to understand the transmission mechanisms, predictions and choose the best control strategies [1,2,3,4,5,6,7]. To take into account the incubation period of the disease some authors point out the importance to introduce time delay in these models which lead them to consider delay differential equations [5,8,9,10].

Fractional differentiation is a generalization of classical differentiation and integration to arbitrary order. Since it naturally include both memory and non local effects, this is quite relevant to model the spread of epidemics. Therefore, large numbers of researchers [4,11,12,13,14,15,16] have started to study epidemic models using the fractional differential equations.

Recently some authors inserted to a fractional-order epidemic model time delay to take into account the incubation period of the disease. Rihan et al [17] investigated a fractional endemic SIR model with time delay and long-rang temporal memory. They studied their stability and proved that Hopf bifurcation appears when the delay passes through some critical value τ^*. Rida & al. [18] provided the qualitative behavior of a fractional order SEI model with logistic growth and time delay. Deng & al. [8] studied the stability of n-dimensional linear fractional systems with multiple time delays and determined a sufficient stability condition for the system. Naresh et al [9] studied the dynamical behavior for a delayed fractional order SIS epidemic model with specific functional response. Owusu-Mensah et al [16] proposed a nonlinear fractional mathematical model to study the COVID-19 epidemic. Different other types of a fractional-order epidemic systems with delays was condidered in [4,6,7,19].

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Following these works we propose in this paper a fractional-order SEIR epidemic model with time delay incorporating a generalized incidence rate function of the form $f(S, I)$. The model that we propose is a generalisation of most of the models mentioned above.

The rest of the paper is organized as follows: In section 2 we give some preliminaries about fractional calculus. In section 3 we give the model and prove some existence and uniqueness results. In section 4 we investigate the existence of both free and endemic steady states in terms of the basic reproduction number. In section 5 we study the local stability of the two steady states of the system. In section 6 we use the Liapunov function to prove global stability of both free and endemic steady states. In section 7 we apply the model to simulate the COVID-19 epidemic in Algeria. Finally we end the paper by a conclusion.

2 | PRELIMINARIES

In this section we recall some fundamental concepts of fractional differential calculus where the derivative is in the Caputo sense.

Definition 1 ([20]). The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C^n(\mathbb{R}^+; \mathbb{R})$ is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha-n+1}} ds,$$

where n is a positive integer such that $\alpha \in (n-1, n)$. Also, the corresponding fractional integral of ordre α with $Re(\alpha) > 0$ is given by

$$I^\alpha_{[0,t]} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(.)$ is the Gamma function.

Lemma 1 ([21]). Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be such that $D^\alpha f(t)$ and $D^\alpha g(t)$ exist almost everywhere and let $a_1, a_2 \in \mathbb{R}$. Then $D^\alpha(a_1 f(t) + a_2 g(t))$ exists almost everywhere, and

$$D^\alpha(a_1 f(t) + a_2 g(t)) = a_1 \ D^\alpha f(t) + a_2 \ D^\alpha g(t).$$

Further the Caputo fractional derivative for a constant function is zero.

Lemma 2 ([22]). Suppose that $f \in C[a, b]$ and $D^\alpha f \in C[a, b]$ with $0 < \alpha \leq 1$. Then there exists $\xi(x) \in [a, x]$, such that

$$f(x) = f(a) + \tfrac{1}{\alpha} D^\alpha f(\xi)(x-a)^\alpha.$$

Based on the previous Lemma we have the following result.

Corollary 1. Suppose that $f \in C([a, b])$ and $D^\alpha f \in C([a, b])$.
If $D^\alpha f(t) \geq 0$, (resp: $D^\alpha f(t) \leqslant 0$) $\forall t \in (a, b)$, then f is nondecreasing (resp: nonincreasing) in $[a, b]$.

Definition 2. The constant point x^* is a steady state of the fractional model

$$D^\alpha x(t) = f(t, x(t)),$$

if and only if $f(t, x^*) = 0$ for all $t > 0$.

Lemma 3 ([13]). Let $\alpha \in (0, 1)$ and consider a continuous function $x : [t_0, \infty) \longrightarrow \mathbb{R}$ satisfying the following condition

$$D^\alpha x(t) + \mu x(t) \leq \nu, \ t \geq t_0, \ \mu, \nu \in \mathbb{R}, \ \mu \neq 0.$$

Then we have the inequality

$$x(t) \leq \left(x(t_0) - \frac{\nu}{\mu} \right) E_\alpha \left(-\mu (t-t_0)^\alpha \right) + \frac{\nu}{\mu},$$

for all $t \geq t_0$, where E_α is the Mittag-Leffler function of one parameter defined by

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}.$$

We can now state the following existence result for fractional differential equations.

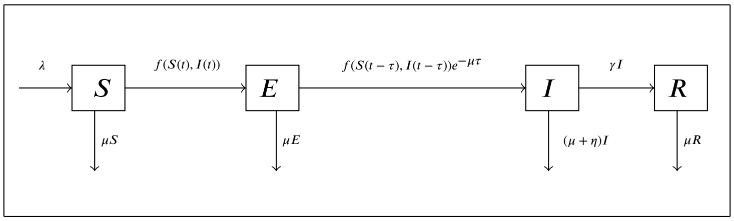

FIGURE 1 The diagram of the model.

Theorem 1 ([23]). Let $\alpha \in (0,1]$, $\Omega \subset \mathbb{R}^n$ a domain and $f : [t_0, \infty) \times \Omega \longrightarrow \mathbb{R}^n$ be a function satisfying the Lipschitz condition on x and consider the following fractional order equation

$$D^\alpha x(t) = f(t, x(t)),\ t > t_0,$$

with the initial condition $x(t_0) = x_0 \in \Omega$. Then the above system has a unique maximal solution.

Lemma 4 ([8]). Let $\tau > 0$, $\alpha \in (0,1]$, A, B two $(n \times n)$ square matrices and $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$. Consider the linear fractional delayed differential system with the Caputo derivative

$$\begin{cases} D^\alpha x(t) = Ax(t) + Bx(t-\tau),\ t > 0, \\ x(t) = \varphi(t),\ t \in [-\tau, 0]. \end{cases} \tag{1}$$

We define the characteristic equation of system (1) by

$$\Delta(s) = \det\left(s^\alpha I_n - A - Be^{-s\tau}\right) = 0.$$

If all the roots of the characteristic equation $\Delta(s) = 0$ have negative real parts, then the zero solution of system (1) is locally asymptotically stable.

Lemma 5 ([8]).
1. If all the eigenvalues λ of the matrix $M = A + B$ satisfy $|\arg(\lambda)| > \alpha\frac{\pi}{2}$ and the characteristic equation $\Delta(s) = 0$ has a no purely imaginary roots for $\tau > 0$ then the zero solution of system (1) is locally asymptotically stable.
2. Suppose $\tau = 0$. If all the eigenvalues λ of M satisfy $|\arg(\lambda)| > \alpha\frac{\pi}{2}$, then the zero solution of (1) is locally asymptotically stable.

Lemma 6 ([14]). Let $x^* \in \Omega \subset \mathbb{R}^n$ be an equilibrium point of the system

$$D^\alpha x(t) = f(t, x(t)),\ t \geq t_0,$$

and let $V(t,x) : [t_0, \infty) \times \Omega \longrightarrow \mathbb{R}$ be a continuously differentiable function such that

$$\begin{aligned} &W_1(x) \leq V(t,x) \leq W_2(x), \\ &D^\alpha V(t,x) \leq -W_3(x), \end{aligned}$$

for $t \geq t_0$ and $x \in \Omega$, where $W_i(x),\ i = 1,2,3$ are continuous and positively defined functions on Ω. Then x^* is uniformly asymptotically stable.

