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RESEARCH ARTICLE

Decomposition of plate displacements via Kirchhoff-Love displacements

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Abstract

In this paper, we show that any displacement of a plate is the sum of a Kirchhoff-Love displacement and two terms, one for shearing and one for warping. Then, the plate is loaded in order to obtain that the bending and shearing contribute the same order of magnitude to the fiber rotations.

KEYWORDS:
linear elasticity, elementary displacement, Kirchhoff-Love displacement, shearing, warping.

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1 | INTRODUCTION

Modeled from the beam theory in the 19th century the theory of thin plates was developed assuming that the fibers of the plate remain non-deformable and perpendicular to the mid-surface (Kirchoof-Love displacements) and neglecting some components of the stress tensor. Later, Mindlin, Timoshenko, Reissner and Uflyang developed the theory of thick plates taking into account the shear (see [1]). From the 3D variational formulation of the elasticity problem for a plate, it has been proven that the limit displacement is of Kirchoff-Love type. The limit of all the components of the stress tensor has been also obtained (see e.g. [3,4]). This justifies the first hypothesis and assumptions.

The aim of this paper is to give an a priori decomposition of a plate displacement as the sum of a Kirchhoff-Love displacement, shearing and warping.

Consider a plate Ω_δ whose mid-surface is a bounded domain ω and whose thickness is 2δ. We show that every displacement $u \in W^{1,p}(\Omega_\delta)$ can be written as

$$u(x) = \underbrace{\begin{pmatrix} \mathcal{U}_1(x') - x_3 \dfrac{\partial \mathcal{U}_3}{\partial x_1}(x') \\ \mathcal{U}_2(x') - x_3 \dfrac{\partial \mathcal{U}_3}{\partial x_2}(x') \\ \mathcal{U}_3(x') \end{pmatrix}}_{\text{Kirchhoff-Love displacement}} + \underbrace{\begin{pmatrix} x_3 \mathfrak{r}_1(x') \\ x_3 \mathfrak{r}_2(x') \\ 0 \end{pmatrix}}_{\text{shearing}} + \underbrace{\bar{u}(x)}_{\text{warping}} \qquad \text{for a.e. } x \text{ in } \Omega_\delta. \tag{1}$$

Here, $\mathcal{U}_m = \mathcal{U}_1 \mathbf{e}_1 + \mathcal{U}_2 \mathbf{e}_2$ is the membrane displacement, it represents the displacement of the mid-surface of the plate, $\mathcal{U}_3$ is the bending. The map $x_3 \in \mathbb{R} \longmapsto x_3 \Big(\dfrac{\partial \mathcal{U}_3}{\partial x_1}(x')\mathbf{e}_1 + \dfrac{\partial \mathcal{U}_3}{\partial x_2}(x')\mathbf{e}_2 \Big)$ stands for a small rotation of the fiber $\{x'\} \times (-\delta, \delta)$ whose axis is directed by $\dfrac{\partial \mathcal{U}_3}{\partial x_2}(x')\mathbf{e}_1 - \dfrac{\partial \mathcal{U}_3}{\partial x_1}(x')\mathbf{e}_2$ and whose angle is approximately equal to the euclidian norm of this vector. Since we are in the framework of small displacements, the symmetric part of the rotation is neglected. After rotation, the fiber remains perpendicular to the plate mid-surface.

The second term in the above writing represents the shear: $x_3 \big(\mathfrak{r}_1(x')\mathbf{e}_1 + \mathfrak{r}_2(x')\mathbf{e}_2 \big)$, that is two small rotations of the fiber

$\{x'\} \times (-\delta, \delta)$, the first one with axis $\mathbf{e}_2$ and angle $-\mathfrak{r}_1(x')$ and axis $\mathbf{e}_1$ and angle $\mathfrak{r}_2(x')$ for the second. The last displacement is the warping, it gives informations on the deformations of the fibers. It satisfies 4 simple relations ($\alpha \in \{1,2\}$)

$$\int_{-\delta}^{\delta} \overline{u}_\alpha(x', x_3)dx_3 = \int_{-\delta}^{\delta} \overline{u}_\alpha(x', x_3)x_3dx_3 = 0 \qquad \text{for a.e. } x' \in \omega. \tag{2}$$

Such a decomposition is of interest only if we can give an order of the different terms that compose it. For a loading of the plate whose elastic strain energy (the square of the L^2 norm of the strain tensor) is of order δ^5, usually the membrane displacement is of order δ^2, the bending of order δ, the rotations of the fibers of order δ^2, $\mathfrak{r}_1$ and $\mathfrak{r}_2$ of order δ^2 and so the shearing is of order δ^3 (all the estimates in Theorem 2).

A few years ago, we introduced other decompositions to study thin structures made up of straight or curbed rods, plates or shells. We have proven that any displacement of a curved or straight rod is the sum of an elementary rod displacement plus a warping (see[6,7,8,9,10,14,15,16]). This decomposition method has been also applied for plates, shells and structures made up plates (see[8,11,19,17]) or structures combining rods and plates (see[12,13,17,25]). Later, the same ideas were applied to decompose the deformations of thin structures (see[21,22,20] for curved rods or shells).

As a general reference on elasticity, we refer the reader to[2,5]. For mathematical modeling of plates we refer to[3,4].
The paper is organized as follows:

- In Section 2 we introduce the main notations and we recall the first result on the decomposition of a plate displacement.
- In Section 3 the new decomposition (1) of a plate displacement is introduced. Theorem 2 give all the estimates of the terms of this decomposition with respect to δ and the L^p norm of the strain tensor. Then, if the plate is fixed on a part of its lateral boundary, Korn-type inequalities are given.
- In Section 4, we choose a sequence of displacements of the clamped plate Ω_δ whose strain tensor has a L^p norm of order $\delta^{2+1/p}$. In Theorem 3, besides the limits of the terms of the decomposition, we give the asymptotic behavior of the strain tensor using the limits of the terms of the decomposition.
- In Section 5 we give an application of our decomposition, we load a straight plate in order to obtain that the bending and the shearing contribute with the same order of magnitude to the rotations of the fibers.
- In Section 6 we conclude this study by giving a shorter decomposition for thin plate displacements (see (45)).
- Sections 7-8 are concerned with calculations needed in Section 3.

In this work, the constants appearing in the estimates will always be independent from δ. As a rule the Latin indices i, j, k and l take values in $\{1,2,3\}$ while the Greek indices α and β in $\{1,2\}$. We also use the Einstein convention of summation over repeated indices.
This paper is inspired by[26] and follows the same lines.

2 | NOTATIONS AND RECALLS

We denote by $|\cdot|$ the euclidian norm of $\mathbb{R}^3$ and by $\cdot$ the associated scalar product.
In this paper, ω denotes a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. We refer ω to an orthonormal frame $\big(O; \mathbf{e}_1, \mathbf{e}_2\big)$. We set $\mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2$. So, the space $\mathbb{R}^3$ is referred to the orthonormal frame $\big(O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\big)$.
Denote

- Ω_δ the plate with mid-surface ω and thickness 2δ

$$\Omega_\delta \doteq \omega \times (-\delta, \delta)$$

- $Y \doteq (0,1)^2$, $Z \doteq (-1/2, 1/2)^2$,

$$\begin{aligned} \mathcal{Z} &= (-1/2, 3/2)^2 = \text{interior}\big(\overline{Z} \cup (\mathbf{e}_1 + \overline{Z}) \cup (\mathbf{e}_2 + \overline{Z}) \cup (\mathbf{e}_1 + \mathbf{e}_2 + \overline{Z})\big), \\ \mathcal{Y} &= (0,2)^2 = \text{interior}\big(\overline{Y} \cup (\mathbf{e}_1 + \overline{Y}) \cup (\mathbf{e}_2 + \overline{Y}) \cup (\mathbf{e}_1 + \mathbf{e}_2 + \overline{Y})\big), \end{aligned} \tag{3}$$

- $\omega_\eta \doteq \{x' \in \mathbb{R}^2 \mid \text{dist}(x',\omega) < \eta\}$ and $\Omega'_\delta = \omega_{3\delta} \times (-\delta,\delta)$
- for every $v \in W^{1,p}(\Omega'_\delta)^3$, $1 \le p \le \infty$,

$$e(v) = \frac{1}{2}\Big((\nabla v)^T + \nabla v\Big), \qquad e_{ij}(v) = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big).$$

$e(v)$ is the 3×3 symmetric matrix whose entries are the $e_{ij}(v)$'s.

Proposition 1. There exists $\delta_0 > 0$ such that for every $\delta \in (0,\delta_0]$, there exists an extension operator P_δ from $W^{1,p}(\Omega_\delta)^3$ into $W^{1,p}(\Omega'_\delta)^3$, $1 < p < \infty$, satisfying

$$\forall u \in W^{1,p}(\Omega_\delta)^3, \quad P_\delta(u) \in W^{1,p}(\Omega'_\delta)^3, \qquad P_\delta(u)_{|\Omega_\delta} = u, \qquad \Big\|e(P_\delta(u))\Big\|_{L^p(\Omega'_\delta)} \le C\|e(u)\|_{L^p(\Omega_\delta)}.$$

The constant does not depend on δ.

Proof. From[24] Lemma 5.22, there exists $\delta'_0 > 0$ (which only depend on the boundary of ω) such that the boundaries $\partial\omega_\eta$, for $\eta \in (0,\delta'_0]$, are uniformly Lipschitz.
Besides, if ω' is a bounded domain with Lipschitz boundary, in[8] Lemma 4.2 we show that there exist $\delta''_0 \in (0,\delta'_0/2]$ (which only depend on the boundary of ω') and for every $\delta \in (0,\delta''_0]$ an extension operator P'_δ from $W^{1,p}\big(\omega' \times (-\delta,\delta)\big)^3$ into $W^{1,p}(\omega'_{2\delta} \times (-\delta,\delta))^3$, $1 < p < \infty$, satisfying

$$\forall u \in W^{1,p}\big(\omega' \times (-\delta,\delta)\big)^3, \qquad \Big\|e(P'_\delta(u))\Big\|_{L^p(\omega'_{2\delta}\times(-\delta,\delta))} \le C\|e(u)\|_{L^p(\Omega_\delta)}$$

where the constant does not depend on δ (it depends on p and $\partial\omega'$, it is the same constant for all the open sets ω_η, $\eta \in (0,\delta''_0]$).
Now, let u be a displacement in $W^{1,p}(\Omega_\delta)^3$, if $\delta \le \delta''_0/2$, using the above result with $\omega' = \omega$, we extend u in oder to obtain a displacement belonging to $W^{1,p}(\omega_{2\delta} \times (-\delta,\delta))^3$. Then, since $2\delta \le \delta''_0$, the open set $\omega_{2\delta}$ has a Lipschitz boundary. So, we can still apply the above extension result. We extend the extension of u and we get a displacement belonging to $W^{1,p}(\omega_{4\delta}\times(-\delta,\delta))^3$. This gives the result of the proposition. □

For simplicity, we will always write u instead of $P_\delta(u)$ the extension of u to the plate Ω'_δ.
Below we recall the definition of an elementary displacement of the plate.

Definition 1. An elementary displacement of the plate Ω'_δ is a displacement $v \in L^1(\Omega_\delta)^3$ written in the form

$$v(x',x_3) = \mathcal{V}(x') + x_3\mathcal{A}(x') \qquad \text{for a.e. } x = (x',x_3) \in \Omega'_\delta.$$

The component $\mathcal{V}$ belongs to $L^1(\omega_{3\delta})^3$ while $\mathcal{A}$ is in $L^1(\omega_{3\delta})^2$, $\mathcal{A} = \mathcal{A}_1\mathbf{e}_1 + \mathcal{A}_2\mathbf{e}_2$.
Here, $\mathcal{V}$ gives the mid-surface displacement and $x_3\mathcal{A}(x')$ represents a "small rotation" of the fiber $\{x'\}\times(-\delta,\delta)$, whose axis is directed by $-\mathcal{A}_2(x')\mathbf{e}_1 + \mathcal{A}_1(x')\mathbf{e}_2$ and whose angle is approximately $|\mathcal{A}(x')|$.

To any displacement $u \in L^1(\Omega'_\delta)^3$ we associate a unique elementary displacement $U^*_{e\ell} \in L^1(\Omega'_\delta)^3$ and a warping $\overline{u}^* \in L^1(\Omega'_\delta)^3$

$$\begin{aligned} u(x) &= U^*_{e\ell}(x) + \overline{u}^*(x) \\ U^*_{e\ell}(x) &= \mathcal{U}^*(x') + x_3\mathcal{R}^*(x') \end{aligned} \qquad \text{for a.e. } x = (x',x_3) \in \Omega'_\delta \tag{4}$$

so that

$$\int_{-\delta}^{\delta} \overline{u}^*(x',x_3)dx_3 = 0, \qquad \int_{-\delta}^{\delta} \overline{u}^*_\alpha(x',x_3)x_3dx_3 = 0 \qquad \text{for a.e. } x' \in \omega_{3\delta}. \tag{5}$$

The above equalities determine $\mathcal{U}^*(x')$ and $\mathcal{R}^*(x')$ in terms of u and integrals over the fiber $\{x'\}\times(-\delta,\delta)$ (see[8] and[24] Chapter 11). We have

$$\mathcal{U}^*(x') = \frac{1}{2\delta}\int_{-\delta}^{\delta} u(x',x_3)dx_3, \quad \mathcal{R}^*(x') = \frac{3}{2\delta^3}\int_{-\delta}^{\delta} x_3\big(u_1(x',x_3)\mathbf{e}_1 + u_2(x',x_3)\mathbf{e}_2\big)dx_3, \quad \text{for a.e. } x' \in \omega_{3\delta}.$$

Theorem 1 (Theorem 4.1 in[8]). Let u be a displacement belonging to $W^{1,p}(\Omega'_\delta)^3$, $1 < p < \infty$, decomposed as (4). The terms $\mathcal{U}^*$, $\mathcal{R}^*$ and $\overline{u}^*$ of this decomposition satisfy

$$
\begin{aligned}
&\|\overline{u}^*\|_{L^p(\Omega'_\delta)} \leq C\delta\|e(u)\|_{L^p(\Omega'_\delta)}, \qquad \|\nabla\overline{u}^*\|_{L^p(\Omega'_\delta)} \leq C\|e(u)\|_{L^p(\Omega'_\delta)},\\
&\delta\|\nabla\mathcal{R}^*\|_{L^p(\omega_{3\delta})} + \|e_{\alpha\beta}(\mathcal{U}^*)\|_{L^p(\omega_{3\delta})} + \Big\|\frac{\partial\mathcal{U}_3^*}{\partial x_\alpha} + \mathcal{R}_\alpha^*\Big\|_{L^p(\omega_{3\delta})} \leq \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega'_\delta)}.
\end{aligned}
\tag{6}
$$

The constant does not depend on δ.

