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RESEARCH ARTICLE

Asymptotic behavior for textiles with loose contact

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Abstract

The paper is dedicated to the modeling and asymptotic investigation of a linear elasticity problem, in the form of variational inequality, for a textile structure. The textile is made of long and thin fibers crossing each others, forming a periodic squared domain. The domain is clamped only partially and an in plane sliding between the fibers is bounded by a contact function, which is chosen to be loose. We also assume a non-penetration condition for the fibers. Both partial clamp and loose contact arise a domain split, leading to different behaviors in each of the four parts. The homogenization is made via unfolding method, with an additional dimension reduction to further simplify the problem. The four cell problems are inequalities heavily coupled by the outer plane macro-micro constraints, while the macroscopic limit problem results to be an inequality of Leray-Lions type with only macro in plane constraints. On both scales, no uniqueness is expected.

KEYWORDS:
Homogenization, periodic unfolding method, dimension reduction, linear elasticity, variational inequality, Leray-Lions problem, contact, non-penetration condition, plates, structure of beams

1 | INTRODUCTION

This paper investigates the linearized elasticity problem for a textile. The domain of the structure is the square $\Omega = (0, L)^2$, made of long individual fibers of length L. The distance between the fibers depends on a small parameter ε, while the cross section of each of them depends on a second parameter asymptotically related to the first one $r \sim \varepsilon$. The fibers periodically cross each others in a fixed pattern forming a woven canvas structure. As we can see in Figure 1 , the domain is split into four parts with respect to the boundary conditions: on Ω_1 the clamp-conditions are set on the left and bottom boundary, while on the remaining left and bottom boundary the fibers are assumed to be glued. Of fundamental importance for this paper are many results already obtained in [14], where the assumptions are identical but where the contact sliding between the fibers is bounded by a strong contact function (of order $g_\varepsilon \sim \varepsilon^3$ or greater) in all three directions. Here, we assume only an in plane contact of order $g_\varepsilon \sim \varepsilon^2$ and a non-penetration condition. Due to the presence of contact cone conditions the elasticity problem is stated as a variational inequality, similar as in [7,14].

Exactly as in [14], small deformations and the sufficient forces applied to the elasticity problem are assumed such that the total energy remains in the linear regime. For elasticity problems in porous plate in Von-Kármán regime one can look into [15], where however the fibers are glued and contact issues not considered, since the structure can be extended to a periodically perforated shell. Forthcoming works will deal with nonlinear regime. In such frame we can recommend for now [13].

Due to the loose contact, results in the clamped parts cannot be extended on the not clamped ones and therefore a domain split is necessary. The homogenization leads to different Leray-Lions type limit problems (see [17]) on each part of the domain. The

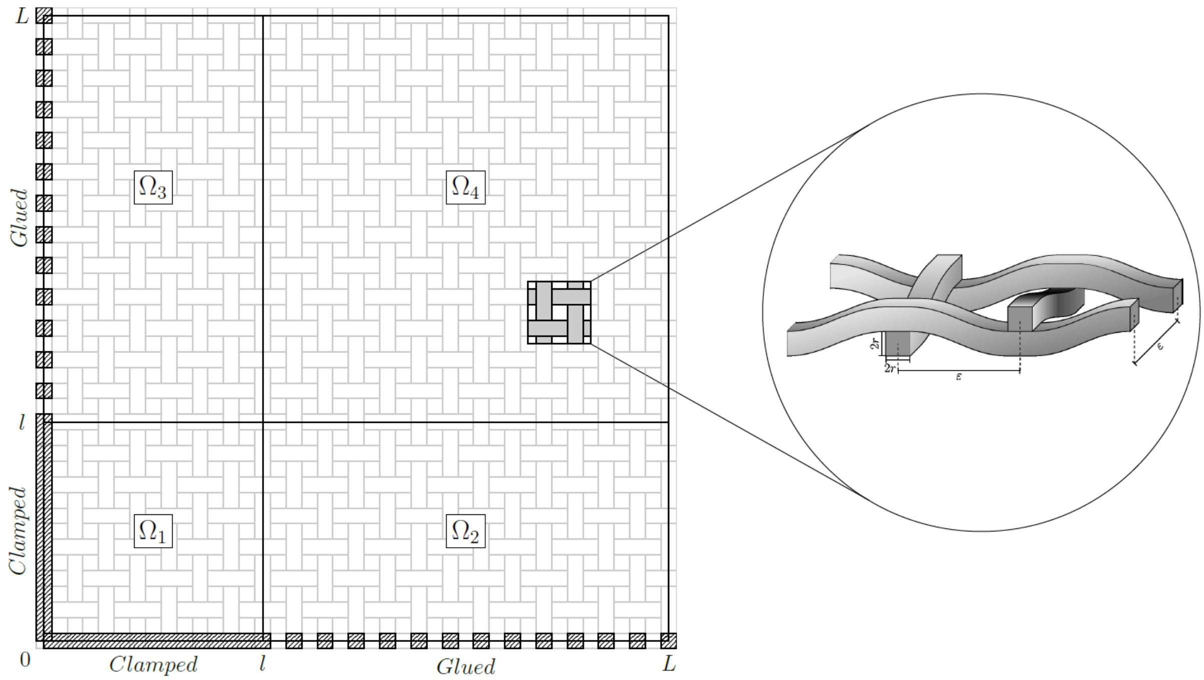

FIGURE 1 The textile domain is split according to partial clamp and glued conditions on the left and bottom boundary. Each cell has a 2ε periodic pattern. The distance between fibers is ε and their cross-section is $r \sim \varepsilon$. The complete structure can be seen as almost 2D.

inequality is maintained in both scales and we get a macroscopic problem with only in plane constraints and a microscopic cell problem with the same macro-micro outer plane conditions already found in [14]. Moreover, the uniqueness of solutions is not preserved in the limit. A physical interpretation of the in plane limit contact (see (82)) gives an idea of how the displacements behave on the four parts of the domain (see Figure 2 left), which is realistic if compared with the real experiments (see Figure 2 right). About numerical works devoted to simulation of textiles with contact sliding, one can look through [1,2,19,20]. The homogenization is made via unfolding method, an equivalent formulation of the two scales convergence. Moreover, a dimension reduction for beams and plates is additionally applied, which takes micro-structure into account and give rise to a representative homogeneous plate model. A first application of the unfolding method for boundary value problems has been done in [4] in periodically perforated domains. For more literature on the unfolding and homogenization in elasticity we refer to [5,18] and the references therein. About dimension reduction of homogeneous plates or rods one can read, for instance, in [3,9,10]. The combination of both is a part of current investigations in [5, Chap. 11] and [12,14,15].
The paper is organized as follows. Section 2 are notations, while Section 3 recalls the main results of [14] for the parameterization of a single curved rod in a fixed and mobile reference frame, giving transformation matrix, symmetric strain tensor and estimates for the decomposition with Q_1 interpolation. In Section 4 the previous results are extended to two beams of rods, one for each direction. Boundary and contact conditions are introduced in the domain, defining the the set of admissible displacements. The elasticity problem is set and existence and uniqueness of solutions are ensured. In Section 5, all the estimates for the displacement fields are given. From the estimates in the clamped parts we port the results in the not clamped ones by using the same techniques of [14], as Korn inequality, Poincaré inequality and Trace theorem. Worthy to note the improved estimates in the third direction from [14] by using the non penetration and periodic oscillation condition with alternating change of the normal sign (see Lemma 6), that allows to estimate the outer plane direction without defining an additional upper bound on the admissible deflection (a gap function $g_{\varepsilon,3}$, see [14]). In Section 6 we pass to the limit for $\varepsilon \to 0$ and get the fields weak convergences by compactness results. Applying the unfolding operators, the limit unfolded fields are found as well as the strain tensors, contact conditions and set of the limit displacements. Section 7 is dedicated to the build of the test-functions which have to satisfy certain properties. Among the others, the strong convergence via unfolding. In Section 8 all the results are summarized and applied: the weak convergence of the displacements together with the strong convergence of the test function imply integral convergence and the limit problem is derived. Existence is ensured by the Stampacchia lemma, a version of Lax-Milgram for closed subsets of Hilbert

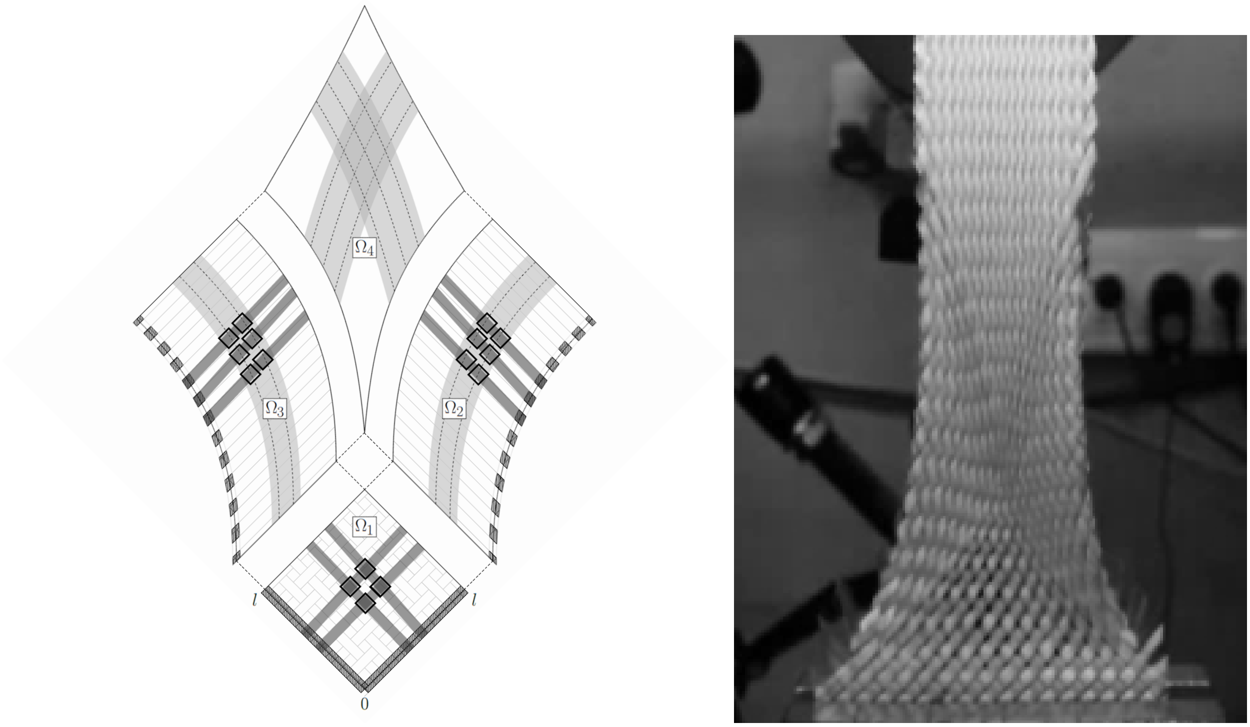

FIGURE 2 Left figure gives a sketch of obtained by analysis yarn's deformations in each textile part. On the right hand-side, a real experiment for tension of the textile with 45^o to the yarn directions is shown.

spaces (see [16]), while no uniqueness is expected. Following the strategy in [5, Section 5.6] the cell problems with correctors are found, while the macroscopic problems result to be of LerayLions type with their respective macroscopic in plane contact conditions. In this paper we use the Einstein convention of summation over repeated indices.

2 | NOTATION

Throughout the paper, the following notation will be used:

- $\Omega \doteq (0, L)^2$, $l > 0$ is a constant. For simplicity we assume that $\dfrac{L}{l} = \dfrac{a}{b}$ where a and b are integers such that $(a, b) = 1$;
- $Y \doteq (0, 1)^2$, $\mathcal{Y} \doteq (0, 2)^2$ are the 1-periodic and 2-periodic cells respectively;
- $\varepsilon \in \mathbb{R}$ is a small parameter such that $2\varepsilon N_\varepsilon = L$ and $2\varepsilon n_\varepsilon = l$, $N_\varepsilon = k_\varepsilon a$, $n_\varepsilon = k_\varepsilon b$ where k_ε is an integer;
- $\mathcal{K}_\varepsilon \doteq \{(p, q) \in \mathbb{N} \times \mathbb{N} \mid (p\varepsilon, q\varepsilon) \in \overline{\Omega}\} = \{0, \dots, 2N_\varepsilon\}^2$ is the set of nodes;
- $\kappa \in (0, 1/3]$ is a fixed constant and r is a parameter related to ε via $r \doteq \kappa\varepsilon$;
- $\omega_\kappa \doteq (-\kappa, \kappa)^2$ is the reference beam cross-section, while the rescaled one is $\omega_r \doteq (-r, r)^2 = (-\kappa\varepsilon, \kappa\varepsilon)^2$;
- $\mathbb{U}_i \doteq \mathbb{U} \cdot \mathbf{e}_i$, $\mathcal{R}_i \doteq \mathcal{R} \cdot \mathbf{e}_i$ for $i \in \{1, 2, 3\}$;
- $\partial_i \doteq \dfrac{\partial}{\partial z_i}$, $\partial_{X_i} \doteq \dfrac{\partial}{\partial X_i}$ denote the partial derivatives with respect to z_i and X_i respectively for $i \in \{1, 2, 3\}$;
- $z \doteq (z_1, z_2, z_3) \in \mathbb{R}^3$ and $z' \doteq (z_1, z_2) \in \mathbb{R}^2$ (if not specified);
- $(\alpha, \beta) \in \{1, 2\}^2$ and $(a, b, c) \in \{0, 1\}^3$ (if not specified);
- C is a real strictly positive constant independent of ε (if not specified).

3 | PRELIMINARIES: PARAMETERIZATION OF A CURVED ROD

Many results of this section have been already proved in[14, Section 3]. We start defining the 2-periodic function

$$\Phi(t) \doteq \begin{cases} -\kappa & \text{if } t \in [0,\kappa], \\ \kappa\Big(6\dfrac{(t-\kappa)^2}{(1-2\kappa)^2} - 4\dfrac{(t-\kappa)^3}{(1-2\kappa)^3} - 1\Big) & \text{if } t \in [\kappa, 1-\kappa], \\ \kappa & \text{if } t \in [1-\kappa, 1], \\ \Phi(2-t) & \text{if } t \in [1,2] \end{cases} \tag{1}$$

and we rescale it to a 2ε-periodic function setting $\Phi_\varepsilon(t) = \varepsilon\Phi\Big(\dfrac{t}{\varepsilon}\Big)$ which is piecewise $C^2(\mathbb{R})$ and overall $C^1(\mathbb{R})$. By definition, such a function satisfies

$$\varepsilon^2\|\Phi''_\varepsilon\|_{L^\infty(\mathbb{R})} + \varepsilon\|\Phi'_\varepsilon\|_{L^\infty(\mathbb{R})} + \|\Phi_\varepsilon\|_{L^\infty(\mathbb{R})} \le C\varepsilon.$$

Dealing with curved rods, the centerline of a rod is parameterized by the function

$$M_\varepsilon(z_1) \doteq z_1\mathbf{e}_1 + \Phi_\varepsilon(z_1)\mathbf{e}_3, \qquad z_1 \in [0,L].$$

This curve has $\mathbf{e}_1$ as mean direction and oscillations direction $\mathbf{e}_3$. We refer the beam to the mobile reference frame (Frenet-Serret), denoted by $\big(\mathbf{t}_\varepsilon, \mathbf{e}_2, \mathbf{n}_\varepsilon\big)$ and defined by

$$\mathbf{t}_\varepsilon \doteq \frac{\partial_1 M_\varepsilon}{|\partial_1 M_\varepsilon|} = \frac{1}{\gamma_\varepsilon}\big(\mathbf{e}_1 + \Phi'_\varepsilon\mathbf{e}_3\big), \qquad \mathbf{n}_\varepsilon \doteq \mathbf{t}_\varepsilon \wedge \mathbf{e}_2 = \frac{1}{\gamma_\varepsilon}\big(-\Phi'_\varepsilon\mathbf{e}_1 + \mathbf{e}_3\big)$$

where $\gamma_\varepsilon \doteq \sqrt{1+(\Phi'_\varepsilon)^2}$. The unit vector fields $\mathbf{t}_\varepsilon$ and $\mathbf{n}_\varepsilon$ belong to $C^1([0,L])^3$. Their derivatives are

$$\frac{d\mathbf{t}_\varepsilon}{dz_1} = c_\varepsilon\gamma_\varepsilon\mathbf{n}_\varepsilon, \qquad \frac{d\mathbf{n}_\varepsilon}{dz_1} = -c_\varepsilon\gamma_\varepsilon\mathbf{t}_\varepsilon$$

where the piecewise continuous function $c_\varepsilon(z_1) \doteq \dfrac{\Phi''_\varepsilon(z_1)}{\gamma^3_\varepsilon(z_1)}$ is the curvature. We denote

$$\mathbf{C}_\varepsilon \doteq \big(\mathbf{t}_\varepsilon \,|\, \mathbf{e}_2 \,|\, \mathbf{n}_\varepsilon\big) \in SO(3)$$

the basis transformation matrix from the fixed frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to the mobile one $(\mathbf{t}_\varepsilon, \mathbf{e}_2, \mathbf{n}_\varepsilon)$. We set the straight reference rod of length L and cross section ω_r

$$P_r \doteq (0,L)\times\omega_r.$$

The curved rod results to be

$$\mathcal{P}_\varepsilon \doteq \psi_\varepsilon(P_r),$$

where the function $\psi_\varepsilon : [0,L]\times\mathbb{R}^2 \to \mathbb{R}^3$ is the transition map from the straight to the curved rod and defined by

$$\psi_\varepsilon(z) \doteq M_\varepsilon(z_1) + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon(z_1).$$

The Jacobian for the changing of coordinates is

$$\eta_\varepsilon(z) \doteq \det\big(\nabla\psi_\varepsilon(z)\big) = \gamma_\varepsilon(z_1)\big(1 - z_3c_\varepsilon(z_1)\big), \qquad \forall z \in \overline{P_r}.$$

As already shown in[14, Remark A.1], there exists $\hat\kappa \in (0,1/3]$ depending on the curvature of the parameterization such that, for every $\kappa \le \hat\kappa$, the Jacobian η_ε is bounded from below and above

$$\frac{1}{C} \le \|\eta_\varepsilon\|_{L^\infty(P_r)} \le C,$$

thus the transformation ψ_ε from P_r onto $\mathcal{P}_\varepsilon$ results to be a diffeomorphism with

$$\nabla\psi_\varepsilon = \mathbf{C}_\varepsilon\begin{pmatrix}\eta_\varepsilon & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{and} \quad \big(\nabla\psi_\varepsilon\big)^{-1} = \begin{pmatrix}\frac{1}{\eta_\varepsilon} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\mathbf{C}^T_\varepsilon. \tag{2}$$

In particular, there exist two constants C_0, C_1 such that for every $\phi \in L^2(\mathcal{P}_\varepsilon)$:

$$C_0\|\phi\circ\psi_\varepsilon\|_{L^2(P_r)} \leq \|\phi\|_{L^2(\mathcal{P}_\varepsilon)} \leq C_1\|\phi\circ\psi_\varepsilon\|_{L^2(P_r)}. \tag{3}$$

This means that the L^2 estimates for a function computed on the straight rod (with respect to the variable z) and the estimates computed on the curved one (with respect to the variable x) will only differ by a constant.

From now on, we will simply denote ϕ the function $\phi\circ\psi_\varepsilon$.

3.1 | The decomposition of displacement

Let $u \in H^1(\mathcal{P}_\varepsilon)^3$ be a displacement. From [11, Theorem 3.1], [10, Lemma 3.2] and proceeding as in [14, Section 3.3] we have the following decomposition:

$$u = U^e + \overline{u}, \qquad \text{a.e. in } \mathcal{P}_\varepsilon \text{ or equivalently in } P_r \tag{4}$$

The quantity $\overline{u} \in H^1(P_r)^3$ is called warping (or reminder term) of the displacement. For a.e. $z_1 \in (0, L)$, it satisfies (see [11])

$$\int_{\omega_r} \overline{u}(z_1, z_2, z_3)dz_2dz_3 = 0, \qquad \int_{\omega_r} \overline{u}(z_1, z_2, z_3) \wedge \big(z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon(z_1)\big)dz_2dz_3 = 0.$$

The quantity $U^e \in H^1(P_r)^3$ is called elementary displacement and it is defined by

$$U^e(z) \doteq \underbrace{\mathbb{U}(z_1) + \mathcal{R}(z_1) \wedge \Phi_\varepsilon(z_1)\mathbf{e}_3}_{\text{middle line displacement}} + \mathcal{R}(z_1) \wedge \big(z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon(z_1)\big)^1$$

where the fields $\mathbb{U}$ and $\mathcal{R}$ belong to $H^1(0, L)^3$. Note that the field describing the middle line displacement is decomposed in a special way according to [14, Definition 3.3]. This is made in order to simplify the usual estimates for an elementary displacement (see [14, Lemma 3.4]). Namely, we get

$$\|\overline{u}\|_{L^2(P_\varepsilon)} \leq Cr\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \qquad \|\nabla\overline{u}\|_{L^2(P_\varepsilon)} \leq C\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \tag{5}$$

$$\|\partial_1\mathcal{R}\|_{L^2(0,L)} \leq \frac{C}{r^2}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \quad \|\partial_1\mathbb{U} - \mathcal{R} \wedge \mathbf{e}_1\|_{L^2(0,L)} \leq \frac{C}{r}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}. \tag{6}$$

If the beam is clamped at one extremity, e.g. $x = 0$, then we have

$$\mathbb{U}(0) = \mathcal{R}(0) = 0. \tag{7}$$

The end of this section is dedicated to a secondary splitting for the fields $\mathbb{U}$ and $\mathcal{R}$. Specifically, given $\mathbb{U}, \mathcal{R} \in H^1(0, L)^3$, we set the unique decomposition

$$\mathbb{U} = \mathbb{U}^{(pwl)} + \mathbb{U}^{(0)}, \qquad \mathcal{R} = \mathcal{R}^{(pwl)} + \mathcal{R}^{(0)}, \qquad \text{a.e. in } (0, L), \tag{8}$$

where $\mathbb{U}^{(pwl)}, \mathcal{R}^{(pwl)} \in W^{1,\infty}(0, L)^3$ coincide with the original functions on each node:

$$\mathbb{U}^{(pwl)}(p\varepsilon) = \mathbb{U}(p\varepsilon), \qquad \mathcal{R}^{(pwl)}(p\varepsilon) = \mathcal{R}(p\varepsilon), \qquad \forall p \in \{0, \dots, 2N_\varepsilon\}$$

and are then extended by Q_1 interpolation in between the nodes, while the functions $\mathbb{U}^{(0)}, \mathcal{R}^{(0)} \in H^1(0, L)^3$ are reminder terms which capture the high oscillations and are by definition zero on the nodes:

$$\mathbb{U}^{(0)}(p\varepsilon) = 0, \qquad \mathcal{R}^{(0)}(p\varepsilon) = 0, \qquad \forall p \in \{0, \dots, 2N_\varepsilon\}.$$

In the following, we recall the estimates of such functions.

Lemma 1. [14, Lemma 3.6] The functions $\mathbb{U}^{(0)}$, $\mathcal{R}^{(0)}$, $\mathbb{U}^{(pwl)}$ and $\mathcal{R}^{(pwl)}$ satisfy

$$\begin{aligned}
&\|\mathcal{R}^{(0)}\|_{L^2(0,L)} + \varepsilon\|d\mathcal{R}^{(0)}\|_{L^2(0,L)} + \varepsilon\|d\mathcal{R}^{(pwl)}\|_{L^2(0,L)} \leq \frac{C}{\varepsilon}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)},\\
&\|\mathbb{U}^{(0)}\|_{L^2(0,L)} + \varepsilon\|d\mathbb{U}^{(0)}\|_{L^2(0,L)} + \varepsilon\|d\mathbb{U}^{(pwl)}\|_{L^2(0,L)} \leq C\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)},\\
&\|d\mathbb{U}^{(pwl)} - \mathcal{R}^{(pwl)} \wedge \mathbf{e}_1\|_{L^2(0,L)} \leq \frac{C}{\varepsilon}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}.
\end{aligned} \tag{9}$$

[1] The map $(z_2, z_3) \mapsto z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon(z_1) + \mathcal{R}(z_1) \wedge \big(z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon(z_1)\big)$ (z_1 fixed) represents a small rotation of the cross section.

Remark 1. On can decompose $\mathbb{U}$ and $\mathcal{R}$ in another way by setting

$$\mathbb{U} = \mathring{\mathbb{U}}^{(pwl)} + \mathring{\mathbb{U}}^{(0)}, \qquad \mathcal{R} = \mathring{\mathcal{R}}^{(pwl)} + \mathring{\mathcal{R}}^{(0)}, \qquad \text{a.e. in } (0, L),$$

where $\mathring{\mathbb{U}}^{(pwl)}$, $\mathring{\mathcal{R}}^{(pwl)} \in W^{1,\infty}(0, L)^3$ coincide with the original functions each second node:

$$\mathring{\mathbb{U}}^{(pwl)}(2p'\varepsilon) = \mathbb{U}(2p'\varepsilon), \quad \mathring{\mathcal{R}}^{(pwl)}(2p'\varepsilon) = \mathcal{R}(2p'\varepsilon), \quad \forall p' \in \{0, \dots, N_\varepsilon\}.$$

Then, proceeding as in the proof of [14, Lemma 3.6], we obtain for these functions the same estimates as those of Lemma 1. In particular, one has

$$\begin{aligned}
&\|\mathcal{R}^{(pwl)} - \mathring{\mathcal{R}}^{(pwl)}\|_{L^2(0,L)} + \varepsilon \left\| d(\mathcal{R}^{(pwl)} - \mathring{\mathcal{R}}^{(pwl)}) \right\|_{L^2(0,L)} \leq \frac{C}{\varepsilon} \|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \\
&\|\mathbb{U}^{(pwl)} - \mathring{\mathbb{U}}^{(pwl)}\|_{L^2(0,L)} + \varepsilon \left\| d(\mathbb{U}^{(pwl)} - \mathring{\mathbb{U}}^{(pwl)}) \right\|_{L^2(0,L)} \leq C \|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}.
\end{aligned} \tag{10}$$

3.2 | Symmetric gradient of the displacement of a curved rod

In this subsection, we give the symmetric gradient of a displacement with respect to the variables (z_1, z_2, z_3) (for more details see [14, Subsection 3.4]).

First, for every $v \in H^1(\mathcal{P}_\varepsilon)^3$, equality $(2)_1$ leads to

$$\nabla_z v = \left(\partial_1 v \middle| \partial_2 v \middle| \partial_3 v\right) = \nabla_x v\, \nabla_z \psi_\varepsilon = \nabla_x v\, \mathbf{C}_\varepsilon \begin{pmatrix} \eta_\varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\mathbf{C}_\varepsilon^T\, \nabla_x v\, \mathbf{C}_\varepsilon = \begin{pmatrix} \frac{1}{\eta_\varepsilon} \partial_1 v \cdot \mathbf{t}_\varepsilon & \partial_2 v \cdot \mathbf{t}_\varepsilon & \partial_3 v \cdot \mathbf{t}_\varepsilon \\ \frac{1}{\eta_\varepsilon} \partial_1 v \cdot \mathbf{e}_2 & \partial_2 v \cdot \mathbf{e}_2 & \partial_3 v \cdot \mathbf{e}_2 \\ \frac{1}{\eta_\varepsilon} \partial_1 v \cdot \mathbf{n}_\varepsilon & \partial_2 v \cdot \mathbf{n}_\varepsilon & \partial_3 v \cdot \mathbf{n}_\varepsilon \end{pmatrix}.$$

As a consequence we get

$$\mathbf{C}_\varepsilon^T e_x(v) \mathbf{C}_\varepsilon = \begin{pmatrix} \frac{1}{\eta_\varepsilon} \partial_1 v \cdot \mathbf{t}_\varepsilon & * & * \\ \frac{1}{2}\Big(\frac{1}{\eta_\varepsilon} \partial_1 v \cdot \mathbf{e}_2 + \partial_2 v \cdot \mathbf{t}_\varepsilon\Big) & \partial_2 v \cdot \mathbf{e}_2 & * \\ \frac{1}{2}\Big(\frac{1}{\eta_\varepsilon} \partial_1 v \cdot \mathbf{n}_\varepsilon + \partial_3 v \cdot \mathbf{t}_\varepsilon\Big) & \frac{1}{2}\Big(\partial_2 v \cdot \mathbf{n}_\varepsilon + \partial_3 v \cdot \mathbf{e}_2\Big) & \partial_3 v \cdot \mathbf{n}_\varepsilon \end{pmatrix}. \tag{11}$$

We denote by $\widetilde{\mathbf{e}}_z(v)$ *the right hand side of the above equality.*

Additionally, in the next sections a vectorial notation for the strain tensor will be used. Indeed, the strain tensor $e_x(v)$ of a displacement $v \in H^1(\mathcal{P}_\varepsilon)^3$ is also written as a column vector with six entries by setting

$$E_x(v) \doteq \big(e_{x,11}(v)\ \ e_{x,22}(v)\ \ e_{x,33}(v)\ \ e_{x,12}(v)\ \ e_{x,13}(v)\ \ e_{x,23}(v)\big)^T. \tag{12}$$

In that way, the symmetric matrix $\widetilde{\mathbf{e}}_z(v) = \mathbf{C}_\varepsilon^T e_x(v) \mathbf{C}_\varepsilon$ is represented by the column vector

$$\widetilde{\mathbf{E}}_z(v) \doteq \big(\widetilde{\mathbf{e}}_{z,11}(v)\ \ \widetilde{\mathbf{e}}_{z,22}(v)\ \ \widetilde{\mathbf{e}}_{z,33}(v)\ \ \widetilde{\mathbf{e}}_{z,12}(v)\ \ \widetilde{\mathbf{e}}_{z,13}(v)\ \ \widetilde{\mathbf{e}}_{z,23}(v)\big)^T. \tag{13}$$

Hence

$$\widetilde{\mathbf{E}}_z(v) \doteq \widetilde{\mathbf{C}}_\varepsilon\, E_x(v),$$

where the matrix $\widetilde{\mathbf{C}}_\varepsilon$ belongs to $C^1([0,L])^{6\times 6}$ and is given by

$$\widetilde{\mathbf{C}}_\varepsilon = \begin{pmatrix} \frac{1}{\gamma_\varepsilon^2} & 0 & \frac{(\Phi'_\varepsilon)^2}{\gamma_\varepsilon^2} & 0 & 2\frac{\Phi'_\varepsilon}{\gamma_\varepsilon^2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{(\Phi'_\varepsilon)^2}{\gamma_\varepsilon^2} & 0 & \frac{1}{\gamma_\varepsilon^2} & 0 & -2\frac{\Phi'_\varepsilon}{\gamma_\varepsilon^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma_\varepsilon} & 0 & \frac{\Phi'_\varepsilon}{\gamma_\varepsilon} \\ -\frac{\Phi'_\varepsilon}{\gamma_\varepsilon^2} & 0 & \frac{\Phi'_\varepsilon}{\gamma_\varepsilon^2} & 0 & \frac{1-(\Phi'_\varepsilon)^2}{\gamma_\varepsilon^2} & 0 \\ 0 & 0 & 0 & -\frac{\Phi'_\varepsilon}{\gamma_\varepsilon} & 0 & \frac{1}{\gamma_\varepsilon} \end{pmatrix}. \tag{14}$$

Below, we use the decomposition (4) of a displacement $u \in H^1(\mathcal{P}_\varepsilon)^3$ to express the matrix $\widetilde{\mathbf{e}}_z(U^e) = \mathbf{C}_\varepsilon^T e_x(U^e)\mathbf{C}_\varepsilon$. Concerning the elementary displacement, we have (see [14, Subsection 3.4]).

$$\widetilde{\mathbf{e}}_z(U^e) = \mathbf{C}_\varepsilon^T e_x(U^e)\mathbf{C}_\varepsilon = \begin{pmatrix} \frac{1}{\eta_\varepsilon}\Big[(\partial_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1) + \partial_1\mathcal{R}\wedge\big(\Phi_\varepsilon\mathbf{e}_3 + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon\big)\Big]\cdot\mathbf{t}_\varepsilon & * & * \\ \frac{1}{2\eta_\varepsilon}\Big[(\partial_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1) + \partial_1\mathcal{R}\wedge\big(\Phi_\varepsilon\mathbf{e}_3 + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon\big)\Big]\cdot\mathbf{e}_2 & 0 & * \\ \frac{1}{2\eta_\varepsilon}\Big[(\partial_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1) + \partial_1\mathcal{R}\wedge\big(\Phi_\varepsilon\mathbf{e}_3 + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon\big)\Big]\cdot\mathbf{n}_\varepsilon & 0 & 0 \end{pmatrix}. \tag{15}$$

Observe that

$$\begin{aligned} \partial_1\mathcal{R}\cdot\Big(\Big(\frac{\Phi_\varepsilon}{\gamma_\varepsilon} + z_3\Big)\mathbf{e}_2 - z_2\mathbf{n}_\varepsilon\Big) &= \Big(\partial_1\mathcal{R}\wedge\big(\Phi_\varepsilon\mathbf{e}_3 + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon\big)\Big)\cdot\mathbf{t}_\varepsilon, \\ -\partial_1\mathcal{R}\cdot\big(z_3\mathbf{t}_\varepsilon + \Phi_\varepsilon\mathbf{e}_1\big) &= \Big(\partial_1\mathcal{R}\wedge\big(\Phi_\varepsilon\mathbf{e}_3 + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon\big)\Big)\cdot\mathbf{e}_2, \\ \partial_1\mathcal{R}\cdot\Big(z_2\mathbf{t}_\varepsilon - \frac{\Phi_\varepsilon\Phi'_\varepsilon}{\gamma_\varepsilon}\mathbf{e}_2\Big) &= \Big(\partial_1\mathcal{R}\wedge\big(\Phi_\varepsilon\mathbf{e}_3 + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon\big)\Big)\cdot\mathbf{n}_\varepsilon. \end{aligned}$$