3 | THE MODEL

Denote by $N(t)$ the total population size at time t. We assume that $N(t)$ is divided into four compartments which are: susceptible individuals $S(t)$, exposed individuals $E(t)$, infected individuals $I(t)$ and recovered individuals $R(t)$ at time t. The susceptible class S consists of individuals who are at risk of catching infection due to close contact with infected individuals. The exposed class E are revealed individuals but not yet infectious. The infected class I consists of individuals who have already caught the disease and they can transfer it to susceptible. the recovered class R consists of individuals who were infected and are now healthy. Denote by λ the recruitment rate of susceptible individuals, μ the death rate of all individuals, η the death rate of infected individuals caused by the disease and γ the transfer rate from infected compartment to recovery compartment. The diagram of the model is given in Figure 1. The spreading dynamic of the epidemic is then governed by the following fractional system

$$\begin{cases} D^{\alpha}S(t) = \lambda - f(S(t), I(t)) - \mu S(t), \\ D^{\alpha}E(t) = f(S(t), I(t)) - f(S(t-\tau), I(t-\tau))e^{-\mu\tau} - \mu E(t), \\ D^{\alpha}I(t) = f(S(t-\tau), I(t-\tau))e^{-\mu\tau} - (\mu+\eta+\gamma)I(t), \\ D^{\alpha}R(t) = \gamma I(t) - \mu R(t), \end{cases} \tag{2}$$

where D^{α} is the Caputo fractional-order derivative with $0 < \alpha \leq 1$. We add to system (2) the following initial conditions

$$S(\theta) = \varphi_1(\theta),\ E(\theta) = \varphi_2(\theta),\ I(\theta) = \varphi_3(\theta),\ \ R(\theta) = \varphi_4(\theta),\ \ \theta \in [-\tau, 0], \tag{3}$$

where $\varphi_i \in C([-\tau, 0]; \mathbb{R})$ are non negative such that $\varphi_i(0) > 0$ for $i = 1, 2, 3, 4$. We assume that the incidence function f is always positive, continuous and satisfy for all $S \geq 0, I \geq 0$ the following conditions [1]

$$\begin{array}{ll} (H1) & f(0, I) = f(S, 0) = 0, \\ (H2) & \frac{\partial f(S,0)}{\partial S} = 0, \\ (H3) & \frac{\partial f(S,I)}{\partial S} > 0, \\ (H4) & \frac{\partial f(S,I)}{\partial I} > 0, \\ (H5) & \frac{\partial^2 f(S,I)}{\partial^2 I} < 0. \end{array} \tag{4}$$

The time delay τ in this model represents the incubation period and the term $f(S(t-\tau), I(t-\tau))e^{-\mu\tau}$ represents the individuals who were exposed at time $t - \tau$ and survive to time t. Denotes by $C = C([-\tau, 0]; \mathbb{R})$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$ equipped with the sup-norm. The non negative cône of C is defined as $C^{+} = C([-\tau, 0], \mathbb{R}_{+})$. The phase space of system (2) is then $C \times C \times C \times C$. Since the first and the third equations are independent of the variable E and R the model can be reduced to the following sub-system

$$\begin{cases} D^{\alpha}S(t) = \lambda - f(S(t), I(t)) - \mu S(t), \\ D^{\alpha}I(t) = f(S(t-\tau), I(t-\tau))e^{-\mu\tau} - (\mu+\eta+\gamma)I(t). \end{cases} \tag{5}$$

In the following we will prove existence and positivity of solutions of system (5).

Lemma 7. There exists a unique solution for the fractional-order system (5) with the initial conditions (3). Furthermore, every solution of system (5)-(3) is positive, bounded and enters some compact attracting set.

Proof. By Theorem 1 system (5) with initial condition (3) have an unique solution on some time interval. To prove that $S \geq 0$, we use contradiction. Assume that there exists a $\varsigma_1 > 0$ such that $S(t) > 0$ for $t \in [0, \varsigma_1)$, $S(\varsigma_1) = 0$ and $S(t) < 0$ for $t \in (\varsigma_1, \varsigma_1 + \epsilon_1]$ with ϵ_1 sufficiently small. From the first equation of system (5), we can see that $D^{\alpha}S(t)|_{t=\varsigma_1} = \lambda > 0$ and hence by Lemma 2 there exists ξ_1 such that

$$S(\varsigma_1 + \epsilon_1) = S(\varsigma_1) + \tfrac{1}{\alpha}D^{\alpha}S(\xi_1)\epsilon_1^{\alpha},$$

where $\varsigma_1 \leq \xi_1 \leq \varsigma_1 + \epsilon_1$. If we choose ϵ_1 sufficiently small we can see $S(\varsigma_1 + \epsilon_1) > 0$ which contradicts the fact that $S(t) < 0$ in $[\varsigma_1, \varsigma_1 + \epsilon_1]$. Hence, we have $S(t) \geq 0$ for $t \geq 0$.
To prove that $I \geq 0$ assume by contradiction that there exists a $\varsigma_2 > 0$ such that $I(t) > 0$ for $t \in [0, \varsigma_2)$, $I(\varsigma_2) = 0$ and $I(t) < 0$ for $t \in (\varsigma_2, \varsigma_2 + \epsilon_2]$ with ϵ_2 sufficiently small. From the second equation of system (5) we have

$$D^{\alpha}I(t)|_{t=\varsigma_2} = f(S(t-\tau), I(t-\tau))e^{-\mu\tau} \geq 0,$$

using the Lemma 2 there exists ξ_2 where

$$I(\varsigma_2 + \epsilon_2) = I(\varsigma_2) + \tfrac{1}{\alpha}D^{\alpha}I(\xi_2)\epsilon_2^{\alpha},$$

where $\varsigma_2 \leq \varsigma_3 \leq \varsigma_2 + \epsilon_2$, and then $I(\varsigma_2 + \epsilon_2) > 0$ which contradicts the fact that $I(t) < 0$ for $t \in [\varsigma_2, \varsigma_2 + \epsilon_2]$.
Now we prove that $S, I > 0$. Let us assume that there exists $t_1 > 0$ such that $S(t_1)$ is the minimum of S and $S(t_1) = 0$, then

$$D^{\alpha}S(t_1) = \lambda > 0,$$

witch implies that $D^{\alpha}S$ is non negative in $[t_1 - \varsigma, t_1 + \varsigma[$ for some $\varsigma > 0$, then by Corollary 1 S is strictly increasing function and hence $S(t_1 - \varsigma) < S(t_1) = 0$ which is a contradiction. We can use a similar argument to prove that $I > 0$.
To prove the boundedness of solutions let us define

$$N(t) = e^{-\mu\tau}S(t-\tau) + I(t),$$

the fractional derivative of $N(t)$ is

$$\begin{aligned} D^\alpha N(t) &\leq e^{-\mu\tau} D^\alpha S(t) + D^\alpha I(t) \\ &\leq \lambda - \mu N(t), \end{aligned}$$

by Lemma 3

$$N(t) \leq \left(N_0 - \tfrac{\lambda}{\mu}\right) E_\alpha(-\mu t^\alpha) + \tfrac{\lambda}{\mu},$$

where $N_0 = S(0) + I(0)$. The last inequality leads to

$$\limsup_{t\longrightarrow+\infty} N(t) \leq \tfrac{\lambda}{\mu}.$$

The solution of system (5) are then uniformly bounded and global. Further the set

$$B = \left\{0 < S + I \leq \tfrac{\lambda}{\mu}\right\},$$

is a positive attracting set for system (5). □

4 | STEADY STATES

To find steady states of system (5) we solve the following system

$$\begin{cases} \lambda - f(S, I) - \mu S = 0, \\ e^{-\mu\tau} f(S, I) - (\mu + \eta + \gamma) I = 0. \end{cases} \tag{6}$$

It is clear that $(S^0, 0)^T$, with $S^0 = \frac{\lambda}{\mu}$ is always a solution of (6). System (5) admits a free steady state $E^0 = (S^0, 0)$. To derive the basic reproduction number of system (5) we use the method of next generation matrix[2,3].

Lemma 8. The basic reproduction number of system (5) is given by

$$\mathcal{R}_0 = \tfrac{e^{-\mu\tau}}{\mu+\eta+\gamma} \tfrac{\partial f}{\partial I}(S^0, 0).$$

Proof. Put $X = (S, I)^T$, then system (5) can be written as follows

$$\begin{aligned} D^\alpha X &= F(X) - V(X), \\ V(X) &= V^-(X) - V^+(X), \end{aligned}$$

where

$$F(X) = \begin{pmatrix} 0 \\ e^{-\mu\tau} f(S, I) \end{pmatrix},$$

and

$$V^+(X) = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \; V_-(X) = \begin{pmatrix} f(S, I) + \mu S \\ (\mu + \eta + \gamma) I \end{pmatrix}.$$

$F(X)$ denote the rate of appearance of new infected individuals in each of the compartments S and I, $V^+(X)$ the rate of transfer of individuals into the compartments S and I by all other means and $V^-(X)$ the rate of transfer of individuals out of the compartments S and I. Let $\mathcal{F}$ and $\mathcal{V}$ be the Jacobian matrices of $F(X), V(X)$ respectively at E^0, then

$$\mathcal{F} = \begin{pmatrix} 0 & 0 \\ e^{-\mu\tau} \frac{\partial f}{\partial S}(S^0, 0) & e^{-\mu\tau} \frac{\partial f}{\partial I}(S^0, 0) \end{pmatrix}, \; \mathcal{V} = \begin{pmatrix} \mu + \frac{\partial f}{\partial S}(S^0, 0) & \frac{\partial f}{\partial I}(S^0, 0) \\ 0 & \mu + \eta + \gamma \end{pmatrix}.$$

Following[3] we define $\mathcal{R}_0$ as the spectral radius of the next generation matrix $\mathcal{F}\mathcal{V}^{-1}$, with $\mathcal{V}$ non-singular as required. Using (H2) we get

$$\mathcal{R}_0 = \rho(\mathcal{F}\mathcal{V}^{-1}) = \tfrac{e^{-\mu\tau}}{\mu+\eta+\gamma} \tfrac{\partial f}{\partial I}(S^0, 0).$$

□

Theorem 2. If $\mathcal{R}_0 > 1$, then system (5) have an unique endemic steady state $E^* = (S^*, I^*)$.