Proof. All the estimates of (6) are the consequences of the ones in[8] Theorem 4.1 except that of $\nabla\mathcal{R}^*$ which is replaced by

$$
\delta\|e_{\alpha\beta}(\mathcal{R}^*)\|_{L^p(\omega_{3\delta})} \leq \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega'_\delta)}.
$$

The above estimate and the $2D$-Korn inequality give a rigid $2D$-displacement $\mathbf{r}(x') = \begin{pmatrix} a_1 - bx_2\\ a_2 + bx_1\end{pmatrix}$, $(a_1, a_2, b) \in \mathbb{R}^3$ such that

$$
\|\mathcal{R}^* - \mathbf{r}\|_{W^{1,p}(\omega_{3\delta})} \leq C\|e_{\alpha\beta}(\mathcal{R}^*)\|_{L^p(\omega_{3\delta})} \leq \frac{C}{\delta^{1+1/p}}\|e(u)\|_{L^p(\Omega'_\delta)}.
$$

Thus, the above and $(6)_5$ yield

$$
\Big\|\frac{\partial\mathcal{U}_3^*}{\partial x_\alpha} + \mathbf{r}_\alpha\Big\|_{L^p(\omega_{3\delta})} \leq \frac{C}{\delta^{1+1/p}}\|e(u)\|_{L^p(\Omega'_\delta)}.
$$

Now, let ϕ_0 be in $\mathcal{D}(\omega)$ such that $\displaystyle\int_\omega \phi_0 dx' = 1$. We have

$$
\int_\omega \Big[\Big(\frac{\partial\mathcal{U}_3^*}{\partial x_1}(x') + a_1 - bx_2\Big)\frac{\partial\phi_0}{\partial x_2} - \Big(\frac{\partial\mathcal{U}_3^*}{\partial x_2}(x') + a_2 + bx_1\Big)\frac{\partial\phi_0}{\partial x_1}\Big]dx' = 2b\int_\omega \phi_0 dx' = 2b.
$$

Besides, the Hölder inequality leads to

$$
\begin{aligned}
&\Big|\int_\omega \Big[\Big(\frac{\partial\mathcal{U}_3^*}{\partial x_1}(x') + a_1 - bx_2\Big)\frac{\partial\phi_0}{\partial x_2} - \Big(\frac{\partial\mathcal{U}_3^*}{\partial x_2}(x') + a_2 + bx_1\Big)\frac{\partial\phi_0}{\partial x_1}\Big]dx'\Big|\\
&\qquad\leq C\Big(\Big\|\frac{\partial\mathcal{U}_3^*}{\partial x_1} + \mathbf{r}_1\Big\|_{L^p(\omega_{3\delta})} + \Big\|\frac{\partial\mathcal{U}_3^*}{\partial x_2} + \mathbf{r}_2\Big\|_{L^p(\omega_{3\delta})}\Big)\|\nabla\phi_0\|_{L^{p'}(\omega)}.
\end{aligned}
$$

So, $|b| \leq \dfrac{C}{\delta^{1+1/p}}\|e(u)\|_{L^p(\Omega'_\delta)}$ which in turn gives $(6)_3$. □

Remark 1. Suppose that the plate is clamped on a part of its lateral boundary

$$
\Gamma_\delta \doteq \gamma \times (-\delta, \delta)
$$

where $\gamma \subset \partial\omega$ is a set with non-null measure. Since the fields $\mathcal{U}^*$ and $\mathcal{R}^*$ are defined via integrals over the fibers, we have

$$
\mathcal{U}^* = 0, \qquad \mathcal{R}^* = 0, \quad \text{a.e. on } \gamma, \qquad \overline{u}^* = 0 \text{ a.e. on } \Gamma_\delta. \tag{7}
$$

3 | DECOMPOSITION OF A PLATE DISPLACEMENT VIA A KIRCHHOFF-LOVE DISPLACEMENT

In this section we decompose every displacement as the sum of a Kirchoof-Love displacement and shearing plus warping. This decomposition suits our purpose better and simplifies the way to obtain estimates and later the asymptotic behaviors of sequences of displacements.

Denote

$$
\begin{aligned}
&\Xi_\delta \doteq \Big\{\xi \in \mathbb{Z}^2 \mid \delta\big(\xi + Y\big) \cap \omega \neq \emptyset\Big\}, \qquad \Xi'_\delta \doteq \Xi_\delta \cup \big(\mathbf{e}_1 + \Xi_\delta\big) \cup \big(\mathbf{e}_2 + \Xi_\delta\big) \cup \big(\mathbf{e}_1 + \mathbf{e}_2 + \Xi_\delta\big),\\
&\widehat{\omega}_\delta \doteq \text{Interior}\bigcup_{\xi\in\Xi_\delta} \delta\big(\xi + \overline{Y}\big).
\end{aligned}
$$

Observe that $\omega \subset \widehat{\omega}_\delta \subset \omega_{3\delta}$ and note that for every $\xi \in \Xi_\delta$, we have

$$
\delta\big(\xi + Z\big) \subset \omega_{3\delta}, \quad \delta\big(\xi + \mathbf{e}_1 + Z\big) \subset \omega_{3\delta}, \quad \delta\big(\xi + \mathbf{e}_2 + Z\big) \subset \omega_{3\delta}, \quad \delta\big(\xi + \mathbf{e}_1 + \mathbf{e}_2 + Z\big) \subset \omega_{3\delta}.
$$

So for every $\xi \in \Xi'_\delta$ we have $\delta\big(\xi + Z\big) \subset \omega_{3\delta}$.
For every $\phi \in L^1(\omega_{3\delta})$ we set

$$\mathcal{M}_\delta(\phi)(t) = \frac{1}{\delta^2}\int_{\delta Z} \phi\big(t+z\big)dz_1dz_2, \quad \forall t=(t_1,t_2)\in\omega_{3\delta} \text{ such that } t+\delta Z\subset \omega_{3\delta},$$

$$\widetilde{\mathcal{M}_\delta}(\phi)(\delta\xi) = \frac{1}{\delta^2}\int_{\delta Y} \phi\big(\delta\xi+z\big)dz_1dz_2 = \mathcal{M}_\delta(\phi)\Big(\delta\xi + \frac{\delta}{2}(\mathbf{e}_1+\mathbf{e}_2)\Big), \quad \forall \xi\in\Xi_\delta.$$

Now, let u be a displacement in $W^{1,p}(\Omega_\delta)^3$, extended in an element belonging to $W^{1,p}(\Omega'_\delta)^3$ and then decomposed as (4).

3.1 | The Kirchhoff-Love displacement associated with u

We first set

$$\mathcal{U}_1 = \mathcal{U}_1^*, \qquad \mathcal{U}_2 = \mathcal{U}_2^* \qquad \text{a.e. in } \omega. \tag{8}$$

Below, we define the third component $\mathcal{U}_3$ in $\widehat{\omega}_\delta$. In the cell $\delta(\xi+\overline{Y})$, $\xi\in\Xi_\delta$, we set

$$\mathcal{U}_3(x') = \Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}\big(x_1-\delta\xi_1, x_2-\delta\xi_2\big), \qquad \forall x'=(x_1,x_2)\in\delta(\xi+\overline{Y}), \quad \xi=(\xi_1,\xi_2)$$

where $\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}$ is given in Section 8

$$\begin{aligned}
\mathbf{A} &= \big(\mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi),\ \mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi+\delta\mathbf{e}_1),\ \mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi+\delta\mathbf{e}_1+\delta\mathbf{e}_2),\ \mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi+\delta\mathbf{e}_2)\big),\\
\mathbf{B} &= -\big(\mathcal{M}_\delta(\mathcal{R}_1^*)(\delta\xi),\ \mathcal{M}_\delta(\mathcal{R}_1^*)(\delta\xi+\delta\mathbf{e}_1),\ \mathcal{M}_\delta(\mathcal{R}_1^*)(\delta\xi+\delta\mathbf{e}_1+\delta\mathbf{e}_2),\ \mathcal{M}_\delta(\mathcal{R}_1^*)(\delta\xi+\delta\mathbf{e}_2)\big),\\
\mathbf{C} &= -\big(\mathcal{M}_\delta(\mathcal{R}_2^*)(\delta\xi),\ \mathcal{M}_\delta(\mathcal{R}_2^*)(\delta\xi+\delta\mathbf{e}_1),\ \mathcal{M}_\delta(\mathcal{R}_2^*)(\delta\xi+\delta\mathbf{e}_1+\delta\mathbf{e}_2),\ \mathcal{M}_\delta(\mathcal{R}_2^*)(\delta\xi+\delta\mathbf{e}_2)\big).
\end{aligned} \tag{9}$$

By construction, $\mathcal{U}_3$ belongs to $W^{2,p}(\widehat{\omega}_\delta)$.
The Kirchhoff-Love displacement associated with u is

$$U_{KL}(x',x_3) \doteq \begin{pmatrix} \mathcal{U}_1(x') - x_3\dfrac{\partial\mathcal{U}_3}{\partial x_1}(x') \\ \mathcal{U}_2(x') - x_3\dfrac{\partial\mathcal{U}_3}{\partial x_2}(x') \\ \mathcal{U}_3(x')\end{pmatrix} \quad \text{for a.e. } x=(x',x_3)=(x_1,x_2,x_3)\in\Omega_\delta.$$

3.2 | The decomposition of the displacement u

Now, we write

$$u(x) = U_{KL}(x) + x_3\mathfrak{r}(x') + \overline{u}(x), \qquad \text{for a.e. } x \text{ in } \Omega_\delta. \tag{10}$$

The above equality defines $\mathfrak{r} = \mathfrak{r}_1\mathbf{e}_1 + \mathfrak{r}_2\mathbf{e}_2$ and $\overline{u}$ by ($\alpha\in\{1,2\}$)

$$\mathfrak{r}_\alpha = \mathcal{R}_\alpha^* + \frac{\partial\mathcal{U}_3}{\partial x_\alpha}, \qquad \overline{u} = \overline{u}^* + \big(\mathcal{U}_3^* - \mathcal{U}_3)\mathbf{e}_3. \tag{11}$$

Theorem 2. The fields $\mathcal{U}_m = \mathcal{U}_1\mathbf{e}_1 + \mathcal{U}_2\mathbf{e}_2$, $\mathcal{U}_3$, $\mathfrak{r}$ and $\overline{u}$ satisfy

$$\mathcal{U}_m,\ \mathfrak{r}\in W^{1,p}(\omega)^2, \quad \mathcal{U}_3\in W^{2,p}(\omega), \quad \overline{u}\in W^{1,p}(\Omega_\delta)^3$$

and the following estimates:

$$\begin{aligned}
&\|e_{\alpha\beta}(\mathcal{U}_m)\|_{L^p(\omega)} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)},\\
&\Big\|\frac{\partial^2\mathcal{U}_3}{\partial x_1^2}\Big\|_{L^p(\omega)} + \Big\|\frac{\partial^2\mathcal{U}_3}{\partial x_2^2}\Big\|_{L^p(\omega)} + \Big\|\frac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_2}\Big\|_{L^p(\omega)} \le \frac{C}{\delta^{1+1/p}}\|e(u)\|_{L^p(\Omega_\delta)},\\
&\|\mathfrak{r}\|_{L^p(\omega)} + \delta\Big\|\nabla\mathfrak{r}\Big\|_{L^p(\omega)} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)},\\
&\|\overline{u}\|_{L^p(\Omega_\delta)} + \delta\|\nabla\overline{u}\|_{L^p(\Omega_\delta)} \le C\delta\|e(u)\|_{L^p(\Omega_\delta)}.
\end{aligned} \tag{12}$$

The constants do not depend on δ.

Proof. Estimate (12) is the consequence of $(6)_3$ and (8).

Below in the estimates the constants do not depend on ξ and δ.

Step 1. We prove $(12)_2$.

From the estimates (46) in Lemma 3 we get for every $\xi \in \Xi_\delta$, $(\alpha,\beta)\in\{1,2\}^2$, (see also $(3)_2$)

$$\begin{aligned}
&\delta^2\Big|\frac{\mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi+\delta\mathbf{e}_\alpha)-\mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi)}{\delta}+\frac{1}{2}\big(\mathcal{M}_\delta(\mathcal{R}_1^*)(\delta\xi+\delta\mathbf{e}_\alpha)+\mathcal{M}_\delta(\mathcal{R}_\alpha^*)(\delta\xi)\big)\Big|^p\\
&\leq\frac{1}{2}\Big(\int_{\delta(\xi+\mathcal{Y})}\Big|\frac{\partial\mathcal{U}_3^*}{\partial x_\alpha}+\mathcal{R}_\alpha^*\Big|^p dx_1dx_2+\delta^p\int_{\delta(\xi+\mathcal{Y})}\Big|\frac{\partial\mathcal{R}_\alpha^*}{\partial x_\alpha}\Big|^p dx_1dx_2\Big)\\
&\delta^2\Big|\frac{\mathcal{M}_\delta(\mathcal{R}_\alpha^*)(\delta\xi+\delta\mathbf{e}_\beta)-\mathcal{M}_\delta(\mathcal{R}_\alpha^*)(\delta\xi)}{\delta}\Big|^p\leq\int_{\delta(\xi+\mathcal{Y})}\Big|\frac{\partial\mathcal{R}_\alpha^*}{\partial x_\beta}\Big|^p dx_1dx_2
\end{aligned}\tag{13}$$

Now, as a consequence of the above estimates, the expressions of the second order partial derivatives of $\mathcal{U}_3$ given in Section 8 we obtain

$$\Big\|\frac{\partial^2\mathcal{U}_3}{\partial x_1^2}\Big\|_{L^p(\omega)}+\Big\|\frac{\partial^2\mathcal{U}_3}{\partial x_2^2}\Big\|_{L^p(\omega)}+\Big\|\frac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_2}\Big\|_{L^p(\omega)}\leq C\Big(\frac{1}{\delta}\Big\|\frac{\partial\mathcal{U}_3^*}{\partial x_\alpha}+\mathcal{R}_\alpha^*\Big\|_{L^p(\omega_{3\delta})}+\|\nabla\mathcal{R}_\alpha\|_{L^p(\omega_{3\delta})}\Big).$$

Then, we get $(12)_2$ thanks to $(6)_3$.