4 | THE TEXTILE STRUCTURE

Let

$$\begin{aligned} P_r^{(1)} &\doteq \big\{z\in\mathbb{R}^3 \;\big|\; z_1\in(0,L),\;\; (z_2,z_3)\in\omega_r\big\}, \\ P_r^{(2)} &\doteq \big\{z\in\mathbb{R}^3 \;\big|\; z_2\in(0,L),\;\; (z_1,z_3)\in\omega_r\big\} \end{aligned}$$

be the straight reference rods in direction $\mathbf{e}_1$ and $\mathbf{e}_2$. For each in plane direction, we define two beams of curved rods by

$$\mathcal{P}_\varepsilon^{(1,q)} \doteq \psi_\varepsilon^{(1,q)}\big(P_r^{(1)}\big), \qquad \mathcal{P}_\varepsilon^{(2,p)} \doteq \psi_\varepsilon^{(2,p)}\big(P_r^{(2)}\big), \qquad (p,q)\in\mathcal{K}_\varepsilon$$

where the diffeomorphisms are defined by

$$\begin{aligned} \psi_\varepsilon^{(1,q)}(z) &\doteq M_\varepsilon^{(1,q)}(z_1) + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon^{(1,q)}(z_1), \\ \psi_\varepsilon^{(2,p)}(z) &\doteq M_\varepsilon^{(2,p)}(z_2) + z_1\mathbf{e}_1 + z_3\mathbf{n}_\varepsilon^{(2,p)}(z_2) \end{aligned}$$

and the middle lines by

$$\begin{aligned} M_\varepsilon^{(1,q)}(z_1) &\doteq z_1\mathbf{e}_1 + q\varepsilon\mathbf{e}_2 + (-1)^{q+1}\Phi_\varepsilon(z_1)\mathbf{e}_3, \\ M_\varepsilon^{(2,p)}(z_2) &\doteq p\varepsilon\mathbf{e}_1 + z_2\mathbf{e}_2 + (-1)^{p}\Phi_\varepsilon(z_2)\mathbf{e}_3. \end{aligned}$$

Note that the quantities $(-1)^{q+1}$ and $(-1)^p$ denote the fact that the woven fibers are alternate, allowing crossing between them. The whole textile structure is given by

$$\mathcal{S}_\varepsilon \doteq \mathcal{S}_\varepsilon^{(1)} \cup \mathcal{S}_\varepsilon^{(2)}, \qquad \mathcal{S}_\varepsilon^{(\alpha)} \doteq \bigcup_{\ell=0}^{2N_\varepsilon-1} \mathcal{P}_\varepsilon^{(\alpha,\ell)}. \tag{16}$$

Each direction has its local mobile frames $(\mathbf{t}_\varepsilon^{(1,q)}, \mathbf{e}_2, \mathbf{n}_\varepsilon^{(1,q)})$ and $(\mathbf{e}_1, \mathbf{t}_\varepsilon^{(2,p)}, \mathbf{n}_\varepsilon^{(2,p)})$ with

$$\begin{aligned} \mathbf{t}_\varepsilon^{(1,q)}(z_1) &= \frac{\partial_1 M_\varepsilon^{(1,q)}}{\left|\partial_1 M_\varepsilon^{(1,q)}\right|}, \qquad \mathbf{n}_\varepsilon^{(1,q)}(z_1) = \mathbf{t}_\varepsilon^{(1,q)}(z_1) \wedge \mathbf{e}_2, \qquad z_1 \in [0, L],\\ \mathbf{t}_\varepsilon^{(2,p)}(z_2) &= \frac{\partial_2 M_\varepsilon^{(2,p)}}{\left|\partial_2 M_\varepsilon^{(2,p)}\right|}, \qquad \mathbf{n}_\varepsilon^{(2,p)}(z_2) = \mathbf{e}_1 \wedge \mathbf{t}_\varepsilon^{(2,p)}(z_2), \qquad z_2 \in [0, L]. \end{aligned}$$

For simplicity, the displacements $u^{(1,q)} \in H^1(\mathcal{P}_\varepsilon^{(1,q)})^3$, $u^{(2,p)} \in H^1(\mathcal{P}_\varepsilon^{(2,p)})^3$ are also referred to their respective straight reference frames with the same names:

$$u^{(1,q)} \in H^1(q\varepsilon\mathbf{e}_2 + P_r^{(1)})^3, \qquad u^{(2,p)} \in H^1(p\varepsilon\mathbf{e}_1 + P_r^{(2)})^3, \qquad (p,q) \in \mathcal{K}_\varepsilon.$$

Hence, we write ($z \in q\varepsilon\mathbf{e}_2 + P_r^{(1)}$ for $u^{(1,q)}$ and $z \in p\varepsilon\mathbf{e}_1 + P_r^{(2)}$ for $u^{(2,p)}$)

$$\begin{aligned} u^{(1,q)}(z) &= \mathbb{U}^{(1,q)}(z_1) + \mathcal{R}^{(1,q)}(z_1) \wedge \Big((-1)^{q+1}\Phi_\varepsilon(z_1)\mathbf{e}_3 + (z_2 - q\varepsilon)\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon^{(1,q)}(z_1)\Big) + \overline{u}^{(1,q)}(z),\\ u^{(2,p)}(z) &= \mathbb{U}^{(2,p)}(z_2) + \mathcal{R}^{(2,p)}(z_2) \wedge \Big((-1)^{p}\Phi_\varepsilon(z_2)\mathbf{e}_3 + (z_1 - p\varepsilon)\mathbf{e}_1 + z_3\mathbf{n}_\varepsilon^{(2,p)}(z_2)\Big) + \overline{u}^{(2,p)}(z). \end{aligned} \tag{17}$$

4.1 | The contact and non penetration condition

The contact between fibers is restricted to the portions where the beams are right above each other. We define such contact domains in the fixed frame $\big(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\big)$ by ($(p,q) \in \mathcal{K}_\varepsilon$)

$$\mathbf{C} \doteq \bigcup_{(p,q)\in\mathcal{K}_\varepsilon} \mathbf{C}_{pq}, \qquad \mathbf{C}_{pq} \doteq \big(C_{pq} \cap \Omega\big) \times \{0\}, \qquad C_{pq} \doteq (p\varepsilon, q\varepsilon) + \omega_r.$$

By (1) and (17), in C_{pq} the displacements reduce for a.e. $(z_1, z_2) \in \omega_r$ to

$$\begin{aligned} u^{(1,q)}(z_1 + p\varepsilon, z_2 + q\varepsilon, (-1)^{p+q+1}r) &= \mathbb{U}^{(1,q)}(p\varepsilon + z_1) + \mathcal{R}^{(1,q)}(p\varepsilon + z_1) \wedge z_2\mathbf{e}_2 + \overline{u}^{(1,q)},\\ u^{(2,p)}(z_1 + p\varepsilon, z_2 + q\varepsilon, (-1)^{p+q}r) &= \mathbb{U}^{(2,p)}(q\varepsilon + z_2) + \mathcal{R}^{(2,p)}(q\varepsilon + z_2) \wedge z_1\mathbf{e}_1 + \overline{u}^{(2,p)}. \end{aligned} \tag{18}$$

The beam-to-beam interaction is characterized by the nonnegative gap-functions $g_{\varepsilon,\alpha}$, describing an admissible in plane sliding. We assume

$$g_{\varepsilon,\alpha} = \varepsilon^2 g_\alpha, \quad g_\alpha \in C(\overline{\Omega}) \quad \text{and therefore} \quad \|g_{\varepsilon,\alpha}\|_{L^2(\Omega)} \le C\varepsilon^2\|g_\alpha\|_{L^\infty(\Omega)}.$$

In the internal part of Ω we do not allow areas where the fibers are glued:

$$\exists\, C_3 > 0 \quad \text{such that} \;\; g_\alpha(z) \ge C_3, \qquad \forall z \in \Omega. \tag{19}$$

As we will see, this condition plays an important role in the build of the contact condition for the test-functions.

We define the in plane contact conditions by setting

$$|u_\alpha^{(1,q)} - u_\alpha^{(2,p)}| \le g_{\varepsilon,\alpha}, \qquad \text{a.e in } \mathbf{C}_{pq}, \;\; \forall (p,q) \in \mathcal{K}_\varepsilon, \tag{20}$$

while the outer plane component

$$0 \le (-1)^{p+q}\big(u_3^{(1,q)} - u_3^{(2,p)}\big) \qquad \text{a.e in } \mathbf{C}_{pq}, \;\; \forall (p,q) \in \mathcal{K}_\varepsilon. \tag{21}$$

only takes into account the fact that the fibers cannot penetrate each others and the oscillating manner of the beams switching in the vertical position.

4.2 | Boundary conditions

In order to properly study the textile behavior, we first split the whole domain Ω in the four domains

$$\Omega = \mathrm{int}\big(\overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3 \cup \overline{\Omega}_4\big)$$

defined by (see Figure 1)

$$\Omega_1 \doteq (0,l)^2, \quad \Omega_2 \doteq (l,L) \times (0,l), \quad \Omega_3 \doteq (0,l) \times (l,L), \quad \Omega_4 \doteq (l,L)^2.$$

In Ω_1 we assume on both lateral boundaries $z_1 = 0$ and $z_2 = 0$ that every displacement equals zero. Given the structure (16), this implies

$$\text{Clamp condition}\quad \begin{cases} u^{(1,q)}_{|z_1=0} = 0 \ \text{ for every } \ q \in \{0,\dots,2n_\varepsilon\}, \\ u^{(2,p)}_{|z_2=0} = 0 \ \text{ for every } \ p \in \{0,\dots,2n_\varepsilon\}. \end{cases} \tag{22}$$

Note that such clamp on Ω_1 affects the behavior of the displacements in the whole Ω and justifies the domain splitting. Indeed, the displacement $u^{(1,q)}$ inherits the clamp condition in $z_1 = 0$ for all $z_2 \in (0, L)$ (and thus in $\Omega_1 \cup \Omega_2$), while the displacement $u^{(2,p)}$ inherits the clamp condition in $z_2 = 0$ for all $z_1 \in (0, L)$ (and thus in $\Omega_1 \cup \Omega_3$).

In the remaining left and bottom boundary, we assume that the fibers are glued:

$$\text{Glued condition}\quad \begin{cases} u^{(1,q)} = u^{(2,0)} \ \text{ a.e. in } \mathbf{C}_{0q}, \ \ q \in \{0,\dots,2N_\varepsilon\}, \\ u^{(1,0)} = u^{(2,p)} \ \text{ a.e. in } \mathbf{C}_{p0}, \ \ p \in \{0,\dots,2N_\varepsilon\}. \end{cases} \tag{23}$$

4.3 | The admissible displacements of the structure

Given the structure, the boundary and contact conditions, the closed convex set of the admissible displacements is denoted by

$$\mathcal{X}_\varepsilon \doteq \Big\{ u = \big(u^{(1,0)},\dots,u^{(1,2N_\varepsilon)}, u^{(2,0)},\dots,u^{(2,2N_\varepsilon)}\big) \in \prod_{q=0}^{2N_\varepsilon} H^1(q\varepsilon \mathbf{e}_2 + P_r^{(1)})^3 \times \prod_{p=0}^{2N_\varepsilon} H^1(p\varepsilon \mathbf{e}_1 + P_r^{(2)})^3 \ \Big| \ u \ \text{ satisfies (20)-(21)-(22)-(23)} \Big\}.$$

We endow the product space

$$\prod_{q=0}^{2N_\varepsilon} H^1(q\varepsilon \mathbf{e}_2 + P_r^{(1)})^3 \times \prod_{p=0}^{2N_\varepsilon} H^1(p\varepsilon \mathbf{e}_1 + P_r^{(2)})^3 \tag{24}$$

with the semi-norm

$$\|u\|_{\mathcal{S}_\varepsilon} \doteq \sqrt{\sum_{q=0}^{2N_\varepsilon} \|\widetilde{\mathbf{e}}_z(u^{(1,q)})\|^2_{L^2(q\varepsilon\mathbf{e}_2+P_r^{(1)})} + \sum_{p=0}^{2N_\varepsilon} \|\widetilde{\mathbf{e}}_z(u^{(2,p)})\|^2_{L^2(p\varepsilon\mathbf{e}_1+P_r^{(2)})}}.$$

By the clamped and glued conditions (22)-(23), we easily check that this semi-norm is in fact a norm, thus $\mathcal{X}_\varepsilon$ is a closed convex subset of the product space (24).

4.4 | The elasticity problem

Due to the contact conditions, the elasticity problem is stated as variational inequality (and in vectorial notation):

$$\begin{cases} \text{Find } u_\varepsilon \in \mathcal{X}_\varepsilon \text{ such that for every } v_\varepsilon \in \mathcal{X}_\varepsilon: \\ \displaystyle\int_{\mathcal{S}_\varepsilon} A_\varepsilon \, E_x(u_\varepsilon) \cdot E_x(u_\varepsilon - v_\varepsilon)\, dx \le \int_{\mathcal{S}_\varepsilon} f_\varepsilon \cdot (u_\varepsilon - v_\varepsilon)\, dx, \end{cases} \tag{25}$$

where $f_\varepsilon \in L^2(\mathcal{S}_\varepsilon)^3$ and where the material elasticity law is incorporated by the matrix A_ε, which satisfies the usual Hooke's law

- $A_\varepsilon \in L^\infty(\mathcal{S}_\varepsilon)^{6\times 6}$;
- A_ε is symmetric;
- A_ε is positive definite and therefore coercive: there exists two constants $C_0, C_1 > 0$ independent of ε such that

$$C_0|\xi|^2 \le A_\varepsilon\, \xi \cdot \xi \le C_1 |\xi|^2 \quad \text{ a.e. in } \ \mathcal{S}_\varepsilon, \quad \forall \xi \in \mathbb{R}^6. \tag{26}$$

Existence and uniqueness of problem (25) are ensured by Stampacchia lemma (see[16]) and the clamped and glued conditions (22)-(23), which do not allow rigid motions in the kernel of the symmetric strain tensor.

In order to switch the elasticity problem from the mobile to the straight reference frame, we first recall the vector-transformation matrices $\widetilde{\mathbf{C}}^{(1,q)}_\varepsilon$ (resp. $\widetilde{\mathbf{C}}^{(2,p)}_\varepsilon$) defined in (14) and we replace Φ_ε by $(-1)^{q+1}\Phi_\varepsilon$ (resp. $(-1)^p\Phi_\varepsilon$). Then, we set

$$\mathbf{A}_\varepsilon = \big(A^{(1,0)}_\varepsilon,\dots,A^{(1,2N_\varepsilon)}_\varepsilon, A^{(2,0)}_\varepsilon,\dots,A^{(2,2N_\varepsilon)}_\varepsilon\big) \in \prod_{q=0}^{2N_\varepsilon} L^\infty(q\varepsilon\mathbf{e}_2 + P_r^{(1)})^{6\times 6} \times \prod_{p=0}^{2N_\varepsilon} L^\infty(p\varepsilon\mathbf{e}_1 + P_r^{(2)})^{6\times 6},$$

where

$$A_\varepsilon^{(1,q)} \doteq \left(\widetilde{\mathbf{C}}_\varepsilon^{(1,q)}\right)^T A_\varepsilon \circ \psi_\varepsilon^{(1,q)}\, \widetilde{\mathbf{C}}_\varepsilon^{(1,q)}, \qquad A_\varepsilon^{(2,p)} \doteq \left(\widetilde{\mathbf{C}}_\varepsilon^{(2,p)}\right)^T A_\varepsilon \circ \psi_\varepsilon^{(2,p)}\, \widetilde{\mathbf{C}}_\varepsilon^{(2,p)}$$

and the forces ($(p,q) \in \mathcal{K}_\varepsilon$)

$$\begin{aligned} F_\varepsilon^{(1,q)} &\doteq f_\varepsilon \circ \psi_\varepsilon^{(1,q)} && \text{a.e. in } q\varepsilon\mathbf{e}_2 + P_r^{(1)}, \\ F_\varepsilon^{(2,p)} &\doteq f_\varepsilon \circ \psi_\varepsilon^{(2,p)} && \text{a.e. in } p\varepsilon\mathbf{e}_1 + P_r^{(2)}. \end{aligned} \tag{27}$$

Following the computation in subsection 3.2 for both directions, we port problem (25) in the straight reference frame:

Find $u_\varepsilon \in \mathcal{X}_\varepsilon$ such that for every $v \in \mathcal{X}_\varepsilon$:

$$\begin{aligned} &\sum_{q=0}^{2N_\varepsilon} \int_{q\varepsilon\mathbf{e}_2 + P_r^{(1)}} A_\varepsilon^{(1,q)}\, \widetilde{\mathbf{E}}_z(u_\varepsilon^{(1,q)}) \cdot \widetilde{\mathbf{E}}_z(u_\varepsilon^{(1,q)} - v^{(1,q)})\, \eta^{(1,q)}\, dz + \sum_{p=0}^{2N_\varepsilon} \int_{p\varepsilon\mathbf{e}_1 + P_r^{(2)}} A_\varepsilon^{(2,p)}\, \widetilde{\mathbf{E}}_z(u_\varepsilon^{(2,p)}) \cdot \widetilde{\mathbf{E}}_z(u_\varepsilon^{(2,p)} - v^{(1,q)})\, \eta^{(2,p)}\, dz \\ &\le \sum_{q=0}^{2N_\varepsilon} \int_{q\varepsilon\mathbf{e}_2 + P_r^{(1)}} F_\varepsilon^{(1,q)} \cdot \left(u_\varepsilon^{(1,q)} - v^{(1,q)}\right) \eta^{(1,q)} dz + \sum_{p=0}^{2N_\varepsilon} \int_{p\varepsilon\mathbf{e}_1 + P_r^{(2)}} F_\varepsilon^{(2,p)} \cdot \left(u_\varepsilon^{(2,p)} - v^{(2,p)}\right) \eta^{(2,p)}\, dz \end{aligned} \tag{28}$$

where $\widetilde{\mathbf{E}}_z(u_\varepsilon^{(1,q)})$ and $\widetilde{\mathbf{E}}_z(u_\varepsilon^{(2,p)})$ are $\mathbb{R}^6$ column vectors (see (12)-(13)) with entries given by (15)-(11) in the respective direction and where the displacements $u_\varepsilon^{(1,q)}$ and $u_\varepsilon^{(2,p)}$ are defined as in (17).

5 | FIELDS ESTIMATES

We want now to estimate the fields involved in problem (28) or, equivalently, those that appear in the representation of respectively strain tensor (see subsection 3.2) and displacement (see (17)). To do so, we first need an extension result.

5.1 | Preliminary decomposition of the displacements

We want to apply the decomposition (8), this time not on a line $[0, L]$ but for the 2D domain Ω. Consider the displacements $u_\varepsilon^{(1,q)}$ and $u_\varepsilon^{(2,p)}$ defined in (17) and set the spaces

$$L_\varepsilon^{(1)} = \bigcup_{q=0}^{2N_\varepsilon} (0, L) \times \{q\varepsilon\}, \qquad L_\varepsilon^{(2)} = \bigcup_{p=0}^{2N_\varepsilon} \{p\varepsilon\} \times (0, L).$$

Given the functions $\mathbb{U}^{(1,q)},\ \mathcal{R}^{(1,q)} \in H^1(L_\varepsilon^{(1)})^3$ and $\mathbb{U}^{(2,p)},\ \mathcal{R}^{(2,p)} \in H^1(L_\varepsilon^{(2)})^3$, $(p,q) \in \mathcal{K}_\varepsilon$, we denote $\mathbb{U}^{(\alpha)}, \mathcal{R}^{(\alpha)} \in W^{1,\infty}(\Omega)^3$ the functions defined on each node $(p\varepsilon, q\varepsilon)$ by:

$$\begin{aligned} \mathbb{U}^{(1)}(p\varepsilon, q\varepsilon) &= \mathbb{U}^{(1,q)}(p\varepsilon), \quad & \mathcal{R}^{(1)}(p\varepsilon, q\varepsilon) &= \mathcal{R}^{(1,q)}(p\varepsilon), \\ \mathbb{U}^{(2)}(p\varepsilon, q\varepsilon) &= \mathbb{U}^{(2,p)}(q\varepsilon), \quad & \mathcal{R}^{(2)}(p\varepsilon, q\varepsilon) &= \mathcal{R}^{(2,p)}(q\varepsilon). \end{aligned}$$

Then, we extend them by Q_1 interpolation on the vertexes of every cell $\varepsilon(p,q) + \varepsilon Y$ included in Ω (see also [14, Subsection 5.1]). Hence, we have

$$\begin{aligned} \mathbb{U}^{(1,q)} &= \mathbb{U}^{(1)}(\cdot, q\varepsilon) + \mathbb{U}_N^{(1)}(\cdot, q\varepsilon), \quad & \mathbb{U}^{(2,p)} &= \mathbb{U}^{(2)}(p\varepsilon, \cdot) + \mathbb{U}_N^{(2)}(p\varepsilon, \cdot), \\ \mathcal{R}^{(1,q)} &= \mathcal{R}^{(1)}(\cdot, q\varepsilon) + \mathcal{R}_N^{(1)}(\cdot, q\varepsilon), \quad & \mathcal{R}^{(2,p)} &= \mathcal{R}^{(2)}(p\varepsilon, \cdot) + \mathcal{R}_N^{(2)}(p\varepsilon, \cdot). \end{aligned} \tag{29}$$

The functions $\mathcal{R}_N^{(\alpha)},\ \mathbb{U}_N^{(\alpha)} \in H^1(L_\varepsilon^{(\alpha)})^3$ are the reminder terms covering the fast oscillations and have zero value on each node. The following lemma recalls the results of Lemma 1 for the new setting.

Lemma 2. [14, Lemma 5.2] The fields $\mathbb{U}_N^{(\alpha)}, \mathcal{R}_N^{(\alpha)}$ satisfy

$$\begin{aligned} &\|\mathbb{U}_N^{(\alpha)}\|_{L^2(L_\varepsilon^{(\alpha)})} + \varepsilon \|\partial_\alpha \mathbb{U}_N^{(\alpha)}\|_{L^2(L_\varepsilon^{(\alpha)})} \le C\|u\|_{S_\varepsilon} \\ &\|\mathcal{R}_N^{(\alpha)}\|_{L^2(L_\varepsilon^{(\alpha)})} + \varepsilon \|\partial_\alpha \mathcal{R}_N^{(\alpha)}\|_{L^2(L_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon}\|u\|_{S_\varepsilon}. \end{aligned} \tag{30}$$

The estimates on the warping (5) are also ported onto the complete structure by (3), leading to

$$\sum_{q=0}^{2N_\varepsilon} \left(\|\overline{u}^{(1,q)}\|^2_{L^2(q\varepsilon\mathbf{e}_2 + P_r^{(1)})} + \varepsilon^2 \|\nabla \overline{u}^{(1,q)}\|^2_{L^2(q\varepsilon\mathbf{e}_2 + P_r^{(1)})} \right) + \sum_{p=0}^{2N_\varepsilon} \left(\|\overline{u}^{(2,p)}\|^2_{L^2(p\varepsilon\mathbf{e}_1 + P_r^{(2)})} + \varepsilon^2 \|\nabla \overline{u}^{(2,p)}\|^2_{L^2(p\varepsilon\mathbf{e}_1 + P_r^{(2)})} \right) \le C\varepsilon^2 \|u\|^2_{S_\varepsilon}. \tag{31}$$

5.2 | Global estimates

These estimates are a direct consequence of (6), decomposition (29) and the extension results in [14, Subsection 5.1].

Lemma 3. [14, Lemma 5.3] One has

$$\begin{aligned}&\|\partial_\alpha \mathcal{R}^{(\alpha)}\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon},\\ &\|\partial_\alpha \mathbb{U}^{(\alpha)} - \mathcal{R}^{(\alpha)} \wedge \mathbf{e}_\alpha\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}\end{aligned} \tag{32}$$

As a consequence of this lemma we get

Lemma 4. The fields $\mathbb{U}^{(1)}$, $\mathbb{U}^{(2)}$, $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ satisfy

$$\begin{aligned}&\|\mathcal{R}^{(1)}\|_{L^2(\Omega_1\cup\Omega_2)} + \|\mathcal{R}^{(2)}\|_{L^2(\Omega_1\cup\Omega_3)} \leq \frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon},\\ &\|\mathbb{U}_1^{(1)}\|_{L^2(\Omega_1\cup\Omega_2)} + \|\mathbb{U}_2^{(2)}\|_{L^2(\Omega_1\cup\Omega_3)} \leq \frac{C}{\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}.\end{aligned} \tag{33}$$

Proof. The proof is a direct consequence of (32), the Poincaré inequality and the clamp conditions (22). □

Set the left and bottom boundary of Ω in the following way:

$$\begin{aligned}&\gamma_2 \doteq [0,l]\times\{0\} \subset \overline{\Omega}_1, && \Gamma_2 \doteq [0,L]\times\{0\} \subset \overline{\Omega_1\cup\Omega_2},\\ &\gamma_1 \doteq \{0\}\times[0,l] \subset \overline{\Omega}_1, && \Gamma_1 \doteq \{0\}\times[0,L] \subset \overline{\Omega_1\cup\Omega_3},\\ &\gamma \doteq \gamma_2\cup\gamma_1 \subset \overline{\Omega}_1, && \Gamma \doteq \Gamma_2\cup\Gamma_1 \subset \overline{\Omega}.\end{aligned} \tag{34}$$

The clamped conditions (22) and (7) give the following boundary conditions on $\mathbb{U}^{(\alpha)}$ and $\mathcal{R}^{(\alpha)}$

$$\mathbb{U}^{(1)} = \mathcal{R}^{(1)} = 0 \quad \text{a.e. on } \gamma_1, \qquad \mathbb{U}^{(2)} = \mathcal{R}^{(2)} = 0 \quad \text{a.e. on } \gamma_2. \tag{35}$$

5.3 | Non penetration condition estimates

Since the displacement switches the vertical position and since two fibers cannot penetrate one into the other, we get a sufficient good estimate for the distance between fibers in the outer plane component without assuming an additional upper bound contact function $g_{\varepsilon,3}$ in (21).

To show this, we start by giving the warping estimates in the contact areas.

Lemma 5. [14, Lemma 5.5] One has

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\left(\|\overline{u}^{(1,q)}\|^2_{L^2(\mathbf{C}_{pq})} + \|\overline{u}^{(2,p)}\|^2_{L^2(\mathbf{C}_{pq})}\right) \leq C\varepsilon\|u\|^2_{S_\varepsilon}$$

The main result is shown in the following lemma and due to the heavy computation, the entire proof is shifted to Appendix 10.

Lemma 6. One has

$$\left\|\mathbb{U}_3^{(1)} - \mathbb{U}_3^{(2)}\right\|_{L^2(\Omega)} + \varepsilon\left\|\mathcal{R}_\alpha^{(1)} - \mathcal{R}_\alpha^{(2)}\right\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}\|u\|_{S_\varepsilon}. \tag{36}$$

We now give all the estimates for the outer plane fields.

Lemma 7. One has

$$\|\mathbb{U}_3^{(\alpha)}\|_{H^1(\Omega)} + \|\mathcal{R}_\alpha^{(\beta)}\|_{H^1(\Omega)} \leq \frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon} \tag{37}$$

Proof. Since the proof is analogous, we just show the estimates for direction $\mathbf{e}_1$. First, it is easy to prove that the following inequalities hold:

$$\begin{aligned}&\|\phi\|^2_{L^2(\Omega)} \leq 2\frac{L}{l}\|\phi\|^2_{L^2(\Omega_1\cup\Omega_3)} + 2L^2\|\partial_1\phi\|^2_{L^2(\Omega)}, && \forall\phi\in L^2(\Omega,\partial_1),\\ &\|\psi\|^2_{L^2(\Omega)} \leq 2\frac{L}{l}\|\psi\|^2_{L^2(\Omega_1\cup\Omega_2)} + 2L^2\|\partial_2\psi\|^2_{L^2(\Omega)}, && \forall\psi\in L^2(\Omega,\partial_2).\end{aligned} \tag{38}$$

Now, from estimates $(32)_1$, (33), (36), the above (38) and the Q_1 interpolation properties, we have

$$\begin{aligned}\|\mathcal{R}_\alpha^{(1)}\|_{L^2(\Omega)} &\leq C\big(\|\mathcal{R}_\alpha^{(1)}\|_{L^2(\Omega_1\cup\Omega_3)} + \|\partial_1\mathcal{R}_\alpha^{(1)}\|_{L^2(\Omega)}\big)\\ &\leq C\big(\|\mathcal{R}_\alpha^{(1)} - \mathcal{R}_\alpha^{(2)}\|_{L^2(\Omega)} + \|\mathcal{R}_\alpha^{(2)}\|_{L^2(\Omega_1\cup\Omega_3)} + \|\partial_1\mathcal{R}_\alpha^{(1)}\|_{L^2(\Omega)}\big) \leq \frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon},\\ \|\partial_2\mathcal{R}_\alpha^{(1)}\|_{L^2(\Omega)} &\leq \frac{C}{\varepsilon}\|\mathcal{R}_\alpha^{(1)} - \mathcal{R}_\alpha^{(2)}\|_{L^2(\Omega)} + C\|\partial_2\mathcal{R}_\alpha^{(2)}\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}.\end{aligned}$$

This, together with $(32)_1$ proves $(32)_2$. We prove now $(37)_1$ for $\alpha = 1$. By $(32)_2$ and $(37)_2$, we first get that

$$\|\partial_1\mathbb{U}_3^{(1)}\|_{L^2(\Omega)} + \|\partial_2\mathbb{U}_3^{(2)}\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon},$$

which together with (36) and the Q_1-interpolation properties gives

$$\|\partial_2\mathbb{U}_3^{(1)}\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon}\|\mathbb{U}_3^{(1)} - \mathbb{U}_3^{(2)}\|_{L^2(\Omega)} + \|\partial_2\mathbb{U}_3^{(2)}\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}$$

and thus an estimate of the gradient of $\mathbb{U}_3^{(1)}$. By the Poincaré inequality and the clamp conditions (22) we get $(37)_1$ for $\alpha = 1$. The proof for $\alpha = 2$ is analogous. □

5.4 | Contact estimates

The remaining fields to estimate are the in plane fields in the not supported areas for the domain, that are, in the domains $\Omega_3 \cup \Omega_4$ for the fields in direction $\mathbf{e}_1$ and $\Omega_2 \cup \Omega_4$ for the fields in direction $\mathbf{e}_2$. In this sense, the 2D Korn's inequality and the fact that estimates (39) allow to switch between the supported direction to the not supported one are used. For more details on this method, see [14, Subsection 5.5].

Set

$$\|g\|_{L^\infty(\Omega)} \doteq \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}.$$

In the following lemma, we recall the difference between the displacements in the in plane components.

Lemma 8. [14, Lemma 5.6] One has

$$\Big\|\mathbb{U}_\alpha^{(1)} - \mathbb{U}_\alpha^{(2)}\Big\|_{L^2(\Omega)} + \varepsilon\Big\|\mathcal{R}_3^{(1)} - \mathcal{R}_3^{(2)}\Big\|_{L^2(\Omega)} \leq C\Big(\varepsilon^2\|g\|_{L^\infty(\Omega)} + \sqrt{\varepsilon}\|u\|_{S_\varepsilon}\Big). \tag{39}$$

Now, proceeding as in [14, Lemma 5.9 and Corollary 5.10] we get the H^1 norms of the in plane fields.

Lemma 9. One has

$$\|\mathbb{U}_\alpha^{(\beta)}\|_{H^1(\Omega)} + \|\mathcal{R}_3^{(\beta)}\|_{L^2(\Omega)} \leq C\Big(\varepsilon\|g\|_{L^\infty(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}\Big). \tag{40}$$

5.5 | Final decomposition of the displacements

Looking at the estimates for each field and the ones concerning their difference (36) and (39), we find convenient to define combined fields.