Proof. Let (S, I) be a solution of (6) such that $I \neq 0$. We have by the first and second equations of (6)

$$\lambda - \mu S = f(S, I) = e^{\mu\tau}(\mu + \eta + \gamma) I,$$

which means

$$S = \tfrac{\lambda - e^{\mu\tau}(\mu+\eta+\gamma) I}{\mu}.$$

It is clear that S exists if and only if $I < \tilde{I} = \frac{\lambda e^{-\mu\tau}}{\mu+\eta+\gamma}$. We then suppose that $0 < I < \tilde{I}$, where I is the solution of the following equation

$$f\left(\frac{\lambda - e^{\mu\tau}(\mu+\eta+\gamma)I}{\mu}, I\right) - e^{\mu\tau}(\mu+\eta+\gamma)I = 0. \tag{7}$$

If $I = 0$ we obtain the free steady state E^0. For $I \neq 0$, let H be the function defined by

$$H(I) = \frac{f\left(\frac{\lambda - e^{\mu\tau}(\mu+\eta+\gamma)I}{\mu}, I\right)}{I} - e^{\mu\tau}(\mu+\eta+\gamma). \tag{8}$$

Using the hypotheses (4) we conclude that the time derivative of H is negative. By the definition of $\mathcal{R}_0$ we have

$$\lim_{I\to 0^+} H(I) = e^{\mu\tau}(\mu+\eta+\gamma)(\mathcal{R}_0 - 1),$$

if $\mathcal{R}_0 > 1$ this leads to

$$\lim_{I\to 0^+} H(I) > 0.$$

On the other hand

$$\lim_{I\to \tilde{I}} H(I) = -e^{\mu\tau}(\mu+\eta+\gamma) < 0.$$

Thus by the fundamental Theorem of algebra there exists an unique positive root $0 < I^* < \tilde{I}$ of (7) and system (5) has an unique endemic steady state $E^* = (S^*, I^*)$. □

5 | LOCAL STABILITY OF STEADY STATES

In this section we study local stability of both free and endemic steady states. Denote by $(\overline{S}, \overline{I})$ one of the two steady states E^0 or E^*. The linearized system (5) around $(\overline{S}, \overline{I})$ takes the following form

$$\begin{cases} D^\alpha S(t) &= -\left(\frac{\partial f(\overline{S},\overline{I})}{\partial S} + \mu\right) S(t) - \frac{\partial f(\overline{S},\overline{I})}{\partial I} I(t), \\ D^\alpha I(t) &= -(\mu+\gamma+\eta)I(t) + e^{-\mu\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial S} S(t-\tau) + e^{-\mu\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I} I(t-\tau). \end{cases} \tag{9}$$

Taking the Laplace transform on both sides of (9) we have

$$\begin{cases} s^\alpha L[S(t)] &= s^{\alpha-1}S(0) - \left(\frac{\partial f(\overline{S},\overline{I})}{\partial S} + \mu\right) L[S(t)] - \frac{\partial f(\overline{S},\overline{I})}{\partial I} L[I(t)], \\ s^\alpha L[I(t)] &= s^{\alpha-1}I(0) - (\mu+\gamma+\eta)L[I(t)] + e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial S}\Big(L[S(t)] \\ & \quad + \int_{-\tau}^0 e^{-st}\varphi_1(t)dt\Big) + e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I}\Big(L[I(t)] + \int_{-\tau}^0 e^{-st}\varphi_3(t)dt\Big), \end{cases}$$

where $L[I(t)]$, $L[S(t)]$ are the Laplace transform of $S(t)$ and $I(t)$ respectively. The above system can be written in the following form

$$A(s)\begin{pmatrix} L[S(t)] \\ L[I(t)] \end{pmatrix} = \begin{pmatrix} B_1(s) \\ B_2(s) \end{pmatrix},$$

with

$$\begin{cases} B_1(s) &= s^{\alpha-1}S(0), \\ B_2(s) &= s^{\alpha-1}I(0) + e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial S}\int_{-\tau}^{0} e^{-st}\varphi_1(t)dt \\ & \quad + e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I}\int_{-\tau}^{0} e^{-st}\varphi_3(t)dt, \end{cases}$$

and

$$A(s) = \begin{pmatrix} s^\alpha + \frac{\partial f(\overline{S},\overline{I})}{\partial S} + \mu & \frac{\partial f(\overline{S},\overline{I})}{\partial I} \\ -e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial S} & s^\alpha - e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I} + \mu+\gamma+\eta \end{pmatrix}.$$

The characteristic polynomial $\Delta(s)$ of $A(s)$ is

$$\begin{aligned} \Delta(s) &= s^{2\alpha} + \left[\frac{\partial f(\overline{S},\overline{I})}{\partial S} + 2\mu - e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I} + \gamma + \eta\right] s^\alpha \\ & \quad + (\mu+\gamma+\eta)\left(\frac{\partial f(\overline{S},\overline{I})}{\partial S} + \mu\right) - \mu e^{-(\mu+s)\tau}\frac{\partial f(\overline{S},\overline{I})}{\partial I}. \end{aligned} \tag{10}$$

5.1 | Local stability of free steady state E^0

Theorem 3. Assume that $\tau = 0$. If $\mathcal{R}_0 < 1$, the free steady state E^0 is locally asymptotically stable.

Proof. For $\tau = 0$, the characteristic matrix of the linearized system (9) evaluated at E^0 takes the form

$$A = \begin{pmatrix} -\frac{\partial f(S^0,0)}{\partial S} - \mu & -\frac{\partial f(S^0,0)}{\partial I}, \\ \frac{\partial f(S^0,0)}{\partial S} & \frac{\partial f(S^0,0)}{\partial I} - (\mu+\gamma+\eta) \end{pmatrix},$$

since $\frac{\partial f(S^0,0)}{\partial S} = 0$, the characteristic polynomial of A is

$$\begin{aligned} P(\lambda) &= \lambda^2 + \Big(2\mu - \frac{\partial f(S^0,0)}{\partial I} + \gamma + \eta\Big)\lambda - \mu\frac{\partial f(S^0,0)}{\partial I} \\ &\quad + \mu(\mu+\gamma+\eta), \\ &= \lambda^2 + \Big(\mu + (\mu+\gamma+\eta)(1-\mathcal{R}_0)\Big)\lambda + \mu(\mu+\gamma+\eta)(1-\mathcal{R}_0). \end{aligned}$$

Since $R_0 < 1$, all the coefficients of P are positive and by Routh-Hurwitz Theorem all the roots λ of P have negative real parts which imply that $|\arg(\lambda)| > \frac{\pi}{2} > \alpha\frac{\pi}{2}$. Using Lemma 5/(2) we conclude that the free steady state is locally asymptotically stable. □

Theorem 4. Assume that $\tau > 0$. If $\mathcal{R}_0 < 1$, then the free steady state E^0 is locally asymptotically stable.