Step 2. We prove $(12)_3$.

First, the Poincaré-Wirtinger inequality applied in the cells $\delta(\xi+Z)$, $\delta(\xi+Y)$ and then in $\delta(\xi+Z)\cup\delta(\xi+Y)$, $\xi\in\Xi_\delta$, allows to compare $\mathcal{M}_\delta(\mathcal{R}_\alpha^*)(\delta\xi)$ and $\widetilde{\mathcal{M}_\delta}(\mathcal{R}_\alpha^*)(\delta\xi)$. We obtain

$$\delta^2|\mathcal{M}_\delta(\mathcal{R}_\alpha^*)(\delta\xi)-\widetilde{\mathcal{M}_\delta}(\mathcal{R}_\alpha^*)(\delta\xi)|^p\leq C\delta^p\|\nabla\mathcal{R}_\alpha^*\|^p_{L^p(\delta(\xi+Z)\cup\delta(\xi+Y))}.\tag{14}$$

Besides, we have

$$\begin{aligned}
\Big\|\frac{\partial\mathcal{U}_3}{\partial x_\alpha}+\mathcal{R}_\alpha^*\Big\|^p_{L^p(\delta(\xi+Y))}\leq&C\Big(\Big\|\mathcal{R}_\alpha^*-\widetilde{\mathcal{M}_\delta}(\mathcal{R}_\alpha^*)(\delta\xi)\Big\|^p_{L^p(\delta(\xi+Y))}\\
&+\delta^2|\widetilde{\mathcal{M}_\delta}(\mathcal{R}_\alpha^*)(\delta\xi)-\mathcal{M}_\delta(\mathcal{R}_\alpha^*)(\delta\xi)|^p+\Big\|\frac{\partial\mathcal{U}_3}{\partial x_\alpha}+\mathcal{M}_\delta(\mathcal{R}_\alpha^*)(\delta\xi)\Big\|^p_{L^p(\delta(\xi+Y))}\Big).
\end{aligned}$$

Then, using equality (48), estimates (13), $(6)_3$ and the above one, after summation over $\xi\in\Xi_\delta$ we obtain

$$\Big\|\frac{\partial\mathcal{U}_3}{\partial x_\alpha}+\mathcal{R}_\alpha^*\Big\|_{L^p(\omega)}\leq\frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)}.\tag{15}$$

We have $\mathfrak{r}_\alpha=\mathcal{R}_\alpha^*+\dfrac{\partial\mathcal{U}_3}{\partial x_\alpha}$ (see (11)), so

$$\|\mathfrak{r}_\alpha\|_{L^p(\omega)}\leq\frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)}.$$

Observe that $\nabla\mathfrak{r}_\alpha=\nabla\mathcal{R}_\alpha^*+\nabla\dfrac{\partial\mathcal{U}_3}{\partial x_\alpha}$, this leads to the estimates of $\nabla\mathfrak{r}$ using $(12)_2$ and $(6)_3$.

Step 3. We prove $(12)_4$.

For a moment, set $\mathfrak{u}=\mathcal{U}_3^*-\mathcal{U}_3$. Thus

$$\frac{\partial\mathcal{U}_3^*}{\partial x_\alpha}+\mathcal{R}_\alpha^*=\mathfrak{r}_\alpha+\frac{\partial\mathfrak{u}}{\partial x_\alpha}$$

which gives

$$\Big\|\frac{\partial\mathfrak{u}}{\partial x_\alpha}+\mathfrak{r}_\alpha\Big\|_{L^p(\omega)}\leq\frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)}.$$

Using $(12)_3$, this leads to

$$\|\nabla\mathfrak{u}\|_{L^p(\omega)}\leq\frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)}.\tag{16}$$

The above together with $(6)_2$ and equality $(11)_2$ yield the estimate of $\nabla\overline{u}$.

First, consider the function $\mathcal{U}_3^\diamondsuit$ defined in the cell $\delta(\xi+Y)$ by

$$\mathcal{U}_3^\diamondsuit(x')=\mathcal{U}_3^*(x')-\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3^*)(\delta\xi)+\widetilde{\mathcal{M}_\delta}(\mathcal{R}_1^*)(\delta\xi)(x_1-\delta\xi_1-\delta/2)+\widetilde{\mathcal{M}_\delta}(\mathcal{R}_2^*)(\delta\xi)(x_2-\delta\xi_2-\delta/2).$$

Applying the Poincaré-Wirtinger inequality leads to

$$\begin{aligned}\|\nabla \mathcal{U}_3^{\Diamond}\|^p_{L^p(\delta(\xi+Y))} &\le C\sum_{\alpha=1}^2 \Big\|\frac{\partial \mathcal{U}_3^*}{\partial x_\alpha}+\widetilde{\mathcal{M}_\delta}(\mathcal{R}^*_\alpha)(\delta\xi)\Big\|^p_{L^p(\delta(\xi+Y))}\\ &\le C\sum_{\alpha=1}^2\Big(\Big\|\frac{\partial \mathcal{U}_3^*}{\partial x_\alpha}+\mathcal{R}^*_\alpha\Big\|^p_{L^p(\delta(\xi+Y))}+\delta^p\|\nabla\mathcal{R}^*_\alpha\|^p_{L^p(\delta(\xi+Y))}\Big)\end{aligned}$$

and

$$\|\mathcal{U}_3^{\Diamond}\|^p_{L^p(\delta(\xi+Y))}\le C\delta^p\sum_{\alpha=1}^2\Big(\Big\|\frac{\partial \mathcal{U}_3^*}{\partial x_\alpha}+\mathcal{R}^*_\alpha\Big\|^p_{L^p(\delta(\xi+Y))}+\delta^p\|\nabla\mathcal{R}^*_\alpha\|^p_{L^p(\delta(\xi+Y))}\Big).$$

Then, consider the function $\mathcal{U}_3^{\triangle}$ defined in the cell $\delta(\xi+Y)$ by

$$\mathcal{U}_3^{\triangle}(x')=\mathcal{U}_3(x')-\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3)(\delta\xi)+\widetilde{\mathcal{M}_\delta}(\mathcal{R}^*_1)(\delta\xi)(x_1-\delta\xi_1-\delta/2)+\widetilde{\mathcal{M}_\delta}(\mathcal{R}^*_2)(\delta\xi)(x_2-\delta\xi_2-\delta/2).$$

Again, thanks to the Poincaré-Wirtinger inequality and (14)-(15) we have

$$\|\mathcal{U}_3^{\triangle}\|^p_{L^p(\delta(\xi+Y))}\le C\delta^p\sum_{\alpha=1}^2\Big(\Big\|\frac{\partial \mathcal{U}_3}{\partial x_\alpha}+\mathcal{R}^*_\alpha\Big\|^p_{L^p(\delta(\xi+\mathcal{Z}))}+\delta^p\|\nabla\mathcal{R}^*_\alpha\|^p_{L^p(\delta(\xi+\mathcal{Z}))}\Big).$$

The above estimates lead to (observe that $\mathfrak{u}=\mathcal{U}_3^*-\mathcal{U}_3=\mathcal{U}_3^{\Diamond}-\mathcal{U}_3^{\triangle}+\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3^*-\mathcal{U}_3)(\delta\xi)$ in $\delta(\xi+Y)$)

$$\Big\|\mathfrak{u}-\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3^*-\mathcal{U}_3)(\delta\xi)\Big\|^p_{L^p(\delta(\xi+Y))}\le C\delta^p\sum_{\alpha=1}^2\Big(\Big\|\frac{\partial \mathcal{U}_3^*}{\partial x_\alpha}+\mathcal{R}^*_\alpha\Big\|^p_{L^p(\delta(\xi+\mathcal{Z}))}+\delta^p\|\nabla\mathcal{R}^*_\alpha\|^p_{L^p(\delta(\xi+\mathcal{Z}))}\Big).\tag{17}$$

It remains to estimate $\Big\|\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3^*-\mathcal{U}_3)(\delta\xi)\Big\|^p_{L^p(\delta(\xi+Y))}$. From equality (49) we obtain

$$\begin{aligned}\delta^2\Big|\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3^*-\mathcal{U}_3)(\delta\xi)\Big|^p\le C\delta^2\Big(&\Big|\frac{\sum_{k=0}^1\sum_{\ell=0}^1\mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi+\delta(k\mathbf{e}_1+\ell\mathbf{e}_2)-4\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3^*)(\delta\xi)}{4}\Big|^p\\ &+\delta^{2p}\Big|\frac{\sum_{\ell=0}^1\mathcal{M}_\delta(\mathcal{R}_1^*)(\delta\xi+\delta\mathbf{e}_1+\delta\ell\mathbf{e}_2)-\widetilde{\mathcal{M}_\delta}(\mathcal{R}^*_1)(\delta\xi+\delta\ell\mathbf{e}_2)}{24\delta}\Big|^p\\ &+\delta^{2p}\Big|\frac{\sum_{k=0}^1\mathcal{M}_\delta(\mathcal{R}_2^*)(\delta\xi+\delta\mathbf{e}_2+\delta k\mathbf{e}_1)-\widetilde{\mathcal{M}_\delta}(\mathcal{R}^*_2)(\delta\xi+\delta k\mathbf{e}_1)}{24\delta}\Big|^p\Big).\end{aligned}\tag{18}$$

The third estimate in (46) implies

$$\begin{aligned}&\delta^{2p+2}\Big|\frac{\sum_{\ell=0}^1\mathcal{M}_\delta(\mathcal{R}_1^*)(\delta\xi+\delta\mathbf{e}_1+\delta\ell\mathbf{e}_2)-\widetilde{\mathcal{M}_\delta}(\mathcal{R}^*_1)(\delta\xi+\delta\ell\mathbf{e}_2)}{24\delta}\Big|^p\\ &+\delta^{2p+2}\Big|\frac{\sum_{k=0}^1\mathcal{M}_\delta(\mathcal{R}_2^*)(\delta\xi+\delta\mathbf{e}_2+\delta k\mathbf{e}_1)-\widetilde{\mathcal{M}_\delta}(\mathcal{R}^*_2)(\delta\xi+\delta k\mathbf{e}_1)}{24\delta}\Big|^p\Big)\le\delta^{2p}\sum_{\alpha=1}^2\|\nabla\mathcal{R}^*_\alpha\|^p_{L^p(\delta(\xi+\mathcal{Z}))}.\end{aligned}\tag{19}$$

Now, observe that

$$\begin{aligned}&\sum_{k=0}^1\sum_{\ell=0}^1\mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi+\delta(k\mathbf{e}_1+\ell\mathbf{e}_2))-4\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3^*)(\delta\xi)\\ =&\big(\mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi+\delta\mathbf{e}_1+\delta\mathbf{e}_2)-2\mathcal{M}_\delta(\mathcal{U}_3^*)\Big(\delta\xi+\frac{\delta}{2}\mathbf{e}_1+\delta\mathbf{e}_2\Big)+\mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi+\delta\mathbf{e}_2)\big)\\ +&\big(\mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi+\delta\mathbf{e}_1)-2\mathcal{M}_\delta(\mathcal{U}_3^*)\Big(\delta\xi+\frac{\delta}{2}\mathbf{e}_1\Big)+\mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi)\big)\\ +2&\Big(\mathcal{M}_\delta(\mathcal{U}_3^*)\Big((\delta\xi+\frac{\delta}{2}\mathbf{e}_1+\delta\mathbf{e}_2\Big)-2\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3^*)(\delta\xi)\big)+\mathcal{M}_\delta(\mathcal{U}_3^*)\Big(\delta\xi+\frac{\delta}{2}\mathbf{e}_1\Big)\Big)\end{aligned}$$

Remind that $\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3^*)(\delta\xi)\big) = \mathcal{M}_\delta(\mathcal{U}_3^*)\Big(\delta\xi + \dfrac{\delta}{2}\mathbf{e}_1 + \dfrac{\delta}{2}\mathbf{e}_2\Big)$. Thanks to $(46)_2$ we get

$$
\begin{aligned}
&\delta^2\Big|\sum_{k=0}^{1}\sum_{\ell=0}^{1}\mathcal{M}_\delta(\mathcal{U}_3^*)(\delta\xi+\delta(k\mathbf{e}_1+\ell\mathbf{e}_2)) - 4\widetilde{\mathcal{M}_\delta}(\mathcal{U}_3^*)(\delta\xi)\Big|^p\\
\le & C\delta^p\sum_{\alpha=1}^{2}\Big(\Big\|\frac{\partial\mathcal{U}_3^*}{\partial x_\alpha}+\mathcal{R}_\alpha^*\Big\|^p_{L^p(\delta(\xi+\mathcal{Z}))} + \delta^p\|\nabla\mathcal{R}_\alpha^*\|^p_{L^p(\delta(\xi+\mathcal{Z}))}\Big).
\end{aligned}
\tag{20}
$$

So, (17)-(18)-(19) and (20) after summation over $\xi\in\Xi_\delta$ lead to the estimate of $\|\mathfrak{u}\|_{L^p(\omega)}$

$$
\|\mathfrak{u}\|_{L^p(\omega)} \le C\delta^{1-1/p}\|e(u)\|_{L^p(\Omega_\delta)}. \tag{21}
$$

Using the above, $(6)_1$ and equality $(11)_2$ completes the proof of $(12)_4$. □

Corollary 1. We have

$$
\|u-U_{KL}\|_{L^p(\Omega_\delta)} \le C\delta\|e(u)\|_{L^p(\Omega_\delta)},\qquad \|\nabla(u-U_{KL})\|_{L^p(\Omega_\delta)} \le C\|e(u)\|_{L^p(\Omega_\delta)}. \tag{22}
$$

The constants do not depend on δ.