First, proceeding as in the proof of [14, Lemma 5.6] we obtain that the glued conditions (23) imply

$$\|\mathbb{U}^{(1)} - \mathbb{U}^{(2)}\|_{L^2(\Gamma)} + \varepsilon\|\mathcal{R}^{(1)} - \mathcal{R}^{(2)}\|_{L^2(\Gamma)} \leq C\|u\|_{S_\varepsilon}. \tag{41}$$

Regarding the outer plane fields, we set the decomposition

$$\begin{aligned}\mathbb{U}_3 &\doteq \frac{1}{2}\big(\mathbb{U}_3^{(1)} + \mathbb{U}_3^{(2)}\big), \qquad &\mathbb{U}_3^{(g)} &\doteq \frac{1}{2}\big(\mathbb{U}_3^{(1)} - \mathbb{U}_3^{(2)}\big),\\ \mathcal{R}_\alpha &\doteq \frac{1}{2}\big(\mathcal{R}_\alpha^{(1)} + \mathcal{R}_\alpha^{(2)}\big), \qquad &\mathcal{R}_\alpha^{(g)} &\doteq \frac{1}{2}\big(\mathcal{R}_\alpha^{(1)} - \mathcal{R}_\alpha^{(2)}\big).\end{aligned} \tag{42}$$

Due to the clamped conditions (35) and estimates (41), one has

$$\|\mathbb{U}_3\|_{L^2(\gamma)} + \varepsilon\|\mathcal{R}_\alpha\|_{L^2(\gamma)} \leq C\|u\|_{S_\varepsilon}. \tag{43}$$

We define the piece-wise linear functions $\mathbb{U}_1$ and $\mathbb{U}_2$ by

$$\mathbb{U}_1 \doteq \mathbb{U}_1^{(2)}(0,\cdot) \qquad \mathbb{U}_2 \doteq \mathbb{U}_2^{(1)}(\cdot,0) \qquad \text{a.e. in } (0,L).$$

The fields $\mathbb{U}_1$, $\mathbb{U}_2$ represent the macroscopic displacements of the mid-surface Ω.
Then, we define the fields $\mathbb{U}_\alpha^{(\mathbf{S})}$, $\mathbb{U}_\alpha^{(\mathbf{B})} \in H^1(\Omega)$ by the equalities

$$\begin{aligned} &\mathbb{U}_1^{(1)} = \mathbb{U}_1 + \mathbb{U}_1^{(\mathbf{S})}, \qquad && \mathbb{U}_2^{(1)} = \mathbb{U}_2 + \mathbb{U}_2^{(\mathbf{B})},\\ &\mathbb{U}_1^{(2)} = \mathbb{U}_1 + \mathbb{U}_1^{(\mathbf{B})}, && \mathbb{U}_2^{(2)} = \mathbb{U}_2 + \mathbb{U}_2^{(\mathbf{S})}, \end{aligned} \tag{44}$$

where $\mathbf{S}$ is for stretching and $\mathbf{B}$ for bending. The function $\mathbb{U}_\alpha^{(\mathbf{S})}$ stands for the relative stretching/compression of the beams whose direction is $\mathbf{e}_\alpha$, while $\mathbb{U}_\alpha^{(\mathbf{B})}$ represents the bending of the beams whose direction is $\mathbf{e}_{3-\alpha}$.
Note that

By the displacement representations (17), the extension operator and the splitting (29), the final decomposition of the displacements becomes ($(p,q) \in \mathcal{K}_\varepsilon$)

$$\begin{aligned} u^{(1,q)}(z) = {} & \left[\begin{pmatrix} \mathbb{U}_1 + \mathbb{U}_1^{(\mathbf{S})} \\ \mathbb{U}_2 + \mathbb{U}_2^{(\mathbf{B})} \\ \mathbb{U}_3 + \mathbb{U}_3^{(g)} \end{pmatrix}(z_1, q\varepsilon) + \mathbb{U}_N^{(1,q)}(z_1)\right] + \left[\begin{pmatrix} \mathcal{R}_1 + \mathcal{R}_1^{(g)} \\ \mathcal{R}_2 + \mathcal{R}_2^{(g)} \\ \mathcal{R}_3^{(1)} \end{pmatrix}(z_1, q\varepsilon) + \mathcal{R}_N^{(1,q)}(z_1)\right] \\ & \wedge \Big((-1)^{q+1}\Phi_\varepsilon(z_1)\mathbf{e}_3 + (z_2 - q\varepsilon)\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon^{(1,q)}(z_1)\Big) + \overline{u}^{(1,q)}(z), \\ u^{(2,p)}(z) = {} & \left[\begin{pmatrix} \mathbb{U}_1 + \mathbb{U}_1^{(\mathbf{B})} \\ \mathbb{U}_2 + \mathbb{U}_2^{(\mathbf{S})} \\ \mathbb{U}_3 - \mathbb{U}_3^{(g)} \end{pmatrix}(z_2, p\varepsilon) + \mathbb{U}_N^{(2,p)}(z_2)\right] + \left[\begin{pmatrix} \mathcal{R}_1 - \mathcal{R}_1^{(g)} \\ \mathcal{R}_2 - \mathcal{R}_2^{(g)} \\ \mathcal{R}_3^{(2)} \end{pmatrix}(z_2, p\varepsilon) + \mathcal{R}_N^{(2,p)}(z_2)\right] \\ & \wedge \Big((-1)^{p}\Phi_\varepsilon(z_2)\mathbf{e}_3 + (z_1 - p\varepsilon)\mathbf{e}_1 + z_3\mathbf{n}_\varepsilon^{(2,p)}(z_2)\Big) + \overline{u}^{(2,p)}(z). \end{aligned} \tag{45}$$

Define the spaces ($\alpha \in \{1,2\}$)

$$L^2(\Omega, \partial_\alpha) \doteq \left\{\phi \in L^2(\Omega) \;\middle|\; \partial_\alpha \phi \in L^2(\Omega)\right\}.$$

In the following Lemma, we present the estimates for such decomposition.

Lemma 10. One has

$$\begin{aligned} &\|\mathbb{U}_\alpha^{(\mathbf{S})}\|_{L^2(\Omega,\partial_\alpha)} + \varepsilon\|\partial_{3-\alpha}\mathbb{U}_\alpha^{(\mathbf{S})}\|_{L^2(\Omega)} \le \frac{C}{\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}, \\ &\|\mathbb{U}_\alpha^{(\mathbf{B})}\|_{L^2(\Omega)} + \varepsilon\|\nabla\mathbb{U}_\alpha^{(\mathbf{B})}\|_{L^2(\Omega)} \le C\Big(\varepsilon^2\|g\|_{L^\infty(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}\Big), \\ &\|\mathbb{U}_\alpha\|_{H^1(0,L)} \le C\Big(\varepsilon\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}\Big), \end{aligned} \tag{46}$$

and

$$\|\mathbb{U}_3\|_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\mathbb{U}_3^{(g)}\|_{L^2(\Omega)} + \|\nabla\mathbb{U}_3^{(g)}\|_{L^2(\Omega)} \le \frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}. \tag{47}$$

For the rotation fields $\mathcal{R}$ one has

$$\begin{aligned} &\|\mathcal{R}_\alpha\|_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\mathcal{R}_\alpha^{(g)}\|_{L^2(\Omega)} + \|\nabla\mathcal{R}_\alpha^{(g)}\|_{L^2(\Omega)} \le \frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}, \\ &\|\mathcal{R}_3^{(\alpha)}\|_{L^2(\Omega,\partial_\alpha)} + \varepsilon\|\partial_{3-\alpha}\mathcal{R}_3^{(\alpha)}\|_{L^2(\Omega)} \le C\Big(\varepsilon\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}\Big). \end{aligned} \tag{48}$$

Proof. We will only show the estimates for $\alpha = 1$, since the case $\alpha = 2$ follows by a symmetric argumentation.
By definition, we have $\partial_1\mathbb{U}_1 = 0$ and thus $\partial_1\mathbb{U}_1^{(\mathbf{S})} = \partial_1\mathbb{U}_1^{(1)}$. This, together with Poincaré inequality, $(32)_1$, (41) and the Q_1 character of interpolated functions, implies that

$$\begin{aligned} \|\mathbb{U}_1^{(\mathbf{S})}\|_{L^2(\Omega,\partial_1)} &\le \|\mathbb{U}_1^{(1)} - \mathbb{U}_1\|_{L^2(\Omega,\partial_1)} + \|\mathbb{U}_1^{(1)} - \mathbb{U}_1^{(2)}\|_{L^2(\Gamma_1)} \\ &\le C\|\partial_1\mathbb{U}_1^{(1)}\|_{L^2(\Omega)} + C\|u\|_{S_\varepsilon} \le \frac{C}{\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}, \\ \|\partial_2\mathbb{U}_1^{(\mathbf{S})}\|_{L^2(\Omega)} &\le \frac{C}{\varepsilon}\|\mathbb{U}_1^{(1)} - \mathbb{U}_1\|_{L^2(\Omega)} \le \frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}, \end{aligned}$$

which proves $(46)_1$.
The above estimates together with (39) lead to

$$\begin{aligned}\|\mathbb{U}_1^{(\mathbf{B})}\|_{L^2(\Omega)} = \|\mathbb{U}_1^{(2)} - \mathbb{U}_1\|_{L^2(\Omega)} &\le \|\mathbb{U}_1^{(2)} - \mathbb{U}_1^{(1)}\|_{L^2(\Omega)} + \|\mathbb{U}_1^{(\mathbf{S})}\|_{L^2(\Omega)}\\ &\le C\Big(\varepsilon^2\|g\|_{L^\infty(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}\Big),\end{aligned}$$

which together with the Q_1 character of interpolated functions proves $(46)_2$.
By the above estimates of $\partial_2\mathbb{U}_1^{(\mathbf{S})}$, (41) and $(40)_1$ we get

$$\begin{aligned}\|\mathbb{U}_1\|_{L^2(\Omega)} &\le \|\mathbb{U}_1^{(\mathbf{S})}\|_{L^2(\Omega)} + \|\mathbb{U}_1^{(1)} - \mathbb{U}_1^{(2)}\|_{L^2(\Gamma_1)} + \|\mathbb{U}_1^{(1)}\|_{L^2(\Omega)}\\ &\le C\Big(\varepsilon\|g\|_{L^\infty(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}\Big),\\ \|\partial_2\mathbb{U}_1\|_{L^2(\Omega)} &\le \|\partial_2\mathbb{U}_1^{(\mathbf{S})}\|_{L^2(\Omega)} + \|\partial_2\mathbb{U}_1^{(1)}\|_{L^2(\Omega)} \le C\Big(\varepsilon\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}\Big)\end{aligned}$$

which again together with the fact that $\partial_1\mathbb{U}_1 = 0$, proves $(46)_3$.
Estimates (47) and $(48)_1$ follow from (36) and (37), while estimate $(48)_2$ follow from $(40)_2$, $(32)_1$ and the Q_1 character of interpolated functions. □

The clamped and glued conditions (35)-(41) yield

$$\mathbb{U}_\alpha^{(\mathbf{S})} = 0 \quad \text{a.e. on } \gamma_\alpha, \qquad \|\mathbb{U}_\alpha\|_{L^2(\gamma_\alpha)} \le C\|u\|_{S_\varepsilon} \tag{49}$$

Now, we can use the remark 1 to define the global fields $\mathring{\mathbb{U}}^{(\alpha)}$, $\mathring{\mathcal{R}}^{(\alpha)} \in W^{1,\infty}(\Omega)^3$ in the following way ($(p,q) \in \mathcal{K}_\varepsilon$ and $(p',q') \in \{0,\dots,N_\varepsilon\}^2$):

$$\begin{aligned}\mathring{\mathbb{U}}^{(1)}(2p'\varepsilon, q\varepsilon) = \mathring{\mathbb{U}}^{(1,q)}(2p'\varepsilon), \quad \mathring{\mathcal{R}}^{(1)}(2p'\varepsilon, q\varepsilon) = \mathring{\mathcal{R}}^{(1,q)}(2p'\varepsilon),\\ \mathring{\mathbb{U}}^{(2)}(p\varepsilon, 2q'\varepsilon) = \mathring{\mathbb{U}}^{(2,p)}(2q'\varepsilon), \quad \mathring{\mathcal{R}}^{(2)}(p\varepsilon, 2q'\varepsilon) = \mathring{\mathcal{R}}^{(2,p)}(2q'\varepsilon).\end{aligned}$$

Then, the estimates (10) give

$$\|\mathbb{U}^{(\alpha)} - \mathring{\mathbb{U}}^{(\alpha)}\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\|u\|_{S_\varepsilon}, \qquad \|\nabla(\mathbb{U}^{(\alpha)} - \mathring{\mathbb{U}}^{(\alpha)})\|_{L^2(\Omega)} \le \frac{C}{\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}. \tag{50}$$

Hence, for the fields $\mathring{\mathbb{U}}^{(\alpha)}$, $\mathring{\mathcal{R}}^{(\alpha)}$ we obtain the same estimates as those in Lemmas 6, 7, 8, 9 and 10 ((47) and (48)).
Set

$$\mathring{\mathbb{U}}_1 \doteq \mathring{\mathbb{U}}_1^{(2)}(0,\cdot), \qquad \mathring{\mathbb{U}}_2 \doteq \mathring{\mathbb{U}}_2^{(1)}(\cdot,0).$$

Lemma 11. One has

$$\begin{aligned}&\|\mathbb{U}^{(\alpha)} - \mathring{\mathbb{U}}^{(\alpha)}\|_{L^2(\Gamma)} \le C\|u\|_{S_\varepsilon},\\ &\|\mathring{\mathbb{U}}_1\|_{H^1(\Omega)} + \|\mathring{\mathbb{U}}_2\|_{H^1(\Omega)} \le C\Big(\varepsilon\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon\sqrt{\varepsilon}}\|u\|_{S_\varepsilon}\Big),\\ &\|\mathbb{U}_\alpha - \mathring{\mathbb{U}}_\alpha\|_{L^2(\Omega)} + \varepsilon\|\nabla(\mathbb{U}_\alpha - \mathring{\mathbb{U}}_\alpha)\|_{L^2(\Omega)} \le C\|u\|_{S_\varepsilon}.\end{aligned} \tag{51}$$

Proof. Estimate $(51)_1$ is an immediate consequence of (50). Then, this estimate and the Q_1 character of these functions yield

$$\|\mathbb{U}_1 - \mathring{\mathbb{U}}_1^{(2)}(0,\cdot)\|_{L^2(0,L)} \le C\|u\|_{S_\varepsilon} \;\Rightarrow\; \|\partial_2(\mathbb{U}_1 - \mathring{\mathbb{U}}_1^{(2)}(0,\cdot))\|_{L^2(0,L)} \le \frac{C}{\varepsilon}\|u\|_{S_\varepsilon}.$$

The above estimates together with $(46)_3$ lead to the estimate $(51)_2$ of $\mathring{\mathbb{U}}_1 = \mathring{\mathbb{U}}_1^{(2)}(0,\cdot)$. Similarly we obtain the estimate of $(51)_2$ of $\mathring{\mathbb{U}}_2 = \mathring{\mathbb{U}}_2^{(1)}(\cdot,0)$. □

5.6 | Assumption on the right hand side

The forces applied must be chosen in order to keep the elasticity problem in a linear regime, since they heavily determine the gradient estimates. Indeed, the coercivity of A_ε applied to problem (25) with $v_\varepsilon = 0$ gives

$$C_0\|u_\varepsilon\|_{S_\varepsilon}^2 \le \int_{S_\varepsilon} A_\varepsilon E_x(u_\varepsilon)\cdot E_x(u_\varepsilon)\,dx \le \Big|\int_{S_\varepsilon} f_\varepsilon\cdot u_\varepsilon\,dx\Big|. \tag{52}$$

Hence, we first set $\widetilde{f}^{(\alpha)} \in H^1(\Omega)^2$ and $f^{(\alpha)} \in H^1(\Omega)^3$ with

$$\widetilde{f}_1^{(\alpha)} = 0 \quad \text{a.e. in } \ \Omega_3 \cup \Omega_4, \qquad \widetilde{f}_2^{(\alpha)} = 0 \quad \text{a.e. in } \ \Omega_2 \cup \Omega_4.$$

and define the forces in the straight reference beam (27) by

$$\begin{aligned} F_\varepsilon^{(1,q)}(z) &\doteq \big(\varepsilon^2 \widetilde{f}_1^{(1)}\mathbf{e}_1 + \varepsilon^2 \widetilde{f}_2^{(1)}\mathbf{e}_2 + \varepsilon^3 f^{(1)}\big)(z_1, q\varepsilon) \quad \text{for a.e. } z \text{ in } q\varepsilon\mathbf{e}_2 + P_r^{(1)}, \\ F_\varepsilon^{(2,p)}(z) &\doteq \big(\varepsilon^2 \widetilde{f}_1^{(2)}\mathbf{e}_1 + \varepsilon^2 \widetilde{f}_2^{(2)}\mathbf{e}_2 + \varepsilon^3 f^{(2)}\big)(p\varepsilon, z_2) \quad \text{for a.e. } z \text{ in } p\varepsilon\mathbf{e}_1 + P_r^{(2)}. \end{aligned}$$

Switching to the straight reference frame (see (28)), we estimate the right hand side using the above forces defined, the final displacements (45) and their estimates in Lemma 10 together with (31). We get

$$\Big|\int_{\mathcal{S}_\varepsilon} f_\varepsilon \cdot u_\varepsilon \, dx\Big| \le C\varepsilon^5 \sum_{\alpha=1}^{2} \big(\|\widetilde{f}^{(\alpha)}\|_{H^1(\Omega)} + \|f^{(\alpha)}\|_{H^1(\Omega)}\big)\Big(\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2\sqrt{\varepsilon}}\|u_\varepsilon\|_{\mathcal{S}_\varepsilon}\Big). \tag{53}$$

Finally, from (52)-(53), we obtain an estimate for the elastic energy

$$\|u_\varepsilon\|_{\mathcal{S}_\varepsilon} \le C\varepsilon^{5/2}\Big(\|g\|_{L^\infty(\Omega)} + \sum_{\alpha=1}^{2}\big(\|\widetilde{f}^{(\alpha)}\|_{H^1(\Omega)} + \|f^{(\alpha)}\|_{H^1(\Omega)}\big)\Big) \le C\varepsilon^{5/2}.$$

6 | ASYMPTOTIC BEHAVIOR OF THE FIELDS

In this section we consider a sequence $\{u_\varepsilon\}_\varepsilon$ of displacements belonging to $\mathcal{X}_\varepsilon$ and satisfying

$$\|u_\varepsilon\|_{\mathcal{S}_\varepsilon} \le C\varepsilon^{5/2}. \tag{54}$$

Applying (54) to the estimates in Lemma 10 and to $(32)_2$ we get

$$\begin{aligned} &\|\mathbb{U}_{\varepsilon,\alpha}\|_{H^1(\Omega)} + \|\mathbb{U}_{\varepsilon,3}\|_{H^1(\Omega)} \le C\varepsilon, \quad \|\mathbb{U}_{\varepsilon,3}^{(g)}\|_{L^2(\Omega)} + \varepsilon\|\nabla\mathbb{U}_{\varepsilon,3}^{(g)}\|_{L^2(\Omega)} \le C\varepsilon, \\ &\|\mathbb{U}_{\varepsilon,\alpha}^{(\mathbf{S})}\|_{L^2(\Omega,\partial_\alpha)} + \varepsilon\|\partial_{3-\alpha}\mathbb{U}_{\varepsilon,\alpha}^{(\mathbf{S})}\|_{L^2(\Omega)} + \|\mathbb{U}_\alpha^{(\mathbf{B})}\|_{L^2(\Omega)} + \varepsilon\|\nabla\mathbb{U}_\alpha^{(\mathbf{B})}\|_{L^2(\Omega)} \le C\varepsilon^2, \\ &\|\mathcal{R}_{\varepsilon,\alpha}\|_{H^1(\Omega)} \le C\varepsilon, \quad \|\mathcal{R}_{\varepsilon,\alpha}^{(g)}\|_{L^2(\Omega)} + \varepsilon\|\nabla\mathcal{R}_{\varepsilon,\alpha}^{(g)}\|_{L^2(\Omega)} \le C\varepsilon^2, \\ &\|\mathcal{R}_{\varepsilon,3}^{(\alpha)}\|_{L^2(\Omega,\partial_\alpha)} + \varepsilon\|\partial_{3-\alpha}\mathcal{R}_{\varepsilon,3}^{(\alpha)}\|_{L^2(\Omega)} \le C\varepsilon, \quad \|\partial_\alpha\mathbb{U}_\varepsilon^{(\alpha)} - \mathcal{R}_\varepsilon^{(\alpha)} \wedge \mathbf{e}_\alpha\|_{L^2(\Omega)} \le C\varepsilon^2. \end{aligned} \tag{55}$$

6.1 | Weak limit of the macroscopic fields

By the boundedness of the sequences (55), compactness results imply the weak convergences of the fields. Denote

$$\begin{aligned} H_{\boldsymbol{\gamma}}^1(\Omega) &\doteq \big\{\phi \in H^1(\Omega) \ \big| \ \phi = 0 \ \text{ a.e. } \boldsymbol{\gamma}\big\}, \\ H_{\boldsymbol{\gamma}}^2(\Omega) &\doteq \big\{\phi \in H^2(\Omega) \ \big| \ \phi = 0 \ \text{ and } \ \nabla\phi = 0 \ \text{ a.e. } \boldsymbol{\gamma}\big\} \end{aligned}$$

and

$$\begin{aligned} L_{(0,l)}^2\big((0,L)_{z_\alpha}\big) &\doteq \big\{\phi \in L^2\big((0,L)_{z_\alpha}\big) \ \big| \ \phi = 0 \ \text{ a.e. in } (0,l)\big\}, \\ H_{(0,l)}^k\big((0,L)_{z_\alpha}\big) &\doteq H^k\big((0,L)_{z_\alpha}\big) \cap L_{(0,l)}^2\big((0,L)_{z_\alpha}\big), \quad k \in \{1,2\}. \end{aligned}$$

The spaces $H_{(0,l)}^1\big((0,L)_{z_\alpha}\big)$ and $H_{(0,l)}^2\big((0,L)_{z_\alpha}\big)$ are for the functions that appear only in the not supported parts in the direction $\mathbf{e}_{3-\alpha}$ that is why they vanish due to the clamp condition in their direction.

Lemma 12. There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\mathbb{U}_1 \in H_{(0,l)}^2\big((0,L)_{z_2}\big)$, $\mathbb{U}_2 \in H_{(0,l)}^2\big((0,L)_{z_1}\big)$, $\mathcal{R}_3^{(\alpha)} \in H_{(0,l)}^1\big((0,L)_{z_\alpha}\big)$, $\mathbb{U}_3 \in H_{\boldsymbol{\gamma}}^2(\Omega)$ and $\mathcal{R}_\alpha \in H_{\boldsymbol{\gamma}}^1(\Omega)$ such that

$$\begin{aligned} &\frac{1}{\varepsilon}\mathbb{U}_{\varepsilon,\alpha} \rightharpoonup \mathbb{U}_\alpha \ \text{ weakly in } H^1(\Omega), \\ &\frac{1}{\varepsilon}\mathbb{U}_{\varepsilon,3} \to \mathbb{U}_3 \ \text{ strongly in } H^1(\Omega), \\ &\frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,\alpha} \rightharpoonup \mathcal{R}_\alpha \ \text{ weakly in } H^1(\Omega) \ \text{ and strongly in } \ L^2(\Omega), \\ &\frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,3}^{(\alpha)} \rightharpoonup \mathcal{R}_3^{(\alpha)} \ \text{ weakly in } L^2(\Omega,\partial_\alpha) \end{aligned} \tag{56}$$

and $\mathcal{Z}^{(\alpha)} \in L^2(\Omega)^3$ such that

$$\frac{1}{\varepsilon^2}\Big(\partial_\alpha \mathbb{U}_\varepsilon^{(\alpha)} - \mathcal{R}_\varepsilon^{(\alpha)} \wedge \mathbf{e}_\alpha\Big) \rightharpoonup \mathcal{Z}^{(\alpha)} \quad \text{weakly in } L^2(\Omega)^3. \tag{57}$$

Moreover, the following identities hold a.e. in Ω:

$$\mathcal{R}_2 = -\partial_1 \mathbb{U}_3, \qquad \mathcal{R}_1 = \partial_2 \mathbb{U}_3, \qquad \mathcal{R}_3^{(1)} = \partial_1 \mathbb{U}_2, \qquad \mathcal{R}_3^{(2)} = -\partial_2 \mathbb{U}_1. \tag{58}$$

Proof. Estimates in Lemma 10 imply the existence of $\mathbb{U}_\alpha,\ \mathcal{R}_\alpha \in H^1(\Omega)$ and $\mathcal{R}_3^{(\alpha)} \in L^2(\Omega, \partial_\alpha)$ such that the convergences $(56)_{1,3,4}$ hold, while (57) is a direct consequence of $(32)_2$. It also exists $\mathbb{U}_3 \in H^1(\Omega)$ such that

$$\frac{1}{\varepsilon}\mathbb{U}_{\varepsilon,3} \rightharpoonup \mathbb{U}_3 \quad \text{weakly in } H^1(\Omega).$$

From (57) we have

$$\frac{1}{\varepsilon}\Big(\partial_\alpha \mathbb{U}_\varepsilon^{(\alpha)} - \mathcal{R}_\varepsilon^{(\alpha)} \wedge \mathbf{e}_\alpha\Big) \to 0 \quad \text{strongly in } L^2(\Omega)^3. \tag{59}$$

Then, the above two convergences and $(56)_3$ yield $(56)_2$. Convergences (56) and (57) lead to the equalities (58). Now (58), the fact that $\mathbb{U}_1$ and $\mathbb{U}_2$ do not depend on z_1 and z_2 respectively and the boundary conditions (35)-(43)-(49)-(54) lead to $\mathbb{U}_1 \in H^2_{(0,l)}\big((0,L)_{z_2}\big)$, $\mathbb{U}_2 \in H^2_{(0,l)}\big((0,L)_{z_1}\big)$, $\mathcal{R}_3^{(\alpha)} \in H^1_{(0,l)}\big((0,L)_{z_\alpha}\big)$, $\mathbb{U}_3 \in H^2_\gamma(\Omega)$ and $\mathcal{R}_\alpha \in H^1_\gamma(\Omega)$. ☐

6.2 | Unfold of the limit fields via in plane unfolding operator

Recall that $\mathcal{Y} = [0,2)^2$ is the 2-periodic reference cell in Ω.

Definition 1 (In plane unfolding operator)**.** For every measurable function ϕ in $L^2(\Omega)$, one defines the measurable function $\mathcal{T}_\varepsilon(\phi)$ in $L^2(\Omega \times \mathcal{Y})$ by

$$\mathcal{T}_\varepsilon(\phi)(z', X') \doteq \phi\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big] + \varepsilon X'\Big) \qquad \text{for a.e. } (z', X') \in \Omega \times \mathcal{Y}.$$

Its properties are the typical ones and can be found in[6]. In particular, this operator satisfies

$$\|\mathcal{T}_\varepsilon(\phi)\|_{L^2(\Omega\times\mathcal{Y})} \leq C\|\phi\|_{L^2(\Omega)}, \qquad \forall \phi \in L^2(\Omega). \tag{60}$$

Moreover, we introduce the mean value operator $\mathcal{M}_\mathcal{Y} \ : \ L^1(\Omega \times \mathcal{Y}) \mapsto L^1(\Omega)$ by

$$\mathcal{M}_\mathcal{Y}(\phi)(\cdot) = \frac{1}{|\mathcal{Y}|}\int_\mathcal{Y} \psi(\cdot, X_1, X_2)dX_1 dX_2 \qquad \forall \psi \in L^1(\Omega \times \mathcal{Y})$$

Denote the set containing the nine nodes of $\mathcal{Y}$ by

$$\mathcal{Y}_K \doteq \{0,1,2\}^2.$$

We define the spaces of special Q_1 interpolates by

$$\begin{aligned}
\mathcal{Q}^1(\mathcal{Y}) &\doteq \big\{\phi \in W^{1,\infty}(\mathcal{Y}) \ \big|\ \phi \text{ is the } Q_1 \text{ interpolated of its values on } \mathcal{Y}_K\big\},\\
\mathcal{Q}^1_{per}(\mathcal{Y}) &\doteq \mathcal{Q}^1(\mathcal{Y}) \cap H^1_{per}(\mathcal{Y}),\\
\mathcal{Q}^1_{per,0}(\mathcal{Y}) &\doteq \big\{\phi \in \mathcal{Q}^1_{per}(\mathcal{Y}) \ \big|\ \mathcal{M}_\mathcal{Y}(\phi) = 0\big\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}^1((0,2)_{X_\alpha}) &\doteq \big\{\phi \in W^{1,\infty}((0,2)_{X_\alpha}) \ \big|\ \phi \text{ is piecewise linear in } [0,1] \text{ and } [1,2]\ \big\},\\
\mathcal{Q}^1_{per}((0,2)_{X_\alpha}) &\doteq \mathcal{Q}^1((0,2)_{X_\alpha}) \cap H^1_{per}((0,2)_{X_\alpha}),\\
\mathcal{Q}^1_{per,0}((0,2)_{X_\alpha}) &\doteq \Big\{\phi \in \mathcal{Q}^1_{per}((0,2)_{X_\alpha}) \ \Big|\ \int_0^2 \phi\, dX_\alpha = 0\Big\}.
\end{aligned}$$

While the unfolding for sequences bounded in $H^1(\Omega)$ and $L^2(\Omega)$ (see[6]) is well known, new results developed in[8] extend the unfolding method to sequences bounded anisotropically, i.e. in $L^2(\Omega, \partial_\alpha)$. Set

$$\begin{aligned}
L^2(\Omega; \mathcal{Q}^1_{per}((0,2)_{X_\alpha})) &\doteq \big\{\phi \in L^2(\Omega; \mathcal{Q}^1_{per}(\mathcal{Y})) \ \big|\ \phi \text{ only depends on } (z', X_\alpha)\big\},\\
L^2(\Omega, \partial_1; \mathcal{Q}^1_{per}((0,2)_{X_2})) &\doteq L^2(\Omega; \mathcal{Q}^1_{per}((0,2)_{X_2})) \cap L^2(\Omega \times (0,2)_{X_2}, \partial_1),\\
L^2(\Omega, \partial_2; \mathcal{Q}^1_{per}((0,2)_{X_1})) &\doteq L^2(\Omega; \mathcal{Q}^1_{per}((0,2)_{X_1})) \cap L^2(\Omega \times (0,2)_{X_1}, \partial_2).
\end{aligned}$$

Moreover, due to the boundary conditions (22)-(23) we set

$$\mathbf{L}^2(\Omega,\partial_\alpha)\doteq\{\phi\in L^2(\Omega,\partial_\alpha)\mid \phi=0 \text{ a.e. on } \Gamma_{3-\alpha}\},$$
$$\mathbf{L}^2(\Omega,\partial_\alpha;\mathcal{Q}^1_{per}((0,2)_{X_{3-\alpha}}))\doteq\Big\{\phi\in L^2(\Omega,\partial_\alpha;\mathcal{Q}^1_{per}((0,2)_{X_{3-\alpha}}))\Big|\ \phi=0 \text{ a.e. on } \Gamma_{3-\alpha}\times(0,2)_{X_{3-\alpha}}\Big\}.$$

We are ready to give the asymptotic behaviour of our unfolded sequences.

Lemma 13. There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\widehat{\mathcal{R}}_\alpha\in L^2(\Omega;\mathcal{Q}^1_{per,0}(\mathcal{Y}))$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathbb{U}_{\varepsilon,\alpha}\big),\ \frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathring{\mathbb{U}}_{\varepsilon,\alpha}\big)\to\mathbb{U}_\alpha \ \text{ strongly in } L^2(\Omega;\mathcal{Q}^1(\mathcal{Y})),\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\nabla\mathbb{U}_{\varepsilon,\alpha}\big),\ \frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\nabla\mathring{\mathbb{U}}_{\varepsilon,\alpha}\big)\rightharpoonup\nabla\mathbb{U}_\alpha \ \text{ weakly in } L^2(\Omega\times\mathcal{Y})^2,\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathbb{U}_{\varepsilon,3}\big)\to\mathbb{U}_3 \ \text{ strongly in } L^2(\Omega;\mathcal{Q}^1(\mathcal{Y})),\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\nabla\mathbb{U}_{\varepsilon,3}\big)\to\nabla\mathbb{U}_3 \ \text{ strongly in } L^2(\Omega\times\mathcal{Y})^2,\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathcal{R}_{\varepsilon,\alpha}\big)\to\mathcal{R}_\alpha \ \text{ strongly in } L^2(\Omega;\mathcal{Q}^1(\mathcal{Y})),\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\nabla\mathcal{R}_{\varepsilon,\alpha}\big)\rightharpoonup\nabla\mathcal{R}_\alpha+\nabla_X\widehat{\mathcal{R}}_\alpha \ \text{ weakly in } L^2(\Omega\times\mathcal{Y})^2
\end{aligned}\tag{61}$$

and $\widehat{\mathbb{U}}^{(\mathbf{S})}_\alpha\in L^2(\Omega\times(0,2)_{X_{3-\alpha}};\mathcal{Q}^1_{per,0}((0,2)_{X_\alpha}))\cap L^2(\Omega;\mathcal{Q}^1(\mathcal{Y}))$, $\mathbb{U}^{(\mathbf{S})}_\alpha$ in the space $\mathbf{L}^2(\Omega,\partial_\alpha;\mathcal{Q}^1_{per}((0,2)_{X_{3-\alpha}}))$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\mathbb{U}^{(\mathbf{S})}_{\varepsilon,\alpha}\big)\rightharpoonup\mathbb{U}^{(\mathbf{S})}_\alpha \ \text{ weakly in } L^2(\Omega;\mathcal{Q}^1(\mathcal{Y})),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_\alpha\mathbb{U}^{(\mathbf{S})}_{\varepsilon,\alpha}\big)\rightharpoonup\partial_\alpha\mathbb{U}^{(\mathbf{S})}_\alpha+\partial_{X_\alpha}\widehat{\mathbb{U}}^{(\mathbf{S})}_\alpha \ \text{ weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathcal{R}^{(\alpha)}_{\varepsilon,3}\big)\rightharpoonup\mathcal{R}^{(\alpha)}_3 \ \text{ weakly in } L^2(\Omega;\mathcal{Q}^1(\mathcal{Y}))
\end{aligned}\tag{62}$$

and also $\mathbb{U}^{(\mathbf{B})}_\alpha\in L^2(\Omega;\mathcal{Q}^1_{per}((0,2)_{X_\alpha}))$, $\widehat{\mathbb{U}}^{(g)}_3$, $\widehat{\mathcal{R}}^{(g)}_\alpha\in L^2(\Omega;\mathcal{Q}^1_{per}(\mathcal{Y}))$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\mathbb{U}^{(\mathbf{B})}_{\varepsilon,\alpha}\big)\rightharpoonup\mathbb{U}^{(\mathbf{B})}_\alpha \ \text{ weakly in } L^2(\Omega;\mathcal{Q}^1(\mathcal{Y})),\\
&\frac{1}{\varepsilon^3}\mathcal{T}_\varepsilon\big(\mathbb{U}^{(g)}_{\varepsilon,3}\big)\rightharpoonup\widehat{\mathbb{U}}^{(g)}_3 \ \text{ weakly in } L^2(\Omega;\mathcal{Q}^1(\mathcal{Y})),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\mathcal{R}^{(g)}_{\varepsilon,\alpha}\big)\rightharpoonup\widehat{\mathcal{R}}^{(g)}_\alpha \ \text{ weakly in } L^2(\Omega;\mathcal{Q}^1(\mathcal{Y})).
\end{aligned}\tag{63}$$

Proof. First, remind that if a sequence $\{\phi_\varepsilon\}_\varepsilon$ is defined as Q_1 interpolate on the nodes (p,q) of $\overline{\Omega}$ (see subsection 5.1), by construction it follows that

$$\{\mathcal{T}_\varepsilon(\phi_\varepsilon)\}_\varepsilon\subset L^p(\Omega;\mathcal{Q}^1(\mathcal{Y})).\tag{64}$$

Due to the estimates (55) and the convergences in Lemma 12, we get $(61)_{1,3,4,5,6}$ by[5, Corollary 1.37, Proposition 1.39, Theorem 1.41]. Since $\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\nabla\mathring{\mathbb{U}}_{\varepsilon,\alpha}\big)$ does not depend on the microscopic variables, its limit (given by $(61)_2$) does not depend on them either. Due to the estimate $(51)_3$, $\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\nabla\mathbb{U}_{\varepsilon,\alpha}\big)$ converges to the same limit.