Proof. From (10) the characteristic equation at E^0 is given by

$$s^{2\alpha} + \Big(2\mu - e^{-(\mu+s)\tau}\frac{\partial f(S^0,0)}{\partial I} + \gamma + \eta\Big)s^\alpha - \mu e^{-(\mu+s)\tau}\frac{\partial f(S^0,0)}{\partial I} + \mu(\mu+\gamma+\eta) = 0. \tag{11}$$

To prove local stability of E^0 we use Lemma 5. Assume by contradiction that the equation (11) has a pair of imaginary roots $s = \omega e^{i\frac{\pi}{2}}$, $\omega > 0$. After substituting s into equation (11), we obtain

$$\omega^{2\alpha}e^{\alpha\pi i} + \Big(2\mu - e^{-\mu\tau}e^{-i\omega\tau}\frac{\partial f(S^0,0)}{\partial I} + \gamma + \eta\Big)\omega^\alpha e^{\frac{\alpha\pi i}{2}} - \mu e^{-\mu\tau}e^{-i\omega\tau}\frac{\partial f(S^0,0)}{\partial I} + \mu(\mu+\gamma+\eta) = 0,$$

separating real and imaginary parts, we have

$$\begin{cases} A_1\cos(\omega\tau) + A_2\sin(\omega\tau) = A_3, \\ A_2\cos(\omega\tau) - A_1\sin(\omega\tau) = A_4, \end{cases} \tag{12}$$

where

$$\begin{aligned} A_1 &= \mu e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I} + \omega^\alpha e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}\cos(\tfrac{\alpha\pi}{2}), \\ A_2 &= \omega^\alpha e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}\sin(\tfrac{\alpha\pi}{2}), \\ A_3 &= \omega^{2\alpha}\cos(\pi\alpha) + \omega^\alpha\Big(2\mu+\gamma+\eta\Big)\cos(\tfrac{\alpha\pi}{2}) + \mu(\mu+\gamma+\eta), \\ A_4 &= \omega^{2\alpha}\sin(\pi\alpha) + \omega^\alpha(2\mu+\gamma+\eta)\sin(\tfrac{\alpha\pi}{2}). \end{aligned}$$

Eliminating τ by squaring and adding the two equations in (12), we obtain

$$\omega^{4\alpha} + A_5\omega^{3\alpha} + A_6\omega^{2\alpha} + A_7\omega^\alpha + A_8 = 0, \tag{13}$$

where

$$\begin{aligned} A_5 &= 2(2\mu+\gamma+\eta)\sin(\pi\alpha)\sin(\tfrac{\alpha\pi}{2}) + 2(2\mu+\gamma+\eta)\cos(\pi\alpha)\cos(\tfrac{\alpha\pi}{2}), \\ A_6 &= -\Big(e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}\Big)^2 + (2\mu+\gamma+\eta)^2 + 2\mu(\mu+\gamma+\eta)\cos(\pi\alpha), \\ A_7 &= 2\Big(-\mu(e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I})^2 + \mu(\mu+\gamma+\eta)(2\mu+\gamma+\eta)\Big)\cos(\tfrac{\alpha\pi}{2}), \\ A_8 &= \mu^2(\mu+\gamma+\eta)^2 - \mu^2\Big(e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}\Big)^2. \end{aligned}$$

We have

$$\begin{aligned} A_5 &= 2(2\mu+\gamma+\eta)\sin(\pi\alpha)\sin(\tfrac{\alpha\pi}{2}) + 2(2\mu+\gamma+\eta)\cos(\pi\alpha)\cos(\tfrac{\alpha\pi}{2}), \\ &= 2(2\mu+\gamma+\eta)\cos(\tfrac{\alpha\pi}{2}), \end{aligned}$$

which implies that $A_5 > 0$. Further we have for A_6

$$\begin{aligned} A_6 &= -\Big(e^{-\mu\tau}\frac{\partial f(S^0,0)}{\partial I}\Big)^2 + (2\mu+\gamma+\eta)^2 + 2\mu(\mu+\gamma+\eta)\cos(\pi\alpha) \\ &= (\mu+\gamma+\eta)^2(1-\mathcal{R}_0^2) + \mu^2 + 2\mu(\mu+\gamma+\eta) + 2\mu(\mu+\gamma+\eta)\cos(\pi\alpha) \\ &\geq (\mu+\gamma+\eta)^2(1-\mathcal{R}_0^2) + \mu^2 + 2\mu(\mu+\gamma+\eta) - 2\mu(\mu+\gamma+\eta) \\ &\geq (\mu+\gamma+\eta)^2(1-\mathcal{R}_0^2) + \mu^2, \end{aligned}$$

hence, if $\mathcal{R}_0 < 1$ we get $A_6 > 0$. For A_7

$$\begin{aligned} A_7 &= 2\mu\Big(-(e^{-\mu\tau}\tfrac{\partial f(S^0,0)}{\partial I})^2 + (\mu+\gamma+\eta)(2\mu+\gamma+\eta)\Big)\cos(\tfrac{\alpha\pi}{2}) \\ &= 2\mu\Big(-(e^{-\mu\tau}\tfrac{\partial f(S^0,0)}{\partial I})^2 + (\gamma+\eta+\mu)^2 + \mu(\mu+\gamma+\eta)\Big)\cos(\tfrac{\alpha\pi}{2}) \\ &= 2\mu\Big((\mu+\gamma+\eta)^2(1-\mathcal{R}_0^2) + \mu(\mu+\gamma+\eta)\Big)\cos(\tfrac{\alpha\pi}{2}), \end{aligned}$$

if $\mathcal{R}_0 < 1$, consequently $A_7 > 0$. Finally

$$\begin{aligned} A_8 &= \mu^2(\mu+\gamma+\eta)^2 - \mu^2\Big(e^{-\mu\tau}\tfrac{\partial f(S^0,0)}{\partial I}\Big)^2 \\ &= \mu^2(\mu+\gamma+\eta)^2(1-\mathcal{R}_0^2). \end{aligned}$$

If $\mathcal{R}_0 < 1$, then $A_8 > 0$. Since $\omega > 0$ we conclude that equation (13) cannot have a positive real root and hence equation (11) has no purely imaginary roots. On the other hand the characteristic equation of the linearized system (9) at E^0 is given by

$$\begin{aligned} \begin{vmatrix} -\mu-\lambda & -\frac{\partial f}{\partial I}(S^0,0) \\ 0 & -(\gamma+\eta+\mu)+e^{-\mu\tau}\frac{\partial f}{\partial I}(S^0,0)-\lambda \end{vmatrix} &= (-\mu-\lambda)\left(e^{-\mu\tau}\frac{\partial f}{\partial I}(S^0,0)-(\gamma+\mu+\eta)-\lambda\right) \\ &= (\lambda+\mu)\left(\lambda-(R^0-1)\right)(\gamma+\mu+\eta) \\ &= 0, \end{aligned}$$

which have two negative real roots $\lambda_1 = -\mu < 0$ and $\lambda_2 = R^0 - 1 < 0$. The condition $|\arg(\lambda_i)| > \alpha\frac{\pi}{2}$ is then satisfied, by Lemma 5/(1) the free steady state is locally asymptotically stable. □

5.2 | Local stability of endemic steady state E^*

We turn now to prove local stability of the endemic steady state E^*.

Theorem 5. Suppose that $\tau = 0$. If $\mathcal{R}_0 > 1$, then the endemic steady sate E^* is locally asymptotically stable if

$$\frac{\partial f}{\partial I}(S^*, I^*) < (\mu+\gamma+\eta). \tag{14}$$

Proof. Put $\tau = 0$. The characteristic equation of system (9) at E^* takes the form

$$\lambda^2 + b\lambda + c = 0, \tag{15}$$

where

$$\begin{aligned} b &= \Big[\tfrac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta - \tfrac{\partial f(S^*,I^*)}{\partial I}\Big], \\ c &= \mu\Big[(\mu+\gamma+\eta) - \tfrac{\partial f(S^*,I^*)}{\partial I}\Big] + (\mu+\gamma+\eta)\tfrac{\partial f(S^*,I^*)}{\partial S}. \end{aligned}$$

By hypothesis (14) we can see that $b > 0, c > 0$ and the Routh-Hurwitz criterion imply that all the roots λ of (15) have negative real parts which means that $|\arg(\lambda)| > \frac{\pi}{2} > \alpha\frac{\pi}{2}$. By Lemma 5/(2) the endemic steady state E^* is locally asymptotically stable. □

Theorem 6. Suppose that $\tau > 0$. If $\mathcal{R}_0 > 1$, then the endemic steady state E^* is locally asymptotically stable if condition (14) holds.