Proof. Estimates (22) are the immediate consequences of (12). □

Remark 2. It worth to note that the Kirchhoff-Love displacement U_{KL} is close to the initial displacement u (see (22)), but we cannot replace u by U_{KL} in an elasticity problem. Shear and warping, even though much smaller than the membrane displacement and the bending, they can not be neglected. If the plate is made up of a homogeneous and isotropic material with Lamé's constants λ and μ, at the limit in the bending or stretching systems, these constants are replaced by the Young modulus E and the Poisson coefficient ν, this is due to the limit shearing and warping (see e.g.[9,10] or the proof of Theorem 4).

Lemma 1. If the plate is clamped on Γ_δ then we have $\mathcal{U}_1 = \mathcal{U}_2 = 0$ a.e. on γ (see Remark 1 and equalities (8)). Moreover, we have

$$
\|\mathcal{U}_3\|_{L^p(\gamma)} \le C\delta^{1-2/p}\|e(u)\|_{L^p(\Omega_\delta)},\qquad \|\nabla\mathcal{U}_3\|_{L^p(\gamma)} \le \frac{C}{\delta^{2/p}}\|e(u)\|_{L^p(\Omega_\delta)}. \tag{23}
$$

The constants do not depend on δ.

Proof. The boundary of ω being Lipschitz, so from (16) and (21) we have the trace result

$$
\|\mathfrak{u}\|^p_{L^p(\partial\omega)} \le \frac{C}{\delta}\|\mathfrak{u}\|^p_{L^p(\omega)} + C\delta^{p-1}\|\nabla\mathfrak{u}\|^p_{L^p(\omega)} \le C\delta^{p-2}\|e(u)\|^p_{L^p(\Omega_\delta)}.
$$

Remind that $\mathfrak{u} = \mathcal{U}_3^* - \mathcal{U}_3$ and $\mathcal{U}_3^* = 0$ a.e. on γ, thus we get $(23)_1$. Similarly, $(12)_3$ yields

$$
\|\mathfrak{r}\|_{L^p(\partial\omega)} \le \frac{C}{\delta^{2/p}}\|e(u)\|_{L^p(\Omega_\delta)}.
$$

Since $\mathfrak{r}_\alpha = \mathcal{R}_\alpha^* + \dfrac{\partial\mathcal{U}_3}{\partial x_\alpha}$ and $\mathcal{R}_\alpha^* = 0$ a.e. on γ, this gives $(23)_2$. □

Remark 3. If γ is an open subset of $\partial\omega$ with a finite number of connected components then we can construct $\mathcal{U}_3$ such that

$$
\mathcal{U}_3 = 0,\qquad \nabla\mathcal{U}_3 = 0\quad \text{a.e. on } \gamma. \tag{24}
$$

Indeed, proceeding as in[24] Lemma 12.1, we can prove that

$$
\|\mathcal{R}^*\|_{L^p(\gamma_\delta^{bl})} \le C\delta\|\nabla\mathcal{R}^*\|_{L^p(\omega)} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)}.
$$

where

$$
\gamma_\delta^{bl} \doteq \big\{x'\in\omega \mid \operatorname{dist}(x',\gamma) < 3\delta\big\}.
$$

Then, we obtain

$$
\|\nabla\mathcal{U}_3^*\|_{L^p(\gamma_\delta^{bl})} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)}.
$$

Hence

$$
\|\mathcal{U}_3^*\|_{L^p(\gamma_\delta^{bl})} \le C\delta^{1-1/p}\|e(u)\|_{L^p(\Omega_\delta)}.
$$

Now, in the cell $\delta(\xi+\overline{Y})$ with $\mathrm{dist}(\delta\xi,\gamma)\le\sqrt{2}\delta$, we replace the values of $A,\ B,\ C$ given by (9) by

$$A=(0,0,0,0),\qquad B=(0,0,0,0),\qquad C=(0,0,0,0).$$

So, the estimates in Theorem 2 are still valid and moreover we have (24).

As a consequence of the above Theorem 2 and Lemma 1, one has

Proposition 2 (Korn type inequalities)**.** Let u be a displacement in $W^{1,p}(\Omega_\delta)$, $1<p<\infty$. Assume the plate clamped on Γ_δ. Then, we have

$$\begin{aligned}
&\|u_1\|_{L^p(\Omega_\delta)}+\|u_2\|_{L^p(\Omega_\delta)}+\delta\|u_3\|_{L^p(\Omega_\delta)}\le C\|e(u)\|_{L^p(\Omega_\delta)},\\
&\sum_{\alpha,\beta=1}^{2}\Big\|\frac{\partial u_\beta}{\partial x_\alpha}\Big\|_{L^p(\Omega_\delta)}+\Big\|\frac{\partial u_3}{\partial x_3}\Big\|_{L^p(\Omega_\delta)}\le C\|e(u)\|_{L^p(\Omega_\delta)},\\
&\sum_{\alpha=1}^{2}\Big(\Big\|\frac{\partial u_3}{\partial x_\alpha}\Big\|_{L^p(\Omega_\delta)}+\Big\|\frac{\partial u_\alpha}{\partial x_3}\Big\|_{L^p(\Omega_\delta)}\Big)\le\frac{C}{\delta}\|e(u)\|_{L^p(\Omega_\delta)}.
\end{aligned}\tag{25}$$

The constants do not depend on δ.

Proof. First, we decompose u as (10). Then, $(12)_1$, the clamped condition satisfied by $\mathcal{U}_1,\mathcal{U}_2$ and the $2D$-Korn inequality lead to

$$\|\mathcal{U}_1\|_{W^{1,p}(\omega)}+\|\mathcal{U}_2\|_{W^{1,p}(\omega)}\le\frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)}.$$

We recall the following classical result: there exists a strictly positive constant C (which only depends on p, $\partial\omega$ and γ) such that

$$\forall\Phi\in W^{1,p}(\omega),\qquad\|\Phi\|_{W^{1,p}(\omega)}\le C\big(\|\nabla\Phi\|_{L^p(\omega)}+\|\Phi\|_{L^p(\gamma)}\big).$$

Estimate $(12)_2$ together with (23) and the above yield

$$\|\nabla\mathcal{U}_3\|_{W^{1,p}(\omega)}\le\frac{C}{\delta^{1+1/p}}\|e(u)\|_{L^p(\Omega_\delta)}\quad\Rightarrow\quad\|\mathcal{U}_3\|_{W^{2,p}(\omega)}\le\frac{C}{\delta^{1+1/p}}\|e(u)\|_{L^p(\Omega_\delta)}.$$

Then, the estimates of the proposition are the consequences of those in $(12)_{3,4}$ and those above. □

4 | ASYMPTOTIC BEHAVIOR OF A SEQUENCE OF DISPLACEMENTS

First, we recall the definition of the dimension reduction operator.

Definition 2. For ϕ measurable function on Ω_δ, the dimension reduction operator $\Pi_\delta(\phi)$ is defined as follows:

$$\Pi_\delta(\phi)(x_1,x_2,X_3)=\phi(x_1,x_2,\delta X_3)\qquad\text{for a.e. }(x_1,x_2,X_3)\in\Omega.$$

$\Pi_\delta(\phi)$ is a measurable function on $\Omega\doteq\omega\times(-1,1)$.

We easily check that

1. for any $\phi\in L^p(\Omega_\delta)$, $1\le p\le\infty$
$$\|\Pi_\delta(\phi)\|_{L^p(\Omega)}=\frac{1}{\delta^{1/p}}\|\phi\|_{L^p(\Omega_\delta)},\tag{26}$$
2. for any $\phi\in W^{1,p}(\Omega_\delta)$, $1\le p\le\infty$
$$\frac{\partial\Pi_\delta(\phi)}{\partial x_1}=\Pi_\delta\Big(\frac{\partial\phi}{\partial x_1}\Big),\qquad\frac{\partial\Pi_\delta(\phi)}{\partial x_2}=\Pi_\delta\Big(\frac{\partial\phi}{\partial x_2}\Big),\qquad\frac{\partial\Pi_\delta(\phi)}{\partial X_3}=\delta\Pi_\delta\Big(\frac{\partial\phi}{\partial x_3}\Big).\tag{27}$$

Let u be a displacement belonging to $W^{1,p}(\Omega_\delta)$, decomposed as (10).
The strain tensor of u is given by the following 3×3 symmetric matrix defined a.e. in Ω_δ by:

$$e(u)=\begin{pmatrix}
e_{11}(\mathcal{U}_m)-x_3\dfrac{\partial^2\mathcal{U}_3}{\partial x_1^2}+x_3e_{11}(\mathfrak{r})+e_{11}(\overline{u}) & * & *\\
e_{12}(\mathcal{U}_m)-x_3\dfrac{\partial\mathcal{U}_3}{\partial x_1\partial x_2}+x_3e_{12}(\mathfrak{r})+e_{12}(\overline{u}) & e_{22}(\mathcal{U}_m)-x_3\dfrac{\partial^2\mathcal{U}_3}{\partial x_2^2}+x_3e_{22}(\mathfrak{r})+e_{22}(\overline{u}) & *\\
\dfrac{1}{2}\mathfrak{r}_1+e_{13}(\overline{u}) & \dfrac{1}{2}\mathfrak{r}_2+e_{23}(\overline{u}) & e_{33}(\overline{u})
\end{pmatrix}\tag{28}$$

where $\mathfrak{r} = \mathfrak{r}_1\mathbf{e}_1 + \mathfrak{r}_2\mathbf{e}_2$, $\mathcal{U}_m = \mathcal{U}_1\mathbf{e}_1 + \mathcal{U}_2\mathbf{e}_2$.

For every $\big(\Phi_m, \Phi_3, \psi, \overline{\Phi}\big) \in W^{1,p}(\omega)^2 \times W^{2,p}(\omega) \times W^{1,p}(\omega)^2 \times L^p(\omega; W^{1,p}(-1,1))^3$ we denote

$$E(\Phi_m, \Phi_3, \psi, \overline{\Phi}) = \begin{pmatrix} e_{11}(\Phi_m) - X_3\dfrac{\partial^2\Phi_3}{\partial x_1^2} & * & * \\ e_{12}(\Phi_m) - X_3\dfrac{\partial\Phi_3}{\partial x_1\partial x_2} & e_{22}(\Phi_m) - X_3\dfrac{\partial^2\Phi_3}{\partial x_2^2} & * \\ \dfrac{1}{2}\psi_1 + \dfrac{1}{2}\dfrac{\partial\overline{\Phi}_1}{\partial X_3} & \dfrac{1}{2}\psi_2 + \dfrac{1}{2}\dfrac{\partial\overline{\Phi}_2}{\partial X_3} & \dfrac{\partial\overline{\Phi}_3}{\partial X_3} \end{pmatrix} \tag{29}$$

where $\Phi_m = \Phi_1\mathbf{e}_1 + \Phi_2\mathbf{e}_2$, $\psi = \psi_1\mathbf{e}_1 + \psi_2\mathbf{e}_2$.

Theorem 3. Let $\{u_\delta\}_\delta$ be a sequence of displacement belonging to $W^{1,p}(\Omega_\delta)$, $1 < p < \infty$, decomposed as (10). Suppose the plate clamped on Γ_δ and

$$\|e(u_\delta)\|_{L^2(\Omega_\delta)} \le C\delta^{2+1/p}$$

where the constant does not depend on δ. Then, there exist a subsequence of $\{\delta\}$, still denoted $\{\delta\}$ and $\mathcal{U}_m = \mathcal{U}_1\mathbf{e}_1 + \mathcal{U}_2\mathbf{e}_2 \in W^{1,p}(\omega)^2$, $\mathcal{U}_3 \in W^{2,p}(\omega)$, $\mathfrak{r} \in L^p(\omega)^2$ and $\overline{U} \in L^p(\omega; W^{1,p}(-1,1))^3$, such that

$$\begin{aligned} &\frac{1}{\delta^2}\mathcal{U}_{m,\delta} \rightharpoonup \mathcal{U}_m \quad \text{weakly in } W^{1,p}(\omega)^2,\\ &\frac{1}{\delta}\mathcal{U}_{3,\delta} \rightharpoonup \mathcal{U}_3 \quad \text{weakly in } W^{2,p}(\omega). \end{aligned} \tag{30}$$

$\mathcal{U}_m$, $\mathcal{U}_3$ satisfy the following boundary conditions:

$$\mathcal{U}_m = 0 \quad \text{a.e. on } \gamma, \qquad \mathcal{U}_3 = 0, \ \ \nabla\mathcal{U}_3 = 0 \quad \text{a.e. on } \gamma.$$

We also have ($\alpha \in \{1,2\}$)

$$\begin{aligned} &\frac{1}{\delta^3}\Pi_\delta(\overline{u}_\delta) \rightharpoonup \overline{U} \quad \text{weakly in } L^p(\omega; W^{1,p}(-1,1))^3,\\ &\frac{1}{\delta^2}\Pi_\delta\Big(\frac{\partial\overline{u}_\delta}{\partial x_\alpha}\Big) \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega)^3,\\ &\frac{1}{\delta^2}\mathfrak{r}_\delta \rightharpoonup \mathfrak{r} \quad \text{weakly in } L^p(\omega)^2, \qquad \frac{1}{\delta}\nabla\mathfrak{r}_\delta \rightharpoonup 0 \quad \text{weakly in } L^p(\omega)^4 \end{aligned} \tag{31}$$

and

$$\begin{aligned} &\frac{1}{\delta^2}\Pi_\delta(u_{\alpha,\delta}) \rightharpoonup \mathcal{U}_1 - X_3\frac{\partial\mathcal{U}_3}{\partial x_\alpha} \quad \text{weakly in } L^p(\omega; W^{1,p}(-1,1)),\\ &\frac{1}{\delta}\Pi_\delta(u_{3,\delta}) \longrightarrow \mathcal{U}_3 \quad \text{strongly in } L^p(\omega; W^{1,p}(-1,1)). \end{aligned} \tag{32}$$

Moreover

$$\frac{1}{\delta^2}\Pi_\delta\big(e(u_\delta)\big) \rightharpoonup E\big(\mathcal{U}_m, \mathcal{U}_3, \mathfrak{r}, \overline{U}\big) \quad \text{weakly in } L^p(\Omega)^6. \tag{33}$$

Proof. Convergences (30)-(31)-(33) are the immediate consequences of the estimates (12), the ones in Proposition 2 and the properties (26)-(27) of the operator Π_δ. Convergences (32) come from those in (30)-(31) and again the properties of the operator Π_δ. □

The limit warping $\overline{U}$ satisfies ($\alpha \in \{1,2\}$)

$$\int_{-1}^{1} \overline{U}_\alpha(\cdot, X_3)dX_3 = \int_{-1}^{1} \overline{U}_\alpha(\cdot, X_3)X_3 dX_3 = 0 \quad \text{a.e. in } \omega.$$

We denote $\mathfrak{W}_p$ the following subspace of $L^p(\omega; W^{1,p}(-1,1))^3$:

$$\mathfrak{W}_p \doteq \Big\{ V \in L^p(\omega; W^{1,p}(-1,1))^3 \mid \int_{-1}^{1} V_\alpha(\cdot, X_3)dX_3 = \int_{-1}^{1} V_\alpha(\cdot, X_3)X_3 dX_3 = 0 \quad \text{a.e. in } \omega \ \ \alpha \in \{1,2\} \Big\}.$$

5 | A PARTICULAR LOADING OF THE PLATE

For simplicity we assume that the plate is made of a homogeneous and isotropic material whose Lamé constants are λ and μ. We also assume that the plate is clamped on its lateral boundary.