Convergences $(62)_{1,2}$ are consequences of[8, Lemma 4.3] and (63) (with the function $\mathbb{U}^{(\mathbf{B})}_\alpha\in L^2(\Omega;\mathcal{Q}^1_{per}(\mathcal{Y}))$) by[5, Theorem 1.36].

There also exists $\widetilde{\mathcal{R}}^{(\alpha)}_3\in L^2(\Omega,\partial_1;\mathcal{Q}^1_{per}((0,2)_{X_{3-\alpha}}))$ (see again[8, Lemma 4.3]) such that

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathcal{R}^{(\alpha)}_{\varepsilon,3}\big)\rightharpoonup\widetilde{\mathcal{R}}^{(\alpha)}_3 \ \text{ weakly in } L^2(\Omega;\mathcal{Q}^1(\mathcal{Y})).$$

From estimate (59) and the above convergences together with (61)-(63), we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}^{(1)}_{\varepsilon,2}-\mathcal{R}^{(1)}_{\varepsilon,3}\big)\rightharpoonup 0=\partial_1\mathbb{U}_2+\partial_{X_1}\mathbb{U}^{(\mathbf{B})}_2-\widetilde{\mathcal{R}}^{(1)}_3 \ \text{ weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}^{(2)}_{\varepsilon,1}+\mathcal{R}^{(2)}_{\varepsilon,3}\big)\rightharpoonup 0=\partial_2\mathbb{U}_1+\partial_{X_2}\mathbb{U}^{(\mathbf{B})}_1+\widetilde{\mathcal{R}}^{(2)}_3 \ \text{ weakly in } L^2(\Omega\times\mathcal{Y}).
\end{aligned}$$

Since $\mathbb{U}_2^{(\mathbf{B})}$ is 2-periodic with respect to X_1 and $\widetilde{\mathcal{R}}_3^{(1)}$ does not depend on X_1 (resp. since $\mathbb{U}_1^{(\mathbf{B})}$ is 2-periodic with respect to X_2 and $\widetilde{\mathcal{R}}_3^{(2)}$ does not depend on X_2) and besides since $\partial_1\mathbb{U}_3+\mathcal{R}_2=0$ (resp. $\partial_2\mathbb{U}_3-\mathcal{R}_1=0$) by (58), we get

$$\partial_{X_1}\mathbb{U}_2^{(\mathbf{B})}=0,\qquad \partial_{X_2}\mathbb{U}_1^{(\mathbf{B})}=0. \tag{65}$$

Hence, the function $\mathbb{U}_2^{(\mathbf{B})}$ (resp. $\mathbb{U}_1^{(\mathbf{B})}$) does not depend on the microscopic variable X_1 (resp. X_2). So, we have $\mathbb{U}_1^{(\mathbf{B})}\in L^2(\Omega;\mathcal{Q}^1_{per}((0,2)_{X_1}))$, $\mathbb{U}_2^{(\mathbf{B})}\in L^2(\Omega;\mathcal{Q}^1_{per}((0,2)_{X_2}))$. Moreover, from the above convergences and (65), we obtain

$$\partial_1\mathbb{U}_2=\widetilde{\mathcal{R}}_3^{(1)},\qquad \partial_2\mathbb{U}_1=-\widetilde{\mathcal{R}}_3^{(2)}. \tag{66}$$

Using (58) and equalities (66), we get $\widetilde{\mathcal{R}}_3^{(1)}=\mathcal{R}_3^{(1)}$ and $\widetilde{\mathcal{R}}_3^{(2)}=\mathcal{R}_3^{(2)}$. This $(62)_3$. □

Now we unfold convergence (57) and the derivatives of $(56)_4$.

Lemma 14. There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\widehat{\mathcal{R}}_3^{(\alpha)}\in L^2(\Omega;\mathcal{Q}^1_{per,0}(\mathcal{Y}))$ such that

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\partial_\alpha\mathcal{R}^{(\alpha)}_{\varepsilon,3}\big)\rightharpoonup\partial_\alpha\mathcal{R}_3^{(\alpha)}+\partial_{X_\alpha}\widehat{\mathcal{R}}_3^{(\alpha)}\ \text{ weakly in } L^2(\Omega\times\mathcal{Y}). \tag{67}$$

There exist $\widehat{\mathbb{U}}_2^{(1)}$, $\widehat{\mathbb{U}}_1^{(2)}$, $\widehat{\mathbb{U}}_3\in L^2(\Omega;\mathcal{Q}^1_{per,0}(\mathcal{Y}))$ and $\widetilde{\mathcal{Z}}_\alpha^{(3-\alpha)}\in L^2(\Omega;\mathcal{Q}^1_{per}((0,2)_{X_\alpha}))$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}^{(1)}_\varepsilon-\mathcal{R}^{(1)}_\varepsilon\wedge\mathbf{e}_1\big)\cdot\mathbf{e}_3\rightharpoonup\mathcal{Z}_3^{(1)}+\partial_{X_1}(\widehat{\mathbb{U}}_3+\widehat{\mathbb{U}}_3^{(g)})+(\widehat{\mathcal{R}}_2+\widehat{\mathcal{R}}_2^{(g)}),\quad\text{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}^{(1)}_\varepsilon-\mathcal{R}^{(1)}_\varepsilon\wedge\mathbf{e}_1\big)\cdot\mathbf{e}_2\rightharpoonup\widetilde{\mathcal{Z}}_2^{(1)}+\partial_{X_1}\widehat{\mathbb{U}}_2^{(1)}-\widehat{\mathcal{R}}_3^{(1)}\quad\text{weakly in } L^2(\Omega\times\mathcal{Y})
\end{aligned} \tag{68}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}^{(2)}_\varepsilon-\mathcal{R}^{(2)}_\varepsilon\wedge\mathbf{e}_2\big)\cdot\mathbf{e}_3\rightharpoonup\mathcal{Z}_3^{(2)}+\partial_{X_2}(\widehat{\mathbb{U}}_3-\widehat{\mathbb{U}}_3^{(g)})-(\widehat{\mathcal{R}}_1-\widehat{\mathcal{R}}_1^{(g)})\quad\text{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}^{(2)}_\varepsilon-\mathcal{R}^{(2)}_\varepsilon\wedge\mathbf{e}_2\big)\cdot\mathbf{e}_1\rightharpoonup\widetilde{\mathcal{Z}}_1^{(2)}+\partial_{X_2}\widehat{\mathbb{U}}_1^{(2)}+\widehat{\mathcal{R}}_3^{(2)}\quad\text{weakly in } L^2(\Omega\times\mathcal{Y}).
\end{aligned} \tag{69}$$

Proof. We start by proving (67) in direction $\mathbf{e}_1$ and $(68)_2$. By estimates (32)-(40)-(55) and the Q_1 interpolation properties, we have

$$\begin{aligned}
&\|\mathbb{U}^{(1)}_{\varepsilon,2}\|_{H^1(\Omega)}\le C\varepsilon,\quad \|\mathcal{R}^{(1)}_{\varepsilon,3}\|_{L^2(\Omega,\partial_1)}+\varepsilon\|\partial_2\mathcal{R}^{(1)}_{\varepsilon,3}\|_{L^2(\Omega)}\le C\varepsilon,\\
&\|\partial_2(\partial_1\mathbb{U}^{(1)}_{\varepsilon,2}-\mathcal{R}^{(1)}_{\varepsilon,3})\|_{L^2(\Omega)}+\varepsilon\|\partial_2(\partial_1\mathcal{R}^{(1)}_{\varepsilon,3})\|_{L^2(\Omega)}\le C\varepsilon.
\end{aligned}$$

Moreover, convergence (57) holds.
Hence, by[8, Lemma 5.4] and (64), there exist $\widehat{\mathcal{R}}_3^{(1)}$, $\widehat{\mathbb{U}}_2^{(1)}\in L^2(\Omega;\mathcal{Q}^1_{per,0}(\mathcal{Y}))$ and $\widetilde{\mathcal{Z}}_2^{(1)}\in L^2(\Omega;\mathcal{Q}^1_{per}((0,2)_{X_2}))$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\partial_1\mathcal{R}^{(1)}_{\varepsilon,3}\big)\rightharpoonup\partial_1\mathcal{R}_3^{(1)}+\partial_{X_1}\widehat{\mathcal{R}}_3^{(1)}\ \text{ weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}^{(1)}_{\varepsilon,2}-\mathcal{R}^{(1)}_{\varepsilon,3}\big)\rightharpoonup\widetilde{\mathcal{Z}}_2^{(1)}+\partial_{X_1}\widehat{\mathbb{U}}_2^{(1)}-\widehat{\mathcal{R}}_3^{(1)}\ \text{ weakly in } L^2(\Omega\times\mathcal{Y}).
\end{aligned}$$

Direction $\mathbf{e}_2$ follows by analogous argumentation, thus (67) and $(68)_2$-$(69)_2$ hold.

Now we prove $(68)_1$-$(69)_1$. By convergences (57)-$(61)_6$ and estimates (47)-$(48)_1$,[5, Lemma 11.11] and property (64) applied to the sequences $(\mathbb{U}_{3,\varepsilon},\mathcal{R}_{2,\varepsilon})$, $(\mathbb{U}_{3,\varepsilon},-\mathcal{R}_{\varepsilon,1})$, there exist $\widehat{\mathbb{U}}_3\in L^2(\Omega;\mathcal{Q}^1_{per,0}(\mathcal{Y}))$ such that, up to a subsequence,

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}_{\varepsilon,3}+\mathcal{R}_{\varepsilon,2}\big)\rightharpoonup\mathcal{Z}_3^{(1)}+\partial_{X_1}\widehat{\mathbb{U}}_3+\widehat{\mathcal{R}}_2\quad\text{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}_{\varepsilon,3}-\mathcal{R}_{\varepsilon,1}\big)\rightharpoonup\mathcal{Z}_3^{(2)}+\partial_{X_2}\widehat{\mathbb{U}}_3-\widehat{\mathcal{R}}_1\quad\text{weakly in } L^2(\Omega\times\mathcal{Y}).
\end{aligned}$$

Hence, by convergences $(63)_{2,3}$ and decompositions (42) we have

$$\begin{aligned}
\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}^{(1)}_{\varepsilon,3}+\mathcal{R}^{(1)}_{\varepsilon,2}\big)&=\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}_{\varepsilon,3}+\mathcal{R}_{\varepsilon,2}\big)+\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}^{(g)}_{\varepsilon,3}\big)+\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\mathcal{R}^{(g)}_{\varepsilon,2}\big)\\
&\rightharpoonup\mathcal{Z}_3^{(1)}+\partial_{X_1}\widehat{\mathbb{U}}_3+\partial_{X_1}\widehat{\mathbb{U}}_3^{(g)}+\widehat{\mathcal{R}}_2+\widehat{\mathcal{R}}_2^{(g)}\quad\text{weakly in } L^2(\Omega\times\mathcal{Y}),\\
\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}^{(2)}_{\varepsilon,3}-\mathcal{R}^{(2)}_{\varepsilon,1}\big)&=\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}_{\varepsilon,3}-\mathcal{R}_{\varepsilon,1}\big)-\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}^{(g)}_{\varepsilon,3}\big)+\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\mathcal{R}^{(g)}_{\varepsilon,1}\big)\\
&\rightharpoonup\mathcal{Z}_3^{(1)}+\partial_{X_1}\widehat{\mathbb{U}}_3-\partial_{X_1}\widehat{\mathbb{U}}_3^{(g)}-\widehat{\mathcal{R}}_1+\widehat{\mathcal{R}}_1^{(g)}\quad\text{weakly in } L^2(\Omega\times\mathcal{Y})
\end{aligned}$$

and thus $(68)_1$-$(69)_1$ hold. □

6.3 | Unfold of the complete structure via global unfolding operator

We note that the limit functions have been unfolded just in the 2D plane Ω. However, the nature of the textile is a 3D structure and that is why a new unfolding operator is introduced. Set

$$Cyl^{(1)} \doteq (0,2)\times(-\kappa,\kappa)\times(-\kappa,\kappa),\qquad Cyl^{(2)} \doteq (-\kappa,\kappa)\times(0,2)\times(-\kappa,\kappa),\qquad Cyl \doteq Cyl^{(1)}\times Cyl^{(2)}.$$

Definition 2 (Global unfolding operator)**.** For every measurable function ϕ in $L^1(S_\varepsilon^{(1)})$ and ψ in $L^1(S_\varepsilon^{(2)})$, one defines the measurable functions $\Pi_\varepsilon^{(1,b)}(\phi)$ in the space $L^1(\Omega\times Cyl^{(1)})$ and $\Pi_\varepsilon^{(2,a)}(\psi)$ in $L^1(\Omega\times Cyl^{(2)})$ respectively by ($(a,b)\in\{0,1\}^2$)

$$\Pi_\varepsilon^{(1,b)}(\phi)(z',X) \doteq \phi\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big]+\varepsilon b\,\mathbf{e}_2+\varepsilon X\Big),\qquad \text{for a.e. } (z',X)\in\Omega\times Cyl^{(1)},$$
$$\Pi_\varepsilon^{(2,a)}(\psi)(z',X) \doteq \psi\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big]+\varepsilon a\,\mathbf{e}_1+\varepsilon X\Big),\qquad \text{for a.e. } (z',X)\in\Omega\times Cyl^{(2)}.$$

Note that this unfolding operator changes the convergence rate, since a dimension reduction is additionally applied.

Lemma 15. For every $\phi\in L^1(S_\varepsilon^{(\alpha)})$ one has

$$\int_{S_\varepsilon^{(\alpha)}}\phi(z)dz \le \frac{\varepsilon}{4}\sum_{c=0}^{1}\int_\Omega\int_{Cyl^{(\alpha)}}\Pi_\varepsilon^{(\alpha,c)}(\phi)(z',X)dz'dX.$$

As a direct consequence, we get that

$$\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\|\Pi_\varepsilon^{(\alpha,c)}(\phi)\|_{L^2(\Omega\times Cyl^{(\alpha)})}\le\frac{C}{\sqrt{\varepsilon}}\|\phi\|_{L^2(S_\varepsilon)},\qquad \forall\phi\in L^2(S_\varepsilon). \tag{70}$$

In order to apply the global unfolding $\Pi_\varepsilon^{(\alpha,c)}$ to the unfolded fields in Lemma 12, note that the in plane operator and the global unfolding operators are related in the following way: for each ϕ defined on $\mathcal{K}_\varepsilon$, its extension $\boldsymbol{\phi}\in W^{1,\infty}(\Omega)$ (see subsection 5.1) satisfies the equalities of the traces

$$\Pi_\varepsilon^{(1,b)}(\boldsymbol{\phi})(z',X_1,0)=\boldsymbol{\phi}\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big]+\varepsilon X_1\mathbf{e}_1+\varepsilon b\mathbf{e}_2\Big)=\mathcal{T}_\varepsilon(\boldsymbol{\phi})(z',X_1,b),\quad \text{for a.e. } (z',X_1)\in\Omega\times(0,2),\quad b\in\{0,1\},$$
$$\Pi_\varepsilon^{(2,a)}(\boldsymbol{\phi})(z',0,X_2)=\boldsymbol{\phi}\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big]+\varepsilon a\mathbf{e}_1+\varepsilon X_2\mathbf{e}_2\Big)=\mathcal{T}_\varepsilon(\boldsymbol{\phi})(z',a,X_2),\quad \text{for a.e. } (z',X_2)\in\Omega\times(0,2),\quad a\in\{0,1\}.$$

Hence, unfolding via $\Pi_\varepsilon^{(\alpha,c)}$ is equivalent to the unfolding via $\mathcal{T}_\varepsilon$ restricted to the beams centerlines in the respective direction. We are ready to give the strain tensor convergences.

Lemma 16. Under the assumptions of Lemma 12, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and fields $\mathfrak{R}^{(\alpha,c)}\in L^2(\Omega;H^1_{per}((0,2)_{X_\alpha}))^3$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon}\Pi_\varepsilon^{(1,b)}\big(\partial_1\mathcal{R}_\varepsilon^{(1,q)}\big)\rightharpoonup\begin{pmatrix}\partial_{12}\mathbb{U}_3\\-\partial_{11}\mathbb{U}_3\\\partial_{11}\mathbb{U}_2\end{pmatrix}+\partial_{X_1}\mathfrak{R}^{(1,b)}\quad\text{weakly in } L^2(\Omega\times Cyl^{(1)})^3,\\
&\frac{1}{\varepsilon}\Pi_\varepsilon^{(2,a)}\big(\partial_2\mathcal{R}_\varepsilon^{(2,p)}\big)\rightharpoonup\begin{pmatrix}\partial_{22}\mathbb{U}_3\\-\partial_{12}\mathbb{U}_3\\-\partial_{22}\mathbb{U}_1\end{pmatrix}+\partial_{X_2}\mathfrak{R}^{(2,a)}\quad\text{weakly in } L^2(\Omega\times Cyl^{(2)})^3
\end{aligned}\tag{71}$$

and $\mathfrak{U}^{(\alpha,c)}\in L^2(\Omega;H^1_{per}((0,2)_{X_\alpha}))^3$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(1,b)}(\partial_1\mathbb{U}_\varepsilon^{(1,q)}-\mathcal{R}_\varepsilon^{(1,q)}\wedge\mathbf{e}_1)\rightharpoonup\begin{pmatrix}\partial_1\mathbb{U}_1^{(\mathbf{S},b)}\\\widetilde{\mathcal{Z}}_2^{(1,b)}\\\mathcal{Z}_3^{(1)}\end{pmatrix}+\partial_{X_1}\mathfrak{U}^{(1,b)}-\mathfrak{R}^{(1,b)}\wedge\mathbf{e}_1\quad\text{weakly in } L^2(\Omega\times Cyl^{(1)})^3,\\
&\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(2,a)}(\partial_2\mathbb{U}_\varepsilon^{(2,p)}-\mathcal{R}_\varepsilon^{(2,p)}\wedge\mathbf{e}_2)\rightharpoonup\begin{pmatrix}\widetilde{\mathcal{Z}}_1^{(2,a)}\\\partial_2\mathbb{U}_2^{(\mathbf{S},a)}\\\mathcal{Z}_3^{(2)}\end{pmatrix}+\partial_{X_2}\mathfrak{U}^{(2,a)}-\mathfrak{R}^{(2,a)}\wedge\mathbf{e}_2\quad\text{weakly in } L^2(\Omega\times Cyl^{(2)})^3,
\end{aligned}\tag{72}$$

where $\mathbb{U}_1^{(\mathbf{S},b)} = \mathbb{U}^{(\mathbf{S})}_{1|X_2=b} \in \mathbf{L}^2(\Omega,\partial_1)$, $\mathbb{U}_2^{(\mathbf{S},a)} = \mathbb{U}^{(\mathbf{S})}_{2|X_1=a} \in \mathbf{L}^2(\Omega,\partial_2)$, $\widetilde{\mathcal{Z}}_2^{(1,b)} = \widetilde{\mathcal{Z}}^{(1)}_{2|X_2=b}$, $\widetilde{\mathcal{Z}}_1^{(2,a)} = \widetilde{\mathcal{Z}}^{(2)}_{1|X_1=a} \in L^2(\Omega)$.

Proof. We just prove direction $\mathbf{e}_1$, since the second one follows the same argumentation. From splitting (29) and the Q_1 extension properties, we first obtain

$$\mathcal{R}_\varepsilon^{(1,q)} = \mathcal{R}_\varepsilon^{(1)} + \mathcal{R}_{\varepsilon,N}^{(1)}, \qquad \mathbb{U}_\varepsilon^{(1,q)} = \mathbb{U}_\varepsilon^{(1)} + \mathbb{U}_{\varepsilon,N}^{(1)}.$$

By estimates (30), there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\widehat{\mathcal{R}}_N^{(1,b)}, \widehat{\mathbb{U}}_N^{(1,b)} \in L^2(\Omega; H^1_{per}((0,2)_{X_1}))^3$ such that

$$\begin{aligned} &\frac{1}{\varepsilon^3}\Pi_\varepsilon^{(1,b)}\big(\mathbb{U}_{\varepsilon,N}^{(1)}\big) \rightharpoonup \widehat{\mathbb{U}}_N^{(1,b)} \ \text{ weakly in } L^2(\Omega; H^1(Cyl^{(1)}))^3,\\ &\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(1,b)}\big(\mathcal{R}_{\varepsilon,N}^{(1)}\big) \rightharpoonup \widehat{\mathcal{R}}_N^{(1,b)} \ \text{ weakly in } L^2(\Omega; H^1(Cyl^{(1)}))^3. \end{aligned} \tag{73}$$

By convergences $(61)_6$, $(63)_3$, (67) and the fact that Q_1 convergences can be restricted to the beams centerlines, we have

$$\begin{aligned} \frac{1}{\varepsilon}\Pi_\varepsilon^{(1,b)}\big(\partial_1\mathcal{R}_\varepsilon^{(1,q)}\big) &= \frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\partial_1\mathcal{R}_\varepsilon^{(1)}\big)_{|X_2=b} + \frac{1}{\varepsilon}\Pi_\varepsilon^{(1,b)}\big(\mathcal{R}_{\varepsilon,N}^{(1)}\big)\\ &\rightharpoonup \partial_1\begin{pmatrix}\mathcal{R}_1\\ \mathcal{R}_2\\ \mathcal{R}_3^{(1)}\end{pmatrix} + \partial_{X_1}\begin{pmatrix}\big(\widehat{\mathcal{R}}_1 + \widehat{\mathcal{R}}_1^{(g)}\big)_{|X_2=b} + \widehat{\mathcal{R}}_{N,1}^{(1,b)}\\ \big(\widehat{\mathcal{R}}_2 + \widehat{\mathcal{R}}_2^{(g)}\big)_{|X_2=b} + \widehat{\mathcal{R}}_{N,2}^{(1,b)}\\ \widehat{\mathcal{R}}^{(1)}_{3|X_2=b} + \widehat{\mathcal{R}}_{N,3}^{(1,b)}\end{pmatrix} \quad \text{weakly in } L^2(\Omega\times Cyl^{(1)})^3. \end{aligned}$$

By convergences $(62)_2$, (68), (69), $(73)_{1,2}$ and the fact that Q_1 convergences can be restricted to the beams centerlines, we get

$$\begin{aligned} \frac{1}{\varepsilon^2}\Pi_\varepsilon^{(1,b)}(\partial_1\mathbb{U}_\varepsilon^{(1,q)} - \mathcal{R}_\varepsilon^{(1,q)}\wedge\mathbf{e}_1) &= \frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\partial_1\mathbb{U}_\varepsilon^{(1)} - \mathcal{R}_\varepsilon^{(1)}\wedge\mathbf{e}_1)_{|X_2=b} + \frac{1}{\varepsilon^2}\Pi_\varepsilon^{(1,b)}(\partial_1\mathbb{U}_{\varepsilon,N}^{(1)} - \mathcal{R}_{\varepsilon,N}^{(1)}\wedge\mathbf{e}_1)\\ &\rightharpoonup \begin{pmatrix}\partial_1\mathbb{U}^{(\mathbf{S})}_{1|X_2=b}\\ \widetilde{\mathcal{Z}}^{(1)}_{2|X_2=b}\\ \mathcal{Z}_3^{(1)}\end{pmatrix} + \begin{pmatrix}\partial_{X_1}\widehat{\mathbb{U}}^{(\mathbf{S})}_{1|X_2=b}\\ \big(\partial_{X_1}\widehat{\mathbb{U}}_2^{(1)} - \widehat{\mathcal{R}}_3^{(1)}\big)_{|X_2=b}\\ \big(\partial_{X_1}(\widehat{\mathbb{U}}_3 + \widehat{\mathbb{U}}_3^{(g)}) + (\widehat{\mathcal{R}}_2 + \widehat{\mathcal{R}}_2^{(g)})\big)_{|X_2=b}\end{pmatrix} + \begin{pmatrix}\partial_{X_1}\widehat{\mathbb{U}}_{N,1}^{(1,b)}\\ \partial_{X_1}\widehat{\mathbb{U}}_{N,2}^{(1,b)} - \widehat{\mathcal{R}}_{N,3}^{(1,b)}\\ \partial_{X_1}\widehat{\mathbb{U}}_{N,3}^{(1,b)} + \widehat{\mathcal{R}}_{N,2}^{(1,b)}\end{pmatrix}\\ &= \begin{pmatrix}\partial_1\mathbb{U}^{(\mathbf{S})}_{1|X_2=b}\\ \widetilde{\mathcal{Z}}^{(1)}_{2|X_2=b}\\ \mathcal{Z}_3^{(1)}\end{pmatrix} + \partial_{X_1}\begin{pmatrix}\widehat{\mathbb{U}}^{(\mathbf{S})}_{1|X_2=b} + \widehat{\mathbb{U}}_{N,1}^{(1,b)}\\ \widehat{\mathbb{U}}^{(1)}_{2|X_2=b} + \widehat{\mathbb{U}}_{N,2}^{(1,b)}\\ (\widehat{\mathbb{U}}_3 + \widehat{\mathbb{U}}_3^{(g)})_{|X_2=b} + \widehat{\mathbb{U}}_{N,3}^{(1,b)}\end{pmatrix} + \begin{pmatrix}(\widehat{\mathcal{R}}_1 + \widehat{\mathcal{R}}_1^{(g)})_{|X_2=b} + \widehat{\mathcal{R}}_{N,1}^{(1,b)}\\ (\widehat{\mathcal{R}}_2 + \widehat{\mathcal{R}}_2^{(g)})_{|X_2=b} + \widehat{\mathcal{R}}_{N,2}^{(1,b)}\\ \widehat{\mathcal{R}}_3^{(1)} + \widehat{\mathcal{R}}_{N,3}^{(1,b)}\end{pmatrix}\wedge\mathbf{e}_1\\ &\qquad \text{weakly in } L^2(\Omega\times Cyl^{(1)})^3. \end{aligned}$$

Since $\mathbb{U}_1^{(\mathbf{S})} \in \mathbf{L}^2(\Omega,\partial_1; Q^1_{per}((0,2)_{X_2}))$ and $\widetilde{\mathcal{Z}}_2^{(1)} \in L^2(\Omega; Q^1_{per}((0,2)_{X_2}))$, their restrictions to the line $\{X_2 = b\}$ are $\mathbb{U}_1^{(\mathbf{S},b)} \in \mathbf{L}^2(\Omega,\partial_1)$ and $\widetilde{\mathcal{Z}}_2^{(1,b)} \in L^2(\Omega)$ respectively. Set

$$\mathfrak{R}^{(1,b)} \doteq \begin{pmatrix}(\widehat{\mathcal{R}}_1 + \widehat{\mathcal{R}}_1^{(g)})_{|X_2=b} + \widehat{\mathcal{R}}_{N,1}^{(1,b)}\\ (\widehat{\mathcal{R}}_2 + \widehat{\mathcal{R}}_2^{(g)})_{|X_2=b} + \widehat{\mathcal{R}}_{N,2}^{(1,b)}\\ \widehat{\mathcal{R}}_3^{(1)} + \widehat{\mathcal{R}}_{N,3}^{(1,b)}\end{pmatrix}, \qquad \mathfrak{U}^{(1,b)} \doteq \begin{pmatrix}\widehat{\mathbb{U}}^{(\mathbf{S})}_{1|X_2=b} + \widehat{\mathbb{U}}_{N,1}^{(1,b)}\\ \widehat{\mathbb{U}}^{(1)}_{2|X_2=b} + \widehat{\mathbb{U}}_{N,2}^{(1,b)}\\ \big(\widehat{\mathbb{U}}_3 + \widehat{\mathbb{U}}_3^{(g)}\big)_{|X_2=b} + \widehat{\mathbb{U}}_{N,3}^{(1,b)}\end{pmatrix}.$$

Since $\widehat{\mathcal{R}}_1$, $\widehat{\mathcal{R}}_2$, $\widehat{\mathcal{R}}_1^{(3)} \in L^2(\Omega; Q^1_{per,0}(\mathcal{Y}))$, $\widehat{\mathcal{R}}_1^{(g)}$, $\widehat{\mathcal{R}}_2^{(g)} \in L^2(\Omega; Q^1_{per}(\mathcal{Y}))$ and the function $\widehat{\mathcal{R}}_N^{(1,b)} \in L^2(\Omega; H^1_{per}((0,2)_{X_1}))^3$, the restriction to the lines $\{X_2 = b\}$ implies that $\mathfrak{R}^{(1,b)} \in L^2(\Omega; H^1_{per}((0,2)_{X_1}))^3$.
Since $\widehat{\mathbb{U}}_1^{(\mathbf{S})} \in L^2(\Omega\times(0,2)_{X_2}; Q^1_{per,0}((0,2)_{X_1})) \cap L^2(\Omega; Q^1(\mathcal{Y}))$, the functions $\widehat{\mathbb{U}}_3$ and $\widehat{\mathbb{U}}_3^{(g)} \in L^2(\Omega; Q^1_{per,0}(\mathcal{Y}))$, $\widehat{\mathbb{U}}_1^{(2)}$, $\widehat{\mathbb{U}}_3^{(g)} \in L^2(\Omega; Q^1_{per}(\mathcal{Y}))$, $\widehat{\mathbb{U}}_N^{(1,b)} \in L^2(\Omega; H^1_{per}((0,2)_{X_1}))^3$, the restriction to the lines $\{X_2 = b\}$ implies that $\mathfrak{U}^{(1,b)} \in L^2(\Omega; H^1_{per}((0,2)_{X_1}))^3$.
Using identities (58) we get $(71)_1$ and $(72)_1$. $\square$

6.4 | The limit of the warping

Set the spaces

$$\overline{\mathbf{W}}^{(1)} \doteq \Big\{(\overline{w}^{(1,0)}, \overline{w}^{(1,1)}) \in H^1(Cyl^{(1)})^{3\times 2} \Big| \text{ 2 periodic with respect to } X_1, \int_{Cyl^{(1)}} \overline{w}^{(1,b)}(\cdot, X)dX_2dX_3 = 0$$
$$\text{and} \int_{Cyl^{(1)}} \overline{w}^{(1,b)}(\cdot, X) \wedge \big((X_2-b)\mathbf{e}_2 + X_3\mathbf{n}^{(1,b)}(X_1)\big)dX_2dX_3 = 0, \;\; b\in\{0,1\}\Big\},$$
$$\overline{\mathbf{W}}^{(2)} \doteq \Big\{(\overline{w}^{(2,0)}, \overline{w}^{(2,1)}) \in H^1(Cyl^{(2)})^{3\times 2} \Big| \text{ 2 periodic with respect to } X_2, \int_{Cyl^{(2)}} \overline{w}^{(2,a)}(\cdot, X)dX_1dX_3 = 0$$
$$\text{and} \int_{Cyl^{(2)}} \overline{w}^{(2,a)}(\cdot, X) \wedge \big((X_1-a)\mathbf{e}_1 + X_3\mathbf{n}^{(2,a)}(X_2)\big)dX_1dX_3 = 0, \;\; a\in\{0,1\}\Big\}.$$

In the following, we show the warping convergences.