Proof. To prove this theorem we use similar arguments as in Theorem 4. By (10) the characteristic equation at E^* is

$$s^{2\alpha} + \Big[\tfrac{\partial f(S^*,I^*)}{\partial S} + 2\mu - e^{-(\mu+s)\tau}\tfrac{\partial f(S^*,I^*)}{\partial I} + \gamma + \eta\Big]s^\alpha + (\mu+\gamma+\eta)\left(\mu + \tfrac{\partial f(S^*,I^*)}{\partial S}\right) - \mu e^{-(\mu+s)\tau}\tfrac{\partial f(S^*,I^*)}{\partial I} = 0. \tag{16}$$

Assume that equation (16) has a purely imaginary root $s = we^{i\frac{\pi}{2}},\ w > 0$. Substituting s in (16) and separating real and imaginary parts we get

$$\begin{cases} B_1\cos(w\tau) + B_2\sin(w\tau) = B_3, \\ B_2\cos(w\tau) - B_1\sin(w\tau) = B_4, \end{cases} \tag{17}$$

where

$$\begin{aligned}
B_1 &= \mu e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I} + e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I} w^{\alpha}\cos(\tfrac{\alpha\pi}{2}),\\
B_2 &= w^{\alpha}\sin(\tfrac{\alpha\pi}{2}) e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I},\\
B_3 &= \omega^{2\alpha}\cos(\alpha\pi) + \omega^{\alpha}\cos(\tfrac{\alpha\pi}{2})\Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big)\\
&\quad + \Big(\mu + \frac{\partial f(S^*,I^*)}{\partial S}\Big)(\mu+\gamma+\eta),\\
B_4 &= \omega^{2\alpha}\sin(\alpha\pi) + \omega^{\alpha}\sin(\tfrac{\alpha\pi}{2})\Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big).
\end{aligned}$$

Adding the squares of both equations (17) give

$$w^{4\alpha} + B_5 w^{3\alpha} + B_6 w^{2\alpha} + B_7 w^{\alpha} + B_8 = 0, \tag{18}$$

where

$$\begin{aligned}
B_5 &= 2\cos(\alpha\pi)\cos(\tfrac{\alpha\pi}{2})\Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big)\\
&\quad +2\sin(\alpha\pi)\sin(\tfrac{\alpha\pi}{2})\Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big),\\
B_6 &= \Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big)^2 - \Big(e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)^2\\
&\quad +2\Big(\mu + \frac{\partial f(S^*,I^*)}{\partial S}\Big)\cos(\alpha\pi)(\mu+\gamma+\eta),\\
B_7 &= 2\Big(\mu + \frac{\partial f(S^*,I^*)}{\partial S}\Big)\cos(\tfrac{\alpha\pi}{2})\Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big)(\mu+\gamma+\eta)\\
&\quad -2\mu\cos(\tfrac{\alpha\pi}{2})\Big(e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)^2,\\
B_8 &= \Big(\mu + \frac{\partial f(S^*,I^*)}{\partial S}\Big)^2(\mu+\gamma+\eta)^2 - \mu^2\Big(e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)^2.
\end{aligned}$$

It is clear that if all the coefficients $B_i,\ i = 5, ..., 8$ are positive then equation (18) cannot have a positive root. Since

$$\begin{aligned}
B_5 &= 2\cos(\alpha\pi)\cos(\tfrac{\alpha\pi}{2})\Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big)\\
&\quad +2\sin(\alpha\pi)\sin(\tfrac{\alpha\pi}{2})\Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big),\\
&= 2\cos(\tfrac{\alpha\pi}{2})\Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big),
\end{aligned}$$

we can see that $B_5 > 0$. On the other hand we have

$$\begin{aligned}
B_6 &= \Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big)^2 - \Big(e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)^2\\
&\quad +2(\mu + \frac{\partial f(S^*,I^*)}{\partial S})\cos(\alpha\pi)(\mu+\gamma+\eta),\\
&> \Big(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta\Big)^2 - \Big(e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)^2\\
&\quad -2(\mu + \frac{\partial f(S^*,I^*)}{\partial S})(\mu+\gamma+\eta),\\
&= \Big(\frac{\partial f(S^*,I^*)}{\partial S} + \mu\Big)^2 + \Big(\mu+\gamma+\eta\Big)^2 - \Big(e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)^2\\
&= \Big(\mu+\gamma+\eta - e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)\Big(\mu+\gamma+\eta + e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)\\
&\quad +\Big(\frac{\partial f(S^*,I^*)}{\partial S} + \mu\Big)^2,
\end{aligned}$$

condition (14) imply that $B_6 > 0$. Further

$$\begin{aligned}
B_7 &= 2\cos(\tfrac{\alpha\pi}{2})\Big[(\mu + \frac{\partial f(S^*,I^*)}{\partial S})(\frac{\partial f(S^*,I^*)}{\partial S} + 2\mu+\gamma+\eta)(\mu+\gamma+\eta)\\
&\quad -\mu\Big(e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)^2\Big],\\
&= 2\cos(\tfrac{\alpha\pi}{2})\Big[(\mu + \frac{\partial f(S^*,I^*)}{\partial S})^2(\mu+\gamma+\eta)\\
&\quad +(\mu + \frac{\partial f(S^*,I^*)}{\partial S})(\mu+\gamma+\eta)^2 - \mu\Big(e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)^2\Big],\\
&= 2\cos(\tfrac{\alpha\pi}{2})\Big[(\mu + \frac{\partial f(S^*,I^*)}{\partial S})^2(\mu+\gamma+\eta) + \frac{f(S^*,I^*)}{\partial S}(\mu+\gamma+\eta)^2\\
&\quad +\mu(\mu+\gamma+\eta)^2 - \mu\Big(e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I}\Big)^2\Big],\\
&= 2\cos(\tfrac{\alpha\pi}{2})\Big[(\mu + \frac{\partial f(S^*,I^*)}{\partial S})^2(\mu+\gamma+\eta) + \frac{f(S^*,I^*)}{\partial S}(\mu+\gamma+\eta)^2\\
&\quad +\mu\Big((\mu+\gamma+\eta - e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I})(\mu+\gamma+\eta + e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I})\Big)\Big],
\end{aligned}$$

and

$$\begin{aligned} B_8 &= (\mu + \tfrac{\partial f(S^*,I^*)}{\partial S})^2\Big(\mu+\gamma+\eta\Big)^2 - \mu^2\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2, \\ &= \Big((\tfrac{\partial f(S^*,I^*)}{\partial S})^2 + 2\mu\tfrac{\partial f(S^*,I^*)}{\partial S}\Big)(\mu+\gamma+\eta)^2 \\ &\quad + \mu^2(\mu+\gamma+\eta)^2 - \mu^2\Big(e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)^2, \\ &= \Big((\tfrac{\partial f(S^*,I^*)}{\partial S})^2 + 2\mu\tfrac{\partial f(S^*,I^*)}{\partial S}\Big)(\mu+\gamma+\eta)^2 \\ &\quad + \mu^2\Big(\mu+\gamma+\eta - e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big)\Big(\mu+\gamma+\eta + e^{-\mu\tau}\tfrac{\partial f(S^*,I^*)}{\partial I}\Big). \end{aligned}$$

Condition (14) yealds that $B_7 > 0$ and $B_8 > 0$. Then we conclude that equation (18) cannot have positive roots. We now check the condition of Lemma 5/(1) about the eigenvalues of the matrix of the linearized system. The characteristic equation of system (9) at E^* is

$$\begin{aligned} P(\lambda) &= \begin{vmatrix} -\Big(\lambda + \frac{\partial f(S^*,I^*)}{\partial S} + \mu\Big) & -\frac{\partial f(S^*,I^*)}{\partial I} \\ e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial S} & -(\gamma+\mu+\eta) + e^{-\mu\tau}\frac{\partial f(S^*,I^*)}{\partial I} - \lambda \end{vmatrix} \\ &= \lambda^2 + \lambda\left((\gamma+\eta+\mu) - e^{-\mu\tau}\frac{\partial f}{\partial I}(S^*,I^*) + \frac{\partial f}{\partial S}(S^*,I^*) + \mu\right) \\ &\quad + \Big(\tfrac{\partial f(S^*,I^*)}{\partial S} + \mu\Big)\left((\gamma+\eta+\mu) - e^{-\mu\tau}\frac{\partial f}{\partial I}(S^*,I^*)\right) + \frac{\partial f}{\partial S}(S^*,I^*)\frac{\partial f}{\partial I}(S^*,I^*)e^{-\mu\tau} \\ &= 0. \end{aligned}$$

According to condition (14) all the coefficients of P are positive and by the Routh-Hurewitz Theorem all the roots have negative real parts. So the condition $|\arg(\lambda)| > \alpha\frac{\pi}{2}$ is satisfyed. By Lemma 5/(1) The endemic steady state E^* is locally asymptotically stable. □

6 | GLOBAL STABILITY

In this section we study global stability of both free and endemic steady states by using the method of Lyapunov function. We prove first global stability of free steady state.