In this section we want to investigate a plate loaded with applied body and surface forces, these forces are chosen so that they do not see the Kirchhoff-Love displacements.

We denote
$$H^1_{\Gamma_\delta}(\Omega_\delta) \doteq \big\{\phi \in H^1(\Omega_\delta) \mid \phi = 0 \ \text{ a.e. on } \partial\omega_\delta \times (-\delta, \delta)\big\},$$
$$a_{ijkl} \doteq \lambda \boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{kl} + \mu\big(\boldsymbol{\delta}_{ik}\boldsymbol{\delta}_{jl} + \boldsymbol{\delta}_{il}\boldsymbol{\delta}_{jk}\big), \qquad \{i, j, k, l\} \in \{1, 2, 3\}^4$$
where $\boldsymbol{\delta}_{ij}$ is the Kronecker symbol.

We recall that there exists a strictly positive constant C such that
$$C|||\zeta|||^2 \le a_{ijkl}\zeta_{ij}\zeta_{kl} \quad \text{for all } 3 \times 3 \text{ symmetric matrices } \zeta \tag{34}$$
where $|||\cdot|||$ is the Frobenius norm.

We denote
$$\sigma_{ij}(v) = a_{ijkl}e_{ij}(v) \qquad \forall v \in H^1(\Omega_\delta).$$
The 3×3 symmetric matrix σ whose entries are the $\sigma_{ij}(v)$ is the stress tensor of v.

We consider the following elasticity problem given in the variational form:
$$\begin{cases} \text{Find } u_\delta \in H^1_{\Gamma_\delta}(\Omega_\delta)^3 \text{ such that } \forall v \in H^1_{\Gamma_\delta}(\Omega_\delta)^3, \\ \displaystyle\int_{\Omega_\delta} \sigma_{ij}(u_\delta)e_{ij}(v)\, dx = \int_{\Omega_\delta} F_\delta \cdot v\, dx + \int_{\partial\Omega_\delta^\pm} G_\delta^\pm \cdot v\, dx' \end{cases} \tag{35}$$

where F_δ belongs to $L^2(\Omega_\delta)^3$, $G_\delta^\pm \in L^2(\omega)^2$ and $\partial\Omega_\delta^\pm \doteq \omega \times \{\pm\delta\}$. The existence and uniqueness of the solution to problem (35) is a classical result.

Now, we suppose that the applied forces are given by
$$F_\delta(x) = f_\delta(x') \quad \text{for a.e. } \ x \in \Omega_\delta, \quad f_\delta \in L^2(\omega)^3.$$
$$G_\delta^+(x') = -G_\delta^-(x') = g_{\delta,1}(x')\mathbf{e}_1 + g_{\delta,2}(x')\mathbf{e}_2, \ \text{ for a.e. } \ x' \in \omega, \quad g_{\delta,1},\ g_{\delta,2} \in L^2(\omega).$$
These forces satisfiy
$$\int_{\Omega_\delta} F_\delta \cdot V_{KL}\, dx + \int_{\partial\Omega_\delta^\pm} G_\delta^\pm \cdot V_{KL}\, dx' = 0$$
for every Kirchhoff-Love displacement $V_{KL} = \begin{pmatrix} \mathcal{V}_1 - x_3\dfrac{\partial \mathcal{V}_3}{\partial x_1} \\ \mathcal{V}_2 - x_3\dfrac{\partial \mathcal{V}_3}{\partial x_2} \\ \mathcal{V}_3 \end{pmatrix}$ belonging to $H^1_{\Gamma_\delta}(\Omega_\delta)^3$.

This first leads to $f_{\delta,1} = f_{\delta,2} = 0$ and then
$$2\delta\int_\omega f_{\delta,3}(x')\mathcal{V}_3(x')dx' - \int_\omega 2\delta\Big(g_{\delta,1}(x')\frac{\partial \mathcal{V}_3}{\partial x_1}(x') + g_{\delta,2}(x')\frac{\partial \mathcal{V}_3}{\partial x_2}(x')\Big)dx' = 0, \qquad \forall \mathcal{V}_3 \in H^2_0(\omega).$$
Hence
$$f_{\delta,3} + \frac{\partial g_{\delta,1}}{\partial x_1} + \frac{\partial g_{\delta,2}}{\partial x_2} = 0 \qquad \text{in } \ H^{-1}(\omega).$$
Let g_1, g_2 be two functions in $H^1(\omega)$. We choose ($\alpha \in \{1, 2\}$)
$$g_{\delta,\alpha} = \delta^2 g_\alpha, \qquad f_3 = -\text{div}(g) = -\Big(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2}\Big), \qquad f_{\delta,3} = \delta^2 f_3.$$

So, for every admissible displacement $u \in H^1_{\Gamma_\delta}(\Omega_\delta)^3$ decomposed as (10), we have

$$
\begin{aligned}
&\int_{\Omega_\delta} F_\delta \cdot u\,dx + \int_{\partial\Omega^\pm_\delta} G^\pm_\delta \cdot u\,dx' \\
=&\delta^2 \int_{\Omega_\delta} f_3 \overline{u}_3 dx + 2\delta^3 \int_\omega g_\alpha \mathfrak{r}_\alpha dx' + \delta^2 \int_\omega g_\alpha \big(\overline{u}_\alpha(x',\delta) - \overline{u}_\alpha(x',-\delta)\big) dx' \\
=&\delta^2 \int_{\Omega_\delta} g_\alpha \frac{\partial \overline{u}_3}{\partial x_\alpha}\,dx + \delta^2 \int_{\Omega_\delta} g_\alpha \mathfrak{r}_\alpha dx + \delta^2 \int_{\Omega_\delta} g_\alpha \frac{\partial \overline{u}_\alpha}{\partial x_3}\,dx \\
=&2\delta^2 \int_{\Omega_\delta} g_\alpha e_{\alpha 3}(u)\,dx.
\end{aligned}
\tag{36}
$$

As a consequence of the above equality and estimates (12) the solution u_δ to problem (35) satisfies

$$
\|e(u_\delta)\|_{L^2(\Omega_\delta)} \leq C\delta^{5/2}
$$

where the constant C does not depend on δ.

Proposition 3. Let u_δ be the solution to problem (35). We decompose u_δ as (10). Then, we first have

$$
\begin{aligned}
\frac{1}{\delta^2}\mathcal{U}_{m,\delta} &\longrightarrow 0 \quad \text{strongly in} \;\; H^1(\omega),\\
\frac{1}{\delta}\mathcal{U}_{3,\delta} &\longrightarrow 0 \quad \text{strongly in} \;\; H^2(\omega),\\
\frac{1}{\delta^3}\Pi_\delta(\overline{u}_{\delta,1}) &\rightharpoonup 0 \quad \text{weakly in} \;\; L^2(\omega; H^1(-1,1)),\\
\frac{1}{\delta^3}\Pi_\delta(\overline{u}_{\delta,i}) &\longrightarrow 0 \quad \text{strongly in} \;\; L^2(\omega; H^1(-1,1)), \quad i \in \{2,3\}.
\end{aligned}
$$

Moreover, there exist $\mathfrak{r}_1,\ \mathfrak{r}_2 \in L^2(\omega)$, such that ($\alpha \in \{1,2\}$)

$$
\frac{1}{\delta^2}\mathfrak{r}_{\alpha,\delta} \rightharpoonup \frac{1}{\mu} g_\alpha \quad \text{weakly in} \;\; L^2(\omega).
$$

Proof. Theorem 3 gives a subsequence of $\{\delta\}$, still denoted $\{\delta\}$ and $\mathcal{U}_m \in H^1_0(\omega)^2$, $\mathcal{U}_3 \in H^2_0(\omega)$, $\overline{U} \in \mathfrak{W}_2$ and $\mathfrak{r}_1,\ \mathfrak{r}_2 \in L^2(\omega)$ such that convergences (30)-(31)-(32) and (33) hold.

We choose $\Phi_m = \Phi_1 \mathbf{e}_1 + \Phi_2 \mathbf{e}_2 \in H^1_0(\omega)^2$, $\Phi_3 \in H^2_0(\omega)$, $\overline{\Phi} \in \mathfrak{W}_2 \cap H^1_\Gamma(\Omega)^3$ and $\psi_1,\ \psi_2 \in H^1_0(\omega)$ where

$$
H^1_\Gamma(\Omega) \doteq \Big\{ V \in H^1(\Omega)^3 \mid V = 0 \;\text{ a.e. on } \partial\omega \times (-1,1) \Big\}.
$$

We define the test displacement ϕ_δ by

$$
\phi_\delta(\cdot, x_3) = \begin{pmatrix}
\delta^2 \Phi_1 - x_3 \delta \dfrac{\partial \Phi_3}{\partial x_1} + x_3 \delta^2 \psi_1 + x_3 \delta^2 \psi_2 + \delta^3 \overline{\Phi}_1\Big(\cdot, \dfrac{x_3}{\delta}\Big) \\
\delta^2 \Phi_2 - x_3 \delta \dfrac{\partial \Phi_3}{\partial x_2} + \delta^3 \overline{\Phi}_2\Big(\cdot, \dfrac{x_3}{\delta}\Big) \\
\delta \Phi_3 + \delta^3 \overline{\Phi}_2\Big(\cdot, \dfrac{x_3}{\delta}\Big)
\end{pmatrix}.
$$

We have (see (28) for the strain tensor of ϕ_δ)

$$
\frac{1}{\delta^2}\Pi_\delta\big(e(\phi_\delta)\big) \longrightarrow \begin{pmatrix}
e_{11}(\Phi_m) - X_3 \dfrac{\partial^2 \Phi_3}{\partial x_1^2} & * & * \\
e_{12}(\Phi_m) - X_3 \dfrac{\partial \Phi_3}{\partial x_1 \partial x_2} & e_{22}(\Phi_m) - X_3 \dfrac{\partial^2 \Phi_3}{\partial x_2^2} & * \\
\dfrac{1}{2}\psi_1 + \dfrac{1}{2}\dfrac{\partial \overline{\Phi}_1}{\partial X_3} & \dfrac{1}{2}\psi_2 + \dfrac{1}{2}\dfrac{\partial \overline{\Phi}_2}{\partial X_3} & \dfrac{\partial \overline{\Phi}_3}{\partial X_3}
\end{pmatrix} \quad \text{strongly in} \;\; L^2(\Omega)^6.
$$

Now, in (35) we choose this test displacement, we transform the left and right hand sides using Π_δ, divide by δ^5 and pass to the limit. Thanks to (36), we obtain

$$\int_\Omega a_{ijkl} E_{ij}\big(\mathcal{U}_m, \mathcal{U}_3, \mathfrak{r}, \overline{U}\big) E_{kl}\big(\Phi_m, \Phi_3, \psi, \overline{\Phi}\big) dx' dX_3 = 2\int_\omega g_\alpha \psi_\alpha dx' + \int_{\Omega_\delta} g_\alpha \frac{\partial \overline{U}_\alpha}{\partial X_3}\, dx' dX_3.$$

This first gives

$$\begin{aligned}&\int_\Omega a_{iikk} E_{ij}\big(\mathcal{U}_m, \mathcal{U}_3, \mathfrak{r}, \overline{U}\big) E_{kk}\big(\Phi_m, \Phi_3, \psi, \overline{\Phi}\big) dx' dX_3\\ &+\mu \int_\Omega \Big(e_{12}(\mathcal{U}_m) - X_3 \frac{\partial \mathcal{U}_3}{\partial x_1 \partial x_2}\Big)\Big(e_{12}(\Phi_m) - X_3 \frac{\partial \Phi_3}{\partial x_1 \partial x_2}\Big)\Big\} dx' dX_3 = 0\end{aligned} \tag{37}$$

and

$$\mu \int_\Omega \Big(\mathfrak{r}_\alpha + \frac{\partial \overline{U}_\alpha}{\partial X_3}\Big)\Big(\psi_\alpha + \frac{\partial \overline{\Phi}_\alpha}{\partial X_3}\Big) dx' dX_3 = \int_\Omega g_\alpha \Big(\psi_\alpha + \frac{\partial \overline{\Phi}_\alpha}{\partial X_3}\Big)\, dx' dX_3. \tag{38}$$

By density of $\mathfrak{W}_2 \cap H^1_\Gamma(\Omega)^3$ in $\mathfrak{W}_2$ and $H^1_0(\omega)$ in $L^2(\omega)$, the above equalities are still satisfied for every $\overline{\Phi} \in \mathfrak{W}_2$ and $\psi_1,\ \psi_2 \in L^2(\omega)$.