Lemma 17. [14, Lemma 7.7] There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\overline{u}^{(1,b)} \in L^2(\Omega; \overline{\mathbf{W}}^{(1)})$, $\overline{u}^{(2,a)} \in L^2(\Omega; \overline{\mathbf{W}}^{(2)})$ such that the following convergences hold:

$$\frac{1}{\varepsilon^3}\Pi_\varepsilon^{(\alpha,c)}(\overline{u}_\varepsilon) \rightharpoonup \overline{u}^{(\alpha,c)} \quad \text{weakly in} \quad L^2(\Omega; H^1(Cyl^{(\alpha)}))^3, \quad c\in\{0,1\}.$$

The strain tensors limits for the warping are directly inherited by (11) and the limit convergences for the reference frame given by Appendix 10, leading to (see also [14, Subsection 7.3])

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(\alpha,c)}\big(\widetilde{\mathbf{e}}_z(\overline{u}_\varepsilon)\big) \rightharpoonup \mathcal{E}_X^{(\alpha,c)}(\overline{u}^{(\alpha,c)}) \quad \text{weakly in} \quad L^2(\Omega\times Cyl^{(\alpha)})^{3\times 3}, \quad c\in\{0,1\}$$

where for every $\phi \in H^1(Cyl^{(1)})^3$ and every $\psi \in H^1(Cyl^{(2)})^3$ we have

$$\mathcal{E}_X^{(1,b)}(\phi) = \begin{pmatrix} \frac{1}{\eta^{(1,b)}}\partial_{X_1}\phi\cdot\mathbf{t}^{(1,b)} & * & * \\ \frac{1}{2}\big(\frac{1}{\eta^{(1,b)}}\partial_{X_1}\phi\cdot\mathbf{e}_2 + \partial_{X_2}\phi\cdot\mathbf{t}^{(1,b)}\big) & \partial_{X_2}\phi\cdot\mathbf{e}_2 & * \\ \frac{1}{2}\big(\frac{1}{\eta^{(1,b)}}\partial_{X_1}\phi\cdot\mathbf{n}^{(1,b)} + \partial_{X_3}\phi\cdot\mathbf{t}^{(1,b)}\big) & \frac{1}{2}\big(\partial_{X_2}\phi\cdot\mathbf{n}^{(1,b)} + \partial_{X_3}\phi\cdot\mathbf{e}_2\big) & \partial_{X_3}\phi\cdot\mathbf{n}^{(1,b)} \end{pmatrix}$$

and

$$\mathcal{E}_X^{(2,a)}(\psi) = \begin{pmatrix} \partial_{X_1}\psi\cdot\mathbf{e}_1 & * & * \\ \frac{1}{2}\big(\partial_{X_1}\psi\cdot\mathbf{t}^{(2,a)} + \frac{1}{\eta^{(2,a)}}\partial_{X_2}\psi\cdot\mathbf{e}_1\big) & \frac{1}{\eta^{(2,a)}}\partial_{X_2}\psi\cdot\mathbf{t}^{(2,a)} & * \\ \frac{1}{2}\big(\partial_{X_1}\psi\cdot\mathbf{n}^{(2,a)} + \partial_{X_3}\psi\cdot\mathbf{e}_1\big) & \frac{1}{2}\big(\frac{1}{\eta^{(2,a)}}\partial_{X_2}\psi\cdot\mathbf{n}^{(2,a)} + \partial_{X_3}\psi\cdot\mathbf{t}^{(2,a)}\big) & \partial_{X_3}\psi\cdot\mathbf{n}^{(2,a)} \end{pmatrix}.$$

6.5 | The limit strain tensor for the elementary displacement

Notation 1. For every $\zeta \in \mathbb{R}^9$, we set

$$\zeta^{(1,0)} \doteq \begin{pmatrix}\zeta_1\\0\\0\end{pmatrix} + \begin{pmatrix}\zeta_5\\-\zeta_6\\\zeta_7\end{pmatrix} \wedge \big(\Phi^{(1,0)}\mathbf{e}_3 + X_2\mathbf{e}_2 + X_3\mathbf{n}^{(1,0)}\big), \quad \zeta^{(1,1)} \doteq \begin{pmatrix}\zeta_2\\0\\0\end{pmatrix} + \begin{pmatrix}\zeta_5\\-\zeta_6\\\zeta_7\end{pmatrix} \wedge \big(\Phi^{(1,1)}\mathbf{e}_3 + (X_2-1)\mathbf{e}_2 + X_3\mathbf{n}^{(1,1)}\big),$$
$$\zeta^{(2,0)} \doteq \begin{pmatrix}0\\\zeta_3\\0\end{pmatrix} + \begin{pmatrix}\zeta_8\\-\zeta_5\\-\zeta_9\end{pmatrix} \wedge \big(\Phi^{(2,0)}\mathbf{e}_3 + X_1\mathbf{e}_1 + X_3\mathbf{n}^{(2,0)}\big), \quad \zeta^{(2,1)} \doteq \begin{pmatrix}0\\\zeta_4\\0\end{pmatrix} + \begin{pmatrix}\zeta_8\\-\zeta_5\\-\zeta_9\end{pmatrix} \wedge \big(\Phi^{(2,1)}\mathbf{e}_3 + (X_1-1)\mathbf{e}_1 + X_3\mathbf{n}^{(2,1)}\big).$$

Accordingly, we define

$$\begin{aligned} &\mathcal{E}_{11}^{(1,b)}(\zeta) = \frac{1}{\eta^{(1,b)}}\zeta^{(1,b)}\cdot\mathbf{t}^{(1,b)}, \quad \mathcal{E}_{12}^{(1,b)}(\zeta) = \frac{1}{2\eta^{(1,b)}}\zeta^{(1,b)}\cdot\mathbf{e}_2, \\ &\mathcal{E}_{13}^{(1,b)}(\zeta) = \frac{1}{2\eta^{(1,b)}}\zeta^{(1,b)}\cdot\mathbf{n}^{(1,b)}, \quad \mathcal{E}_{23}^{(1,b)}(\zeta) = \mathcal{E}_{22}^{(1,b)}(\zeta) = \mathcal{E}_{33}^{(1,b)}(\zeta) = 0, \end{aligned} \tag{74}$$

and

$$\mathcal{E}_{22}^{(2,a)}(\zeta) = \frac{1}{\eta^{(2,a)}}\zeta^{(2,a)} \cdot \mathbf{e}_2, \quad \mathcal{E}_{12}^{(2,a)}(\zeta) = \frac{1}{2\eta^{(2,a)}}\zeta^{(2,a)} \cdot \mathbf{t}^{(2,a)},$$
$$\mathcal{E}_{23}^{(2,a)}(\zeta) = \frac{1}{2\eta^{(2,a)}}\zeta^{(2,a)} \cdot \mathbf{n}^{(2,a)}, \quad \mathcal{E}_{11}^{(2,a)}(\zeta) = \mathcal{E}_{13}^{(2,a)}(\zeta) = \mathcal{E}_{33}^{(2,a)}(\zeta) = 0. \tag{75}$$

First note that the strain tensor admits a weak limit in the form of a weak convergent subsequence. Indeed, from (54) and (70) we have

$$\left\| \frac{1}{\varepsilon^2}\Pi_\varepsilon^{(\alpha,c)}\big(\widetilde{\mathbf{e}}_z(u_\varepsilon)\big) \right\|_{L^2(\Omega\times Cyl^{(\alpha)})} \le \frac{1}{\varepsilon^{5/2}}\|u_\varepsilon\|_{\mathcal{S}_\varepsilon} \le C.$$

Set

$$\widehat{\mathbf{W}}^{(1)} \doteq \big\{(\widehat{w}^{(1,0)}, \widehat{w}^{(1,1)}) \in H^1(Cyl^{(1)})^{3\times 2} \;\big|\; 2 \text{ periodic with respect to } X_1\big\},$$
$$\widehat{\mathbf{W}}^{(2)} \doteq \big\{(\widehat{w}^{(2,0)}, \widehat{w}^{(2,1)}) \in H^1(Cyl^{(2)})^{3\times 2} \;\big|\; 2 \text{ periodic with respect to } X_2\big\}.$$

We first consider the direction $\mathbf{e}_1$. By Lemma 17, the representation of the limit strain tensor (15) together with Lemma 16 and the limit mobile reference frame given by Appendix 10, the limit strain tensor is split in two main parts ($b \in \{0,1\}$)

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(1,b)}\big(\widetilde{\mathbf{e}}_z(u_\varepsilon)\big) \rightharpoonup \mathcal{E}^{(1,b)}(\partial\mathbf{U}) + \mathcal{E}_X^{(1,b)}(\widehat{u}^{(1,b)}) \;\text{ weakly in }\; L^2(\Omega; H^1(Cyl^{(1)}))^{3\times 3} \tag{76}$$

where $\mathcal{E}^{(1,b)}$ is given by (74) but with ζ replaced by

$$\partial\mathbf{U} = \big(\partial_1\mathbb{U}_1^{(\mathbf{S},0)}, \partial_1\mathbb{U}_1^{(\mathbf{S},1)}, \partial_2\mathbb{U}_2^{(\mathbf{S},0)}, \partial_2\mathbb{U}_2^{(\mathbf{S},1)}, \partial_{12}\mathbb{U}_3, \partial_{11}\mathbb{U}_3, \partial_{11}\mathbb{U}_2, \partial_{22}\mathbb{U}_3, \partial_{22}\mathbb{U}_1\big), \tag{77}$$

while $\mathcal{E}_X^{(1,b)}(\widehat{u}^{(1,b)})$ is the symmetric gradient of the displacement $\widehat{u}^{(1,b)}$ defined by

$$\widehat{u}^{(1,b)} \doteq \mathfrak{U}^{(1,b)} + \big(\mathcal{Z}_3^{(1)}\mathbf{e}_2 - \widetilde{\mathcal{Z}}_2^{(1,b)}\mathbf{e}_3 + \mathfrak{R}^{(1,b)}\big) \wedge \big(\Phi^{(1,b)}\mathbf{e}_3 + X_3\mathbf{n}^{(1,b)} + (X_2 - b)\mathbf{e}_2\big) + \overline{u}^{(1,b)}.$$

We have $\widehat{u}^{(1,b)} \in L^2(\Omega; \widehat{\mathbf{W}}^{(1)})$.

Concerning direction $\mathbf{e}_2$, the same argumentation applies and the limit strain tensor becomes ($a \in \{0,1\}$)

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(2,a)}\big(\widetilde{\mathbf{e}}_z(u_\varepsilon)\big) \rightharpoonup \mathcal{E}^{(2,a)}(\partial\mathbf{U}) + \mathcal{E}_X^{(2,a)}(\widehat{u}^{(2,a)}) \text{ weakly in }\; L^2(\Omega; H^1(Cyl^{(2)}))^{3\times 3},$$

where $\mathcal{E}^{(2,a)}$ is given (75) but with ζ replaced by (77), while $\mathcal{E}_X^{(2,a)}(\widehat{u}^{(2,a)})$ is the symmetric gradient of the displacement $\widehat{u}^{(2,a)}$ defined by

$$\widehat{u}^{(2,a)} \doteq \mathfrak{U}^{(2,a)} + \big(- \mathcal{Z}_3^{(2)}\mathbf{e}_1 + \widetilde{\mathcal{Z}}_1^{(2,a)}\mathbf{e}_3 + \mathfrak{R}^{(2,a)}\big) \wedge \big(\Phi^{(2,a)}\mathbf{e}_3 + X_3\mathbf{n}^{(2,a)} + (X_1 - a)\mathbf{e}_1\big) + \overline{u}^{(2,a)}.$$

We have $\widehat{u}^{(2,a)} \in L^2(\Omega; \widehat{\mathbf{W}}^{(2)})$.

6.6 | Unfold of the contact conditions via contact unfolding operator

To obtain the limit contact conditions, it is necessary to introduce a third unfolding operator defined on the contact areas. Set the contact area by

$$\mathbf{C}_{ab} \doteq a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + \omega_\kappa.$$

Definition 3 (Contact unfolding operator)**.** For every measurable function ϕ in $L^2(\mathbf{C})$, we define the measurable functions $T_\varepsilon^{\mathbf{C}_{ab}}(\phi) \in L^2(\Omega\times\omega_\kappa)$ by

$$T_\varepsilon^{\mathbf{C}_{ab}}(\phi)(z', X') \doteq \phi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon\begin{pmatrix} a \\ b\end{pmatrix} + \varepsilon\begin{pmatrix} X_1' \\ X_2'\end{pmatrix}\right) \qquad \text{for a.e. } (z', X') \in \Omega\times\omega_\kappa.$$

Let $\phi \in L^1(S_\varepsilon^{(1)})$, $\psi \in L^1(S_\varepsilon^{(2)})$ and $\varphi \in L^2(\Omega)$. This operator is related to the previous ones via the identities (a.e. $(z', X') \in \Omega\times\omega_\kappa$):

$$T_\varepsilon^{\mathbf{C}_{ab}}(\phi)(z', X_1', X_2') = \Pi_\varepsilon^{(1,b)}(\phi)(z', a + X_1', X_2', (-1)^{a+b+1}\kappa),$$
$$T_\varepsilon^{\mathbf{C}_{ab}}(\psi)(z', X_1', X_2') = \Pi_\varepsilon^{(2,a)}(\psi)(z', X_1', b + X_2', (-1)^{a+b}\kappa). \tag{78}$$

In particular, from (60) and $(78)_3$ we have that

$$\sum_{a,b=0}^{1} \|T_\varepsilon^{\mathbf{C}_{ab}}(\varphi)\|_{L^2(\Omega\times\omega_\kappa)} \le C\|\varphi\|_{L^2(\mathbf{C})}, \qquad \forall\varphi \in L^2(\mathbf{C}).$$

By (18)-(29), the displacements decomposition in the contact areas becomes a.e. $(z_1, z_2) \in \omega_r$:

$$\begin{aligned} u_\varepsilon^{(1,q)}(z_1 + p\varepsilon, z_2 + q\varepsilon, (-1)^{p+q+1} r) =& \mathbb{U}_\varepsilon^{(1)}(p\varepsilon + z_1, q\varepsilon) + \mathcal{R}_\varepsilon^{(1)}(p\varepsilon + z_1, q\varepsilon) \wedge z_2 \mathbf{e}_2 \\ &+ \mathbb{U}_{\varepsilon,N}^{(1,q)}(p\varepsilon + z_1) + \mathcal{R}_{\varepsilon,N}^{(1,q)}(p\varepsilon + z_1) \wedge z_2 \mathbf{e}_2 + \overline{u}_\varepsilon^{(1,q)}, \\ u_\varepsilon^{(2,p)}(z_1 + p\varepsilon, z_2 + q\varepsilon, (-1)^{p+q} r) =& \mathbb{U}_\varepsilon^{(2)}(p\varepsilon, q\varepsilon + z_2) + \mathcal{R}_\varepsilon^{(2)}(p\varepsilon, q\varepsilon + z_2) \wedge z_1 \mathbf{e}_1 \\ &+ \mathbb{U}_{\varepsilon,N}^{(2,p)}(q\varepsilon + z_2) + \mathcal{R}_{\varepsilon,N}^{(2,p)}(q\varepsilon + z_2) \wedge z_1 \mathbf{e}_1 + \overline{u}_\varepsilon^{(2,p)}. \end{aligned} \tag{79}$$

We estimate now the difference in the contact areas.

Lemma 18. The displacements satisfy

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|u_{\varepsilon,\alpha}^{(1,q)} - u_{\varepsilon,\alpha}^{(2,p)}\|^2_{L^2(\mathbf{C}_{pq})} \le C\varepsilon^4, \quad \sum_{(p,q)\in\mathcal{K}_\varepsilon} \|u_{\varepsilon,3}^{(1,q)} - u_{\varepsilon,3}^{(2,p)}\|^2_{L^2(\mathbf{C}_{pq})} \le C\varepsilon^6. \tag{80}$$

Proof. Estimate $(80)_2$ follows the lines of[14, Lemma 7.8], using the results of Lemma 6.

We prove now $(80)_1$ for $\alpha = 1$, since the proof for $\alpha = 2$ is similar. Observe that decomposition (44) in $\mathbf{C}_{pq}$ and the Q_1 interpolation properties yield (see[14, Lemma 5.1])

$$\begin{aligned} &\mathbb{U}_{\varepsilon,1}^{(1)}(p\varepsilon + z_1, q\varepsilon) = \mathbb{U}_{\varepsilon,1}(q\varepsilon) + \mathbb{U}_{\varepsilon,1}^{(\mathbf{S})}(p\varepsilon + z_1, q\varepsilon), \\ &\mathbb{U}_{\varepsilon,1}^{(2)}(p\varepsilon, q\varepsilon + z_2) = \mathbb{U}_{\varepsilon,1}(q\varepsilon) + z_2 \partial \mathbb{U}_{\varepsilon,1}(q\varepsilon + z_2) + \mathbb{U}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon, q\varepsilon + z_2). \end{aligned}$$

Then, the above writings (79) give

$$\begin{aligned} &u_\varepsilon^{(1,q)}(z_1 + p\varepsilon, z_2 + q\varepsilon, (-1)^{p+q+1} r) - u_\varepsilon^{(2,p)}(z_1 + p\varepsilon, z_2 + q\varepsilon, (-1)^{p+q} r) \\ &= \left[\mathbb{U}_{\varepsilon,1}^{(\mathbf{S})}(p\varepsilon + z_1, q\varepsilon) - z_2 \partial \mathbb{U}_{\varepsilon,1}(q\varepsilon + z_2) - \mathbb{U}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon, q\varepsilon + z_2)\right] \\ &\quad + \left[\overline{u}_{\varepsilon,1}^{(1,q)} - \overline{u}_{\varepsilon,1}^{(2,p)} + \mathbb{U}_{\varepsilon,N,1}^{(1,q)}(p\varepsilon + z_1) - \mathbb{U}_{\varepsilon,N,1}^{(2,p)}(q\varepsilon + z_2) - z_2 \mathcal{R}_{\varepsilon,3}^{(1)}(p\varepsilon + z_1, q\varepsilon) - z_2 \mathcal{R}_{\varepsilon,N,3}^{(1,q)}(p\varepsilon + z_1)\right] \end{aligned} \tag{81}$$

Then, $(80)_1$ follows by estimates (55) and Lemmas 2-5 together with (54). ☐

We finally give the limit contact conditions.

Lemma 19. The in plane limit contact conditions are

$$\begin{aligned} &|\mathbb{U}_1^{(\mathbf{S},b)} - \mathbb{U}_1^{(\mathbf{B},a)}| + \kappa \Big|\partial_2 \mathbb{U}_1 + \partial_1 \mathbb{U}_2\Big| \le g_1 \quad \text{a.e. in } \Omega, \\ &|\mathbb{U}_2^{(\mathbf{S},a)} - \mathbb{U}_2^{(\mathbf{B},b)}| + \kappa \Big|\partial_2 \mathbb{U}_1 + \partial_1 \mathbb{U}_2\Big| \le g_2 \quad \text{a.e. in } \Omega. \end{aligned} \tag{82}$$

The outer plane limit contact conditions are a.e. in $\Omega \times \mathbf{C}_{ab}$

$$0 \le (-1)^{a+b}\Big[\frac{(X_1 - a)^2}{2}\partial_{11}\mathbb{U}_3 - \frac{(X_2 - b)^2}{2}\partial_{22}\mathbb{U}_3 + \widehat{u}_3^{(1,b)}(\cdot, X_1, X_2 - b, (-1)^{a+b+1}\kappa) - \widehat{u}_3^{(2,a)}(\cdot, X_1 - a, X_2, (-1)^{a+b}\kappa)\Big]. \tag{83}$$

Proof. The outer plane limit contact condition (83) follows the same lines of[14, Section 7.5] and taking into account that

$$(X_1', X_2') = (X_1 - a, X_2 - b) \qquad \text{with } (X_1', X_2') \in \omega_\kappa, \quad (X_1, X_2) \in \mathbf{C}_{ab}.$$

Now we turn into (82) and we consider the first component. Applying the contact unfolding operator to (81) and due to the estimates in Lemmas 2-5 together with (54), we have that

$$\frac{1}{\varepsilon^2} T_\varepsilon^{\mathbf{C}_{ab}}\Big[\mathbb{U}_{\varepsilon,N,1}^{(1,q)}(p\varepsilon + z_1) - \mathbb{U}_{\varepsilon,N,1}^{(2,p)}(q\varepsilon + z_2) - z_2 \mathcal{R}_{\varepsilon,N,3}^{(1,q)}(p\varepsilon + z_1) + \overline{u}_{\varepsilon,1}^{(1,q)}(z') - \overline{u}_{\varepsilon,1}^{(2,p)}(z')\Big] \to 0 \quad \text{strongly in } L^2(\Omega \times \omega_\kappa).$$

Then, convergences $(61)_2$, $(62)_{1,3}$, $(63)_1$ and equalities $(58)_3$ yield

$$\begin{aligned} &\frac{1}{\varepsilon^2} T_\varepsilon^{\mathbf{C}_{ab}}\Big[\mathbb{U}_{\varepsilon,1}^{(\mathbf{S})}(p\varepsilon + z_1, q\varepsilon) - \mathbb{U}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon, q\varepsilon + z_2) - z_2 \partial_2 \mathbb{U}_{\varepsilon,1}(q\varepsilon + z_2) - z_2 \mathcal{R}_{\varepsilon,3}^{(1)}(p\varepsilon + z_1, q\varepsilon)\Big] \\ &\rightharpoonup \mathbb{U}_1^{(\mathbf{S})}(\cdot, b) - \mathbb{U}_1^{(\mathbf{B})}(\cdot, a) - X_2'\Big(\partial_2 \mathbb{U}_1 + \mathcal{R}_3^{(1)}\Big) \\ &\quad = \mathbb{U}_1^{(\mathbf{S})}(\cdot, b) - \mathbb{U}_1^{(\mathbf{B})}(\cdot, a) - X_2'(\partial_2 \mathbb{U}_1 + \partial_1 \mathbb{U}_2) \quad \text{weakly in } L^2(\Omega \times \omega_\kappa). \end{aligned}$$

Hence, the in plane condition in the first component a.e $z' \in \Omega$ and every $X' \in \omega_\kappa$ is

$$|\mathbb{U}_1^{(\mathbf{S})}(z', b) - \mathbb{U}_1^{(\mathbf{B})}(z', a) - X_2'(\partial_2 \mathbb{U}_1(z_2) + \partial_1 \mathbb{U}_2(z_1))| \le g_1(z').$$

By the admissible choices $X_2' = \pm\kappa$, the above inequality becomes

$$|\mathbb{U}_1^{(\mathbf{S})}(z', b) - \mathbb{U}_1^{(\mathbf{B})}(z', a)| + \kappa|\partial_2\mathbb{U}_1(z_2) + \partial_1\mathbb{U}_2(z_1)| \le g_1(z') \text{ for a.e. } z' \in \Omega.$$

Then, due to the above inequality we get $(82)_{1,2}$ in the first component. The second one follows a symmetric argumentation. $\square$

6.7 | The displacements limit set

Denote

$$\begin{aligned}
\mathcal{X}_M &\doteq H^2_{(0,l)}\big((0,L)_{z_2}\big) \times H^2_{(0,l)}\big((0,L)_{z_1}\big) \times H^2_{\gamma}(\Omega),\\
\mathcal{X}_S &\doteq \mathbf{L}^2(\Omega, \partial_1)^2 \times \mathbf{L}^2(\Omega, \partial_2)^2, \qquad \mathcal{X}_B \doteq L^2(\Omega)^4,\\
\mathcal{X}_m &\doteq L^2(\Omega; \widehat{\mathbf{W}}^{(1)}) \times L^2(\Omega; \widehat{\mathbf{W}}^{(2)}),
\end{aligned}$$

where $\mathcal{X}_M$ is the space of the macroscopic functions that appear in the strain tensors, $\mathcal{X}_B$ is the space of the relative macroscopic bendings functions that appear in the contact and right hand side of the problem, $\mathcal{X}_m$ is the space that gathers all the microscopic fields. In particular, the functions belonging to their respective spaces are defined by

$$\begin{aligned}
&\mathbb{V} \doteq \big(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3\big) \in \mathcal{X}_M,\\
&\mathbb{V}^{(\mathbf{S})} \doteq \big(\mathbb{V}_1^{(\mathbf{S},0)}, \mathbb{V}_1^{(\mathbf{S},1)}, \mathbb{V}_2^{(\mathbf{S},0)}, \mathbb{V}_2^{(\mathbf{S},1)}\big) \in \mathcal{X}_S,\\
&\mathbb{V}^{(\mathbf{B})} \doteq \big(\mathbb{V}_1^{(\mathbf{B},0)}, \mathbb{V}_1^{(\mathbf{B},1)}, \mathbb{V}_2^{(\mathbf{B},0)}, \mathbb{V}_2^{(\mathbf{B},1)}\big) \in \mathcal{X}_B,\\
&\widehat{v} \doteq \big(\widehat{v}^{(1,0)}, \widehat{v}^{(1,1)}, \widehat{v}^{(2,0)}, \widehat{v}^{(2,1)}\big) \in \mathcal{X}_m.
\end{aligned} \tag{84}$$

Adding the limit contact conditions (82) and (83), we finally define the limit set of admissible displacements by

$$\begin{aligned}
\mathcal{X} \doteq \Big\{ \big(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})}, \widehat{v}\big) \in \mathcal{X}_M \times \mathcal{X}_S \times \mathcal{X}_B \times \mathcal{X}_m \;\Big|\; & |\mathbb{U}_1^{(\mathbf{S},b)} - \mathbb{U}_1^{(\mathbf{B},a)}| + \kappa|\partial_2\mathbb{U}_1 + \partial_1\mathbb{U}_2| \le g_1 \text{ a.e. in } \Omega,\\
& |\mathbb{U}_2^{(\mathbf{S},a)} - \mathbb{U}_2^{(\mathbf{B},b)}| + \kappa|\partial_2\mathbb{U}_1 + \partial_1\mathbb{U}_2| \le g_2 \text{ a.e. in } \Omega,\\
\text{and a.e. in } \Omega \times \mathbf{C}_{ab} \quad & 0 \le (-1)^{a+b}\Big[\frac{(X_1 - a)^2}{2}\partial_{11}\mathbb{V}_3 - \frac{(X_2 - b)^2}{2}\partial_{22}\mathbb{V}_3\\
+\widehat{v}_3^{(1,b)}(\cdot, X_1, X_2 - b, (-1)^{a+b+1}\kappa) - \widehat{v}_3^{(2,a)}(\cdot, X_1 - a, X_2, (-1)^{a+b}\kappa)\Big], \;& (a,b) \in \{0,1\}^2 \Big\}.
\end{aligned}$$

Note that $\mathcal{X}$ is a closed convex subset of the Hilbert space $\mathcal{X}_M \times \mathcal{X}_S \times \mathcal{X}_B \times \mathcal{X}_m$ endowed with the product norm.

7 | BUILD OF THE TEST-FUNCTIONS

First, define the spaces

$$\begin{aligned}
\mathcal{C}_M &\doteq C^3(\overline{\Omega})^3 \cap \mathcal{X}_M, \qquad \mathcal{C}_S \doteq C^2(\overline{\Omega})^4 \cap \mathcal{X}_S, \quad \mathcal{C}_B \doteq C^2(\overline{\Omega})^4 \cap \mathcal{X}_B,\\
\mathcal{C}_m &\doteq W^{1,\infty}(\Omega; \widehat{\mathbf{W}}^{(1)}) \times W^{1,\infty}(\Omega; \widehat{\mathbf{W}}^{(2)}).
\end{aligned}$$

Accordingly to (84), we take $(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})}, \widehat{v}) \in \mathcal{C}_M \times \mathcal{C}_S \times \mathcal{C}_B \times \mathcal{C}_m$. We also assume the following additional boundary conditions (see (34)):

$$\begin{aligned}
&\mathbb{V}_1^{(\mathbf{S},0)} = 0 \quad \text{a.e. on } \boldsymbol{\gamma}_1 \cup \Gamma_2, \qquad \mathbb{V}_2^{(\mathbf{S},0)} = 0 \quad \text{a.e. on } \Gamma_1 \cup \boldsymbol{\gamma}_2,\\
&\mathbb{V}_1^{(\mathbf{B},0)} = 0, \;\; \nabla\mathbb{V}_1^{(\mathbf{B},a)} = 0 \quad \text{a.e. on } \boldsymbol{\gamma}_1 \cup \Gamma_2,\\
&\mathbb{V}_2^{(\mathbf{B},0)} = 0, \;\; \nabla\mathbb{V}_2^{(\mathbf{B},b)} = 0 \quad \text{a.e. on } \Gamma_1 \cup \boldsymbol{\gamma}_2,\\
&\widehat{v}^{(\alpha,c)} = 0 \;\; \text{a.e. on } \;\partial\Omega \times \overline{Cyl^{(\alpha)}}.
\end{aligned}$$

$$z' = [z'] + \{z'\}, \qquad [z'] \in \mathbb{Z}^2, \quad \{z'\} \in (0,1)^2, \quad \text{for a.e. } z' \in \mathbb{R}^2.$$

We set

$$\begin{aligned}
C_k &\doteq \big[(k-\kappa)\varepsilon, (k+\kappa)\varepsilon\big] \quad k \in \{0, \dots, 2N_\varepsilon\},\\
I_k &\doteq \big[(k+\kappa)\varepsilon, (k+1-\kappa)\varepsilon\big], \quad k \in \{0, \dots, 2N_\varepsilon - 1\}.
\end{aligned}$$

We define the different test functions for $z \in q\varepsilon\mathbf{e}_2 + P_r^{(1)}$, $q \in \{0, \dots, 2N_\varepsilon\}$ (the reference beams of direction $\mathbf{e}_1$). First, the functions $\mathbb{V}_{\varepsilon,\alpha}^{(1,q)}$, $\mathbb{V}_{\varepsilon,3}^{(1,q)} \in W^{2,\infty}(0,L)$ are defined by

$$\mathbb{V}_{\varepsilon,1}^{(1,q)}(z_1) \doteq \mathbb{V}_1(q\varepsilon), \quad \text{constant for a.e. } z_1 \in [0,L],$$

$$\mathbb{V}_{\varepsilon,2}^{(1,q)}(z_1) \doteq \begin{cases} \mathbb{V}_2(p\varepsilon) + (z_1 - p\varepsilon)\partial_1\mathbb{V}_2(p\varepsilon) + \dfrac{1}{2}(z_1 - p\varepsilon)^2\partial_{11}\mathbb{V}_2(p\varepsilon) & \text{if } z_1 \in C_p, \\ \text{cubic interpolated} & \text{if } z_1 \in I_p, \end{cases}$$

$$\mathbb{V}_{\varepsilon,3}^{(1,q)}(z_1) \doteq \begin{cases} \mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_1 - p\varepsilon)\partial_1\mathbb{V}_3(p\varepsilon, q\varepsilon) + \dfrac{1}{2}(z_1 - p\varepsilon)^2\partial_{11}\mathbb{V}_3(p\varepsilon, q\varepsilon) & \text{if } z_1 \in C_p, \\ \text{cubic interpolated} & \text{if } z_1 \in I_p \end{cases}$$

then $\mathbb{V}_{\varepsilon,1}^{(\mathbf{S},q)}$, $(\partial_2\mathbb{V}_3)_\varepsilon^{(1,q)} \in W^{1,\infty}(0,L)$, $\mathbb{V}_{\varepsilon,2}^{(\mathbf{B},p)} \in W^{2,\infty}(0,L)$, (below $b \equiv q \bmod 2$)

$$\mathbb{V}_{\varepsilon,1}^{(\mathbf{S},q)}(z_1) \doteq \begin{cases} \mathbb{V}_{\varepsilon,1}^{(\mathbf{S},b)}(p\varepsilon, q\varepsilon) & \text{if } z_1 \in C_p, \\ \text{linear interpolated} & \text{if } z_1 \in I_p, \end{cases}$$

$$(\partial_2\mathbb{V}_3)_\varepsilon^{(1,q)}(z_1) \doteq \begin{cases} \partial_2\mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_1 - p\varepsilon)\partial_{12}\mathbb{V}_3(p\varepsilon, q\varepsilon) & \text{if } z_1 \in C_p, \\ \text{linear interpolated} & \text{if } z_1 \in I_p, \end{cases}$$

$$\mathbb{V}_{\varepsilon,2}^{(\mathbf{B},q)}(z_1) \doteq \begin{cases} \mathbb{V}_2^{(\mathbf{S},a)}(p\varepsilon, q\varepsilon) + (z_1 - p\varepsilon)\partial_1\mathbb{V}_2^{(\mathbf{S},a)}(p\varepsilon, q\varepsilon) & \text{if } z_1 \in C_p, \\ \text{cubic interpolated} & \text{if } z_1 \in I_p \end{cases}$$

and $\widehat{v}_\varepsilon^{(1,q)} \in W^{1,\infty}\big((0,L) \times (q\varepsilon - \kappa\varepsilon, q\varepsilon + \kappa\varepsilon) \times (-\kappa\varepsilon, \kappa\varepsilon)\big)^3$

$$\widehat{v}_\varepsilon^{(1,q)}(z) \doteq \begin{cases} \widehat{v}^{(1,b)}\Big(p\varepsilon, q\varepsilon, 2\Big\{\dfrac{z_1}{2\varepsilon}\Big\}, \dfrac{z_2 - q\varepsilon}{\varepsilon}, \dfrac{z_3}{\varepsilon}\Big) & \text{if } z_1 \in C_p, \\ \text{linear interpolated with respect to the first variable,} & \text{if } z_1 \in I_p, \end{cases}$$

Now, we define the different test functions for $z \in p\varepsilon\mathbf{e}_1 + P_r^{(2)}$, $p \in \{0, \dots, 2N_\varepsilon\}$ (the reference beams of direction $\mathbf{e}_2$). First, the functions $\mathbb{V}_{\varepsilon,\alpha}^{(2,p)}$, $\mathbb{V}_{\varepsilon,3}^{(2,p)} \in W^{2,\infty}(0,L)$ are defined by

$$\mathbb{V}_{\varepsilon,1}^{(2,p)}(z_2) \doteq \begin{cases} \mathbb{V}_1(q\varepsilon) + (z_2 - q\varepsilon)\partial_2\mathbb{V}_1(q\varepsilon) + \dfrac{1}{2}(z_2 - q\varepsilon)^2\partial_{22}\mathbb{V}_1(q\varepsilon) & \text{if } z_2 \in C_q, \\ \text{cubic interpolated} & \text{if } z_2 \in I_q, \end{cases}$$

$$\mathbb{V}_{\varepsilon,2}^{(2,p)}(z_2) \doteq \mathbb{V}_2(p\varepsilon), \quad \text{constant for a.e. } z_2 \in [0,L],$$

$$\mathbb{V}_{\varepsilon,3}^{(2,p)}(z_2) \doteq \begin{cases} \mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_2 - q\varepsilon)\partial_2\mathbb{V}_3(p\varepsilon, q\varepsilon) + \dfrac{1}{2}(z_2 - q\varepsilon)^2\partial_{22}\mathbb{V}_3(p\varepsilon, q\varepsilon), & \text{if } z_2 \in C_q, \\ \text{cubic interpolated} & \text{if } z_2 \in I_q \end{cases}$$

then $\mathbb{V}_{\varepsilon,2}^{(\mathbf{S},p)}$, $(\partial_1\mathbb{V}_3)_\varepsilon^{(2,p)} \in W^{1,\infty}(0,L)$, $\mathbb{V}_{\varepsilon,1}^{(\mathbf{B},p)} \in W^{2,\infty}(0,L)$ (below $a \equiv p \bmod 2$)

$$\mathbb{V}_{\varepsilon,2}^{(\mathbf{S},p)}(z_2) \doteq \begin{cases} \mathbb{V}_{\varepsilon,2}^{(\mathbf{S},a)}(p\varepsilon, q\varepsilon) & \text{if } z_2 \in C_q, \\ \text{linear interpolated} & \text{if } z_2 \in I_q, \end{cases}$$

$$(\partial_1\mathbb{V}_3)_\varepsilon^{(2,p)}(z_2) \doteq \begin{cases} \partial_1\mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_2 - q\varepsilon)\partial_{12}\mathbb{V}_3(p\varepsilon, q\varepsilon) & \text{if } z_2 \in C_q, \\ \text{linear interpolated} & \text{if } z_2 \in I_q, \end{cases}$$

$$\mathbb{V}_{\varepsilon,1}^{(\mathbf{B},p)}(z_2) \doteq \begin{cases} \mathbb{V}_1^{(\mathbf{B},a)}(p\varepsilon, q\varepsilon) + (z_2 - q\varepsilon)\partial_2\mathbb{V}_2^{(\mathbf{B},a)}(p\varepsilon, q\varepsilon) & \text{if } z_2 \in C_q, \\ \text{cubic interpolated} & \text{if } z_2 \in I_q \end{cases}$$

and $\widehat{v}_\varepsilon^{(2,p)} \in W^{1,\infty}\big((p\varepsilon-\kappa\varepsilon,p\varepsilon+\kappa\varepsilon)\times(0,L)\times(-\kappa\varepsilon,\kappa\varepsilon)\big)^3$

$$\widehat{v}_\varepsilon^{(2,p)}(z) \doteq \begin{cases} \widehat{v}^{(2,a)}\Big(p\varepsilon, q\varepsilon, \dfrac{z_1-p\varepsilon}{\varepsilon}, 2\Big\{\dfrac{z_2}{2\varepsilon}\Big\}, \dfrac{z_3}{\varepsilon}\Big), & \text{if } z_2\in C_q,\\ \text{linear interpolated with respect to the second variable}, & \text{if } z_2 \in I_q. \end{cases}$$

Now, we compose the test displacements v_ε in the directions $\mathbf{e}_1$ and $\mathbf{e}_2$ by

$$\begin{aligned} v_\varepsilon^{(1,q)}(z) &= V_\varepsilon^{e(1,q)}(z) + \varepsilon^3\widehat{v}_\varepsilon^{(1,q)}(z), \quad z\in q\varepsilon\mathbf{e}_2 + P_r^{(1)},\\ v_\varepsilon^{(2,p)}(z) &= V_\varepsilon^{e(2,p)}(z) + \varepsilon^3\widehat{v}_\varepsilon^{(2,p)}(z), \quad z\in p\varepsilon\mathbf{e}_1 + P_r^{(2)}, \end{aligned} \tag{85}$$

where the elementary displacements are defined by

$$\begin{aligned} V_\varepsilon^{e(1,q)}(z) \doteq{}& \begin{pmatrix} \varepsilon\mathbb{V}_{\varepsilon,1}^{(1,q)}(z_1) + \varepsilon^2\mathbb{V}_{\varepsilon,1}^{(\mathbf{S},q)}(z_1)\\ \varepsilon\mathbb{V}_{\varepsilon,2}^{(1,q)}(z_1) + \varepsilon^2\mathbb{V}_{\varepsilon,2}^{(\mathbf{B},q)}(z_1)\\ \varepsilon\mathbb{V}_{\varepsilon,3}^{(1,q)}(z_1)\end{pmatrix} + \begin{pmatrix} \varepsilon(\partial_2\mathbb{V}_3)_\varepsilon^{(1,q)}(z_1)\\ -\varepsilon\partial_1(\mathbb{V}_{\varepsilon,3}^{(1,q)})(z_1)\\ \varepsilon\partial_1(\mathbb{V}_{\varepsilon,2}^{(1,q)}(z_1)) + \varepsilon^2\partial_1(\mathbb{V}_{\varepsilon,2}^{(\mathbf{B},q)})(z_1)\end{pmatrix}\\ &\wedge\Big(\varepsilon\Phi^{(1,b)}\Big(2\Big\{\frac{z_1}{2\varepsilon}\Big\}\Big)\mathbf{e}_3 + \big(z_2-q\varepsilon\big)\mathbf{e}_2 + z_3\mathbf{n}^{(1,b)}\Big(2\Big\{\frac{z_1}{2\varepsilon}\Big\}\Big)\Big),\\ V_\varepsilon^{e(2,p)}(z) \doteq{}& \begin{pmatrix} \varepsilon\mathbb{V}_{\varepsilon,1}^{(2,p)}(z_2) + \varepsilon^2\mathbb{V}_{\varepsilon,1}^{(\mathbf{B},p)}(z_2)\\ \varepsilon\mathbb{V}_{\varepsilon,2}^{(2,p)}(z_2) + \varepsilon^2\mathbb{V}_{\varepsilon,2}^{(\mathbf{S},p)}(z_2)\\ \varepsilon\mathbb{V}_{\varepsilon,3}^{(2,p)}(z_2)\end{pmatrix} + \begin{pmatrix} \varepsilon\partial_2(\mathbb{V}_{\varepsilon,3}^{(2,p)})(z_2)\\ -\varepsilon(\partial_1\mathbb{V}_3)_\varepsilon^{(2,p)}(z_1)\\ -\varepsilon\partial_2(\mathbb{V}_{\varepsilon,1}^{(2,p)}(z_2)) - \varepsilon^2\partial_2(\mathbb{V}_{\varepsilon,1}^{(\mathbf{B},p)})(z_2)\end{pmatrix}\\ &\wedge\Big(\varepsilon\Phi^{(2,a)}\Big(2\Big\{\frac{z_2}{2\varepsilon}\Big\}\Big)\mathbf{e}_3 + \big(z_1-p\varepsilon\big)\mathbf{e}_1 + z_3\mathbf{n}^{(2,a)}\Big(2\Big\{\frac{z_2}{2\varepsilon}\Big\}\Big)\Big). \end{aligned}$$

7.1 | The limit strain tensors for the test-functions

The limit of the unfolded strain tensor is an immediate consequence of the unfolding operator properties and of the regularity of the test functions (see also[14, Lemma 8.1]). We easily obtain

$$\frac{1}{\varepsilon^2}\Pi^{(\alpha,c)}\big(\widetilde{\mathbf{e}}_z(v_\varepsilon)\big) \to \mathcal{E}^{(\alpha,c)}(\partial\mathbf{V}) + \mathcal{E}_X^{(\alpha,c)}(\widehat{v}^{(\alpha,c)}) \quad \text{strongly in } L^2(\Omega\times Cyl^{(\alpha)})^{3\times3} \tag{86}$$

where $\mathcal{E}^{(1,b)}$ and $\mathcal{E}^{(2,a)}$ are respectively given by (74) and (75) but with ζ replaced by

$$\partial\mathbf{V} \doteq \big(\partial_1\mathbb{V}_1^{(\mathbf{S},0)}, \partial_1\mathbb{V}_1^{(\mathbf{S},1)}, \partial_2\mathbb{V}_2^{(\mathbf{S},0)}, \partial_2\mathbb{V}_2^{(\mathbf{S},1)}, \partial_{12}\mathbb{V}_3, \partial_{11}\mathbb{V}_3, \partial_{11}\mathbb{V}_2, \partial_{22}\mathbb{V}_3, \partial_{22}\mathbb{V}_1\big). \tag{87}$$

7.2 | The initial contact conditions for the test-functions

Regarding the outer plane component, conditions (21) are satisfied by construction of the test displacements (see also[14, Section 8.1]).

First, observe that by construction, the glued conditions (23) are satisfied.

Now, we check the in plane contact conditions (20). We set

$$\mathbf{N} \doteq \|\partial_{11}\mathbb{V}_2\|_{L^\infty(\Omega)} + \|\partial_{22}\mathbb{V}_1\|_{L^\infty(\Omega)} + \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\Big(\|\partial_\alpha\mathbb{V}_{3-\alpha}^{(\mathbf{B},c)}\|_{L^\infty(\Omega)} + \sum_{\beta=1}^{2}\|\widehat{v}_\beta^{(\alpha,c)}\|_{L^\infty(\Omega\times Cyl^{(\alpha)})}\Big).$$

Below, we replace the test displacements $v_{\varepsilon,\alpha}^{(1,q)}$ and $v_{\varepsilon,\alpha}^{(2,p)}$ in the in plane components by $\lambda_\varepsilon^* v_{\varepsilon,\alpha}^{(1,q)}$ and $\lambda_\varepsilon^* v_{\varepsilon,\alpha}^{(2,p)}$, where $\lambda_\varepsilon^* \doteq 1 - C^*\varepsilon$ where C^* is a nonnegative constant that will be assigned later. Concerning the difference of the displacements in the in plane components, we get (remind that $\kappa < 1$)

$$\begin{aligned} &v_{\varepsilon,\alpha}^{(1,q)}(z', (-1)^{a+b+1}\kappa\varepsilon) - v_{\varepsilon,\alpha}^{(2,p)}(z', (-1)^{a+b+1}\kappa\varepsilon)\\ &= \varepsilon^2\begin{pmatrix} \mathbb{V}_1^{(\mathbf{S},b)}(p\varepsilon,q\varepsilon) - \mathbb{V}_{\varepsilon,1}^{(\mathbf{B},a)}(p\varepsilon,q\varepsilon) - \frac{z_2-q\varepsilon}{\varepsilon}\big(\partial_1\mathbb{V}_2(p\varepsilon) + \partial_2\mathbb{V}_1(q\varepsilon)\big)\\ \mathbb{V}_2^{(\mathbf{B},b)}(p\varepsilon,q\varepsilon) - \mathbb{V}_2^{(\mathbf{S},a)}(p\varepsilon,q\varepsilon) + \frac{z_1-p\varepsilon}{\varepsilon}\big(\partial_1\mathbb{V}_2(p\varepsilon) + \partial_2\mathbb{V}_1(q\varepsilon)\big)\end{pmatrix}\\ &\quad + \varepsilon^3\begin{pmatrix} -\frac{z_2-q\varepsilon}{\varepsilon}\partial_2\mathbb{V}_1^{(\mathbf{B},a)}(p\varepsilon,q\varepsilon) - \frac{(z_1-p\varepsilon)(z_2-q\varepsilon)}{\varepsilon^2}\partial_{11}\mathbb{V}_2(p\varepsilon) - \frac{(z_2-q\varepsilon)^2}{2\varepsilon^2}\partial_{22}\mathbb{V}_1(q\varepsilon) + \widehat{v}_1^{(1,b)} - \widehat{v}_1^{(2,a)}\\ \frac{z_1-p\varepsilon}{\varepsilon}\partial_1\mathbb{V}_2^{(\mathbf{B},b)}(p\varepsilon,q\varepsilon) + \frac{(z_1-p\varepsilon)(z_2-q\varepsilon)}{\varepsilon^2}\partial_{22}\mathbb{V}_1(q\varepsilon) + \frac{(z_1-p\varepsilon)^2}{2\varepsilon^2}\partial_{11}\mathbb{V}_2(p\varepsilon) + \widehat{v}_2^{(1,b)} - \widehat{v}_2^{(2,a)}\end{pmatrix}. \end{aligned}$$

Regarding the in plane contact in original, we have a.e. in C_{pq} that

$$
\begin{aligned}
&\lambda_\varepsilon^*\begin{pmatrix} |v_{\varepsilon,1}^{(1,q)}(z',(-1)^{a+b+1}\kappa\varepsilon)-v_{\varepsilon,1}^{(2,p)}(z',(-1)^{a+b}\kappa\varepsilon)| \\ |v_{\varepsilon,2}^{(1,q)}(z',(-1)^{a+b+1}\kappa\varepsilon)-v_{\varepsilon,2}^{(2,p)}(z',(-1)^{a+b}\kappa\varepsilon)|\end{pmatrix}\\
&\le \lambda_\varepsilon^*\varepsilon^2\left(\left|\begin{matrix}\mathbb{V}_1^{(\mathbf{S},b)}(p\varepsilon,q\varepsilon)-\mathbb{V}_{\varepsilon,1}^{(\mathbf{B},a)}(p\varepsilon,q\varepsilon)-\frac{z_2-q\varepsilon}{\varepsilon}\left(\partial_1\mathbb{V}_2(p\varepsilon)+\partial_2\mathbb{V}_1(q\varepsilon)\right)\\ \mathbb{V}_2^{(\mathbf{B},a)}(p\varepsilon,q\varepsilon)-\mathbb{V}_2^{(\mathbf{S},b)}(q\varepsilon,p\varepsilon)+\frac{z_1-p\varepsilon}{\varepsilon}\left(\partial_1\mathbb{V}_2(p\varepsilon)+\partial_2\mathbb{V}_1(q\varepsilon)\right)\end{matrix}\right|\right)+\varepsilon^3\mathbf{N}\begin{pmatrix}1\\1\end{pmatrix}\\
&\le \lambda_\varepsilon^*\varepsilon^2\begin{pmatrix}g_1(p\varepsilon,q\varepsilon)\\ g_2(p\varepsilon,q\varepsilon)\end{pmatrix}+\varepsilon^3\mathbf{N}\begin{pmatrix}1\\1\end{pmatrix}\le\varepsilon^2\begin{pmatrix}g_1(p\varepsilon,q\varepsilon)\\ g_2(p\varepsilon,q\varepsilon)\end{pmatrix}+\varepsilon^3\mathbf{N}\begin{pmatrix}1\\1\end{pmatrix}-C^*C_3\varepsilon^3\begin{pmatrix}1\\1\end{pmatrix}\le\varepsilon^2\begin{pmatrix}g_1(p\varepsilon,q\varepsilon)\\ g_2(p\varepsilon,q\varepsilon)\end{pmatrix}
\end{aligned}
$$

where the last inequality is true only if (recall property (19) of g_α) we take the value $C^*=\mathbf{N}/C_3$. Hence, the in plane contact conditions (20) are satisfied.
Also, by construction of the test displacements, the glued conditions (23) are satisfied.

8 | THE UNFOLDED LIMIT PROBLEM

In this section, all tools and results developed in this paper are summarized and lead to the homogenization of the textile. However, since in problem (28) the vectorial notation is used, we have to write the limit strain tensors in such notation. From (74)-(75), we define the column vectors with six entries $E^{(\alpha,c)}\in L^2(\Omega)^6$, $E_X^{(\alpha,c)}\in L^2(\Omega;H^1(Cyl^{(\alpha)}))^6$ by

$$
\begin{aligned}
E^{(\alpha,c)}(\partial\mathbf{U})&=\left(\mathcal{E}_{11}^{(\alpha,c)}\ \mathcal{E}_{22}^{(\alpha,c)}\ \mathcal{E}_{33}^{(\alpha,c)}\ \mathcal{E}_{12}^{(\alpha,c)}\ \mathcal{E}_{13}^{(\alpha,c)}\ \mathcal{E}_{23}^{(\alpha,c)}\right)^T,\\
E_X^{(\alpha,c)}(\widehat{u}^{(\alpha,c)})&=\left(\mathcal{E}_{X,11}^{(\alpha,c)}\ \mathcal{E}_{X,22}^{(\alpha,c)}\ \mathcal{E}_{X,33}^{(\alpha,c)}\ \mathcal{E}_{X,12}^{(\alpha,c)}\ \mathcal{E}_{X,13}^{(\alpha,c)}\ \mathcal{E}_{X,23}^{(\alpha,c)}\right)^T.
\end{aligned}\tag{88}
$$

Theorem 1. Let $u_\varepsilon\in\mathcal{X}_\varepsilon$ be a solution of problem (28). We assume that the sequence $\{\mathbf{A}_\varepsilon\}_\varepsilon$ satisfies the assumptions in Subsection 4.4 and that there exist $A^{(\alpha,c)}\in L^\infty(Cyl^{(\alpha)})^{6\times6}$ such that

$$
\Pi_\varepsilon^{(\alpha,c)}\left(\mathbf{A}_\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\right)(z',X)\to A^{(\alpha,c)}(X)\quad\text{for a.e. }(z',X)\in\Omega\times Cyl^{(\alpha)}.\tag{89}
$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, such that $(\mathbb{U},\widetilde{\mathbb{U}},\widehat{u})\in\mathcal{X}$ is a solution of the unfolded limit problem:
Find $(\mathbb{U},\mathbb{U}^{(\mathbf{S})},\mathbb{U}^{(\mathbf{B})},\widehat{u})\in\mathcal{X}$ such that for every $(\mathbb{V},\mathbb{V}^{(\mathbf{S})},\mathbb{V}^{(\mathbf{B})},\widehat{v})\in\mathcal{X}$:

$$
\begin{aligned}
&\sum_{\alpha=1}^2\sum_{c=0}^1\int_{\Omega\times Cyl^{(\alpha)}}A^{(\alpha,c)}\left(E^{(\alpha,c)}(\partial\mathbf{U})+E_X^{(\alpha,c)}(\widehat{u})\right)\cdot\left(E^{(\alpha,c)}\left(\partial\mathbf{U}-\partial\mathbf{V}\right)+E_X^{(\alpha,c)}(\widehat{u}-\widehat{v})\right)\boldsymbol{\eta}^{(\alpha,c)}dz'dX\\
&\le\mathbf{C}_0(\kappa)\sum_{\beta=1}^2\int_\Omega\left(f_\alpha^{(\beta)}(\mathbb{U}_\alpha-\mathbb{V}_\alpha)+f_3^{(\beta)}(\mathbb{U}_3-\mathbb{V}_3)\right)dz'-\mathbf{C}_1(\kappa)\sum_{\beta=1}^2\int_\Omega\widetilde{f}_\alpha^{(\beta)}(\partial_\alpha\mathbb{U}_3-\partial_\alpha\mathbb{V}_3)dz'\\
&\quad+\frac{\mathbf{C}_0(\kappa)}{2}\sum_{c=0}^1\int_\Omega\left(\widetilde{f}_\alpha^{(\alpha)}\left(\mathbb{U}_\alpha^{(\mathbf{S},c)}-\mathbb{V}_\alpha^{(\mathbf{S},c)}\right)+\widetilde{f}_\alpha^{(3-\alpha)}\left(\mathbb{U}_\alpha^{(\mathbf{B},c)}-\mathbb{V}_\alpha^{(\mathbf{B},c)}\right)\right)dz',
\end{aligned}\tag{90}
$$

where $\partial\mathbf{U}$ and $\partial\mathbf{V}$ are defined in (77) and (87) respectively and

$$
\mathbf{C}_0(\kappa)\doteq4\kappa^2\int_0^2\boldsymbol{\gamma}(t)dt,\qquad\mathbf{C}_1(\kappa)\doteq4\kappa^2\int_0^2\Phi(t)\boldsymbol{\gamma}(t)dt\tag{91}
$$

Moreover, the solution is not unique.

Proof. Concerning the limits of the unfolded frame, as well as the integration over the textile domain, we refer to Appendix 10.

First, from convergences (76)-(6.5)-(86) written in vectorial notation (see (88)), convergence (89) and[6, Corollary 2.12] we get

$$
\begin{aligned}
&\frac{1}{\varepsilon^5}\sum_{s=0}^{2N_\varepsilon}\int_{s\varepsilon\mathbf{e}_\alpha+P_r^{(\alpha)}} A_\varepsilon^{(\alpha,s)}\widetilde{\mathbf{E}}_z(u_\varepsilon^{(\alpha,s)})\cdot\widetilde{\mathbf{E}}_z(v_\varepsilon^{(\alpha,s)})\,\eta^{(\alpha,s)}\,dz\\
&\to\sum_{c=0}^{1}\int_{\Omega\times Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\partial\mathbf{U})+E_X^{(\alpha,c)}(\widehat{u}^{(\alpha,c)})\big)\cdot\big(E^{(\alpha,c)}(\partial\mathbf{V})+E_X^{(\alpha,c)}(\widehat{v}^{(\alpha,c)})\big)\boldsymbol{\eta}^{(\alpha,c)}\,dz'dX,
\end{aligned}
\tag{92}
$$

where u_ε is the solution to problem (28) and v_ε is the test function defined as in (85). By convergences (76)-(6.5), the weak lower semicontinuity of the convex functionals, problem (28) and[6, Corollary 2.12] we have

$$
\begin{aligned}
&\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{\Omega\times Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\partial\mathbf{U})+E_X^{(\alpha,c)}(\widehat{u}^{(\alpha,c)})\big)\cdot\big(E^{(\alpha,c)}(\partial\mathbf{U})+E_X^{(\alpha,c)}(\widehat{u}^{(\alpha,c)})\boldsymbol{\eta}^{(\alpha,c)}\big)dz'dX\\
&\le\liminf_{\varepsilon\to0}\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\sum_{s=0}^{2N_\varepsilon}\int_{s\varepsilon\mathbf{e}_\alpha+P_r^{(\alpha)}} A_\varepsilon^{(\alpha,s)}\widetilde{\mathbf{E}}_z(u_\varepsilon^{(\alpha,s)})\cdot\widetilde{\mathbf{E}}_z(u_\varepsilon^{(\alpha,s)})\,\eta^{(\alpha,s)}dz\\
&=\liminf_{\varepsilon\to0}\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\sum_{s=0}^{2N_\varepsilon}\int_{s\varepsilon\mathbf{e}_\alpha+P_r^{(\alpha)}} F_\varepsilon^{(\alpha,s)}\cdot u_\varepsilon^{(\alpha,s)}(z)\,\eta^{(\alpha,s)}dz.
\end{aligned}
\tag{93}
$$

We prove now that the last term in (93) converges. By assumptions on the forces in subsection 5.6, the definition of displacement (45) and convergences in Lemma 13 we have (see Lemma 23 for the details)

$$
\begin{aligned}
\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\sum_{s=0}^{2N_\varepsilon}\int_{s\varepsilon\mathbf{e}_\alpha+P_r^{(\alpha)}} F_\varepsilon^{(\alpha,s)}\cdot u_\varepsilon^{(\alpha,s)}\,\eta^{(\alpha,s)}\,dz\to&\,\mathbf{C}_0(\kappa)\sum_{\beta=1}^{2}\Big(\int_\Omega f_\alpha^{(\beta)}\,\mathbb{U}_\alpha\,dz'+\int_\Omega f_3^{(\beta)}\,\mathbb{U}_3\,dz'\Big)-\mathbf{C}_1(\kappa)\sum_{\beta=1}^{2}\int_\Omega\widetilde{f}_\alpha^{(\beta)}\,\partial_\alpha\mathbb{U}_3\,dz'\\
&+\frac{\mathbf{C}_0(\kappa)}{2}\sum_{c=0}^{1}\Big(\int_\Omega\big(\widetilde{f}_\alpha^{(\alpha)}\,\mathbb{U}_\alpha^{(\mathbf{S},c)}+\widetilde{f}_\alpha^{(3-\alpha)}\,\mathbb{U}_\alpha^{(\mathbf{B},c)}\big)\,dz'\Big).
\end{aligned}
\tag{94}
$$

At last, again by assumptions on the forces in subsection 5.6, the definition of test displacement (85) and convergences in Section 7 we obtain the limit of

$$
\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\sum_{s=0}^{2N_\varepsilon}\int_{s\varepsilon\mathbf{e}_\alpha+P_r^{(\alpha)}} F_\varepsilon^{(\alpha,s)}\cdot v_\varepsilon^{(\alpha,s)}\,\eta^{(\alpha,s)}\,dz
$$

replacing in (94) the functions $\mathbb{U},\widetilde{\mathbb{U}}$ by $\mathbb{V},\widetilde{\mathbb{V}}$. Hence, inequality (90) follows due to (92), (93) and (94). A density argument gives (90) for any test function in $\mathcal{X}$.

The existence of solutions for problem (90) is a direct consequence of the bilinearity, boundedness and coercivity (since from (26) and (89) one obtains

$$
C_0|\xi|^2\le A^{(\alpha,c)}(X)\,\xi\cdot\xi\le C_1|\xi|^2\quad\text{for a.e. }\ X\in Cyl^{(\alpha)}\quad\text{and}\quad\forall\xi\in\mathbb{R}^6)
$$

together with the Stampacchia Lemma.

Concerning uniqueness, assume that $(\mathbb{U},\mathbb{U}^{(\mathbf{S})},\mathbb{U}^{(\mathbf{B})},\widehat{u})$, $(\mathbb{U}',\mathbb{U}^{(\mathbf{S})'},\mathbb{U}^{(\mathbf{B})'},\widehat{u}')$ are both solutions of (90). By the two inequalities given by (90) with $(\mathbb{U},\mathbb{U}^{(\mathbf{S})},\mathbb{U}^{(\mathbf{B})},\widehat{u})$ as a solution and $(\mathbb{U}',\mathbb{U}^{(\mathbf{S})'},\mathbb{U}^{(\mathbf{B})'},\widehat{u}')$ as a test function and vice versa, we get that

$$
\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{\Omega\times Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\partial\mathbf{U}-\partial\mathbf{U}')+E_X^{(\alpha,c)}(\widehat{u}^{(\alpha,c)}-\widehat{u}^{(\alpha,c)'})\big)\cdot\big(E^{(\alpha,c)}(\partial\mathbf{U}-\partial\mathbf{U}')+E_X^{(\alpha,c)}(\widehat{u}^{(\alpha,c)}-\widehat{u}^{(\alpha,c)'})\big)\boldsymbol{\eta}^{(\alpha,c)}dzdX\le0.
$$

By the coercivity of $A^{(\alpha,c)}$, this implies

$$
\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\Big\|E^{(\alpha,c)}(\partial\mathbf{U}-\partial\mathbf{U}')+E_X^{(\alpha,c)}(\widehat{u}^{(\alpha,c)}-\widehat{u}^{(\alpha,c)'})\Big\|_{L^2(\Omega\times Cyl^{(\alpha)})}=0.
$$

Lemma 20 below gives $\widehat{u}^{(\alpha,c)'} = \widehat{u}^{(\alpha,c)} + \mathbf{r}^{(\alpha,c)}$, where $\mathbf{r}^{(\alpha,c)}$ are 4 rigid displacements, and $\partial\mathbf{U} = \partial\mathbf{U}'$, which together with the limit boundary conditions (see the definition of $\mathcal{X}$) imply that

$$\mathbb{U} = \mathbb{U}', \quad \mathbb{U}^{(\mathbf{S})} = \mathbb{U}^{(\mathbf{S})'}.$$

Concerning the fields $\mathbb{U}^{(\mathbf{B})}$ and $\mathbb{U}^{(\mathbf{B})'}$, the limit contact conditions (see again the definition of $\mathcal{X}$) and the fact that $\mathbb{U}^{(\mathbf{S})} = \mathbb{U}^{(\mathbf{S})'}$ imply that a.e. $z \in \Omega$ and $(a,b) \in \{0,1\}^2$:

$$\begin{aligned}\left|\mathbb{U}_1^{(\mathbf{B},a)} - \mathbb{U}_1^{(\mathbf{B},a)'}\right| &\le \left|\mathbb{U}_1^{(\mathbf{S},b)} - \mathbb{U}_1^{(\mathbf{B},a)}\right| + \left|\mathbb{U}_1^{(\mathbf{S},b)'} - \mathbb{U}_1^{(\mathbf{B},a)'}\right| \le 2g_1,\\ \left|\mathbb{U}_2^{(\mathbf{B},b)} - \mathbb{U}_2^{(\mathbf{B},b)'}\right| &\le \left|\mathbb{U}_2^{(\mathbf{S},a)} - \mathbb{U}_2^{(\mathbf{B},b)}\right| + \left|\mathbb{U}_2^{(\mathbf{S},a)'} - \mathbb{U}_2^{(\mathbf{B},b)'}\right| \le 2g_2.\end{aligned}$$

Hence, the proof is complete. □

8.1 | The microscopic cell problem

For every $\zeta \in \mathbb{R}^9$, we introduce the displacements $\widehat{W}_\zeta = (\widehat{W}_\zeta^{(1,b)}, \widehat{W}_\zeta^{(2,a)})$ by

$$\widehat{W}_\zeta^{(1,b)}(X_1) = \zeta_6\theta(X_1)\mathbf{n}^{(1,b)}(X_1), \quad \widehat{W}_\zeta^{(2,a)}(X_2) = \zeta_8\theta(X_2)\mathbf{n}^{(2,a)}(X_2) \tag{95}$$

where $\theta \in C^1_{per}(\mathbb{R})$ is a 2-periodic function satisfying

$$\theta(t) = \frac{1}{2}(t-c)^2 \ \text{ a.e. in } [c-\kappa, c+\kappa], \quad c \in \{0,1,2\}.$$

We define the convex subset $\widehat{\mathbf{W}}_\zeta$ of $\widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$ by

$$\begin{aligned}\widehat{\mathbf{W}}_\zeta \doteq \Big\{ (\widehat{w}^{(1)}, \widehat{w}^{(2)}) \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)} \ \Big|\ 0 \le (-1)^{a+b}\Big(\big(\widehat{W}_\zeta^{(1,b)}(X_1) + \widehat{w}_3^{(1,b)}(X_1, X_2 - b, (-1)^{a+b+1}\kappa)\big)\\ - \big(\widehat{W}_\zeta^{(2,a)}(X_2) + \widehat{w}_3^{(2,a)}(X_1 - a, X_2, (-1)^{a+b}\kappa)\big)\Big) \text{ a.e. on } \mathbf{C}_{ab}\Big\}.\end{aligned} \tag{96}$$

Note that this set includes the micro-macro outer plane contact conditions (83).

Lemma 20. Let ζ be in $\mathbb{R}^9$ and $\widehat{v} \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$ satisfying

$$E(\zeta) + E_X(\widehat{v}) = 0. \tag{97}$$

Then $\zeta = 0$ and $\widehat{v}^{(1,b)}$, $\widehat{v}^{(2,a)}$ are periodic rigid displacements

$$\begin{aligned}\widehat{v}^{(1,b)}(X) &= \mathbf{a}^{(1,b)} + \big(b^{(1,b)}\mathbf{e}_1 - \Phi^{(1,b)}(X_1)\mathbf{e}_3\big) \wedge \big((X_2 - b)\mathbf{e}_2 + X_3\mathbf{n}(X_1)\big) \text{ in } Cyl^{(1)},\\ \widehat{v}^{(2,a)}(X) &= \mathbf{a}^{(2,a)} + \big(b^{(2,a)}\mathbf{e}_2 - \Phi^{(2,a)}(X_2)\mathbf{e}_3\big) \wedge \big((X_1 - a)\mathbf{e}_1 + X_3\mathbf{n}(X_2)\big) \text{ in } Cyl^{(2)},\end{aligned} \tag{98}$$

where $\mathbf{a}^{(1,b)}$, $\mathbf{a}^{(2,a)}$ belong to $\mathbb{R}^3$ and $\mathbf{b}^{(1,b)}$, $\mathbf{b}^{(2,a)}$ belong to $\mathbb{R}$.

Proof. The solution of the equation (97) is given by

$$\begin{aligned}\widehat{v}^{(1,b)} &= \mathcal{A}^{(1,b)} + \mathcal{B}^{(1,b)} \wedge \big((X_2 - b)\mathbf{e}_2 + X_3\mathbf{n}(X_1)\big),\\ \widehat{v}^{(2,a)} &= \mathcal{A}^{(2,a)} + \mathcal{B}^{(2,a)} \wedge \big((X_1 - a)\mathbf{e}_1 + X_3\mathbf{n}(X_2)\big)\end{aligned}$$

with (see (74)-(75))

$$\mathcal{B}^{(1,b)}(X_1) = \mathbf{b}^{(1,b)} - (X_1 - 1)\begin{pmatrix}\zeta_5\\ -\zeta_6\\ \zeta_7\end{pmatrix} - \Phi^{(1,b)}(X_1)\mathbf{e}_3,$$

$$\mathcal{B}^{(2,a)}(X_1) = \mathbf{b}^{(2,a)} - (X_2 - 1)\begin{pmatrix}\zeta_8\\ -\zeta_5\\ -\zeta_9\end{pmatrix} - \Phi^{(2,a)}(X_2)\mathbf{e}_3$$

and

$$\mathcal{A}^{(1,b)}(X_1) = \mathbf{a}^{(1,b)} + (X_1 - 1)\left[\mathbf{b}^{(1,b)} \wedge \mathbf{e}_1 - \zeta_{1+b}\mathbf{e}_1\right] - \frac{1}{2}(X_1 - 1)^2 \begin{pmatrix} \zeta_5 \\ -\zeta_6 \\ \zeta_7 \end{pmatrix} \wedge \mathbf{e}_1,$$

$$\mathcal{A}^{(2,a)}(X_2) = \mathbf{a}^{(2,a)} + (X_2 - 1)\left[\mathbf{b}^{(2,a)} \wedge \mathbf{e}_2 - \zeta_{3+a}\mathbf{e}_2\right] - \frac{1}{2}(X_2 - 1)^2 \begin{pmatrix} \zeta_8 \\ -\zeta_5 \\ -\zeta_9 \end{pmatrix} \wedge \mathbf{e}_2,$$

where $\mathbf{b}^{(1,b)},\ \mathbf{a}^{(1,b)},\ \mathbf{b}^{(2,a)},\ \mathbf{a}^{(2,a)}$ belong to $\mathbb{R}^3$.

First, note that the functions $X_1 \mapsto (X_1 - 1)^2$ and $X_2 \mapsto (X_2 - 1)^2$ can be extended to 2-periodic functions. Then, the periodicity of $\mathcal{A}^{(1,b)}$ and $\mathcal{B}^{(1,b)}$ (resp. $\mathcal{A}^{(2,a)}$ and $\mathcal{B}^{(2,a)}$) with respect to X_1 (resp. X_2) yields $\zeta_1 = \zeta_2 = \ldots = \zeta_9 = 0$ and thus $\zeta = 0$. Furthermore, one gets $\mathbf{b}^{(1,b)} \wedge \mathbf{e}_1$ and $\mathbf{b}^{(2,a)} \wedge \mathbf{e}_2 = 0$. This leads to the expression (98) of $\hat{v}^{(1,b)}$ and $\hat{v}^{(2,a)}$. $\square$

Now, replacing $\mathbb{U}$ and $\mathbb{V}$ by a unique ζ in problem (90) leads to the following microscopic cell problem:

$$\begin{aligned} &\text{For } \zeta \text{ in } \mathbb{R}^9, \text{ find } \hat{\chi} \in \widehat{\mathbf{W}}_\zeta \text{ such that for every } \hat{v} \in \widehat{\mathbf{W}}_\zeta : \\ &\sum_{\alpha=1}^{2}\sum_{c=0}^{1} \int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\zeta) + E_X^{(\alpha,c)}(\hat{\chi})\big) \cdot E_X^{(\alpha,c)}(\hat{\chi} - \hat{v})\,\boldsymbol{\eta}^{(\alpha,c)}\,dX \le 0. \end{aligned} \tag{99}$$

Applying the change of functions

$$\begin{aligned} &\hat{\hat{\chi}}^{(1,b)}(\zeta,\cdot) = \hat{\chi}^{(1,b)}(\zeta,\cdot) - \widehat{W}_\zeta^{(1,b)}, \qquad \hat{\hat{\chi}}^{(2,a)}(\zeta,\cdot) = \hat{\chi}^{(2,a)}(\zeta,\cdot) - \widehat{W}_\zeta^{(2,a)}, \\ &\hat{\hat{v}}^{(1,b)}(\zeta,\cdot) = \hat{v}^{(1,b)}(\zeta,\cdot) - \widehat{W}_\zeta^{(1,b)}, \qquad \hat{\hat{v}}^{(2,a)}(\zeta,\cdot) = \hat{v}^{(2,a)}(\zeta,\cdot) - \widehat{W}_\zeta^{(2,a)}, \end{aligned} \tag{100}$$

we transform the above problem in an equivalent one:

$$\begin{aligned} &\text{For } \zeta \text{ in } \mathbb{R}^9, \text{ find } \hat{\hat{\chi}}(\zeta,\cdot) \in \widehat{\mathbf{W}}_0 \text{ such that for every } \hat{\hat{v}} \in \widehat{\mathbf{W}}_0 : \\ &\sum_{\alpha=1}^{2}\sum_{c=0}^{1} \int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\left(E^{(\alpha,c)}(\zeta) + E_X^{(\alpha,c)}(\widehat{W}_\zeta) + E_X^{(\alpha,c)}(\hat{\hat{\chi}}(\zeta,\cdot))\right) \cdot E_X^{(\alpha,c)}(\hat{\hat{\chi}}(\zeta,\cdot) - \hat{\hat{v}})\,\boldsymbol{\eta}^{(\alpha,c)}\,dX \le 0, \end{aligned} \tag{101}$$

where $\widehat{\mathbf{W}}_0$ is the set $\widehat{\mathbf{W}}_\zeta$ with $\zeta = 0$ (see (95)-(96)).

This problem admits solutions by the Stampacchia lemma (see [16]). Moreover, if $\hat{\hat{\chi}}(\zeta,\cdot)$, $\tilde{\tilde{\chi}}(\zeta,\cdot)$ are both solutions of (101), then there exist 4 rigid displacements $\mathbf{r}_\zeta^{(\alpha,c)} \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$ such that

$$\hat{\hat{\chi}}^{(\alpha,c)}(\zeta,\cdot) = \mathbf{r}_\zeta^{(\alpha,c)} + \tilde{\tilde{\chi}}^{(\alpha,c)}(\zeta,\cdot). \tag{102}$$

Indeed, we can consider problem (101) with $\hat{\hat{\chi}}(\zeta,\cdot)$ as solution and $\tilde{\tilde{\chi}}(\zeta,\cdot)$ as test-function, then the same problem with $\tilde{\tilde{\chi}}(\zeta,\cdot)$ as solution and $\hat{\hat{\chi}}(\zeta,\cdot)$ as the test function. Summing up both inequalities leads to

$$\sum_{\alpha=1}^{2}\sum_{c=0}^{1} \int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\,E_X^{(\alpha,c)}(\tilde{\tilde{\chi}}(\zeta,\cdot) - \hat{\hat{\chi}}(\zeta,\cdot)) \cdot E_X^{(\alpha,c)}(\tilde{\tilde{\chi}}(\zeta,\cdot) - \hat{\hat{\chi}}(\zeta,\cdot))\,\boldsymbol{\eta}^{(\alpha,c)}\,dX \le 0,$$

from where we get that $E_X(\hat{\hat{\chi}}) = E_X(\tilde{\tilde{\chi}})$, since by coercivity the above quantity is also nonnegative. Hence, (102) follows by Lemma 20.

Lemma 21. The map $\zeta \in \mathbb{R}^9 \mapsto E_X(\hat{\hat{\chi}}(\zeta,\cdot)) \in L^2(Cyl)^6$ is continuous. Moreover, there exists a constant C independent on ζ such that

$$\forall \zeta \in \mathbb{R}^9, \qquad \left\| E_X(\hat{\hat{\chi}}(\zeta,\cdot)) \right\|_{L^2(Cyl)} \le C|\zeta|. \tag{103}$$

Proof. Consider problem (101) with ζ (resp ξ) as parameter and with $\widehat{\widehat{\chi}}(\xi,\cdot)$ (resp. $\widehat{\widehat{\chi}}(\zeta,\cdot)$) as test-function. Taking the difference, we obtain

$$
\begin{aligned}
&C_0\Big\|E_X(\widehat{\widehat{\chi}}(\zeta,\cdot)) - E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\xi,\cdot))\Big\|^2_{L^2(Cyl)}\\
&\le \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\zeta,\cdot)) - E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\xi,\cdot))\big)\cdot\big(E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\zeta,\cdot)) - E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\xi,\cdot))\big)\boldsymbol{\eta}^{(\alpha,c)}\,dX\\
&\le \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\zeta,\cdot))\cdot\big(E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\zeta,\cdot)) - E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\xi,\cdot))\big)\boldsymbol{\eta}^{(\alpha,c)}\,dX\\
&\quad+\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\xi,\cdot))\cdot\big(E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\xi,\cdot)) - E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\zeta,\cdot))\big)\boldsymbol{\eta}^{(\alpha,c)}\,dX\\
&\le -\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\zeta) + E^{(\alpha,c)}(\widehat{W}_\zeta)\big)\cdot\big(E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\zeta,\cdot)) - E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\xi,\cdot))\big)\boldsymbol{\eta}^{(\alpha,c)}\,dX\\
&\quad-\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\xi) + E^{(\alpha,c)}(\widehat{W}_\xi)\big)\cdot\big(E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\xi,\cdot)) - E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\zeta,\cdot))\big)\boldsymbol{\eta}^{(\alpha,c)}\,dX\\
&\le \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\xi-\zeta) + E^{(\alpha,c)}(\widehat{W}_{\xi-\zeta})\big)\cdot\big(E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\zeta,\cdot)) - E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\xi,\cdot))\big)\boldsymbol{\eta}^{(\alpha,c)}\,dX\\
&\le C\Big\|E^{(\alpha,c)}(\zeta-\xi) + E_X^{(\alpha,c)}(\widehat{W}_{\zeta-\xi})\Big\|_{L^2(Cyl)}\Big\|E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\zeta,\cdot)) - E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\xi,\cdot))\Big\|_{L^2(Cyl)}.
\end{aligned}
$$

Hence,

$$
\Big\|E_X(\widehat{\widehat{\chi}}(\zeta,\cdot)) - E_X(\widehat{\widehat{\chi}}(\xi,\cdot))\Big\|_{L^2(Cyl)} \le C\Big\|E^{(\alpha,c)}(\zeta-\xi) + E_X^{(\alpha,c)}(\widehat{W}_{\zeta-\xi})\Big\|_{L^2(Cyl)} \le C|\zeta-\xi| \tag{104}
$$

and thus the continuity of $\zeta\in\mathbb{R}^9 \mapsto E_X(\widehat{\widehat{\chi}}(\zeta,\cdot)) \in L^2(Cyl)^6$ for the strong topology of $L^2(Cyl)^6$ is proved. Since $E_X(\widehat{\widehat{\chi}}(0,\cdot)) = 0$, the above inequality also proves (103). $\square$

Since the cell problem (99) has been solved, we can now define the homogenizing operator and its main properties.

Proposition 1. Under the assumptions of Theorem 1, the function A^{hom} defined by ($n\in\{1,\dots,9\}$)

$$
A_n^{hom}(\zeta) \doteq \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\left(E^{(\alpha,c)}(\zeta) + E_X^{(\alpha,c)}(\widehat{W}_\zeta) + E_X^{(\alpha,c)}(\widehat{\widehat{\chi}}(\zeta,X))\right)\cdot E^{(\alpha,c)}(\mathbf{e}_n)\,\boldsymbol{\eta}^{(\alpha,c)}\,dX
$$

with $\widehat{\widehat{\chi}}(\zeta,\cdot)$ a solution to problem (101) is continuous and monotone.

Proof. First note that the map $\zeta\in\mathbb{R}^9 \mapsto E_X(\widehat{\widehat{\chi}}(\zeta,\cdot)) \in L^2(Cyl)^6$ is continuous by Lemma 21. Hence, the map $\zeta\in\mathbb{R}^9 \mapsto A^{hom}(\zeta)\in\mathbb{R}^9$ is continuous. Moreover, due to (104), it is a Lipschitzian map.

Now we prove the monotonicity. The change of functions (100) and the coercivity of the matrix A lead to

$$
\begin{aligned}
&\big(A^{hom}(\zeta) - A^{hom}(\xi)\big)\cdot(\zeta-\xi)\\
&= \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\zeta-\xi) + E_X^{(\alpha,c)}(\hat{\chi}(\zeta,\cdot)-\hat{\chi}(\xi,\cdot))\big)\cdot E^{(\alpha,c)}(\zeta-\xi)\,\boldsymbol{\eta}^{(\alpha,c)}\,dX\\
&\geq C\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} \Big|E^{(\alpha,c)}(\zeta-\xi) + E_X^{(\alpha,c)}(\hat{\chi}(\zeta,\cdot)-\hat{\chi}(\xi,\cdot))\Big|^2\boldsymbol{\eta}^{(\alpha,c)}\,dX\\
&\quad -\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\zeta) + E_X^{(\alpha,c)}(\hat{\chi}(\zeta,\cdot))\big)\cdot\big(E_X^{(\alpha,c)}(\hat{\chi}(\zeta,\cdot)-\hat{\chi}(\xi,\cdot))\big)\boldsymbol{\eta}^{(\alpha,c)}\,dX\\
&\quad -\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\xi) + E_X^{(\alpha,c)}(\hat{\chi}(\xi,\cdot))\big)\cdot\big(E_X^{(\alpha,c)}(\hat{\chi}(\xi,\cdot)-\hat{\chi}(\zeta,\cdot))\big)\boldsymbol{\eta}^{(\alpha,c)}\,dX.
\end{aligned}
$$

The first integral is nonnegative, as well as the other two by (99) with the choice of $\hat{\chi}(\xi,\cdot)$ and $\hat{\chi}(\zeta,\cdot)$ as test functions respectively. Hence, for every $(\zeta,\xi)\in\mathbb{R}^9$, we get

$$
\big(A^{hom}(\zeta) - A^{hom}(\xi)\big)\cdot(\zeta-\xi) \geq C\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} \Big|E^{(\alpha,c)}(\zeta-\xi) + E_X^{(\alpha,c)}(\hat{\chi}(\zeta,\cdot)-\hat{\chi}(\xi,\cdot))\Big|^2 dX \geq 0
$$

and thus the monotonicity of A^{hom} is proved. □

At last, we show the strict monotonicity of the homogenizing operator.

Lemma 22. There exists a constant $C_1 > 0$ such that

$$
\forall \zeta\in\mathbb{R}^9,\qquad A^{hom}(\zeta)\cdot\zeta \geq C_1|\zeta|^2. \tag{105}
$$

Proof. Step 1. In this step we prove that there exists a constant $C_1 > 0$ such that for all $\zeta\in\mathbb{R}^9$ it holds

$$
\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} \Big|E^{(\alpha,c)}(\zeta) + E_X^{(\alpha,c)}(\widehat{W}_\zeta) + E_X^{(\alpha,c)}\big(\hat{\hat{\chi}}(\zeta,\cdot)\big)\Big|^2 dX \geq C_1|\zeta|^2. \tag{106}
$$

Suppose by contradiction that the statement does not hold. Hence, for every $n\in\mathbb{N}^*$ there exist $\zeta_n\in\mathbb{R}^9\backslash\{0\}$ such that

$$
\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} \Big|E^{(\alpha,c)}(\zeta_n) + E_X^{(\alpha,c)}(\widehat{W}_{\zeta_n}) + E_X^{(\alpha,c)}\big(\hat{\hat{\chi}}(\zeta_n,\cdot)\big)\Big|^2 dX \leq \frac{1}{n}|\zeta_n|^2.
$$

Dividing by $|\zeta_n|^2$ on both sides we get

$$
\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} \Big|E^{(\alpha,c)}\Big(\frac{\zeta_n}{|\zeta_n|}\Big) + E_X^{(\alpha,c)}\Big(\widehat{W}_{\frac{\zeta_n}{|\zeta_n|}}\Big) + \frac{1}{|\zeta_n|}E_X^{(\alpha,c)}\big(\hat{\hat{\chi}}(\zeta_n,\cdot)\big)\Big|^2 dX \leq \frac{1}{n}. \tag{107}
$$

The sequence $\Big\{\dfrac{\zeta_n}{|\zeta_n|}\Big\}_n$ is bounded in $\mathbb{R}^9$ while $\Big\{\dfrac{1}{|\zeta_n|}E_X\big(\hat{\hat{\chi}}(\zeta_n,\cdot)\big)\Big\}_n$ is bounded in $L^2(Cyl)^6$ by Lemma 21. Therefore, by Korn's Inequality and Lemma 21, there exist periodic rigid displacements $\mathbf{r}_n = (\mathbf{r}_n^{(1,b)},\mathbf{r}_n^{(2,a)})\in\widehat{\mathbf{W}}^{(1)}\times\widehat{\mathbf{W}}^{(2)}$ such that

$$
\Big\|\frac{1}{|\zeta_n|}\hat{\hat{\chi}}(\zeta_n,\cdot)-\mathbf{r}_n\Big\|_{H^1(Cyl)} \leq \Big\|\frac{1}{|\zeta_n|}E_X\big(\hat{\hat{\chi}}(\zeta_n,\cdot)\big)\Big\|_{L^2(Cyl)} \leq C
$$

and therefore the sequence $\Big\{\dfrac{1}{|\zeta_n|}\hat{\hat{\chi}}(\zeta_n,\cdot)-\mathbf{r}_n\Big\}_n$ is uniformly bounded in $\widehat{\mathbf{W}}^{(1)}\times\widehat{\mathbf{W}}^{(2)}$. Hence, there exist a subsequence of $\{n\}$, still denoted $\{n\}$, $\zeta\in\mathbb{R}^9$ with $|\zeta|=1$ and $\chi_0\in\widehat{\mathbf{W}}^{(1)}\times\widehat{\mathbf{W}}^{(2)}$ such that

$$
\frac{\zeta_n}{|\zeta_n|}\to\zeta,\quad \frac{1}{|\zeta_n|}\hat{\hat{\chi}}(\zeta_n,\cdot)-\mathbf{r}_n \rightharpoonup \chi_0 \quad\text{weakly in}\quad \widehat{\mathbf{W}}^{(1)}\times\widehat{\mathbf{W}}^{(2)}.
$$

By the weak convergences above and the weak lower semicontinuity, passing to the limit in (107) gives

$$\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} \left|E^{(\alpha,c)}(\zeta)+E_X^{(\alpha,c)}\big(\widehat{W}_\zeta\big)+E_X^{(\alpha,c)}\big(\chi_0\big)\right|^2 dX$$

$$\le \liminf_{n\to+\infty}\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} \left|E^{(\alpha,c)}\Big(\frac{\zeta_n}{|\zeta_n|}\Big)+E_X^{(\alpha,c)}\Big(\widehat{W}_{\frac{\zeta_n}{|\zeta_n|}}\Big)+E_X^{(\alpha,c)}\Big(\frac{1}{|\zeta_n|}\widehat{\hat{\chi}}(\zeta_n,X)\Big)\right|^2 dX\le 0$$

and thus

$$0=E(\zeta)+E_X(\widehat{W}_\zeta)+E_X\big(\chi_0\big)=E(\zeta)+E_X\big(\hat{\chi}_0\big),\quad \hat{\chi}_0=\widehat{W}_\zeta+\chi_0\in\widehat{\mathbf{W}}^{(1)}\times\widehat{\mathbf{W}}^{(2)}.$$

Lemma 20 gives $\zeta=0$, which is a contradiction. Hence (106) is proved.

Step 2. In this step we prove the thesis of the lemma.

Again reintroducing the change of functions (100) we have

$$\begin{aligned}A^{hom}(\zeta)\cdot\zeta&=\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\zeta)+E_X^{(\alpha,c)}(\widehat{W}_\zeta)+E_X^{(\alpha,c)}(\widehat{\hat{\chi}}(\zeta,\cdot))\big)\cdot E^{(\alpha,c)}(\zeta)\,\boldsymbol{\eta}^{(\alpha,c)}\,dX\\&=\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\zeta)+E_X^{(\alpha,c)}(\hat{\chi}(\zeta,\cdot))\big)\cdot\big(E^{(\alpha,c)}(\zeta)+E_X^{(\alpha,c)}(\hat{\chi}(\zeta,\cdot))\big)\,\boldsymbol{\eta}^{(\alpha,c)}dX\\&\quad-\sum_{\alpha=1}^{2}\sum_{c=0}^{1}\int_{Cyl^{(\alpha)}} A^{(\alpha,c)}\big(E^{(\alpha,c)}(\zeta)+E_X^{(\alpha,c)}(\hat{\chi}(\zeta,\cdot))\big)\cdot E_X^{(\alpha,c)}(\hat{\chi}(\zeta,\cdot))\,\boldsymbol{\eta}^{(\alpha,c)}\,dX.\end{aligned}$$

The second integral is nonnegative by problem (101), while in the first one we apply (106) and the coercivity of the matrices $A^{(\alpha,c)}$. We conclude thereby that for every $\zeta\in\mathbb{R}^9$ there exists a constant $C_1>0$ such that (105) is satisfied. □

8.2 | The macroscopic problem

Denote

$$\begin{aligned}\mathcal{X}^H\doteq\Big\{\big(\mathbb{V},\mathbb{V}^{(\mathbf{S})},\mathbb{V}^{(\mathbf{B})}\big)\in\mathcal{X}_M\times\mathcal{X}_S\times\mathcal{X}_B\;\Big|\;&|\mathbb{V}_1^{(\mathbf{S},b)}-\mathbb{V}_1^{(\mathbf{B},a)}|+\kappa\big|\partial_2\mathbb{V}_1+\partial_1\mathbb{V}_2\big|\le g_1\ \text{ a.e. in }\ \Omega,\\&|\mathbb{V}_2^{(\mathbf{S},a)}-\mathbb{V}_2^{(\mathbf{B},b)}|+\kappa\big|\partial_2\mathbb{V}_1+\partial_1\mathbb{V}_2\big|\le g_2\ \text{ a.e. in }\ \Omega,\ (a,b)\in\{0,1\}^2\Big\}\end{aligned}$$

Theorem 2. Let the assumptions of Theorem 1 hold, let A^{hom} be as in Proposition 1, let $\partial\mathbf{U}$ and $\partial\mathbf{V}$ be as in (77) and (87) respectively and let $\mathbf{C}_0(\kappa)$ and $\mathbf{C}_1(\kappa)$ be as in (91).

Then, the homogenized problem

Find $\big(\mathbb{U},\mathbb{U}^{(\mathbf{S})},\mathbb{U}^{(\mathbf{B})}\big)\in\mathcal{X}^H$ such that

$$\begin{aligned}\int_\Omega A^{hom}(\partial\mathbf{U})\cdot(\partial\mathbf{U}-\partial\mathbf{V})\,dz'\le&\,\mathbf{C}_0(\kappa)\sum_{\beta=1}^{2}\Big(\int_\Omega f_\alpha^{(\beta)}(\mathbb{U}_\alpha-\mathbb{V}_\alpha)dz'+\int_\Omega f_3^{(\beta)}(\mathbb{U}_3-\mathbb{V}_3)dz'\Big)\\&+\frac{\mathbf{C}_0(\kappa)}{2}\sum_{c=0}^{1}\Big(\int_\Omega \widetilde{f}_\alpha^{(\alpha)}\big(\mathbb{U}_\alpha^{(\mathbf{S},c)}-\mathbb{V}_\alpha^{(\mathbf{S},c)}\big)+\widetilde{f}_\alpha^{(3-\alpha)}\big(\mathbb{U}_\alpha^{(\mathbf{B},c)}-\mathbb{V}_\alpha^{(\mathbf{B},c)}\big)\,dz'\Big)\\&-\mathbf{C}_1(\kappa)\sum_{\beta=1}^{2}\int_\Omega \widetilde{f}_\alpha^{(\beta)}(\partial_\alpha\mathbb{U}_3-\partial_\alpha\mathbb{V}_3)dz',\qquad\forall\big(\mathbb{V},\mathbb{V}^{(\mathbf{S})},\mathbb{V}^{(\mathbf{B})}\big)\in\mathcal{X}^H\end{aligned}\tag{108}$$

admits solutions. Moreover, such a solution is not unique.

Proof. The existence of solutions for problem (108) is a direct consequence of the continuity, boundedness (see Proposition 1) and coercivity (see Lemma 22) of the homogenizing operator A^{hom}, together with the Stampacchia Lemma.

Concerning uniqueness, assume that $\big(\mathbb{U},\mathbb{U}^{(\mathbf{S})},\mathbb{U}^{(\mathbf{B})}\big)$, $\big(\mathbb{U}',\mathbb{U}^{(\mathbf{S})'},\mathbb{U}^{(\mathbf{B})'}\big)$ are both solutions of (108). Then, by the two inequalities given by (108) with $\left(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}, \mathbb{U}^{(\mathbf{B})}\right)$ as a solution and $\left(\mathbb{U}', \mathbb{U}^{(\mathbf{S})'}, \mathbb{U}^{(\mathbf{B})'}\right)$ as a test function and vice versa, we get that

$$\int_{\Omega} \left(A^{hom}(\partial \mathbf{U}) - A^{hom}(\partial \mathbf{U}')\right) \cdot (\partial \mathbf{U} - \partial \mathbf{U}')\, dz' \leq 0,$$

which together with the monotonicity of A^{hom} implies that the above quantity is also nonnegative. Hence, by the coercivity of A^{hom} this implies that $\partial \mathbf{U} = \partial \mathbf{U}'$ in the L^2 sense, thus the not uniqueness of the solution is shown in the same fashion as in the proof of Theorem 1. □

The operator structure of the homogenized problem is known as the *Leray–Lions* operator.

9 | CONCLUSIONS

As a conclusion, we can give and approximation of the displacements in the direction beams $\mathbf{e}_1$ and $\mathbf{e}_2$

$$u^{(1,q)}(z) \approx \underbrace{\begin{pmatrix} \varepsilon \mathbb{U}_1 + \varepsilon^2 \mathbb{U}_1^{(\mathbf{S},b)} \\ \varepsilon \mathbb{U}_2 + \varepsilon^2 \mathbb{U}_2^{(\mathbf{B},b)} \\ \varepsilon \mathbb{U}_3 \end{pmatrix} (z_1, q\varepsilon) + \begin{pmatrix} \varepsilon \partial_2 \mathbb{U}_3 \\ -\varepsilon \partial_1 \mathbb{U}_3 \\ \varepsilon \partial_1 \mathbb{U}_2 \end{pmatrix} (z_1, q\varepsilon) \wedge (-1)^{q+1} \Phi_\varepsilon(z_1) \mathbf{e}_3}_{\text{middle line displacement}}$$
$$+ \begin{pmatrix} \varepsilon \partial_2 \mathbb{U}_3 \\ -\varepsilon \partial_1 \mathbb{U}_3 \\ \varepsilon \partial_1 \mathbb{U}_2 \end{pmatrix} (z_1, q\varepsilon) \wedge \left((z_2 - q\varepsilon)\mathbf{e}_2 + z_3 \mathbf{n}_\varepsilon^{(1,q)}(z_1)\right) + \varepsilon^3 \widehat{u}^{(1,q)}(z),$$

$$u^{(2,p)}(z) \approx \underbrace{\begin{pmatrix} \varepsilon \mathbb{U}_1 + \varepsilon^2 \mathbb{U}_2^{(\mathbf{B},a)} \\ \varepsilon \mathbb{U}_2 + \varepsilon^2 \mathbb{U}_2^{(\mathbf{S},a)} \\ \varepsilon \mathbb{U}_3 \end{pmatrix} (z_2, p\varepsilon) + \begin{pmatrix} \varepsilon \partial_2 \mathbb{U}_3 \\ -\varepsilon \partial_1 \mathbb{U}_3 \\ \varepsilon \partial_2 \mathbb{U}_1 \end{pmatrix} (z_2, p\varepsilon) \wedge (-1)^{p} \Phi_\varepsilon(z_2) \mathbf{e}_3}_{\text{middle line displacement}}$$
$$+ \begin{pmatrix} \varepsilon \partial_2 \mathbb{U}_3 \\ -\varepsilon \partial_1 \mathbb{U}_3 \\ \varepsilon \partial_2 \mathbb{U}_1 \end{pmatrix} (z_2, p\varepsilon) \wedge \left((z_1 - p\varepsilon)\mathbf{e}_1 + z_3 \mathbf{n}_\varepsilon^{(2,p)}(z_2)\right) + \varepsilon^3 \widehat{u}^{(2,p)}(z).$$

We note that, even though the glued conditions in the definition of elasticity problem before the limit (28) do not allow in plane rigid motions of the displacements and therefore ensure uniqueness, this characteristic is not preserved in the limit problem (90) and more precisely neither in the microscopic scale (see (102)) nor in the macroscopic scale (see the proof of Theorem 1).

In fact, this behavior could have been already expected once the limit contact conditions were found. Indeed, assume $(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}, \mathbb{U}^{(\mathbf{B})}), (\mathbb{U}', \mathbb{U}^{(\mathbf{S})'}, \mathbb{U}^{(\mathbf{B})'}) \in \mathcal{X}$ such that $(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}) = (\mathbb{U}', \mathbb{U}^{(\mathbf{S})'})$. Consider direction $\mathbf{e}_1$. By (82), we would have

$$\begin{cases} |\mathbb{U}_1^{(\mathbf{S},b)} - \mathbb{U}_1^{(\mathbf{B},a)}| \leq g_1 - \kappa \left| \partial_2 \mathbb{U}_1 + \partial_1 \mathbb{U}_2 \right| & \text{a.e. in } \Omega, \\ |\mathbb{U}_1^{(\mathbf{S},b)'} - \mathbb{U}_1^{(\mathbf{B},a)'}| \leq g_1 - \kappa \left| \partial_2 \mathbb{U}_1' + \partial_1 \mathbb{U}_2' \right| & \text{a.e. in } \Omega. \end{cases}$$

It is then clear that $\mathbb{U}^{(\mathbf{B})} = \mathbb{U}^{(\mathbf{B})'}$ if and only if

$$g_1(z_1, z_2) - \kappa \left| \partial_2 \mathbb{U}_1(z_2) + \partial_1 \mathbb{U}_2(z_1) \right| = 0, \qquad \text{a.e. } (z_1, z_2) \in \Omega.$$

But this is in general not true: with the admissible choice of $g_1(z_1, z_2) = \kappa$ for a.e. $(z_1, z_2) \in \Omega$ and by the fact that $\mathbb{U}_1$ only depends on z_2 (and $\mathbb{U}_2$ only on z_1) such equality would be satisfied if and only if $\mathbb{U}_1(z_2) = C_1 z_2$, $\mathbb{U}_2(z_1) = C_2 z_1$ for a.e. $(z_1, z_2) \in \Omega$, $C_1 + C_2 = \pm 1$. But this is impossible due to the boundary conditions.

Hence, uniqueness is not ensured and in none of the 4 parts of the domain Ω_1-Ω_4.

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None reported.

Conflict of interest

The authors declare no potential conflict of interests.

10 | APPENDIX

10.1 | Proof of Lemma 6

Proof. We set for a.e. $z' = (z_1, z_2)$ in ω_r:

$$\begin{aligned}
\mathbf{u}^{(1,q)}(p\varepsilon + z_1, q\varepsilon + z_2) &= u_\varepsilon^{(1,q)}(z_1 + p\varepsilon, z_2 + q\varepsilon, (-1)^{p+q+1} r),\\
\mathbf{u}^{(2,p)}(p\varepsilon + z_1, q\varepsilon + z_2) &= u_\varepsilon^{(2,p)}(z_1 + p\varepsilon, z_2 + q\varepsilon, (-1)^{p+q} r),\\
\overline{\mathbf{u}}^{(1,q)}(p\varepsilon + z_1, q\varepsilon + z_2) &= \overline{u}_\varepsilon^{(1,q)}(z_1 + p\varepsilon, z_2 + q\varepsilon, (-1)^{p+q+1} r),\\
\overline{\mathbf{u}}^{(2,p)}(p\varepsilon + z_1, q\varepsilon + z_2) &= \overline{u}_\varepsilon^{(2,p)}(z_1 + p\varepsilon, z_2 + q\varepsilon, (-1)^{p+q} r)
\end{aligned}$$

Step 1. In this step, we rewrite the displacements in the contact areas as (for a.e. $z' = (z_1, z_2)$ in ω_r)

$$\begin{aligned}
\mathbf{u}^{(1,q)}(p\varepsilon + z_1, q\varepsilon + z_2) &= \mathbb{U}^{(1,q)}(p\varepsilon) + \mathcal{R}^{(1,q)}(p\varepsilon) \wedge \big(z_1\mathbf{e}_1 + z_2\mathbf{e}_2\big) + Q_{p,q}^{(1)}(z'),\\
\mathbf{u}^{(2,p)}(p\varepsilon + z_1, q\varepsilon + z_2) &= \mathbb{U}^{(2,p)}(q\varepsilon) + \mathcal{R}^{(2,p)}(q\varepsilon) \wedge \big(z_1\mathbf{e}_1 + z_2\mathbf{e}_2\big) + Q_{p,q}^{(2)}(z'),
\end{aligned} \tag{109}$$

where the reminder terms $Q_{p,q}^{(\alpha)}$ are estimated by

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|Q_{p,q}^{(\alpha)}\|_{L^2(\omega_r)}^2 \le C\varepsilon \|u\|_{\mathcal{S}_\varepsilon}^2. \tag{110}$$

Remind the decomposition of the displacements in the contact parts given by (18). For a.e. $z' = (z_1, z_2)$ in ω_r, we rewrite them as in (109), where $Q_{p,q}^{(\alpha)}$ are defined by

$$\begin{aligned}
Q_{p,q}^{(1)}(z') &\doteq \big(\mathbb{U}^{(1,q)}(p\varepsilon + z_1) - \mathbb{U}^{(1,q)}(p\varepsilon) - \mathcal{R}^{(1,q)}(p\varepsilon) \wedge z_1\mathbf{e}_1\big) + \big(\mathcal{R}^{(1,q)}(p\varepsilon + z_1) - \mathcal{R}^{(1,q)}(p\varepsilon)\big) \wedge z_2\mathbf{e}_2 + \overline{\mathbf{u}}^{(1,q)}(p\varepsilon + z_1, q\varepsilon + z_2),\\
Q_{p,q}^{(2)}(z') &\doteq \big(\mathbb{U}^{(2,p)}(q\varepsilon + z_2) - \mathbb{U}^{(2,p)}(q\varepsilon) - \mathcal{R}^{(2,p)}(q\varepsilon) \wedge z_2\mathbf{e}_2\big) + \big(\mathcal{R}^{(2,p)}(q\varepsilon + z_2) - \mathcal{R}^{(2,p)}(q\varepsilon)\big) \wedge z_1\mathbf{e}_1 + \overline{\mathbf{u}}^{(2,p)}(p\varepsilon + z_1, q\varepsilon + z_2).
\end{aligned}$$

We want now to prove (110) and due to the symmetrical behavior, we will just estimate $Q_{p,q}^{(1)}$. We first have that

$$\begin{aligned}
\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|Q_{p,q}^{(1)}\|_{L^2(\omega_r)}^2 = \sum_{(p,q)\in\mathcal{K}_\varepsilon} \Big(&\int_{\omega_r} \Big| \int_0^{z_1} \partial_1\mathbb{U}^{(1,q)}(p\varepsilon + t) - \mathcal{R}^{(1,q)}(p\varepsilon) \wedge \mathbf{e}_1 dt \Big|^2 dz' \\
&+ \int_{\omega_r} z_2^2 \Big| \int_0^{z_1} \partial_1\mathcal{R}^{(1,q)}(p\varepsilon + t) dt \Big|^2 dz' \Big) + \sum_{(p,q)\in\mathcal{K}_\varepsilon} \|\overline{u}^{(1,q)}\|_{L^2(\omega_r)}^2.
\end{aligned}$$

Using Jensen's inequality on each term in the parenthesis, we get

$$\begin{aligned}
&\sum_{(p,q)\in\mathcal{K}_\varepsilon} \int_{\omega_r} z_2^2 \Big| \int_0^{z_1} \partial_1\mathcal{R}^{(1,q)}(p\varepsilon + t) dt \Big|^2 dz' \le C\varepsilon^5 \sum_{q=0}^{2N_\varepsilon - 1} \|\partial_1\mathcal{R}^{(1,q)}\|_{L^2(0,L)}^2,\\
&\sum_{(p,q)\in\mathcal{K}_\varepsilon} \int_{\omega_r} \Big| \int_0^{z_1} \partial_1\mathbb{U}^{(1,q)}(p\varepsilon + t) - \mathcal{R}_2^{(1,q)}(p\varepsilon) \wedge \mathbf{e}_1 dt \Big|^2 dz' \\
&\le \sum_{(p,q)\in\mathcal{K}_\varepsilon} \int_{\omega_r} z_1 \int_0^{z_1} \Big(\Big|\partial_1\mathbb{U}^{(1,q)}(p\varepsilon + t) - \mathcal{R}^{(1,q)}(p\varepsilon + t) \wedge \mathbf{e}_1\Big|^2 + t \int_0^t \Big|\partial_1\mathcal{R}_2^{(1,q)}(p\varepsilon + s)\Big|^2 ds \Big) dt dz' \\
&\le C \sum_{q=0}^{2N_\varepsilon - 1} \Big(\varepsilon^3 \|\partial_1\mathbb{U}^{(1,q)} - \mathcal{R}^{(1,q)} \wedge \mathbf{e}_1\|_{L^2(0,L)}^2 + \varepsilon^5 \|\partial_1\mathcal{R}_2^{(1,q)}\|_{L^2(0,L)}^2 \Big).
\end{aligned}$$

By estimates $(9)_{1,3}$ and Lemma 5, we get (110).

Step 2. By the non penetration condition (21) *in the contact parts of the cell* $(p\varepsilon, q\varepsilon) + \varepsilon Y$*, for a.e.* $z' = (z_1, z_2)$ *in* ω_r *we show that we have*

$$\begin{aligned}
0 \le (-1)^{p+q} \Big[& \big(\mathfrak{u}_3^{(1,q)}(p\varepsilon + z_1, q\varepsilon + z_2) - \mathfrak{u}_3^{(2,p)}(p\varepsilon + z_1, q\varepsilon + z_2)\big) \\
& + \big(\mathfrak{u}_3^{(2,p+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + z_2) - \mathfrak{u}_3^{(1,q)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + z_2)\big) \\
& + \big(\mathfrak{u}_3^{(1,q+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + \varepsilon + z_2) - \mathfrak{u}_3^{(2,p+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + \varepsilon + z_2)\big) \\
& + \big(\mathfrak{u}_3^{(2,p)}(p\varepsilon + z_1, q\varepsilon + \varepsilon + z_2) - \mathfrak{u}_3^{(1,q+1)}(p\varepsilon + z_1, q\varepsilon + \varepsilon + z_2)\big)\Big] \\
= (-1)^{p+q} \Big[& \varepsilon \mathcal{R}_2^{(1)}(p\varepsilon, q\varepsilon) - \varepsilon \mathcal{R}_1^{(2)}(p\varepsilon + \varepsilon, q\varepsilon) - \varepsilon \mathcal{R}_2^{(1)}(p\varepsilon, q\varepsilon + \varepsilon) + \varepsilon \mathcal{R}_1^{(2)}(p\varepsilon, q\varepsilon)\Big] \\
& + (-1)^{p+q} \Big[R_{p,q}^{(1)}(z') + R_{p+1,q}^{(2)}(z') + R_{p,q+1}^{(1)}(z') + R_{p,q}^{(2)}(z')\Big],
\end{aligned} \tag{111}$$

where the reminder terms $R_{p,q}^{(\alpha)}$, $R_{p,q+1}^{(1)}$ *and* $R_{p+1,q}^{(2)}$ *are estimated by*

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \| R_{p,q}^{(\alpha)}\|_{L^2(\omega_r)}^2 + \| R_{p,q+1}^{(1)}\|_{L^2(\omega_r)}^2 + \| R_{p+1,q}^{(2)}\|_{L^2(\omega_r)}^2 \le C\varepsilon \|u\|_{S_\varepsilon}^2. \tag{112}$$

Indeed, by the non penetration condition (21) on the vertexes of the cell $(p\varepsilon, q\varepsilon) + \varepsilon Y$ and pairing the involved terms in a different way, we get a.e. $z' = (z_1, z_2)$ in ω_r that

$$\begin{aligned}
0 \le (-1)^{p+q} \Big[& \big(\mathfrak{u}_3^{(1,q)}(p\varepsilon + z_1, q\varepsilon + z_2) - \mathfrak{u}_3^{(2,p)}(p\varepsilon + z_1, q\varepsilon + z_2)\big) \\
& + \big(\mathfrak{u}_3^{(2,p+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + z_2) - \mathfrak{u}_3^{(1,q)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + z_2)\big) \\
& + \big(\mathfrak{u}_3^{(1,q+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + \varepsilon + z_2) - \mathfrak{u}_3^{(2,p+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + \varepsilon + z_2)\big) \\
& + \big(\mathfrak{u}_3^{(2,p)}(p\varepsilon + z_1, q\varepsilon + \varepsilon + z_2) - \mathfrak{u}_3^{(1,q+1)}(p\varepsilon + z_1, q\varepsilon + \varepsilon + z_2)\big)\Big] \\
= (-1)^{p+q} \Big[& \big(\mathfrak{u}_3^{(1,q)}(p\varepsilon + z_1, q\varepsilon + z_2) - \mathfrak{u}_3^{(1,q)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + z_2)\big) \\
& + \big(\mathfrak{u}_3^{(2,p+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + z_2) - \mathfrak{u}_3^{(2,p+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + \varepsilon + z_2)\big) \\
& + \big(\mathfrak{u}_3^{(1,q+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + \varepsilon + z_2) - \mathfrak{u}_3^{(1,q+1)}(p\varepsilon + z_1, q\varepsilon + \varepsilon + z_2)\big) \\
& + \big(\mathfrak{u}_3^{(2,p)}(p\varepsilon + z_1, q\varepsilon + \varepsilon + z_2) - \mathfrak{u}_3^{(2,p)}(p\varepsilon + z_1, q\varepsilon + z_2)\big)\Big].
\end{aligned}$$

Then, the right hand side of the above equality is rewritten in the following way:

$$\begin{aligned}
&\mathfrak{u}_3^{(1,q)}(p\varepsilon + z_1, q\varepsilon + z_2) - \mathfrak{u}_3^{(1,q)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + z_2) = \varepsilon \mathcal{R}_2^{(1)}(p\varepsilon, q\varepsilon) + R_{p,q}^{(1)}(z'), \\
&\mathfrak{u}_3^{(2,p+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + z_2) - \mathfrak{u}_3^{(2,p+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + \varepsilon + z_2) = -\varepsilon \mathcal{R}_1^{(2)}(p\varepsilon + \varepsilon, q\varepsilon) + R_{p+1,q}^{(2)}(z') \\
&\mathfrak{u}_3^{(1,q+1)}(p\varepsilon + \varepsilon + z_1, q\varepsilon + \varepsilon + z_2) - \mathfrak{u}_3^{(1,q+1)}(p\varepsilon + z_1, q\varepsilon + \varepsilon + z_2) = -\varepsilon \mathcal{R}_2^{(1)}(p\varepsilon, q\varepsilon + \varepsilon) + R_{p,q+1}^{(1)}(z') \\
&\mathfrak{u}_3^{(2,p)}(p\varepsilon + z_1, q\varepsilon + \varepsilon + z_2) - \mathfrak{u}_3^{(2,p)}(p\varepsilon + z_1, q\varepsilon + z_2) = \varepsilon \mathcal{R}_1^{(2)}(p\varepsilon, q\varepsilon) + R_{p,q}^{(2)}(z'),
\end{aligned}$$

where $R_{p,q}^{(1)}(z')$, $R_{p,q}^{(2)}(z')$ are defined by $(z' = (z_1, z_2))$

$$\begin{aligned}
R_{p,q}^{(1)} \doteq & \big(\mathbb{U}_3^{(1,q)}(p\varepsilon) - \mathbb{U}_3^{(1,q)}(p\varepsilon + \varepsilon) - \varepsilon \mathcal{R}_2^{(1,q)}(p\varepsilon)\big) + \big(\mathcal{R}_1^{(1,q)}(p\varepsilon) - \mathcal{R}_1^{(1,q)}(p\varepsilon + \varepsilon)\big) z_2 \\
& - \big(\mathcal{R}_2^{(1,q)}(p\varepsilon) - \mathcal{R}_2^{(1,q)}(p\varepsilon + \varepsilon)\big) z_1 + Q_{p,q}^{(1)} - Q_{p+1,q}^{(1)}, \\
R_{p,q}^{(2)} \doteq & \big(\mathbb{U}_3^{(2,p)}(q\varepsilon) - \mathbb{U}_3^{(2,p)}(q\varepsilon + \varepsilon) + \varepsilon \mathcal{R}_1^{(2,p)}(q\varepsilon)\big) + \big(\mathcal{R}_1^{(2,p)}(q\varepsilon) - \mathcal{R}_1^{(2,p)}(q\varepsilon + \varepsilon)\big) z_2 \\
& - \big(\mathcal{R}_2^{(2,p)}(q\varepsilon) - \mathcal{R}_2^{(2,p)}(q\varepsilon + \varepsilon)\big) z_1 + Q_{p,q}^{(2)} - Q_{p,q+1}^{(2)}
\end{aligned}$$

and $R_{p+1,q}^{(2)}$, $R_{p,q+1}^{(1)}$ are referred from the above defined. we have now to prove (112) and due to the symmetrical behavior, we will just estimate $R_{p,q}^{(1)}$. We first have

$$\begin{aligned}
\sum_{(p,q)\in\mathcal{K}_\varepsilon} \| R_{p,q}^{(1)}\|_{L^2(\omega_r)}^2 = \sum_{(p,q)\in\mathcal{K}_\varepsilon} \Big(& \int_{\omega_r} \Big| \int_0^\varepsilon \partial_1 \mathbb{U}_3^{(1,q)}(p\varepsilon + t) - \mathcal{R}_2^{(1,q)}(p\varepsilon) dt \Big|^2 dz' + \int_{\omega_r} z_2^2 \Big| \int_0^\varepsilon \partial_1 \mathcal{R}_1^{(1,q)}(p\varepsilon + t) dt \Big|^2 dz' \\
& + \int_{\omega_r} z_1^2 \Big| \int_0^\varepsilon \partial_1 \mathcal{R}_2^{(1,q)}(p\varepsilon + t) dt \Big|^2 dz' \Big) + \sum_{(p,q)\in\mathcal{K}_\varepsilon} \| Q_{p,q}^{(1)}\|_{L^2(\omega_r)}^2 + \sum_{(p,q)\in\mathcal{K}_\varepsilon} \| Q_{p+1,q}^{(1)}\|_{L^2(\omega_r)}^2.
\end{aligned}$$

Using Jensen's inequality on each term in the parenthesis, we get

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\int_{\omega_r} z_2^2\Big|\int_0^\varepsilon \partial_1\mathcal{R}_1^{(1,q)}(p\varepsilon+t)dt\Big|^2 dz' \le C\varepsilon^5\sum_{q=0}^{2N_\varepsilon-1}\|\partial_1\mathcal{R}_1^{(1,q)}\|^2_{L^2(0,L)},$$

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\int_{\omega_r} z_1^2\Big|\int_0^\varepsilon \partial_1\mathcal{R}_2^{(1,q)}(p\varepsilon+t)dt\Big|^2 dz' \le C\varepsilon^5\sum_{q=0}^{2N_\varepsilon-1}\|\partial_1\mathcal{R}_2^{(1,q)}\|^2_{L^2(0,L)},$$

$$\begin{aligned}
&\sum_{(p,q)\in\mathcal{K}_\varepsilon}\int_{\omega_r}\Big|\int_0^\varepsilon \partial_1\mathbb{U}_3^{(1,q)}(p\varepsilon+t)-\mathcal{R}_2^{(1,q)}(p\varepsilon)dt\Big|^2 dz'\\
&\le Cr^2\sum_{(p,q)\in\mathcal{K}_\varepsilon}\Big(\Big|\int_0^\varepsilon \partial_1\mathbb{U}^{(1,q)}(p\varepsilon+t)-\mathcal{R}^{(1,q)}(p\varepsilon+t)\wedge\mathbf{e}_1dt\Big|^2+\Big|\int_0^\varepsilon\int_0^\varepsilon \partial_1\mathcal{R}_2^{(1,q)}(p\varepsilon+s)dsdt\Big|^2\Big)\\
&\le C\varepsilon^2\sum_{q=0}^{2N_\varepsilon-1}\Big(\varepsilon\|\partial_1\mathbb{U}^{(1,q)}-\mathcal{R}^{(1,q)}\wedge\mathbf{e}_1\|^2_{L^2(0,L)}+\varepsilon^3\|\partial_1\mathcal{R}_2^{(1,q)}\|^2_{L^2(0,L)}\Big).
\end{aligned}$$

By estimates $(9)_{1,3}$ and (110) we get (112) for $R^{(1)}_{p,q}$.

Step 3. In this step we prove that

$$\begin{aligned}
&\sum_{k=p}^{p+1}\sum_{\ell=q}^{q+1}\Big|\big(\mathbb{U}_3^{(1)}-\mathbb{U}_3^{(2)}\big)(k\varepsilon,\ell\varepsilon)-z_1\big(\mathcal{R}_2^{(1)}-\mathcal{R}_2^{(2)}\big)(k\varepsilon,\ell\varepsilon)+z_2\big(\mathcal{R}_1^{(1)}-\mathcal{R}_1^{(1)}\big)(k\varepsilon,\ell\varepsilon)\big|\\
&\le(-1)^{p+q}\big[\varepsilon\mathcal{R}_2^{(1)}(p\varepsilon,q\varepsilon)-\varepsilon\mathcal{R}_1^{(2)}(p\varepsilon+\varepsilon,q\varepsilon)-\varepsilon\mathcal{R}_2^{(1)}(p\varepsilon,q\varepsilon+\varepsilon)+\varepsilon\mathcal{R}_1^{(2)}(p\varepsilon,q\varepsilon)\big]+S_{p,q}(z'),
\end{aligned}\tag{113}$$

where the reminder term $S_{p,q}$ is estimated by

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\|S_{p,q}\|^2_{L^2(\omega_r)}\le C\varepsilon\|u\|^2_{S_\varepsilon}.\tag{114}$$

We first note that in equality (111) the left hand side is positive. Hence, we replace the left-hand side by (109) and take the modulus. Applying Step 1 on the left hand side and Step 2 on the right hand side, we get a.e $z'\in\omega_r$ that

$$\begin{aligned}
&\sum_{k=p}^{p+1}\sum_{\ell=q}^{q+1}\Big|\big(\mathbb{U}_3^{(1)}-\mathbb{U}_3^{(2)}\big)(k\varepsilon,\ell\varepsilon)-z_1\big(\mathcal{R}_2^{(1)}-\mathcal{R}_2^{(2)}\big)(k\varepsilon,\ell\varepsilon)+z_2\big(\mathcal{R}_1^{(1)}-\mathcal{R}_1^{(1)}\big)(k\varepsilon,\ell\varepsilon)+(Q^{(1)}_{k,\ell,3}-Q^{(2)}_{k,\ell,3})(z')\Big|\\
&=(-1)^{p+q}\big[\varepsilon\mathcal{R}_2^{(1)}(p\varepsilon,q\varepsilon)-\varepsilon\mathcal{R}_1^{(2)}(p\varepsilon+\varepsilon,q\varepsilon)-\varepsilon\mathcal{R}_2^{(1)}(p\varepsilon,q\varepsilon+\varepsilon)+\varepsilon\mathcal{R}_1^{(2)}(p\varepsilon,q\varepsilon)\big]\\
&\quad+(-1)^{p+q}\big[R^{(1)}_{p,q}(z')+R^{(2)}_{p+1,q}(z')-R^{(1)}_{p,q+1}(z')-R^{(2)}_{p,q}(z')\big].
\end{aligned}$$

In particular, the above equality is rewritten in the form (113) with $S_{p,q}$ defined by

$$S_{p,q}\doteq(-1)^{p+q}\Big(R^{(1)}_{p,q}+R^{(2)}_{p+1,q}-R^{(1)}_{p,q+1}-R^{(2)}_{p,q}\Big)+\sum_{k=p}^{p+1}\sum_{\ell=q}^{q+1}\Big|(Q^{(1)}_{k,\ell,3}-Q^{(2)}_{k,\ell,3})\Big|.$$

Estimate (114) is a direct consequence of estimates (110) and (112) since

$$\begin{aligned}
\sum_{(p,q)\in\mathcal{K}_\varepsilon}\|S_{p,q}\|^2_{L^2(\omega_r)}&\le\sum_{(p,q)\in\mathcal{K}_\varepsilon}\Big(\|R^{(\alpha)}_{p,q}\|^2_{L^2(\omega_r)}+\|R^{(1)}_{p,q+1}\|^2_{L^2(\omega_r)}+\|R^{(2)}_{p+1,q}\|^2_{L^2(\omega_r)}+\|Q^{(\alpha)}_{p,q}\|^2_{L^2(\omega_r)}\Big)\\
&\le C\varepsilon\|u\|^2_{S_\varepsilon}.
\end{aligned}$$

Step 4. In this step we prove the thesis of the lemma.

Starting from inequality (113) of Step 3, we replace (p,q) by $(2p,2q)$, $(2p+1,2q)$, $(2p,2q+1)$ and $(2p+1,2q+1)$. For a.e

$z' \in \omega_r$, we obtain

$$
\begin{aligned}
&\sum_{k=2p}^{2p+1}\sum_{\ell=2q}^{2q+1}\Big|\big(\mathbb{U}_3^{(1)}-\mathbb{U}_3^{(2)}\big)(k\varepsilon,\ell\varepsilon)-z_1\big(\mathcal{R}_2^{(1)}-\mathcal{R}_2^{(2)}\big)(k\varepsilon,\ell\varepsilon)+z_2\big(\mathcal{R}_1^{(1)}-\mathcal{R}_1^{(1)}\big)(k\varepsilon,\ell\varepsilon)\Big|\\
&+\sum_{k=2p}^{2p+1}\sum_{\ell=2q+1}^{2q+2}\Big|\big(\mathbb{U}_3^{(1)}-\mathbb{U}_3^{(2)}\big)(k\varepsilon,\ell\varepsilon)-z_1\big(\mathcal{R}_2^{(1)}-\mathcal{R}_2^{(2)}\big)(k\varepsilon,\ell\varepsilon)+z_2\big(\mathcal{R}_1^{(1)}-\mathcal{R}_1^{(1)}\big)(k\varepsilon,\ell\varepsilon)\Big|\\
&+\sum_{k=2p+1}^{2p+2}\sum_{\ell=2q}^{2q+1}\Big|\big(\mathbb{U}_3^{(1)}-\mathbb{U}_3^{(2)}\big)(k\varepsilon,\ell\varepsilon)-z_1\big(\mathcal{R}_2^{(1)}-\mathcal{R}_2^{(2)}\big)(k\varepsilon,\ell\varepsilon)+z_2\big(\mathcal{R}_1^{(1)}-\mathcal{R}_1^{(1)}\big)(k\varepsilon,\ell\varepsilon)\Big|\\
&+\sum_{k=2p+1}^{2p+2}\sum_{\ell=2q+1}^{2q+2}\Big|\big(\mathbb{U}_3^{(1)}-\mathbb{U}_3^{(2)}\big)(k\varepsilon,\ell\varepsilon)-z_1\big(\mathcal{R}_2^{(1)}-\mathcal{R}_2^{(2)}\big)(k\varepsilon,\ell\varepsilon)+z_2\big(\mathcal{R}_1^{(1)}-\mathcal{R}_1^{(1)}\big)(k\varepsilon,\ell\varepsilon)\Big|\\
&\le\sum_{k=2p}^{2p+1}\sum_{\ell=2q}^{2q+1}(-1)^{k+\ell}\big[\varepsilon\mathcal{R}_2^{(1)}(k\varepsilon,\ell\varepsilon)-\varepsilon\mathcal{R}_1^{(2)}(k\varepsilon+\varepsilon,\ell\varepsilon)-\varepsilon\mathcal{R}_2^{(1)}(k\varepsilon,\ell\varepsilon+\varepsilon)+\varepsilon\mathcal{R}_1^{(2)}(k\varepsilon,\ell\varepsilon)\big]\\
&\quad+S_{2p,2q}(z')+S_{2p+1,2q}(z')+S_{2p,2q+1}(z')+S_{2p+1,2q+1}(z').
\end{aligned}
\tag{115}
$$

We set

$$
T_{p,q}(z')\doteq\sum_{k=2p}^{2p+1}\sum_{\ell=2q}^{2q+1}(-1)^{k+\ell}\big[\varepsilon\mathcal{R}_2^{(1)}(k\varepsilon,\ell\varepsilon)-\varepsilon\mathcal{R}_1^{(2)}(k\varepsilon+\varepsilon,\ell\varepsilon)-\varepsilon\mathcal{R}_2^{(1)}(k\varepsilon,\ell\varepsilon+\varepsilon)+\varepsilon\mathcal{R}_1^{(2)}(k\varepsilon,\ell\varepsilon)\big]
$$

and we want to prove that this term has a sufficient good estimate:

$$
\sum_{(p,q)\in\mathcal{K}_\varepsilon}\|T_{p,q}\|^2_{L^2(\omega_r)}\le C\varepsilon\|u\|^2_{S_\varepsilon}. \tag{116}
$$

Indeed, writing down the sum and pairing the terms, we get that

$$
\begin{aligned}
T_{p,q}(z')\doteq&\,\varepsilon\big[\mathcal{R}_2^{(1)}(2p\varepsilon,2q\varepsilon)-\mathcal{R}_2^{(1)}(2p\varepsilon+\varepsilon,2q\varepsilon)\big]+\varepsilon\big[\mathcal{R}_2^{(1)}(2p\varepsilon+\varepsilon,2q\varepsilon+\varepsilon)-\mathcal{R}_2^{(1)}(2p\varepsilon,2q\varepsilon+\varepsilon)\big]\\
&+\varepsilon\big[\mathcal{R}_2^{(1)}(2p\varepsilon+\varepsilon,2q\varepsilon+\varepsilon)-\mathcal{R}_2^{(1)}(2p\varepsilon,2q\varepsilon+\varepsilon)\big]+\varepsilon\big[\mathcal{R}_2^{(1)}(2p\varepsilon,2q\varepsilon+2\varepsilon)-\mathcal{R}_2^{(1)}(2p\varepsilon+\varepsilon,2q\varepsilon+2\varepsilon)\big]\\
&-\varepsilon\big[\mathcal{R}_1^{(2)}(2p\varepsilon+\varepsilon,2q\varepsilon)-\mathcal{R}_1^{(2)}(2p\varepsilon+\varepsilon,2q\varepsilon+\varepsilon)\big]-\varepsilon\big[\mathcal{R}_1^{(2)}(2p\varepsilon+2\varepsilon,2q\varepsilon+\varepsilon)-\mathcal{R}_1^{(2)}(2p\varepsilon+2\varepsilon,2q\varepsilon)\big]\\
&-\varepsilon\big[\mathcal{R}_1^{(2)}(2p\varepsilon,2q\varepsilon+\varepsilon)-\mathcal{R}_1^{(2)}(2p\varepsilon,2q\varepsilon)\big]-\varepsilon\big[\mathcal{R}_1^{(2)}(2p\varepsilon+\varepsilon,2q\varepsilon)-\mathcal{R}_1^{(2)}(2p\varepsilon+\varepsilon,2q\varepsilon+\varepsilon)\big].
\end{aligned}
$$

By estimate $(9)_1$ we prove (116) since

$$
\begin{aligned}
\sum_{(p,q)\in\mathcal{K}_\varepsilon}\|T_{p,q}\|^2_{L^2(\omega_r)}=\varepsilon^2\sum_{(p,q)\in\mathcal{K}_\varepsilon}\Big(&\int_{\omega_r}\Big|\int_0^\varepsilon-\partial_1\mathcal{R}_2^{(1,2q)}(2p\varepsilon+t)+2\partial_1\mathcal{R}_2^{(1,2q+1)}(2p\varepsilon+t)-\partial_1\mathcal{R}_2^{(1,2q+2)}(2p\varepsilon+t)dt\Big|^2dz'\\
&+\int_{\omega_r}\Big|\int_0^\varepsilon-\partial_2\mathcal{R}_1^{(2,2p)}(2q\varepsilon+t)+2\partial_2\mathcal{R}_1^{(2,2p+1)}(2q\varepsilon+t)-\partial_2\mathcal{R}_1^{(2,2p+2)}(2q\varepsilon+t)dt\Big|^2dz'\Big)\\
\le C\varepsilon^5\Big(&\sum_{q=0}^{2N_\varepsilon-1}\|\partial_1\mathcal{R}_2^{(1,q)}\|^2_{L^2(0,L)}+\sum_{p=0}^{2N_\varepsilon-1}\|\partial_2\mathcal{R}_1^{(2,p)}\|^2_{L^2(0,L)}\Big)\le C\varepsilon\|u\|^2_{S_\varepsilon}.
\end{aligned}
$$

Taking the L^2 norm in the left-hand side of (115) and applying (114)-(116) on the right hand side, we have that

$$
\sum_{(p,q)\in\mathcal{K}_\varepsilon}\Big(\varepsilon^2\Big|\big(\mathbb{U}_3^{(1)}-\mathbb{U}_3^{(2)}\big)(p\varepsilon,q\varepsilon)\Big|^2+\varepsilon^4\Big|\big(\mathcal{R}_\alpha^{(1)}-\mathcal{R}_\alpha^{(2)}\big)(p\varepsilon,q\varepsilon)\Big|^2\Big)\le C\varepsilon\|u\|^2_{S_\varepsilon}
$$

and thus dividing by r^2 and applying the Q_1 interpolation properties in subsection 5.1 we prove (36). □

10.2 | Asymptotic behavior of the reference frame

We will only consider direction $\mathbf{e}_1$, since the second follows a symmetric argumentation.

First, for every $z_2\in(q\varepsilon-\kappa\varepsilon,q\varepsilon+\kappa\varepsilon)$, one has $q=\Big[\dfrac{z_2}{\varepsilon}+\dfrac{1}{2}\Big]=2\Big[\dfrac{z_2}{2\varepsilon}\Big]+b$ where $b=0$ if q is even and $b=1$ if q is odd.

Then, by definition of $\Pi_\varepsilon^{(1,b)}$ (see (2)) we have $\Pi_\varepsilon^{(1,b)}\big((-1)^q\big) = (-1)^b$.
We are now ready to unfold the mobile reference frame. The unfolding of the oscillating function $(-1)^{(q+1)}\Phi_\varepsilon$ is

$$\frac{1}{\varepsilon}\Pi^{(1,b)}\big((-1)^{(q+1)}\Phi_\varepsilon\big) = (-1)^{1+b}\Phi \qquad \text{a.e. in } \Omega\times Cyl^{(1)}$$

where Φ is given in (1).
Moreover, we have $\boldsymbol{\gamma} = \sqrt{1+\big(\partial_{X_1}\Phi\big)^2}$ a.e. in $\Omega\times Cyl^{(1)}$ and

$$\begin{aligned}
&\Pi_\varepsilon^{(1,b)}(\mathbf{t}_\varepsilon^{(1,q)}) = \mathbf{t}^{(1,b)} = \frac{1}{\boldsymbol{\gamma}}\big(\mathbf{e}_1 + \partial_{X_1}\Phi^{(1,b)}\mathbf{e}_3\big),\ \Pi_\varepsilon^{(1,b)}(\mathbf{n}_\varepsilon^{(1,q)}) = \mathbf{n}^{(1,b)} = \frac{1}{\boldsymbol{\gamma}}\big(-\partial_{X_1}\Phi^{(1,b)}\mathbf{e}_1 + \mathbf{e}_3\big),\\
&\varepsilon\Pi_\varepsilon^{(1,b)}(c_\varepsilon^{(1,q)}) = \boldsymbol{c}^{(1,b)} = \frac{\partial^2_{X_1}\Phi^{(1,b)}}{\boldsymbol{\gamma}^3}, \quad \Pi_\varepsilon^{(1,b)}(\eta_\varepsilon^{(1,q)}) = \boldsymbol{\eta}^{(1,b)} = \boldsymbol{\gamma}\,\big(1 - X_3\boldsymbol{c}^{(1,b)}\big),\\
&\Pi_\varepsilon^{(1,b)}(\nabla\psi_\varepsilon^{(1,q)}) = \big(\boldsymbol{\eta}^{(1,b)}\mathbf{t}^{(1,b)}\ \big|\ \mathbf{e}_2\ \big|\ \mathbf{n}^{(1,b)}\big),
\end{aligned}$$

Lemma 23. One has the following values for the integrals:

$$\begin{aligned}
&\int_{Cyl^{(1)}} \boldsymbol{\eta}^{(1,b)}dX = 4\kappa^2\int_0^2 \boldsymbol{\gamma}\,dX_1,\\
&\int_{Cyl^{(1)}} \big((-1)^{b+1}\Phi^{(1,b)}\mathbf{e}_3 + X_2\mathbf{e}_2 + X_3\mathbf{n}^{(1,b)}\big)\boldsymbol{\eta}^{(1,b)}dX = 4\kappa^2\Big(\int_0^2 \boldsymbol{\gamma}\,\Phi\,dX_1\Big)\mathbf{e}_3.
\end{aligned} \tag{117}$$

Proof. First, due to the definition of $\boldsymbol{\eta}$ and the symmetries of the cross-sections with respect to the lines $X_2 = 0$ and $X_3 = 0$, we immediately get $(117)_1$.
Regarding $(117)_2$, again by symmetry (the symmetries of the cross-sections with respect to the lines $X_2 = 0$ and $X_3 = 0$), we first get that

$$\begin{aligned}
&\int_{Cyl^{(1)}} \big((-1)^{b+1}\Phi^{(1,b)}\mathbf{e}_3 + X_2\mathbf{e}_2 + X_3\mathbf{n}^{(1,b)}\big)\boldsymbol{\eta}^{(1,b)}dX\\
&= 4\kappa^2\Big(\int_0^2 (-1)^{b+1}\Phi^{(1,b)}\boldsymbol{\gamma}dX_1\Big)\mathbf{e}_3 + \frac{4\kappa^4}{3}\Big(\int_0^2 \partial_{X_1}\Phi^{(1,b)}\boldsymbol{c}^{(1,b)}dX_1\Big)\mathbf{e}_1 - \frac{4\kappa^4}{3}\Big(\int_0^2 \boldsymbol{c}^{(1,b)}dX_1\Big)\mathbf{e}_3
\end{aligned}$$

By the fact that $\Phi^{(1,b)} = (-1)^{1+b}\Phi$, the first parenthesis gives the RHS of $(117)_2$, while the second and third parenthesis can be integrated and are equal to zero since Φ is 2-periodic and satisfies $\partial_{X_1}\Phi(0) = \partial_{X_1}\Phi(1) = \partial_{X_1}\Phi(2) = 0$. $\square$

References

1. G. Barbagallo, P. Boisse, M. D'Agostino, A. Madeo: Continuum and discrete models for unbalanced woven fabrics. International Journal of Solids and Structures 94-95 (2015).

2. P. Boisse, F. Dumont, N. Hamila, E. Vidal-Sallé: Simulation of wrinkling during textile composite reinforcement forming. Influence of tensile, in plane shear and bending stiffnesses. Composites Science and Technology 71 (5), Elsevier, (2011) pp. 683.

3. P. G. Ciarlet: *Mathematical Elasticity: Theory of Plates. Studies in Mathematics and its Applications.* North-Holland(1997).

4. D. Cioranescu, A. Damlamian, P. Donato, G. Griso, R. Zaki: The periodic unfolding method in domains with holes. SIAM J. Math. Anal. 44 (2012) pp. 718-760.

5. D. Cioranescu, A. Damlamian and G. Griso: *The Periodic Unfolding Method. Theory and Applications to Partial Differential Problems.* Springer, Singapore (2018). 10.1007/978-981-13-3032-2.

6. D. Cioranescu, A. Damlamian, G. Griso: The periodic unfolding method in homogenization. SIAM J. Math. Anal. 40 (4), 2008, 1585-1620.

7. D. Cioranescu, A. Damlamian, J. Orlik: Two-scale analysis for homogenization of multi-scale contact problems in elasticity. Asymptotic Analysis 82 (2013).

8. R. Falconi, G. Griso, J. Orlik: Periodic unfolding for anisotropically bounded sequences. Preprint at HAL: https://hal.archives-ouvertes.fr/hal-03532607

9. G. Griso: Asymptotic behavior of curved rods by the unfolding method. Mathematical Methods in the Applied Sciences 27 (2004) pp. 2081-2110.

10. G. Griso: Asymptotic behavior of structures made of curved rods. Analysis and Applications. 6 (2008) pp. 11-22.

11. G. Griso: Decomposition of displacements of thin structures. J. Math. Pures Appl. 89 (2008) pp. 199-233.

12. G. Griso, M. Hauck, J. Orlik: Asymptotic analysis for periodic perforated shells. ESAIM: M2AN. Vol. 55, Issue 1, 1-36 (2021), https://doi.org/10.1051/m2an/2020067

13. G. Griso, L. Khilkova, J. Orlik , O. Sivak: Asymptotic behavior of stable structures made of beams, J. Elast. 143 (2), 239-299 (2021).

14. G. Griso, J. Orlik, S. Wackerle: Asymptotic Behavior for Textiles, SIAM J. Math. Anal., (2020).

15. G. Griso, J. Orlik, S. Wackerle: Asymptotic Behavior for Textiles in von-Kármán regime, J.M.P.A. 144, 164-193 (2020).

16. D. Kinderlehrer, G. Stampacchia: *An Introduction to Variational Inequalities and Their Applications. Society for Industrial and Applied Mathematics.* Society for Industrial and Applied Mathematics (2000).

17. J. Leray, J.-L. Lions: Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder. Bulletin de la Société Mathématique de France, Volume 93 (1965) pp. 97-107.

18. O. A. Oleinik, A.S. Shamaev, G. A. Yosifian: Mathematical Problems in Elasticity and Homogenization. Studies in Mathematics and its Applications 26 (1992).

19. J. Orlik, G. Panasenko, V. Shiryaev: Optimization of textile-like materials via homogenization and Beam approximations. SIAM Journal on Multiscale Modeling and Simulation, 14 (2016) pp. 637-667.

20. J. Orlik, V. Shiryaev: A one-dimensional computational model for hyperelastic string structures with Coulomb friction. Mathematical Methods in the Applied Sciences (2016).

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