6.1 | Global stability of free steady state E^0

Theorem 7. If $\mathcal{R}_0 < 1$ then the free steady state E^0 is globally asymptotically stable.

Proof. Define the Lyapunov function V as follows

$$V(t) = S(t) - S^0 - \int_0^t \tfrac{f(S^0,I(t))}{f(S(\theta),I(t))}d\theta + e^{\mu\tau}I(t) + \int_{t-\tau}^t f(S(\theta),I(\theta))d\theta,$$

it is clear that V is non-negative defined function at E^0. We have

$$D^\alpha V(t) \le \Big(1 - \tfrac{f(S^0,I(t))}{f(S(t),I(t))}\Big) D^\alpha S(t) + e^{\mu\tau}D^\alpha I(t) + f(S(t),I(t)) - f(S(t-\tau),I(t-\tau)).$$

By using the two equations of system (8) we have

$$\begin{aligned} D^\alpha V(t) &\le \Big(1 - \tfrac{f(S^0,I(t))}{f(S(t),I(t))}\Big)\Big(\lambda - \mu S(t) - f(S(t),I(t))\Big) + f(S(t-\tau,I(t-\tau)) \\ &\quad - e^{\mu\tau}(\mu+\gamma+\eta)I(t) + f(S(t),I(t)) - f(S(t-\tau),I(t-\tau)), \end{aligned}$$

since $\lambda = \mu S^0$, thus

$$D^\alpha V(t) \le \mu\Big(1 - \tfrac{f(S^0,I(t))}{f(S(t),I(t))}\Big)(S^0 - S(t)) + f(S^0,I(t)) - (\mu+\eta+\gamma)e^{\mu\tau}I(t),$$

the condition $\frac{\partial^2 f(S(t),I(t))}{\partial^2 I(t)} < 0$ yealds that

$$f(S^0,I(t)) < I(t)\frac{\partial f(S^0,I(t))}{\partial I(t)} < I(t)\frac{\partial f(S^0,0)}{\partial I(t)},$$

and hence

$$
\begin{aligned}
D^\alpha V(t) &\le \mu\left(1-\frac{f(S^0,I(t))}{f(S(t),I(t))}\right)(S^0-S(t)) + I(t)\frac{\partial f(S^0,0)}{\partial I(t)} - (\mu+\eta+\gamma)e^{\mu\tau}I(t),\\
&\le \mu\left(1-\frac{f(S^0,I(t))}{f(S(t),I(t))}\right)(S^0-S(t)) + (\mu+\eta+\gamma)e^{\mu\tau}(\mathcal{R}_0-1)I(t).
\end{aligned}
$$

Since f is an increasing function in the first variable, we can see that

$$
\begin{aligned}
\frac{f(S^0,I(t))}{f(S(t),I(t))} \ge 1, \ \forall\ S^0 \ge S(t),\\
\frac{f(S^0,I(t))}{f(S(t),I(t))} \le 1, \ \forall\ S^0 \le S(t),
\end{aligned}
$$

so $\left(1-\frac{f(S^0,I(t))}{f(S(t),I(t))})(S^0-S(t)\right) \le 0$. If $\mathcal{R}_0 < 1$, then

$$
D^\alpha V(t) \le -W_3,
$$

where

$$
W_3 = \mu(1-\frac{f(S^0,I(t))}{f(S(t),I(t))})(S(t)-S^0) + (\mu+\eta+\gamma)e^{\mu\tau}(1-\mathcal{R}_0)I(t) \ge 0.
$$

According to Lemma 6 and since the free steady state E^0 is locally asymptotically stable then by the Lasalle invariance principle it is globally asymptotically stable. □

6.2 | Global stability of endemic steady state E^*

Theorem 8. Assume that $\mathcal{R}_0 > 1$ and the condition (14) holds. Then the endemic steady state E^* is globally asymptotically stable if the following inequality hold

$$
\left(\frac{f(S^*,I^*)}{f(S,I^*)} - \frac{f(S^*,I)}{f(S,I)}\right)\left(1-\frac{f(S,I)}{f(S^*,I^*)}\right) \le 0,\ \forall S, I > 0.
$$

Proof. Let $H(x) = x - 1 - \ln x$. Note that $H : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ has a strict global minimum at $x = 1$.
Define the Lyapunov function as follows

$$
\begin{aligned}
L(t) = {} & S(t) - S^* - \int_{S^*}^{S(t)} \frac{f(S^*,I(t))}{f(\theta,I(t))}d\theta + e^{\mu\tau}\left(I(t) - I^* - \int_{I^*}^{I(t)}\frac{f(S(t),I^*)}{f(S(t),\theta)}d\theta\right)\\
&+\imath\int_0^\tau\left(\frac{f(S(t-\theta),I(t-\theta))}{f(S^*,I^*)} - 1 - \ln\frac{f(S(t-\theta),I(t-\theta))}{f(S^*,I^*)}\right)d\theta,
\end{aligned}
$$

where

$$
\imath = f(S^*,I^*).
$$

The fractional derivative of L satisfies

$$
\begin{aligned}
D^\alpha L(t) \le {} & \left(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\right)D^\alpha S(t) + e^{\mu\tau}\left(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\right)D^\alpha I(t)\\
&+\imath\Big[-\frac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)} + \frac{f(S(t),I(t))}{f(S^*,I^*)} + \ln\left(\frac{f(S(t-\tau),I(t-\tau)}{f(S(t),I(t))}\right)\Big],
\end{aligned} \tag{19}
$$

using the first equation of (5) we get

$$
\left(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\right)D^\alpha S(t) = \left(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\right)(\lambda - \mu S(t) - f(S(t),I(t))).
$$

The first equation of (6) leads to

$$
\lambda = \mu S^* + f(S^*,I^*),
$$

that is,

$$
\begin{aligned}
\big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\big)D^\alpha S(t) = {} & \big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\big)\big(\mu(S^*-S(t)) + f(S^*,I^*)\\
&-f(S(t),I(t))\big),
\end{aligned}
$$

thus,

$$
\begin{aligned}
\big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\big)D^\alpha S(t) = {} & \mu S^*\big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\big)(1-\frac{S(t)}{S^*})\\
&+f(S^*,I^*)\big(1-\frac{f(S^*,I(t))}{f(S(t),I(t))}\big)\big(1-\frac{f(S(t),I(t))}{f(S^*,I^*)}\big).
\end{aligned} \tag{20}
$$

Since

$$
e^{\mu t}\big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\big)D^\alpha I(t) = \big(1-\frac{f(S(t),I^*)}{f(S(t),I(t))}\big)\Big(f(S(t-\tau),I(t-\tau)) - e^{\mu\tau}(\mu+\gamma+\eta)I(t)\Big),
$$

using the second equation of system (6) we have

$$
\frac{f(S^*,I^*)}{I^*} = (\mu+\eta+\gamma)e^{\mu\tau},
$$

which leads to,

$$e^{\mu t}\big(1-\tfrac{f(S(t),I^*)}{f(S(t),I(t))}\big)D^\alpha I(t) \;=\; f(S^*,I^*)\big(1-\tfrac{f(S(t),I^*)}{f(S(t),I(t))}\big)\Big(\tfrac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}-\tfrac{I(t)}{I^*}\Big),$$

hence,

$$\begin{aligned} e^{\mu t}\left(1-\tfrac{f(S(t),I^*)}{f(S(t),I(t))}\right)D^\alpha I(t) \;=\; & f(S^*,I^*)\big(1-\tfrac{f(S(t),I^*)}{f(S(t),I(t))}\big)\Big(\tfrac{f(S(t),I(t))}{f(S(t),I^*)}-\tfrac{I(t)}{I^*}\Big)\\ & +f(S^*,I^*)\big(1-\tfrac{f(S(t),I^*)}{f(S(t),I(t))}\big)\Big(\tfrac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}-\tfrac{f(S(t),I(t))}{f(S(t),I^*)}\Big). \end{aligned} \tag{21}$$