So, from (37) we get $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}_3 = 0$ and $\overline{U}_3 = 0$ (up to a function belonging to $L^2(\omega)$) since (34) and (37) imply

$$\begin{aligned}&\sum_{\alpha,\beta=1}^2 \Big(\big\|e_{\alpha\beta}(\mathcal{U}_m)\big\|^2_{L^2(\Omega)} + \Big\|\frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta}\Big\|^2_{L^2(\Omega)}\Big) + \Big\|\frac{\partial \overline{U}_3}{\partial X_3}\Big\|^2_{L^2(\Omega)}\\ &\le C\Big(\sum_{\alpha,\beta=1}^2 \Big\|e_{\alpha\beta}(\mathcal{U}_m) - X_3 \frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta}\Big\|^2_{L^2(\Omega)} + \Big\|\frac{\partial \overline{U}_3}{\partial X_3}\Big\|^2_{L^2(\Omega)}\Big) \le 0.\end{aligned}$$

Then, (38) gives ($\alpha \in \{1,2\}$)

$$\mathfrak{r}_\alpha = \frac{1}{\mu} g_\alpha \quad \text{a.e. in } \omega, \qquad \overline{U}_\alpha = 0 \quad \text{a.e. in } \Omega.$$

Since the limit problems admit a unique solution, the whole sequences of fields converge towards their limit. As usual we prove the strong convergence of the strain tensor which in turn gives the strong convergences in the proposition. □

The displacement

$$u^{ap}_\delta(x) = \delta^2 x_3\big(\mathfrak{r}_1(x')\mathbf{e}_1 + \mathfrak{r}_2(x')\mathbf{e}_2\big) \qquad \text{for a.e. } x \in \Omega_\delta$$

is an approximation of the solution u_δ to problem (35). Below, we give an error estimate.

Lemma 2. Assume g_1 and $g_2 \in H^1_0(\omega)^1$, then we have

$$\|e(u_\delta - u^{ap}_\delta)\| \le C\delta^{7/2}(\|g_1\|_{H^1(\omega)} + \|g_2\|_{H^1(\omega)}). \tag{39}$$

The constant is independent of δ.

Proof. First observe that under the assumption of the lemma, u^{ap}_δ is an admissible displacement of the plate. We first have

$$\|e_{\alpha\beta}(u^{ap}_\delta)\|_{L^2(\Omega_\delta)} \le C\delta^{7/2}(\|g_1\|_{H^1(\omega)} + \|g_2\|_{H^1(\omega)}) \tag{40}$$

and

$$e_{13}(u^{ap}_\delta) = \frac{\delta^2}{2}\mathfrak{r}_1, \quad e_{23}(u^{ap}_\delta) = \frac{\delta^2}{2}\mathfrak{r}_2, \quad e_{33}(u^{ap}_\delta) = 0.$$

Now, let ϕ be a displacement in $H^1_{\Gamma_\delta}(\Omega_\delta)^3$, from (36) we have

$$4\mu \int_{\Omega_\delta} \big(e_{13}(u^{ap}_\delta)e_{13}(\phi) + e_{23}(u^{ap}_\delta)e_{23}(\phi)\big) dx = 2\delta^2 \int_{\Omega_\delta} g_\alpha e_{\alpha 3}(\phi)\, dx.$$

Hence

$$\int_{\Omega_\delta} \sigma_{ij} e_{ij}(u_\delta - u^{ap}_\delta) e_{kl}(\phi)\, dx = -\int_{\Omega_\delta} \sigma_{\alpha\beta}(u^{ap}_\delta) e_{\alpha\beta}(\phi)\, dx \tag{41}$$

which in turn due to (40) give (39). □

[1] If we only assume g_1 and $g_2 \in H^1(\omega)$ we can prove that $\|e(u_\delta - u^{ap}_\delta)\| \le C\delta^3(\|g_1\|_{H^1(\omega)} + \|g_2\|_{H^1(\omega)})$.

Theorem 4. Under the assumption of Lemma 2, we have

$$\begin{aligned}
&\frac{1}{\delta^3}\Pi_\delta(u_{\delta,1}) \to -X_3\frac{\partial \mathcal{U}_3^\diamondsuit}{\partial x_1} + X_3\mathfrak{r}_1 \quad \text{strongly in } L^2(\omega; H^1(-1,1)),\\
&\frac{1}{\delta^3}\Pi_\delta(u_{\delta,2}) \to -X_3\frac{\partial \mathcal{U}_3^\diamondsuit}{\partial x_2} + X_3\mathfrak{r}_2 \quad \text{strongly in } L^2(\omega; H^1(-1,1)),\\
&\frac{1}{\delta^2}\Pi_\delta(u_{\delta,3}) \to \mathcal{U}_3^\diamondsuit \quad \text{strongly in } L^2(\omega; H^1(-1,1))
\end{aligned}$$

where $\mathcal{U}_3^\diamondsuit \in H_0^2(\omega)$ is the unique solution to

$$\int_\omega \Big((1-\nu)\Big(\frac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_\alpha \partial x_\beta} - e_{\alpha\beta}(\mathfrak{r})\Big)\frac{\partial^2 \Phi_3}{\partial x_\alpha \partial x_\beta} + \nu\big(\Delta \mathcal{U}_3^\diamondsuit - e_{\alpha\alpha}(\mathfrak{r})\big)\Delta\Phi_3\Big)dx' = 0, \qquad \forall \Phi_3 \in H_0^2(\omega).$$

$\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ the Poisson coefficient.
Moreover we have the following strong convergence in $L^2(\Omega)^6$:

$$\frac{1}{\delta^3}\Pi_\delta\big(e(u_\delta - u_\delta^{ap})\big) \to X_3\begin{pmatrix} -\dfrac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_1^2} + e_{11}(\mathfrak{r}) & -\dfrac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_1 \partial x_2} + e_{12}(\mathfrak{r}) & 0\\ -\dfrac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_1 \partial x_2} + e_{12}(\mathfrak{r}) & -\dfrac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_2^2} + e_{22}(\mathfrak{r}) & 0\\ 0 & 0 & \dfrac{\lambda}{\lambda+2\mu}\big(\Delta\mathcal{U}_3^\diamondsuit - e_{11}(\mathfrak{r}) - e_{22}(\mathfrak{r})\big)\end{pmatrix}.$$

Proof. We decompose $u_\delta - u_\delta^{ap}$ as (10), we write

$$u_\delta(x) - u_\delta^{ap}(x) = U_{KL,\delta}^\diamondsuit(x) + x_3\mathfrak{r}_\delta^\diamondsuit(x') + \overline{u}_\delta^\diamondsuit(x), \quad \text{for a. e. } x \in \Omega_\delta.$$

Due to Theorems 2 and 3, there exist a subsequence of $\{\delta\}$, still denoted $\{\delta\}$ and $\mathcal{U}_1^\diamondsuit,\ \mathcal{U}_2^\diamondsuit \in H_0^1(\omega)$, $\mathcal{U}_3^\diamondsuit \in H_0^2(\omega)$, $\overline{U}^\diamondsuit \in \mathfrak{W}_2$ and $\mathfrak{r}^\diamondsuit \in L^2(\omega)^2$ such that

$$\begin{aligned}
&\frac{1}{\delta^3}\mathcal{U}_{\delta,m}^\diamondsuit \rightharpoonup \mathcal{U}_m^\diamondsuit \quad \text{weakly in } H_0^1(\omega),\\
&\frac{1}{\delta^2}\mathcal{U}_{\delta,3}^\diamondsuit \rightharpoonup \mathcal{U}_3^\diamondsuit \quad \text{weakly in } H_0^2(\omega),\\
&\frac{1}{\delta^4}\Pi_\delta(\overline{u}_\delta^\diamondsuit) \rightharpoonup \overline{U}^\diamondsuit \quad \text{weakly in } L^2(\omega; H^1(-1,1))^3, \qquad \frac{1}{\delta^3}\Pi_\delta\Big(\frac{\partial \overline{u}_\delta^\diamondsuit}{\partial x_\alpha}\Big) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega)^3,\\
&\frac{1}{\delta^3}\mathfrak{r}_\delta^\diamondsuit \rightharpoonup \mathfrak{r}^\diamondsuit \quad \text{weakly in } L^2(\omega)^2, \qquad \frac{1}{\delta^2}\nabla\mathfrak{r}_\delta^\diamondsuit \rightharpoonup 0 \quad \text{weakly in } L^2(\omega)^2
\end{aligned}$$

and

$$\frac{1}{\delta^3}\Pi_\delta\big(e(u_\delta - u_\delta^{ap})\big) \rightharpoonup E\big(\mathcal{U}_m^\diamondsuit, \mathcal{U}_3^\diamondsuit, \mathfrak{r}^\diamondsuit, \overline{U}^\diamondsuit\big) \quad \text{weakly in } L^p(\Omega)^6. \tag{42}$$

Now, in (41) we choose the test displacement introduced in the proof of Proposition 3, we transform the left and right hand sides using Π_δ, divide by δ^6 and pass to the limit. We obtain

$$\begin{aligned}
&\int_\Omega a_{ijkl}E_{ij}\big(\mathcal{U}_m^\diamondsuit, \mathcal{U}_3^\diamondsuit, \mathfrak{r}^\diamondsuit, \overline{U}^\diamondsuit\big)\, E_{kl}\big(\Phi_m, \Phi_3, \psi, \overline{\Phi}\big)dx'dX_3\\
&= -\int_\Omega a_{\alpha\beta\alpha'\beta'}e_{\alpha\beta}(\mathfrak{r})X_3E_{\alpha'\beta'}\big(\Phi_m, \Phi_3, \psi, \overline{\Phi}\big)dx'dX_3.
\end{aligned} \tag{43}$$

By density of $\mathfrak{W}_2 \cap H_\Gamma^1(\Omega)^3$ in $\mathfrak{W}_2$ and $H_0^1(\omega)$ in $L^2(\omega)$, the above equality is still satisfied for every $\overline{\Phi} \in \mathfrak{W}_2$ and $\psi \in L^2(\omega)^2$. The above equality yields

$$\mu\int_\Omega \Big(\mathfrak{r}_\alpha^\diamondsuit + \frac{\partial \overline{U}_\alpha^\diamondsuit}{\partial X_3}\Big)\Big(\psi_\alpha + \frac{\partial \overline{\Phi}_\alpha}{\partial X_3}\Big)dx'dX_3 = 0$$

Hence $\mathfrak{r}_\alpha^\diamondsuit = 0$ and $\overline{U}_\alpha^\diamondsuit = 0$.
Equation (43) also gives

$$\begin{aligned}
&\int_\Omega \Big\{ \Big[(\lambda+2\mu)\Big(e_{11}(\mathcal{U}_m^\diamondsuit) - X_3\frac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_1^2}\Big) + \lambda\Big(e_{22}(\mathcal{U}_m^\diamondsuit) - X_3\frac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_2^2}\Big) + \lambda\frac{\partial \overline{U}_3^\diamondsuit}{\partial X_3}\Big)\Big]\Big(e_{11}(\Phi_m) - X_3\frac{\partial^2 \Phi_3}{\partial x_1^2}\Big)\\
&+\Big[\lambda\Big(e_{11}(\mathcal{U}_m^\diamondsuit) - X_3\frac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_1^2}\Big) + (\lambda+2\mu)\Big(e_{22}(\mathcal{U}_m^\diamondsuit) - X_3\frac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_2^2}\Big) + \lambda\frac{\partial \overline{U}_3^\diamondsuit}{\partial X_3}\Big)\Big]\Big(e_{22}(\Phi_m) - X_3\frac{\partial^2 \Phi_3}{\partial x_2^2}\Big)\\
&+\Big[\lambda\Big(e_{11}(\mathcal{U}_m^\diamondsuit) - X_3\frac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_1^2}\Big) + \lambda\Big(e_{22}(\mathcal{U}_m^\diamondsuit) - X_3\frac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_2^2}\Big) + (\lambda+2\mu)\frac{\partial \overline{U}_3^\diamondsuit}{\partial X_3}\Big)\Big]\frac{\partial \overline{\Phi}_3}{\partial X_3}\\
&+\mu\Big(e_{12}(\mathcal{U}_m^\diamondsuit) - X_3\frac{\partial \mathcal{U}_3^\diamondsuit}{\partial x_1\partial x_2}\Big)\Big(e_{12}(\Phi_m) - X_3\frac{\partial \Phi_3}{\partial x_1\partial x_2}\Big)\Big\} dx'dX_3 = -\int_\Omega a_{\alpha\beta\alpha'\beta'}e_{\alpha\beta}(\mathfrak{r})X_3E_{\alpha'\beta'}\big(\Phi_m,\Phi_3,\psi,\overline{\Phi}\big)dx'dX_3.
\end{aligned} \tag{44}$$

In (44) we choose $\Phi = 0$, this yields

$$\frac{\partial \overline{U}_3^\diamondsuit}{\partial X_3} = -\frac{\lambda}{\lambda+2\mu}\big(e_{11}(\mathcal{U}_m^\diamondsuit) + e_{22}(\mathcal{U}_m^\diamondsuit) - X_3\Delta\mathcal{U}_3^\diamondsuit\big).$$

Replacing $\dfrac{\partial \overline{U}_3^\diamondsuit}{\partial X_3}$ in (44) leads to

$$\begin{aligned}
&\frac{E}{1-\nu^2}\int_\omega \Big((1-\nu)e_{\alpha\beta}(\mathcal{U}_m^\diamondsuit)e_{\alpha\beta}(\Phi_m) + \nu e_{\alpha\alpha}(\mathcal{U}_m^\diamondsuit)e_{\alpha\alpha}(\Phi_m)\Big)dx'\\
&+\frac{E}{3(1-\nu^2)}\int_\omega \Big((1-\nu)\Big(\frac{\partial^2 \mathcal{U}_3^\diamondsuit}{\partial x_\alpha\partial x_\beta} - e_{\alpha\beta}(\mathfrak{r})\Big)\frac{\partial^2\Phi_3}{\partial x_\alpha\partial x_\beta} + \nu\big(\Delta\mathcal{U}_3^\diamondsuit - e_{\alpha\alpha}(\mathfrak{r})\big)\Delta\Phi_3)\Big)dx' = 0
\end{aligned}$$

where $E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is the Young modulus. Hence, $\mathcal{U}_m^\diamondsuit = 0$ and we get the equation satisfied by $\mathcal{U}_3^\diamondsuit$.
Since the limit problem admits a unique solution, the whole sequences of the different fields converge towards their limit. As usual we prove the strong convergence of the strain tensor which in turn gives the strong convergences in the theorem. ☐

As a consequence of Proposition 3 and Theorem 4, regarding the stress tensors, of u_δ and $u_\delta - u_\delta^{ap}$ we have the following strong convergences in $L^2(\Omega)^6$:

$$\frac{1}{\delta^2}\Pi_\delta\big(\sigma(u_\delta)\big) \longrightarrow \frac{1}{2}\begin{pmatrix} 0 & 0 & g_1\\ 0 & 0 & g_2\\ g_1 & g_2 & 0\end{pmatrix},$$

$$\frac{1}{\delta^3}\Pi_\delta\big(\sigma(u_\delta - u_\delta^{ap})\big) \longrightarrow \frac{E}{1-\nu^2}X_3\begin{pmatrix} -\dfrac{\partial^2\mathcal{U}_3^\diamondsuit}{\partial x_1^2} + e_{11}(\mathfrak{r}) & 2(1-\nu)\Big(-\dfrac{\partial^2\mathcal{U}_3^\diamondsuit}{\partial x_1\partial x_2} + e_{12}(\mathfrak{r})\Big) & 0\\ 2(1-\nu)\Big(-\dfrac{\partial^2\mathcal{U}_3^\diamondsuit}{\partial x_1\partial x_2} + e_{12}(\mathfrak{r})\Big) & -\dfrac{\partial^2\mathcal{U}_3^\diamondsuit}{\partial x_2^2} + e_{22}(\mathfrak{r}) & 0\\ 0 & 0 & 0\end{pmatrix}.$$

6 | CONCLUSION

If we are dealing with a very thin plate, it will be better to replace the decomposition (1) with a shorter one. So, any displacement $u \in W^{1,p}(\Omega_\delta)^3$, $1 < p < \infty$, is also decomposed as

$$u(x) = \underbrace{\begin{pmatrix} \mathcal{U}_1(x') - x_3 \dfrac{\partial \mathcal{U}_3}{\partial x_1}(x') \\ \mathcal{U}_2(x') - x_3 \dfrac{\partial \mathcal{U}_3}{\partial x_2}(x') \\ \mathcal{U}_3(x') \end{pmatrix}}_{\text{Kirchhoff-Love displacement}} + \underbrace{\widetilde{u}(x)}_{\text{residual displacement}} \qquad \text{for a.e. } x \text{ in } \Omega_\delta. \tag{45}$$

The residual displacement is

$$\widetilde{u}(x) = x_3\big(\mathfrak{r}_1(x')\mathbf{e}_1 + \mathfrak{r}_2(x')\mathbf{e}_2\big) + \overline{u}(x) \quad \text{for a.e. } x \text{ in } \Omega_\delta.$$

It satisfies the following two conditions:

$$\int_{-\delta}^{\delta} \widetilde{u}_1(x', x_3)dx_3 = \int_{-\delta}^{\delta} \widetilde{u}_2(x', x_3)dx_3 = 0 \qquad \text{for a.e. } x' \in \omega.$$

As immediate consequence of Theorem 2 we have

Theorem 5. The fields $\mathcal{U}_m = \mathcal{U}_1\mathbf{e}_1 + \mathcal{U}_2\mathbf{e}_2$, $\mathcal{U}_3$ and $\widetilde{u}$ satisfy

$$\mathcal{U}_m \in W^{1,p}(\omega)^2, \quad \mathcal{U}_3 \in W^{2,p}(\omega), \quad \widetilde{u} \in W^{1,p}(\Omega_\delta)^3$$

and the following estimates:

$$\begin{aligned}
&\|e_{\alpha\beta}(\mathcal{U}_m)\|_{L^p(\omega)} \leq \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)},\\
&\Big\|\frac{\partial^2 \mathcal{U}_3}{\partial x_1^2}\Big\|_{L^p(\omega)} + \Big\|\frac{\partial^2 \mathcal{U}_3}{\partial x_2^2}\Big\|_{L^p(\omega)} + \Big\|\frac{\partial^2 \mathcal{U}_3}{\partial x_1 \partial x_2}\Big\|_{L^p(\omega)} \leq \frac{C}{\delta^{1+1/p}}\|e(u)\|_{L^p(\Omega_\delta)},\\
&\|\widetilde{u}\|_{L^p(\Omega_\delta)} + \delta\|\nabla \widetilde{u}\|_{L^p(\Omega_\delta)} \leq C\delta\|e(u)\|_{L^p(\Omega_\delta)}.
\end{aligned}$$

The constants do not depend on δ.

If the plate is clamped on Γ_δ, then the estimates of Lemma 1 are still valid and we have $\mathcal{U}_m = 0$ on γ. Of course, Proposition 2 is also still valid. Proceeding as in [11] the above decomposition (45) can be extended to structures made up of a large number of plates.

7 | A LEMMA

Lemma 3. Let Φ and Ψ two functions belonging to $W^{1,p}\big((0,2\delta)\times(0,\delta)\big)$, $1 \leq p < \infty$. We have

$$\begin{aligned}
&\Big|\frac{\mathcal{M}_\delta(\Phi)(\delta) - \mathcal{M}_\delta(\Phi)(0)}{\delta} + \frac{1}{2}\big(\mathcal{M}_\delta(\Psi)(0) + \mathcal{M}_\delta(\Psi)(\delta)\big)\Big|^p \leq \frac{C}{\delta^2}\Big(\int_0^{2\delta}\int_0^{\delta}\Big|\frac{\partial \Phi}{\partial x} + \Psi\Big|^p dxdy + \delta^p \int_0^{2\delta}\int_0^{\delta}\Big|\frac{\partial \Psi}{\partial x}\Big|^p dxdy\Big),\\
&\Big|\frac{\mathcal{M}_\delta(\Phi)(\delta) - 2\mathcal{M}_\delta(\Phi)(\delta/2) + \mathcal{M}_\delta(\Phi)(0)}{\delta}\Big|^p \leq \frac{C}{\delta^2}\Big(\int_0^{2\delta}\int_0^{\delta}\Big|\frac{\partial \Phi}{\partial x} + \Psi\Big|^p dxdy + \delta^p \int_0^{2\delta}\int_0^{\delta}\Big|\frac{\partial \Psi}{\partial x}\Big|^p dxdy\Big),\\
&\Big|\frac{\mathcal{M}_\delta(\Psi)(\delta) - \mathcal{M}_\delta(\Psi)(0)}{\delta}\Big|^p \leq \frac{C}{\delta^2}\int_0^{2\delta}\int_0^{\delta}\Big|\frac{\partial \Psi}{\partial x}\Big|^p dxdy.
\end{aligned} \tag{46}$$

where for every $\Theta \in L^1\big((0,2\delta)\times(0,\delta)\big)$

$$\mathcal{M}_\delta(\Theta)(t)=\frac{1}{\delta^2}\int_0^\delta\int_0^\delta \Theta(s_1+t,s_2)ds_1ds_2,\qquad t\in[0,\delta].$$

The constants depend only on p.

Proof. *Step 1.* A preliminary result.

Let ϕ and ψ be two functions belonging to $C^1([0,2\delta])$. In this step we prove that ($1\le p<\infty$)

$$\begin{aligned}
&\Big|\frac{m_\delta(\phi)(\delta)-m_\delta(\phi)(0)}{\delta}+m_\delta(\psi)(0)\Big|^p\le\frac{2^{p-1}}{\delta}\Big(\int_0^{2\delta}\Big|\frac{d\phi}{dx}(t)+\psi(t)\Big|^p dt+\delta^p\int_0^{2\delta}\Big|\frac{d\psi}{dx}(t)\Big|^p dt\Big),\\
&\Big|\frac{m_\delta(\phi)(\delta)-2m_\delta(\phi)(\delta/2)+m_\delta(\phi)(0)}{\delta}\Big|^p\le\frac{2^{p+3}}{\delta}\Big(\int_0^{2\delta}\Big|\frac{d\phi}{dx}(t)+\psi(t)\Big|^p dt+\delta^p\int_0^{2\delta}\Big|\frac{d\psi}{dx}(t)\Big|^p dt\Big),\\
&\Big|\frac{m_\delta(\psi)(\delta)-m_\delta(\psi)(0)}{\delta}\Big|^p\le\frac{1}{\delta}\int_0^{2\delta}\Big|\frac{d\psi}{dx}(t)\Big|^p dt
\end{aligned}\tag{47}$$

where for every $\theta\in L^1(0,2\delta)$

$$m_\delta(\theta)(t)=\frac{1}{\delta}\int_0^\delta\theta(t+s)ds,\qquad t\in[0,\delta].$$

We prove $(47)_1$. We have

$$\phi(x+\delta)-\phi(x)=\int_0^\delta\frac{d\phi}{dx}(x+t)dt\qquad\forall x\in[0,\delta].$$

So

$$\int_0^\delta\big(\phi(x+\delta)-\phi(x)\big)dx+\delta\int_0^\delta\psi(x)dx=\int_0^\delta\int_0^\delta\big(\psi(x)-\psi(x+t)\big)dxdt+\int_0^\delta\int_0^\delta\Big(\frac{d\phi}{dx}(x+t)+\psi(x+t)\Big)dxdt.$$

Above, the first termof the RHS is

$$\int_0^\delta\int_0^\delta\big(\psi(x+t)-\psi(x)\big)dxdt=\int_0^\delta\int_0^\delta\int_x^{x+t}\frac{d\psi}{dx}(s)dsdxdt.$$

Hence, using the Holdër inequality

$$\Big|\int_0^\delta\int_0^\delta\big(\psi(x+t)-\psi(t)\big)dt\Big|^p\le\delta^{3p-1}\int_0^{2\delta}\Big|\frac{d\psi}{dx}(s)\Big|^p ds.$$

The second term in the RHS is bounded by

$$\Big|\int_0^\delta\int_0^\delta\Big(\frac{d\phi}{dx}(x+t)+\psi(x+t)\Big)dxdt\Big|^p\le\delta^{2p-1}\int_0^{2\delta}\Big|\frac{d\phi}{dx}(s)+\psi(s)\Big|^p ds$$

Finally, we get $(47)_1$.

We prove $(47)_2$. We have

$$m_\delta(\phi)(\delta)-m_\delta(\phi)(\delta/2)=\frac{1}{\delta}\int_0^\delta\big(\phi(s+\delta)-\phi(s+\delta/2)\big)ds=\frac{1}{\delta}\int_0^\delta\int_{s+\delta/2}^{s+\delta}\frac{d\phi}{dx}(t)dtds,$$

$$m_\delta(\phi)(\delta/2)-m_\delta(\phi)(0)=\frac{1}{\delta}\int_0^\delta\big(\phi(s+\delta/2)-\phi(s)\big)ds=\frac{1}{\delta}\int_0^\delta\int_s^{s+\delta/2}\frac{d\phi}{dx}(t)dtds,$$

Thus

$$\begin{aligned}m_\delta(\phi)(\delta)-2m_\delta(\phi)(\delta/2)+m_\delta(\phi)(0)=&\frac{1}{\delta}\int_0^\delta\int_{s+\delta/2}^{s+\delta}\Big(\frac{d\phi}{dx}(t)+\psi(t)\Big)dtds-\frac{1}{\delta}\int_0^\delta\int_s^{s+\delta/2}\Big(\frac{d\phi}{dx}(t)+\psi(t)\Big)dtds\\&-\frac{1}{\delta}\int_0^\delta\int_s^{s+\delta/2}\big(\psi(t+\delta/2)-\psi(t)\big)dtds.\end{aligned}$$

Then, the Hölder inequality leads to

$$\begin{aligned}\Big|m_\delta(\phi)(\delta)-2m_\delta(\phi)(\delta/2)+m_\delta(\phi)(0)\Big|^p\le&\frac{3^{p-1}}{\delta^p}\frac{\delta^{2p-2}}{2^{p-1}}\int_0^\delta\int_0^{2\delta}\Big|\frac{d\phi}{dx}(t)+\psi(t)\Big|^pdtds+\frac{3^{p-1}}{\delta^p}\frac{\delta^{2p-2}}{2^{p-1}}\int_0^\delta\int_0^{2\delta}\Big|\frac{d\phi}{dx}(t)+\psi(t)\Big|^pdtds\\&+\frac{3^{p-1}}{\delta^p}\frac{\delta^{3p-3}}{4^{p-1}}\int_0^\delta\int_0^{2\delta}\int_0^{2\delta}\Big|\frac{d\psi(y)}{dx}\Big|^pdydtds.\end{aligned}$$

The above inequality yields $(47)_2$.

We prove $(47)_3$. We start with

$$\delta\big(m_\delta(\psi(\delta)-m_\delta(\psi(0))\big)=\int_0^\delta\big(\psi(\delta+t)-\psi(t)\big)dt=\int_0^\delta\int_t^{\delta+t}\frac{d\psi}{dx}(s)dsdt.$$

Then, using Hölder inequality, this yields

$$\Big|\delta\big(m_\delta(\psi(\delta)-m_\delta(\psi(0))\big)\Big|^p\le\delta^{2p-1}\int_0^{2\delta}\Big|\frac{d\psi}{dx}(s)\Big|^pdsdt.$$

Hence, $(47)_3$ is proved.

Step 2. We prove the inequality of the lemma.

We first choose two functions Φ and Ψ belonging to $C^1\big([0,2\delta]\times[0,\delta]\big)$. From (47), we have

$$\Big|\frac{\frac{1}{\delta}\int_0^\delta\Phi(x+\delta,y)dx-\frac{1}{\delta}\int_0^\delta\Phi(x,y)dx}{\delta}+\frac{1}{\delta}\int_0^\delta\Psi(x,y)dx\Big|^p\le\frac{1}{\delta}\Big(\int_0^{2\delta}\Big|\frac{\partial\Phi}{\partial x}(x,y)+\Psi(x,y)\Big|^pdx+\delta^p\int_0^{2\delta}\Big|\frac{\partial\Psi}{\partial x}(x,y)\Big|^pdx\Big).$$

Now, using Hölder inequality we obtain

$$\begin{aligned}\Big|\frac{\mathcal{M}_\delta(\Phi)(\delta)-\mathcal{M}_\delta(\Phi)(0)}{\delta}+\mathcal{M}_\delta(\Psi)(0)\Big|^p&\le\frac{1}{\delta}\int_0^\delta\Big|\frac{\frac{1}{\delta}\int_0^\delta\Phi(x+\delta,y)dx+\frac{1}{\delta}\int_0^\delta\Phi(x,y)dx}{\delta}+\frac{1}{\delta}\int_0^\delta\Psi(x,y)dx\Big|^pdy\\&\le\frac{1}{\delta^2}\Big(\int_0^{2\delta}\int_0^\delta\Big|\frac{\partial\Phi}{\partial x}(x,y)+\Psi(x,y)\Big|^pdxdy+\delta^p\int_0^{2\delta}\int_0^\delta\Big|\frac{\partial\Psi}{\partial x}(x,y)\Big|^pdxdy\Big)\end{aligned}$$

Thanks to a symmetry argument, we also obtain

$$\Big|\frac{\mathcal{M}_\delta(\Phi)(\delta)-\mathcal{M}_\delta(\Phi)(0)}{\delta}+\mathcal{M}_\delta(\Psi)(\delta)\Big|^p \le \frac{1}{\delta^2}\Big(\int_0^{2\delta}\int_0^{\delta}\Big|\frac{\partial\Phi}{\partial x}(x,y)-\Psi(x,y)\Big|^p dxdy+\delta^p\int_0^{2\delta}\int_0^{\delta}\Big|\frac{\partial\Psi}{\partial x}(x,y)\Big|^p dxdy\Big).$$

From the two inequalities above, we obtain $(46)_1$ for Φ and Ψ in $C^1\big([0,2\delta]\times[0,\delta]\big)$. Similarly, we show $(46)_{2,3}$ starting from $(47)_{2,3}$. A density argument gives the estimates for every Φ and Ψ belonging to $W^{1,p}\big((0,2\delta)\times(0,\delta)\big)$, $1\le p<\infty$. □

8 | THE FUNCTION $\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}$ (SEE ALSO[18])

Denote Q_0, Q_1, $\mathring{Q}_0$ and $\mathring{Q}_1$ the following polynomial functions ($t\in[0,1]$)

$$Q_0(t)=\frac{(2t+\delta)(t-\delta)^2}{\delta^3},\qquad \mathring{Q}_0(t)=\frac{t(t-\delta)^2}{\delta^2},$$
$$Q_1(t)=\frac{t^2(3\delta-2t)}{\delta^3},\qquad \mathring{Q}_1(t)=\frac{t^2(t-\delta)}{\delta^2},$$
$$P(t)=\frac{6t(t-\delta)}{\delta^2},\qquad R(t)=\frac{12t-6\delta}{\delta}.$$

Note that

$$Q_1(t)=Q_0(\delta-t),\qquad \mathring{Q}_1(t)=-\mathring{Q}_0(\delta-t),\qquad \mathring{Q}_0(t)=\frac{P(t)}{6}(t-\delta),\qquad \mathring{Q}_1(t)=\frac{P(t)}{6}t,$$
$$\frac{dQ_0}{dt}(t)=\frac{6t(t-\delta)}{\delta^3}=\frac{1}{\delta}P(t),\qquad \frac{dQ_1}{dt}(t)=\frac{6t(\delta-t)}{\delta^3}=-\frac{1}{\delta}P(t),$$
$$\frac{d\mathring{Q}_0}{dt}(t)=\frac{(t-\delta)(3t-\delta)}{\delta^2}=\frac{1}{2}P(t)+\frac{\delta-t}{\delta},\qquad \frac{d\mathring{Q}_1}{dt}(t)=\frac{t(3t-2\delta)}{\delta^2}=\frac{1}{2}P(t)+\frac{t}{\delta}.$$

Set

$$\mathbf{A}=\big(A_{0,0},\,A_{1,0},\,A_{1,1},\,A_{0,1}\big),\qquad \mathbf{B}=\big(B_{0,0},\,B_{1,0},\,B_{1,1},\,B_{0,1}\big),\qquad \mathbf{C}=\big(C_{0,0},\,C_{1,0},\,C_{1,1},\,C_{0,1}\big).$$

We define the polynomial function $\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}\in W^{2,\infty}(\delta Y)$ by

$$\begin{aligned}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,x_2)=&A_{0,0}P_{0,0}(x_1,x_2)+A_{0,1}P_{0,1}(x_1,x_2)+A_{1,0}P_{1,0}(x_1,x_2)+A_{1,1}P_{1,1}(x_1,x_2)\\&+B_{0,0}d_1P_{0,0}(x_1,x_2)+B_{1,0}d_1P_{1,0}(x_1,x_2)+B_{0,1}d_1P_{0,1}(x_1,x_2)+B_{1,1}d_1P_{1,1}(x_1,x_2)\\&+C_{0,0}d_2P_{0,0}(x_1,x_2)+C_{0,1}d_2P_{0,1}(x_1,x_2)+C_{1,0}d_2P_{10}(x_1,x_2)+C_{1,1}d_2P_{1,1}(x_1,x_2)\end{aligned}$$

where for all $(x_1,x_2)\in[0,\delta]^2$

$$\begin{aligned}P_{0,0}(x_1,x_2)&=Q_0(x_1)Q_0(x_2), & P_{0,1}(x_1,x_2)&=Q_0(x_1)Q_1(x_2),\\P_{1,0}(x_1,x_2)&=Q_1(x_1)Q_0(x_2), & P_{1,1}(x_1,x_2)&=Q_1(x_1)Q_1(x_2),\\d_1P_{0,0}(x_1,x_2)&=\mathring{Q}_0(x_1)Q_0(x_2), & d_1P_{0,1}(x_1,x_2)&=\mathring{Q}_0(x_1)Q_1(x_2),\\d_1P_{1,0}(x_1,x_2)&=\mathring{Q}_1(x_1)Q_0(x_2), & d_1P_{1,1}(x_1,x_2)&=\mathring{Q}_1(x_1)Q_1(x_2),\\d_2P_{0,0}(x_1,x_2)&=Q_0(x_1)\mathring{Q}_0(x_2), & d_2P_{0,1}(x_1,x_2)&=Q_0(x_1)\mathring{Q}_1(x_2),\\d_2P_{1,0}(x_1,x_2)&=Q_1(x_1)\mathring{Q}_0(x_2), & d_2P_{1,1}(x_1,x_2)&=Q_1(x_1)\mathring{Q}_1(x_2).\end{aligned}$$

By construction, we have

$$\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(k\delta,p\delta)=A_{k,p},\qquad \partial_{x_1}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(k\delta,p\delta)=B_{k,p},\qquad \partial_{x_2}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(k\delta,p\delta)=C_{k,p},\qquad (k,p)\in\{0,1\}^2.$$

Moreover, $(k,p)\in\{0,1\}^2$

- $\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,p\delta)$ only depends on $A_{0,p}$, $A_{1,p}$, $B_{0,p}$ and $B_{1,p}$,
- $\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(k\delta,x_2)$ only depends on $A_{k,0}$, $A_{k,1}$, $C_{k,0}$ and $C_{k,1}$,
- $\partial_{x_1}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,p\delta)$ only depends on $A_{0,p}$, $A_{1,p}$, $B_{0,p}$ and $B_{1,p}$,
- $\partial_{x_1}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(k\delta,x_2)$ only depends on $B_{k,0}$ and $B_{k,1}$,
- $\partial_{x_2}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,p\delta)$ only depends on $C_{0,p}$ and $C_{1,p}$,

- $\partial_{x_2}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(k\delta, x_2)$ only depends on $A_{k,0}$, $A_{k,1}$, $C_{k,0}$ and $C_{k,1}$,

Now, observe that $\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}$ can be rewritten as

$$\begin{aligned}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,x_2) =&\big(A_{0,0}Q_0(x_1)+A_{1,0}Q_1(x_1)+B_{0,0}\mathring{Q}_0(x_1)+B_{1,0}\mathring{Q}_1(x_1)\big)Q_0(x_2)\\ &+\big(A_{0,1}Q_0(x_1)+A_{1,1}Q_1(x_1)+B_{0,1}\mathring{Q}_0(x_1)+B_{1,1}\mathring{Q}_1(x_1)\big)Q_1(x_2)\\ &+\big(C_{0,0}\mathring{Q}_0(x_2)+C_{0,1}\mathring{Q}_1(x_2)\big)Q_0(x_1)+\big(C_{1,0}\mathring{Q}_0(x_2)+C_{1,1}\mathring{Q}_1(x_2)\big)Q_1(x_1)\\ =&\big(A_{0,0}Q_0(x_2)+A_{0,1}Q_1(x_2)+C_{0,0}\mathring{Q}_0(x_2)+C_{0,1}\mathring{Q}_1(x_2)\big)Q_0(x_1)\\ &+\big(A_{1,0}Q_0(x_2)+A_{1,1}Q_1(x_2)+C_{1,0}\mathring{Q}_0(x_2)+C_{1,1}\mathring{Q}_1(x_2)\big)Q_1(x_1)\\ &+\big(B_{0,0}\mathring{Q}_0(x_1)+B_{1,0}\mathring{Q}_1(x_1)\big)Q_0(x_2)+\big(B_{0,1}\mathring{Q}_0(x_1)+B_{1,1}\mathring{Q}_1(x_1)\big)Q_1(x_2).\end{aligned}$$

We have

$$\begin{aligned}\partial_{x_1}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,x_2) =&\Big[\Big(\frac{A_{0,0}-A_{1,0}}{\delta}+\frac{1}{2}(B_{0,0}+B_{1,0})\Big)P(x_1)+B_{0,0}\frac{\delta-x_1}{\delta}+B_{1,0}\frac{x_1}{\delta}\Big]Q_0(x_2)\\ &+\Big[\Big(\frac{A_{0,1}-A_{1,1}}{\delta}+\frac{1}{2}(B_{0,1}+B_{1,1})\Big)P(x_1)+B_{0,1}\frac{\delta-x_1}{\delta}+B_{1,1}\frac{x_1}{\delta}\Big]Q_1(x_2)\\ &+\Big((C_{0,0}-C_{1,0})(x_2-\delta)+(C_{0,1}-C_{1,1})x_2\Big)\frac{P(x_2)}{6}P(x_1)\end{aligned}$$

and

$$\begin{aligned}\partial_{x_2}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,x_2) =&\Big[\Big(\frac{A_{0,0}-A_{0,1}}{\delta}+\frac{1}{2}(C_{0,0}+C_{0,1})\Big)P(x_2)+C_{0,0}\frac{\delta-x_2}{\delta}+C_{0,1}\frac{x_2}{\delta}\Big]Q_0(x_1)\\ &+\Big[\Big(\frac{A_{1,0}-A_{1,1}}{\delta}+\frac{1}{2}(C_{1,0}+C_{1,1})\Big)P(x_2)+C_{1,0}\frac{\delta-x_2}{\delta}+C_{1,1}\frac{x_2}{\delta}\Big]Q_1(x_1)\\ &+\Big((B_{0,0}-B_{0,1})(x_1-\delta)+(B_{1,0}-B_{1,1})x_1\Big)\frac{P(x_1)}{6}P(x_2).\end{aligned}$$

Now, since $Q_0(t)+Q_1(t)=1$ we obtain

$$\begin{aligned}\partial_{x_1}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,x_2)-B_{0,0} =&\Big[\Big(\frac{A_{0,0}-A_{1,0}}{\delta}+\frac{1}{2}(B_{0,0}+B_{1,0})\Big)P(x_1)+(B_{1,0}-B_{0,0})\frac{x_1}{\delta}\Big]Q_0(x_2)\\ &+\Big[\Big(\frac{A_{0,1}-A_{1,1}}{\delta}+\frac{1}{2}(B_{0,1}+B_{1,1})\Big)P(x_1)+(B_{0,1}-B_{0,0})\frac{\delta-x_1}{\delta}\\ &\qquad+\big((B_{1,1}-B_{1,0})+(B_{1,0}-B_{0,0})\big)\frac{x_1}{\delta}\Big]Q_1(x_2)\\ &+\Big((C_{0,0}-C_{1,0})(x_2-\delta)+(C_{0,1}-C_{1,1})x_2\Big)\frac{P(x_2)}{6}P(x_1).\end{aligned}\tag{48}$$

The second order partial derivatives are

$$\begin{aligned}\partial^2_{x_1x_1}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,x_2) =&\Big[\frac{1}{\delta}\Big(\frac{A_{0,0}-A_{1,0}}{\delta}+\frac{1}{2}(B_{0,0}+B_{1,0})\Big)R(x_1)+\frac{B_{1,0}-B_{0,0}}{\delta}\Big]Q_0(x_2)\\ &+\Big[\frac{1}{\delta}\Big(\frac{A_{0,1}-A_{1,1}}{\delta}+\frac{1}{2}(B_{0,1}+B_{1,1})\Big)R(x_1)+\frac{B_{1,1}-B_{0,1}}{\delta}\Big]Q_1(x_2)\\ &+\Big(\frac{C_{0,0}-C_{1,0}}{\delta}(x_2-\delta)+\frac{C_{0,1}-C_{1,1}}{\delta}x_2\Big)\frac{P(x_2)}{6}R(x_1)\end{aligned}$$

and a similar expression for $\partial^2_{x_2x_2}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,x_2)$ and

$$\begin{aligned}&\partial^2_{x_1x_2}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,x_2)\\ &=\frac{1}{\delta}\Big(\frac{A_{0,0}-A_{1,0}}{\delta}+\frac{1}{2}(B_{0,0}+B_{1,0})\Big)P(x_1)P(x_2)+\Big(\frac{B_{0,0}-B_{0,1}}{\delta}\frac{\delta-x_1}{\delta}+\frac{B_{1,0}-B_{1,1}}{\delta}\frac{x_1}{\delta}\Big)P(x_2)\\ &-\frac{1}{\delta}\Big(\frac{A_{0,1}-A_{1,1}}{\delta}+\frac{1}{2}(B_{0,1}+B_{1,1})\Big)P(x_1)P(x_2)\\ &+\Big((C_{0,0}-C_{1,0})\frac{x_2-\delta}{\delta}+(C_{0,1}-C_{1,1})\frac{x_2}{\delta}\Big)\frac{R(x_2)}{6}P(x_1)+\Big(\frac{C_{0,0}-C_{1,0}}{\delta}+\frac{C_{0,1}-C_{1,1}}{\delta}\Big)\frac{P(x_2)}{6}R(x_1).\end{aligned}$$

We also have

$$\begin{aligned}&\frac{1}{\delta^2}\int\limits_{\delta Y}\Phi_{\mathbf{A},\mathbf{B},\mathbf{C}}(x_1,x_2)dx_1dx_2\\&=\frac{A_{0,0}+A_{0,1}+A_{1,0}+A_{1,1}}{4}+\frac{B_{0,0}-B_{1,0}+B_{0,1}-B_{1,1}}{24}\delta+\frac{C_{0,0}+C_{1,0}-C_{0,1}-C_{1,1}}{24}\delta.\end{aligned}\tag{49}$$

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