Substituting the equations (20) and (21) into (19) we get

$$\begin{aligned} D^\alpha L(t) \;\le\; & \mu S^*\big(1-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}\big)(1-\tfrac{S(t)}{S^*})\\ & +f(S^*,I^*)\big(1-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}\big)\big(1-\tfrac{f(S(t),I(t))}{f(S^*,I^*)}\big)\\ & +f(S^*,I^*)\big(1-\tfrac{f(S(t),I^*)}{f(S(t),I(t))}\big)\Big(\tfrac{f(S(t),I(t))}{f(S(t),I^*)}-\tfrac{I(t)}{I^*}\Big)\\ & +f(S^*,I^*)\big(1-\tfrac{f(S(t),I^*)}{f(S(t),I(t))}\big)\Big(\tfrac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}-\tfrac{f(S(t),I(t))}{f(S(t),I^*)}\Big)\\ & +f(S^*,I^*)\Big[-\tfrac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}+\tfrac{f(S(t),I(t))}{f(S^*,I^*)}+\ln\tfrac{f(S(t-\tau),I(t-\tau))}{f(S(t),I(t))}\Big]. \end{aligned} \tag{22}$$

Put

$$\begin{aligned} A \;=\; & \big(1-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}\big)\big(1-\tfrac{f(S(t),I(t))}{f(S^*,I^*)}\big)\\ & +\big(1-\tfrac{f(S(t),I^*)}{f(S(t),I(t))}\big)\Big(\tfrac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}-\tfrac{f(S(t),I(t))}{f(S(t),I^*)}\Big)\\ & +\Big[-\tfrac{f(S(t-\tau),I(t-\tau))}{f(S^*,I^*)}+\tfrac{f(S(t),I(t))}{f(S^*,I^*)}+\ln\tfrac{f(S(t-\tau),I(t-\tau))}{f(S(t),I(t))}\Big], \end{aligned}$$

then (22) become

$$\begin{aligned} D^\alpha L(t) \;\le\; & \mu S^*\big(1-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}\big)(1-\tfrac{S(t)}{S^*})\\ & +f(S^*,I^*)\big(1-\tfrac{f(S(t),I^*)}{f(S(t),I(t))}\big)\Big(\tfrac{f(S(t),I(t))}{f(S(t),I^*)}-\tfrac{I(t)}{I^*}\Big)+Af(S^*,I^*), \end{aligned} \tag{23}$$

and after some calculations

$$\begin{aligned} A \;=\; & 2-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}+\tfrac{f(S^*,I(t))}{f(S^*,I^*)}-\tfrac{f(S(t),I(t))}{f(S(t),I^*)}-\tfrac{f(S(t-\tau),I(t-\tau))f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\\ & +\ln(\tfrac{f(S(t-\tau),I(t-\tau)}{f(S(t),I(t))})\\ =\; & 2-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}+\tfrac{f(S^*,I(t))}{f(S^*,I^*)}-\tfrac{f(S(t),I(t))}{f(S(t),I^*)}-\tfrac{f(S(t-\tau),I(t-\tau))f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\\ & +\ln(\tfrac{f(S(t-\tau),I(t-\tau))f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)})+\ln(\tfrac{f(S^*,I^*)}{f(S(t),I^*)})+\tfrac{f(S^*,I^*)}{f(S(t),I^*)}-\tfrac{f(S^*,I^*)}{f(S(t),I^*)}\\ =\; & -\tfrac{f(S(t-\tau),I(t-\tau))f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}+1+\ln(\tfrac{f(S(t-\tau),I(t-\tau)f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)})\\ & -\tfrac{f(S^*,I^*)}{f(S(t),I^*)}+1+\ln(\tfrac{f(S^*,I^*)}{f(S(t),I^*)})-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}+\tfrac{f(S^*,I(t))}{f(S^*,I^*)}\\ & -\tfrac{f(S(t),I(t))}{f(S(t),I^*)}+\tfrac{f(S^*,I^*)}{f(S(t),I^*)},\\ =\; & -H\Big(\tfrac{f(S(t-\tau),I(t-\tau)f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\Big)-H\Big(\tfrac{f(S^*,I^*)}{f(S(t),I^*)}\Big)-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}+\tfrac{f(S^*,I(t))}{f(S^*,I^*)}\\ & -\tfrac{f(S(t),I(t))}{f(S(t),I^*)}+\tfrac{f(S^*,I^*)}{f(S(t),I^*)}, \end{aligned}$$

but

$$\begin{aligned} & -\tfrac{f(S^*,I(t))}{f(S(t),I(t))}+\tfrac{f(S^*,I(t))}{f(S^*,I^*)}-\tfrac{f(S(t),I(t))}{f(S(t),I^*)}+\tfrac{f(S^*,I^*)}{f(S(t),I^*)}\\ & =\Big(\tfrac{f(S^*,I^*)}{f(S(t),I^*)}-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(1-\tfrac{f(S(t),I(t))}{f(S^*,I^*)}\Big), \end{aligned}$$

thus

$$\begin{aligned} A \;=\; & -H\Big(\tfrac{f(S(t-\tau),I(t-\tau)f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\Big)-H\Big(\tfrac{f(S^*,I^*)}{f(S(t),I^*)}\Big)\\ & +\Big(\tfrac{f(S^*,I^*)}{f(S(t),I^*)}-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(1-\tfrac{f(S(t),I(t))}{f(S^*,I^*)}\Big), \end{aligned} \tag{24}$$

substituting (24) into (23), we gets

$$\begin{aligned} D^\alpha L(t) \;\le\; & \mu S^*\big(1-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}\big)(1-\tfrac{S(t)}{S^*})\\ & +f(S^*,I^*)\Big\{\big(1-\tfrac{f(S(t),I^*)}{f(S(t),I(t))}\big)\Big(\tfrac{f(S(t),I(t))}{f(S(t),I^*)}-\tfrac{I(t)}{I^*}\Big)\\ & -H\Big(\tfrac{f(S(t-\tau),I(t-\tau)f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\Big)-H\Big(\tfrac{f(S^*,I^*)}{f(S(t),I^*)}\Big)\\ & +\Big(\tfrac{f(S^*,I^*)}{f(S(t),I^*)}-\tfrac{f(S^*,I(t))}{f(S(t),I(t))}\Big)\Big(1-\tfrac{f(S(t),I(t))}{f(S^*,I^*)}\Big)\Big\}. \end{aligned} \tag{25}$$

Since f is a monotonically increasing function with respect to S, this implies that

$$\begin{aligned} & \tfrac{f(S^*,I(t))}{f(S(t),I(t))}\ge 1,\ \forall\ S^*\ge S(t),\\ & \tfrac{f(S^*,I(t))}{f(S(t),I(t))}\le 1,\ \forall\ S^*\le S(t), \end{aligned}$$

and then

$$\left(1 - \frac{f(S^*,I(t))}{f(S(t),I(t))}\right)\left(1 - \frac{S(t)}{S^*}\right) \leq 0,$$

using the hypothesis (4) we get

$$\left(1 - \frac{f(S(t),I^*)}{f(S(t),I(t))}\right)\left(\frac{f(S(t),I(t))}{f(S(t),I^*)} - \frac{I(t)}{I^*}\right) \leq 0.$$

If

$$\left(\frac{f(S^*,I^*)}{f(S(t),I^*)} - \frac{f(S^*,I(t))}{f(S(t),I(t))}\right)\left(1 - \frac{f(S(t),I(t))}{f(S^*,I^*)}\right) \leq 0,$$

we can conclude that

$$D^\alpha L(t) \leq -W_3,$$

where

$$\begin{aligned} W_3 \;=\; & \mu S^*\left(\frac{f(S^*,I(t))}{f(S(t),I(t))} - 1\right)\left(1 - \frac{S(t)}{S^*}\right) \\ & + f(S^*,I^*)\Bigg\{\left(\frac{f(S(t),I^*)}{f(S(t),I(t))} - 1\right)\left(\frac{f(S(t),I(t))}{f(S(t),I^*)} - \frac{I(t)}{I^*}\right) \\ & + H\left(\frac{f(S(t-\tau),I(t-\tau)f(S(t),I^*)}{f(S(t),I(t))f(S^*,I^*)}\right) + H\left(\frac{f(S^*,I^*)}{f(S(t),I^*)}\right) \\ & + \left(\frac{f(S^*,I^*)}{f(S(t),I^*)} - \frac{f(S^*,I(t))}{f(S(t),I(t))}\right)\left(\frac{f(S(t),I(t))}{f(S^*,I^*)} - 1\right)\Bigg\} \geq 0. \end{aligned}$$

By the Lasalle invariance principle and Lemma 6 we conclude that the endemic steady state is globally asymptotically stable. □

Remark 1. The condition

$$\left(\frac{f(S^*,I^*)}{f(S,I^*)} - \frac{f(S^*,I)}{f(S,I)}\right)\left(1 - \frac{f(S,I)}{f(S^*,I^*)}\right) \leq 0,$$

is satisfied by the most used incidence functions such as $f(S,I) = \beta S^n I^m$ and $f(S,I) = \frac{\beta S^n I^m}{1+aS}$ with $a > 0$ and $n, m \geq 0$. More generally the condition is also satisfied by the incidence function of the type $f(S,I) = \dfrac{\beta S I}{\psi(I)}$ where ψ is a concave function.

7 | NUMERICAL SIMULATIONS

In this section we give some numerical simulations to illustrate our results. As an example we take the incidence function $f(S,I) = \beta S I$ which leads to the following model

$$\begin{cases} D^\alpha S(t) \;=\; \lambda - \beta S(t)I(t) - \mu S(t), \\ D^\alpha E(t) \;=\; \beta S(t)I(t) - \beta S(t-\tau)I(t-\tau)e^{-\mu\tau} - \mu E(t), \\ D^\alpha I(t) \;=\; \beta S(t-\tau)I(t-\tau)e^{-\mu\tau} - (\mu+\eta+\gamma)I(t), \\ D^\alpha R(t) \;=\; \gamma I(t) - \mu R(t). \end{cases} \tag{26}$$

We use the SEIR model (26) to describe the disease of COVID-19 in algeria at the beginning of the infection. Since it was reported that the first case of COVID-19 epidemic in Algeria dates from 25 February 2020[24] we run then our simulations from this date. The basic reproduction number in this case is given by $\mathcal{R}_0 = \beta \frac{e^{-\mu\tau}}{\mu+\eta+\gamma}\frac{\lambda}{\mu}$. Since we conduct our simulations on a short period of time (some months) we can assume that the death rate mortality of infected individuals $\eta = 0$. The parameters of system (26) are given in Table 1 and are taken from[25].

We first consider the case with classic derivative $\alpha = 1$ and without delay $\tau = 0$. Numerical simulations give the graphics in figure 2. As described in[25] we observe a peak of infected individuals after 90 days from the beginning of the infection.

TABLE 1 Parameters and values of model (26).

Parameters	meaning	values
λ	recruitment rate of susceptible individuals	10^{-5}
β	transmission rate per infectious individuals	2.1
μ	Death rate of all individuals	10^{-5}
γ	transfer rate of infected individuals to recovery compartment	0.1
η	Death rate of infected individuals caused by the disease	0
τ	Incubation period	2 – 14 (days)

Source: [25]

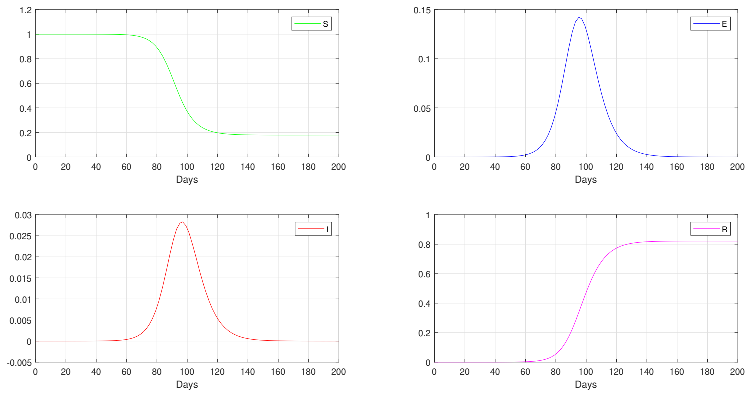

FIGURE 2 Model (26) in the case $\alpha = 1$ and without delay $\tau = 0$.

In figure 3 we have plotted the solutions of system (26) in the case $\alpha = 0.9$ and $\tau = 5$ days. We can observe a peak of infected individuals of around 140 days after the beginning of the infection which corresponds to late July 2020.

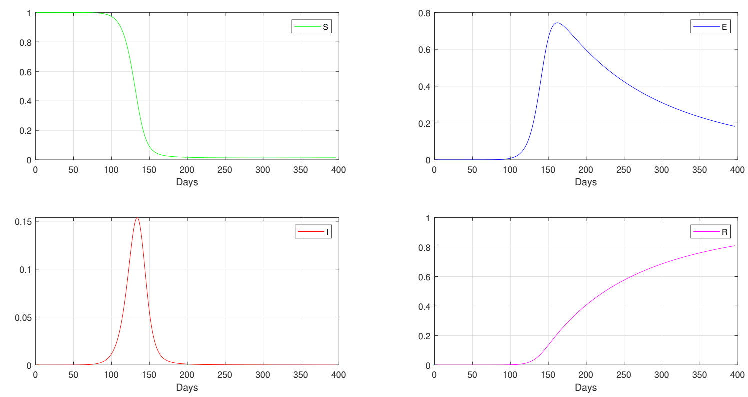

FIGURE 3 Solutions of model (26) in the case $\tau = 5$ and $\alpha = 0.9$.

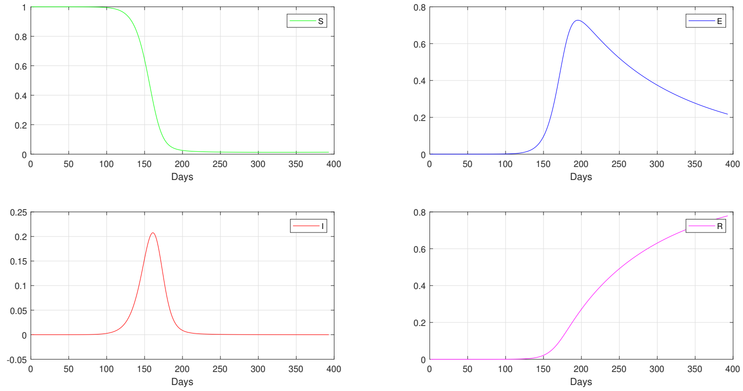

FIGURE 4 Solutions of model (26) in the case $\tau = 8$ days.

In figure 4 we have plotted the solutions in the case $\tau = 8$ days and $\alpha = 0.9$ and in figure 5 we have taken $\tau = 10$ days. We observe a peak of infected individuals around 160 days after the beginning of the infection which corresponds to late July. Since the WHO data[26] show a peak of the epidemic of COVID-19 in Algeria at 24 July, 2020, we can conclude that the model (26) with fractional derivative and time delay describes the epidemic more precisely than the one with classic derivative and without delay.

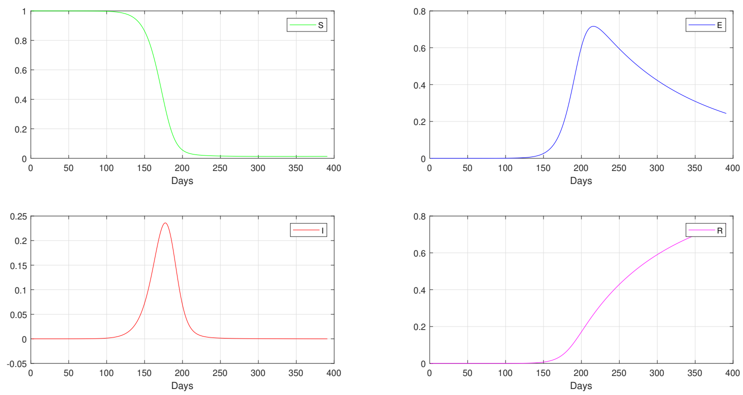

FIGURE 5 Solutions of model (26) in the case $\tau = 10$ days.

8 | CONCLUSION

In this paper we have investigated a fractional-order SEIR epidemic model with general incidence rate function and time delay. The basic reproduction number is computed using the method of next generation matrix. We have obtained necessary and sufficient conditions for local stability of both free and endemic steady states. Using the method of Liapunov function we have proved the global stability of the two steady states under some conditions on the incidence function. The system is used to describe the COVID-19 epidemic in Algeria at the beginning of the appearance of the epidemic in late February 2020. Rather than the case without delay and with classic derivative our simulations predict the first peak of the epidemic around late July 2020 which is consistent with WHO data. We can then conclude that the introduction of time delay and fractional derivative in epidemic models can describe more really the spreading of an epidemic. The model we have proposed is a generalisation of many other models in literature and can be used to describe many other types of epidemics.

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How to cite this article: