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Asymptotic analysis of spectral problems in thick junctions with the branched fractal structure

Taras A. Mel'nyk*

[1]Department of Mathematical Physics, Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, Volodymyrska st., 64/13, Kyiv 01601, Ukraine

Correspondence
*Taras A. Mel'nyk, Corresponding address. Email: melnyk@imath.kiev.ua

Present Address
Taras Shevchenko National University of Kyiv, Volodymyrska st., 64/13, Kyiv 01601, Ukraine

Abstract

A spectral problem is considered in a domain Ω_ε that is the union of a domain Ω_0 and a lot of thin trees situated ε-periodically along some manifold on the boundary of Ω_0. The trees have finite number of branching levels. The perturbed Robin boundary condition $\partial_\nu u^\varepsilon + \varepsilon^{\alpha_i} k_{i,m} u^\varepsilon = 0$ is given on the ith branching layer; $\{\alpha_i\}$ are real parameters. The asymptotic analysis of this problem is made as $\varepsilon \to 0$, i.e., when the number of the thin trees infinitely increases and their thickness vanishes. In particular, the Hausdorff convergence of the spectrum to the spectrum of the corresponding nonstandard homogenized spectral problem is proved, the leading terms of asymptotics are constructed, and the corresponding asymptotic estimates are justified for the eigenvalues and eigenfunctions.

KEYWORDS:
spectral problem; asymptotic approximation; homogenization; fractal structure

1 | INTRODUCTION

It is an interesting problem to study the behaviour of eigenvalues and eigenfunctions of spectral problems in perturbed domains, since eigenvalues are among the most successful tools of applied mathematics, e.g, in quantum mechanics, vibration analysis, magnetohydrodynamics, control theory and many other fields (see, e.g.,[1,2,3]). In addition, different unexpected phenomena appear in the asymptotic behaviour of the spectrum.

In this article, we begin to study spectral problems in thick junctions of a new type, namely *thick junctions with the branched structure or thick fractal junctions*. Such a thick junction is a union of some domain, which is called the junction's body, and a large number of thin trees situated ε-periodically along some manifold on the boundary of the junction's body. The trees has finite number of branching levels. The small parameter ε characterizes the distance between neighboring thin branches and also their thickness. To simplify calculations, here we consider the case of a 2-D thick fractal junction (see Fig 1). For the first time such domains were considered in the author's paper[4], where the asymptotic behavior of the solution to a semi-linear parabolic problem was studied.

Various constructions of thick junction type are successfully used in nanotechnologies, microtechnique, modern engineering constructions (microstrip radiator, efficient sensors (inertial, biological, chemical), micro-fractal constructions: fractal antennas, fractal transistors, fractal heat radiators and so on). Many biological systems have such structures, e.g., root systems, nervous systems, intestine linings. A fairly complete review on this topic is presented in the monograph[5], where different asymptotic methods and approaches developed intensively during the last two decades are discussed.

In the recent article[6], different applications of tree structures were shown and what mathematical issues related to trees are of interest in different fields. In the present paper, the influence of a thick tree structure on the asymptotic behaviour of the spectrum will be studied as the number of attached thin trees increases infinitely and their thickness vanishes.

The paper is organized as follows. After the statement of the problem in Sec. 2, the leading terms of asymptotic expansions for the eigenfunctions are formally constructed in Sec. 3. The asymptotics consists of the outer expansions both in the junction's body and in each thin branches as well as the inner expansions in a neighborhood both of the joint zone and each branching levels. Then, using the method of matched asymptotic expansions[7], the corresponding non-standard homogenized spectral problem is derived in subsection 3.4 and its spectrum is determined in Sec. 4 (the most remarkable thing in the spectrum structure is the presence of *gaps*). Asymptotic approximations for the eigenfunctions are constructed in Sec. 5. Section 6 deals with the justification of the asymptotics. For this we use a special approach and prove the Hausdorff convergence of the spectrum and asymptotic estimates both for the eigenvalues and eigenfunctions. The central place in this approach is the construction of a special multi-sheeted extension operator for the eigenfunctions (see Th.2). The main results are formulated in § 6.1. In Conclusion 7 the obtained results are analyzed and research perspectives are considered. The results were reported at the conference[8].

2 | STATEMENT OF THE PROBLEM

Let Ω_0 be a bounded domain in $\mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega_0$ and $\Omega_0 \subset \{x := (x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0\}$. Let $\partial\Omega_0$ contain the segment $I_0 = \{x : \ x_1 \in [0, a], \ x_2 = 0\}$. We assume that there exists a positive number δ_0 such that

$$\Omega_0 \cap \{x : \ 0 < x_2 < \delta_0\} = \{x : \ x_1 \in (0, a), \ \ x_2 \in (0, \delta_0)\}.$$

Let a, l_1, l_2, l_3 be positive numbers, $h_0, h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2}, h_{2,3}, h_{2,4}$ be fixed numbers from the interval $(0, 1)$ and

$$h_{1,1} + h_{1,2} < h_0, \quad h_{2,1} + h_{2,2} < h_{1,1}, \quad h_{2,3} + h_{2,4} < h_{1,2}.$$

Let us also introduce a small parameter $\varepsilon = \frac{a}{N}$, where N is a large positive integer.

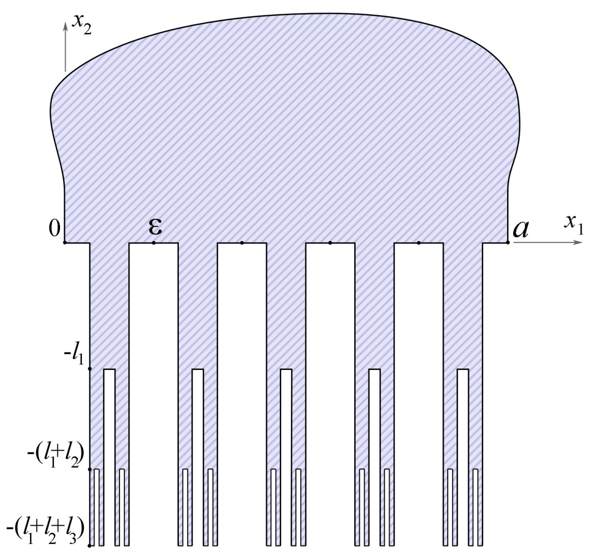

FIGURE 1 Thick junction Ω_ε with the branched structure

A model thick fractal junction Ω_ε (Fig. 1) consists of the junction's body Ω_0,

- a large number of the thin rods

$$G_j^{(0)}(\varepsilon) = \left\{ x : \ \left| x_1 - \varepsilon(j + \frac{1}{2}) \right| < \frac{\varepsilon h_0}{2}, \quad x_2 \in (-l_1, 0] \right\}, \quad j \in \{0, 1, \dots, N-1\},$$

from the zero layer $G^{(0)}_\varepsilon := \bigcup_{j=0}^{N-1} G^{(0)}_j(\varepsilon)$,

- a large number of the thin rods

$$G_j^{(1,m)}(\varepsilon) = \left\{ x: \ \big|x_1 - \varepsilon(j + b_{1,m})\big| < \frac{\varepsilon h_{1,m}}{2}, \quad x_2 \in \big(-(l_2 + l_1), -l_1\big] \right\}, \quad j \in \{0, 1, \dots, N-1\},$$

from the first branching layer consisting of two classes $G^{(1,m)}_\varepsilon := \bigcup_{j=0}^{N-1} G_j^{(1,m)}(\varepsilon)$, $m \in \{1, 2\}$, where

$$b_{1,1} = \frac{1 - h_0 + h_{1,1}}{2}, \quad b_{1,2} = \frac{1 + h_0 - h_{1,2}}{2}, \tag{1}$$

- and a large number of the thin rods

$$G_j^{(2,m)}(\varepsilon) = \left\{ x: \ \big|x_1 - \varepsilon(j + b_{2,m})\big| < \frac{\varepsilon h_{2,m}}{2}, \ x_2 \in \big(-(l_3 + l_2 + l_1), -(l_2 + l_1)\big] \right\}, \quad j \in \{0, 1, \dots, N-1\},$$

from the second branching layer consisting of four classes

$$G^{(2,m)}_\varepsilon := \bigcup_{j=0}^{N-1} G_j^{(2,m)}(\varepsilon), \quad m \in \{1, 2, 3, 4\},$$

where

$$b_{2,1} = \frac{1 - h_0 + h_{2,1}}{2}, \quad b_{2,2} = \frac{1 - h_0 + 2h_{1,1} - h_{2,2}}{2}, \quad b_{2,3} = \frac{1 + h_0 - 2h_{1,2} + h_{2,3}}{2}, \quad b_{2,4} = \frac{1 + h_0 - h_{2,4}}{2}. \tag{2}$$

Thus, $\Omega_\varepsilon = \Omega_0 \bigcup G^{(0)}_\varepsilon \bigcup G^{(1)}_\varepsilon \bigcup G^{(2)}_\varepsilon$, where

$$G^{(1)}_\varepsilon = \bigcup_{m=1}^{2} G^{(1,m)}_\varepsilon, \quad G^{(2)}_\varepsilon = \bigcup_{m=1}^{4} G^{(2,m)}_\varepsilon.$$

The small parameter ε characterizes the distance between neighboring thin branches and also their thickness. Precisely, each branch (rod) $G_j^{(i,m)}(\varepsilon)$ has small cross-section of size $\mathcal{O}(\varepsilon)$ and constant height. In addition, at fixed $j \in \{0, 1, \dots, N-1\}$ the branches $G_j^{(0)}(\varepsilon)$, $\{G_j^{(1,m)}(\varepsilon)\}_{m=1}^2$, $\{G_j^{(2,m)}(\varepsilon)\}_{m=1}^4$ form the tree with two branching levels at $x_2 = -l_1$ and $x_2 = -l_1 - l_2$. These trees are ε-periodically distributed along the segment I_0.

In Ω_ε we consider the following spectral problem:

$$\begin{cases} -\Delta_x u^\varepsilon \ = \ \lambda^\varepsilon \, u^\varepsilon & \text{in } \Omega_\varepsilon, \\ -\partial_\nu u^\varepsilon \ = \ \varepsilon^{\alpha_i} \, k_{i,m} \, u^\varepsilon & \text{on } \Upsilon^{(i,m)}_\varepsilon, \ \ i = 0, 1, 2, \\ \partial^p_{x_1} u^\varepsilon|_{x_1 = 0} \ = \ \partial^p_{x_1} u^\varepsilon|_{x_1 = a}, & x_2 \in [0, \delta_0], \ p = 0, 1, \\ u_\varepsilon \ = \ 0 & \text{on } \ \Gamma_1, \\ \partial_\nu u_\varepsilon \ = \ 0 & \text{on } \ \partial\Omega_\varepsilon \setminus (\Upsilon_\varepsilon \cup \Gamma_0 \cup \Gamma_1) \\ [u^\varepsilon]\Big|_{x_2 = -\sum_{n=0}^i l_n} \ = \ \big[\partial_{x_2} u^\varepsilon\big]\Big|_{x_2 = -\sum_{n=0}^i l_n} = 0 & \text{on } Q^{(i)}_\varepsilon, \ \ i = 0, 1, 2, \end{cases} \tag{3}$$

where ∂_ν is the outward normal derivative; the brackets in the last line denote the jump of the enclosed quantities; $\Upsilon^{(i,m)}_\varepsilon$ is the union of vertical boundaries of the thin rods $G^{(i,m)}_\varepsilon$,

$$Q^{(i)}_\varepsilon = \overline{G^{(i)}_\varepsilon} \cap \Big\{ x \in \mathbb{R}^2 : \ x_2 = -\sum_{n=0}^{i} l_n \Big\}, \quad l_0 = 0, \quad i \in \{0, 1, 2\};$$

parameters $\{\alpha_i\}_{i=0}^2$ are greater or equal to 1; the constants $\{k_{i,m}\}$ are positive ($k_{0,0} =: k_0$);

$$\Gamma_0 := \big\{ x : \ x_1 = 0, \ \ x_2 \in [0, \delta_0] \big\} \bigcup \big\{ x : \ x_1 = a, \ \ x_2 \in [0, \delta_0] \big\},$$

Γ_1 is a curve on $\partial\Omega_0 \setminus \Gamma_0$ and its length $\ell_{\Gamma_1} > 0$,

$$\Upsilon_\varepsilon = \Upsilon^{(0)}_\varepsilon \bigcup \big(\Upsilon^{(1,1)}_\varepsilon \cup \Upsilon^{(1,2)}_\varepsilon \big) \bigcup \big(\Upsilon^{(2,1)}_\varepsilon \cup \ldots \cup \Upsilon^{(2,4)}_\varepsilon \big).$$

Remark 1. Hereafter we use the following shortening:

$$\{x_2 = -\sum\nolimits_{n=0}^{i} l_n\} := \{x \in \mathbb{R}^2 : \ x_2 = -\sum\nolimits_{n=0}^{i} l_n\};$$

also if the index $i = 0$, then the index m is absent and notation $G_\varepsilon^{(0,m)}$ means $G_\varepsilon^{(0)}$ and $\Upsilon_\varepsilon^{(0,m)} =: \Upsilon_\varepsilon^{(0)}$.

In the space

$$\mathcal{H}_\varepsilon := \big\{ u \in H^1(\Omega_\varepsilon) : \ u(0,x_2) = u(a,x_2), \ x_2 \in [0,\delta_0]; \ \ u|_{\Gamma_1} = 0 \big\},$$

where the restriction of a function to a part of the boundary is understood in the sense of the trace in the Sobolev space, we introduce a new norm $\|\cdot\|_\varepsilon$ generated by the scalar product

$$\langle u, v \rangle_\varepsilon = \int\limits_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx + \varepsilon^{\alpha_0} k_0 \int\limits_{\Upsilon_\varepsilon^{(0)}} u \, v \, dx_2 + \sum_{i=1}^{2} \varepsilon^{\alpha_i} \sum_{m=1}^{i+\lfloor \frac{i+2}{2} \rfloor} k_{i,m} \int\limits_{\Upsilon_\varepsilon^{(i,m)}} u \, v \, dx_2,$$

(here $\lfloor t \rfloor$ is the integer part of a real number t), which is uniformly equivalent to the standard norm, i.e., there exist constants $C_1 > 0, \ C_2 > 0$ and ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$ and $v \in \mathcal{H}_\varepsilon$:

$$C_1 \|v\|_{H^1(\Omega_\varepsilon)} \le \|v\|_\varepsilon \le C_2 \|v\|_{H^1(\Omega_\varepsilon)}. \tag{4}$$

The proof is similar to the proof of[9, Lemma 1].

Definition 1. A number λ^ε is called an eigenvalue of the problem (3) if there exists a function $u^\varepsilon \in \mathcal{H}_\varepsilon \setminus \{0\}$ such that

$$\langle u^\varepsilon, \varphi \rangle_\varepsilon = \lambda^\varepsilon \left(u^\varepsilon, \varphi \right)_{L^2(\Omega_\varepsilon)} \quad \forall \varphi \in \mathcal{H}_\varepsilon. \tag{5}$$

The function u^ε is called an eigenfunction that corresponds to λ^ε.

The spectral problem (3) is equivalent to the spectral problem

$$A_\varepsilon u^\varepsilon = \tfrac{1}{\lambda^\varepsilon} u^\varepsilon \quad \text{in } \ \mathcal{H}_\varepsilon,$$

where the operator $A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$ is defined by the equality

$$\langle A_\varepsilon u, v \rangle_\varepsilon = \big(u, v \big)_{L^2(\Omega_\varepsilon)} \quad \forall \, u, v \in \mathcal{H}_\varepsilon. \tag{6}$$

It is easy to verify that the operator A_ε is self-adjoint, positive, and compact.

Thus, for each fixed value of ε all eigenvalues of problem (3) can be ordered as follows

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \le \ldots \le \lambda_n^\varepsilon \le \cdots \to +\infty \quad \text{as} \quad n \to +\infty, \tag{7}$$

where each eigenvalue is counted as many times as its multiplicity. The corresponding eigenfunctions $\{u_n^\varepsilon\}_{n\in\mathbb{N}}$, which belong to $\mathcal{H}_\varepsilon$, are orthonormalized as follows

$$(u_n^\varepsilon, u_m^\varepsilon)_{L^2(\Omega_\varepsilon)} = \delta_{n,m}, \quad \{n, \, m\} \in \mathbb{N}. \tag{8}$$

The aim is to study the asymptotic behaviour of $\{\lambda_n^\varepsilon\}_{n\in\mathbb{N}}$ and $\{u_n^\varepsilon\}_{n\in\mathbb{N}}$ as $\varepsilon \to 0$, i.e., when the number of attached thin trees increases infinitely and their thickness vanishes.

Remark 2. It is clear that the thin rods from the zero layer $G_\varepsilon^{(0)}$ fill out the rectangle $D_0 = (0,a) \times (-l_1, 0)$ in the limit as ε tends to zero; thin rods from each of two classes $G_\varepsilon^{(1,1)}$ and $G_\varepsilon^{(1,2)}$ of the first branching layer fill out the rectangle $D_1 = (0,a) \times \big(-l_2 - l_1, -l_1 \big)$; and thin rods from each of four classes $G_\varepsilon^{(2,m)}$, $m \in \{1,2,3,4\}$, of the second branching layer fill out the rectangle $D_2 = (0,a) \times \big(-l_3 - l_2 - l_1, -l_2 - l_1 \big)$ in the limit.

Of course, one can consider thick fractal junctions, in which thin rods from different classes have variable thickness and different lengths, i.e., the length of the rods can differ both depending on the class from the same branching layer and in the class itself (see, e.g.,[5], where boundary-values problems were studied in thick multi-level junctions with different structures).

3 | FORMAL ASYMPTOTIC EXPANSIONS

3.1 | Estimates for eigenvalues

By virtue of the minimax principle for eigenvalues and the right inequality in (4), we deduce

$$\lambda_n^\varepsilon = \min_{E\in\mathbf{E}_n}\ \max_{v\in E,\, v\neq 0} \frac{\|v\|_\varepsilon^2}{\|v\|^2_{L^2(\Omega_\varepsilon)}} \le C_2^2 \min_{E\in\mathbf{E}_n}\ \max_{v\in E,\, v\neq 0}\left(\frac{\int_{\Omega_\varepsilon}|\nabla v|^2\,dx}{\int_{\Omega_\varepsilon} v^2\,dx} + 1\right)$$
$$\le C_2^2 \max_{0\neq v\in\mathcal{L}_n} \frac{\int_{\Omega_\varepsilon}|\nabla v|^2\,dx}{\int_{\Omega_\varepsilon} v^2\,dx} + C_2^2 \le C_2^2\tau_n \max_{0\neq v\in\mathcal{L}_n}\frac{\int_{D_0} v^2\,dx}{\int_{G_\varepsilon^{(0)}} v^2\,dx} + C_2^2 \le C_1(n). \tag{9}$$

Here $\mathcal{L}_n$ is the n-dimensional subspace of $\mathcal{H}_\varepsilon$, which is spanned on functions $\{\widehat{\phi}_k\}_{k=1}^n$ such that $\widehat{\phi}_k = 0$ in $\Omega_\varepsilon \setminus G_\varepsilon^{(0)}$ and $\widehat{\phi}_k = \phi_k$ in $G_\varepsilon^{(0)}$, where $\{\phi_k\}_{k=1}^n$ are eigenfunctions of the Laplace operator in D_0 with the Neumann conditions on the vertical sides of D_0 and the Dirichlet ones on the other parts.

Taking into account (8) and the left inequality in (4), we obtain from the integral identity (5) the lower estimates for the eigenvalues:

$$\lambda_n^\varepsilon = \lambda_n^\varepsilon\|u_n^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = \|u_n^\varepsilon\|_\varepsilon^2 \ge C_1^2\|u_n^\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \ge C_1^2\|u_n^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = C_1^2. \tag{10}$$

Based on estimates (9) and (10), we seek the leading terms of the asymptotics for λ_n^ε in the form (hereafter the index n is omitted)

$$\lambda^\varepsilon \approx \mu_0 + \varepsilon\mu_1 + \ldots \tag{11}$$

3.2 | Outer expansions

Asymptotic ansatzes for u_n^ε (hereafter the index n is omitted) are as follows:

$$u^\varepsilon(x) \approx v_0^+(x) + \varepsilon v_1^+(x) + \ldots \quad \text{in the junction's body } \Omega_0, \tag{12}$$

and in each thin rod $G_j^{(i,m)}(\varepsilon)$

$$u^\varepsilon(x) \approx v_0^{(i,m)}(x) + \varepsilon v_1^{(i,m)}(x, \tfrac{x_1}{\varepsilon} - j) + \varepsilon^2 v_2^{(i,m)}(x, \tfrac{x_1}{\varepsilon} - j) + \ldots \tag{13}$$

Here $i \in \{0,1,2\}$; if $i=0$, then $m=0$; and $m \in \{1,\ldots,i+\lfloor\frac{i+2}{2}\rfloor\}$ in the other cases; $j\in\{0,\ldots,N-1\}$. The asymptotic series (12) and (13) are usually called *outer expansions*.

Substituting (12) and (11) in the equation of the problem (3), in the boundary conditions on $\partial\Omega_0$ and collecting coefficients at ε^0, we get

$$\begin{cases} -\Delta_x\, v_0^+(x) = \mu_0\, v_0^+(x), & x\in\Omega_0 \\ \partial_{x_1}^p v_0^+|_{x_1=0} = \partial_{x_1}^p v_0^+|_{x_1=a}, & x_2\in[0,\delta_0],\quad p=0,1 \\ v_0^+ = 0, & x\in\Gamma_1 \\ \partial_\nu v_0^+ = 0, & x\in\partial\Omega_0\setminus(\Gamma_0\cup\Gamma_1\cup I_0). \end{cases} \tag{14}$$

Now let us find limit relations in each rectangle D_i, $i\in\{0,1,2\}$ (see Remark 2). For this we fix indexes i, m and j. Using Taylor series with respect to the variable x_1 at the point $x_1 = \varepsilon(j+b_{i,m})$ (points $\{b_{i,m}\}$ are defined in (1) and (2), $b_{0,m} = b_0 = \frac{1}{2}$) and passing to the "fast" variable $\xi_1 = \varepsilon^{-1}x_1$, we rewrite (13) in the form

$$u^\varepsilon \approx v_0^{(i,m)}\big(\varepsilon(j+b_{i,m}), x_2\big) + \varepsilon V_1^{(i,m,j)}(\xi_1,x_2) + \varepsilon V_2^{(i,m,j)}(\xi_1,x_2) + \ldots, \tag{15}$$

where

$$V_1^{(i,m,j)} = v_1^{(i,m)}\big(\varepsilon(j+b_{i,m}), x_2, \xi_1 - j\big) + (\xi_1 - j - b_{i,m})\frac{\partial v_0^{(i,m)}}{\partial x_1}\big(\varepsilon(j+b_{i,m}), x_2\big),$$

$$V_2^{(i,m,j)} = v_2^{(i,m)}\big(\varepsilon(j+b_{i,m}), x_2, \xi_1 - j\big) + (\xi_1 - j - b_{i,m})\frac{\partial v_1^{(i,m)}}{\partial x_1}\big(\varepsilon(j+b_{i,m}), x_2, \xi_1 - j\big)$$
$$+ \frac{(\xi_1 - j - b_{i,m})^2}{2}\,\frac{\partial^2 v_0^{(i,m)}}{\partial x_1^2}\big(\varepsilon(j+b_{i,m}), x_2\big).$$

Substituting (15) into (3) instead of u^ε and taking into account the view of the Laplace operator in the variables ξ_1 and x_2 ($\Delta_{\xi_1,x_2} = \varepsilon^{-2}\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial x_2^2}$), the collection of coefficients of the same power of ε gives one dimensional boundary value problems with respect to ξ_1 :

$$\partial^2_{\xi_1\xi_1} V_1^{(i,m,j)}(\xi_1, x_2) = 0, \quad \xi_1 \in I_{h_{i,m}}(b_{i,m}); \qquad \partial_{\xi_1} V_1^{(i,m,j)}(b_{i,m} \pm \tfrac{h_{i,m}}{2}, x_2) = 0, \tag{16}$$

where $\partial_{\xi_1} = \frac{\partial}{\partial \xi_1}$, $\partial^2_{\xi_1\xi_1} = \frac{\partial^2}{\partial \xi_1^2}$ and $I_{h_{i,m}}(b_{i,m}) = \big(b_{i,m} - \frac{h_{i,m}}{2}, b_{i,m} + \frac{h_{i,m}}{2}\big)$.

From (16) it follows that $V_1^{(i,m,j)}$ is independent of ξ_1. Since we look only for the first terms of the asymptotics, we can regard that it is zero. Thus,

$$v_1^{(i,m)}\big(\varepsilon(j + b_{i,m}), x_2, \xi_1 - j\big) = -\big(\xi_1 - j - b_{i,m}\big)\, \partial_{x_1} v_0^{(i,m)}\big(\varepsilon(j + b_{i,m}), x_2\big). \tag{17}$$

The problem for the function $V_2^{(i,m,j)}$ is as follows:

$$\begin{cases} -\partial^2_{\xi_1\xi_1} V_2^{(i,m,j)} = \big(\partial^2_{x_2x_2} v_0^{(i,m)}(x) + \mu_0 v_0^{(i,m)}(x)\big)\Big|_{x_1=\varepsilon(j+b_{i,m})}, & \xi_1 \in I_{h_{i,m}}(b_{i,m}), \\ \partial_{\xi_1} V_2^{(i,m,j)}\big(\xi_1, x_2\big)\Big|_{\xi_1 = b_{i,m} \pm \frac{h_{i,m}}{2}} = \mp \delta_{\alpha_i,1} k_{i,m} v_0^{(i,m)}(\varepsilon(j + b_{i,m}), x_2). \end{cases} \tag{18}$$

The solvability condition for (18) is given by the differential equation

$$-h_{i,m}\, \partial^2_{x_2x_2} v_0^{(i,m)}(x) + 2\delta_{\alpha_i,1} k_{i,m} v_0^{(i,m)}(x) = \mu_0 h_{i,m}\, v_0^{(i,m)}(x) \tag{19}$$

with respect to variables x_2 and at the fixed value of $x_1 = \varepsilon(j + b_{i,m})$. Here $\delta_{\alpha_i,1}$ is Kronecker's symbols (recall that $\alpha_i \geq 1$).

Since the points $\{x_1 = \varepsilon(j+b_{i,m}) : \ j = 0, \dots, N-1\}$ form the ε-net in the interval $(0, a)$, we can extend all equations obtained above on N segments to the rectangle D_i as $\varepsilon \to 0$. Thus, we get one ordinary differential equation

$$-h_0\, \partial^2_{x_2x_2} v_0^{(0)}(x) + 2\delta_{\alpha_0,1} k_0 v_0^{(0)}(x) = \mu_0 h_0 v_0^{(0)}(x) \ \text{ in } D_0; \tag{20}$$

two differential equations ($m \in \{1, 2\}$)

$$-h_{1,m}\, \partial^2_{x_2x_2} v_0^{(1,m)}(x) + 2\delta_{\alpha_1,1} k_{1,m} v_0^{(1,m)}(x) = \mu_0 h_{1,m} v_0^{(1,m)}(x) \ \text{ in } D_1; \tag{21}$$

and four differential equations ($m \in \{1, 2, 3, 4\}$)

$$-h_{2,m}\, \partial^2_{x_2x_2} v_0^{(2,m)}(x) + 2\delta_{\alpha_2,1} k_{2,m} v_0^{(2,m)}(x) = \mu_0 h_{2,m} v_0^{(2,m)}(x) \ \text{ in } D_2. \tag{22}$$

Due to the Neumann condition on the bases

$$Q_\varepsilon^{(3)} = \overline{\Omega}_\varepsilon \cap \{x : \ x_2 = -(l_1 + l_2 + l_3)\},$$

we obtain the following boundary conditions for functions $\{v_0^{(2,m)}\}$:

$$\partial_{x_2} v_0^{(2,m)}\big(x_1, -(l_1 + l_2 + l_3)\big) = 0, \quad m = 1, 2, 3, 4. \tag{23}$$

3.3 | Construction of inner expansions

To find transmission conditions in the joint zone I_0 and in each branching zones $I_1 = \{x : \ x_1 \in (0, a), \ x_2 = -l_1\}$ and $I_2 = \{x : \ x_1 \in (0, a), \ x_2 = -(l_1 + l_2)\}$, we use the method of matched asymptotic expansions for the outer expansions (12), (13) and inner ones in neighborhoods of I_0, I_1 and I_2.

3.3.1 | Inner expansion in a neighborhood of I_0

In a neighborhood of the joint zone I_0 we introduce the "rapid" coordinates $\xi = (\xi_1, \xi_2)$, where $\xi_1 = \varepsilon^{-1}x_1$ and $\xi_2 = \varepsilon^{-1}x_2$. Passing to $\varepsilon = 0$, we see that the rod $G_0^{(0)}(\varepsilon)$ transforms into the semi-infinite strip

$$\Pi^-_{h_0} = \Big(\frac{1}{2} - \frac{h_0}{2}, \frac{1}{2} + \frac{h_0}{2}\Big) \times (-\infty, 0];$$

the domain Ω_0 transforms into the first quadrant $\{\xi : \ \xi_1 > 0, \ \xi_2 > 0\}$. Taking into account the periodic structure of Ω_ε in a neighborhood of I_0, we take the following cell of periodicity $\Pi_0 = \Pi^-_{h_0} \cup \Pi^+$ (Fig. 2), where junction-layer problems will be considered; here $\Pi^+ = (0, 1) \times (0, +\infty)$. Obviously, their solutions must be 1-periodic in ξ_1, i.e.,

$$\partial^p_{\xi_1} Z \ _{|\xi_1=0} = \partial^p_{\xi_1} Z \ _{|\xi_1=1} \quad \text{on } \partial\Pi^+ \cap \{\xi : \ \xi_2 > 0\}, \ p = 0, 1.$$

We propose the following ansatz for the inner asymptotic expansion in a neighborhood of $I_0 \cap \Omega_\varepsilon$:

$$u^\varepsilon \approx v_0^+(x_1,0) + \varepsilon\Big(Z_1^{(0)}\big(\tfrac{x}{\varepsilon}\big)\, \partial_{x_1} v_0^+(x_1,0) + Z_2^{(0)}\big(\tfrac{x}{\varepsilon}\big)\, \partial_{x_2} v_0^+(x_1,0)\Big) + \ldots \qquad (24)$$

Substituting (24) in the differential equations of problem (3) and in the corresponding boundary conditions, taking into account that the Laplace operator takes the following form $\varepsilon^{-2}\Delta_\xi$ in the coordinates ξ and collecting the coefficients of the same power of ε, we arrive the following junction-layer problems for the coefficients $Z_1^{(0)}$ and $Z_2^{(0)}$:

$$\begin{aligned} -\Delta_\xi\, Z_p^{(0)}(\xi) &= 0, && \xi \in \Pi_0, \\ \partial_{\xi_2} Z_p^{(0)}(\xi_1,0) &= 0, && \xi_1 \in (0,1) \setminus \big(\tfrac{1}{2} - \tfrac{h_0}{2}, \tfrac{1}{2} - \tfrac{h_0}{2}\big), \\ \partial_{\xi_1} Z_p^{(0)}(\xi) &= -\delta_{p,1}, && \xi \in \partial\Pi_{h_0}^- \cap \{\xi :\ \xi_2 < 0\}, \qquad p = 1,2. \end{aligned} \qquad (25)$$

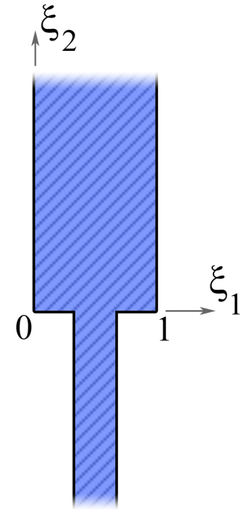

FIGURE 2 Domain Π_0

The existence and the main asymptotic relations for solutions to problems (25) can be obtained from general results about the asymptotic behavior of solutions to elliptic problems in domains with different exits to infinity[10,11]. However, in some cases one can define more exactly the asymptotic relations and detect other properties of solutions (see[9, Lemma 4.1 and Corollary 4.1], [12]). From those results it follows the proposition.

Proposition 1. There exist unique solutions $Z_1^{(0)}, Z_2^{(0)} \in H^1_{loc,\xi_2}(\Pi_0)$ to problems (25) respectively, which have the following differentiable asymptotics

$$Z_1^{(0)}(\xi) = \begin{cases} \mathcal{O}(\exp(-2\pi\xi_2)), & \xi_2 \to +\infty, \\ \Big(-\xi_1 + \dfrac{1}{2}\Big) + \mathcal{O}(\exp(\pi h_0^{-1}\xi_2)), & \xi_2 \to -\infty, \end{cases} \qquad (26)$$

$$Z_2^{(0)}(\xi) = \begin{cases} \xi_2 + \mathcal{O}(\exp(-2\pi\xi_2)), & \xi_2 \to +\infty, \\ \dfrac{\xi_2}{h_0} + C_2 + \mathcal{O}(\exp(\pi h_0^{-1}\xi_2)), & \xi_2 \to -\infty, \end{cases} \qquad (27)$$

Moreover, $Z_1^{(0)}$ is odd in ξ_1 and $Z_2^{(0)}$ is even in ξ_1 with respect to $\frac{1}{2}$.

Recall that a function Z belongs to the Sobolev space $H^1_{loc,\xi_2}(\Pi_0)$ if for every $R > 0$ the function $Z \in H^1(\Pi_0 \cap \{\xi :\ |\xi_2| < R\})$.

3.3.2 | Inner expansion in a neighborhood of the first branching zone I_1

In a neighborhood of I_1 we introduce the "rapid" coordinates $\xi = (\xi_1, \xi_2)$, where $\xi_1 = \varepsilon^{-1}x_1$ and $\xi_2 = \varepsilon^{-1}(x_2 + l_1)$. Passing to $\varepsilon = 0$, we see that the rod $G_0^{(0)}(\varepsilon)$ transforms into the semi-infinite strip

$$\Pi_{h_0}^+ = \Big(\frac{1}{2} - \frac{h_0}{2}, \frac{1}{2} + \frac{h_0}{2}\Big) \times (0, +\infty)$$

and rods $G_0^{(1,m)}(\varepsilon)$, $m \in \{1,2\}$, transform into the semi-infinite strips

$$\Pi_{1,m}^- = \Big(b_{1,m} - \frac{h_{1,m}}{2}, b_{1,m} + \frac{h_{1,m}}{2}\Big) \times (-\infty, 0], \qquad m \in \{1,2\},$$

respectively. Considering the periodic structure of Ω_ε in a neighborhood of I_1, we take the following cell of periodicity $\Pi_1 = \Pi_{h_0}^+ \cup \Pi_{1,1}^- \cup \Pi_{1,2}^-$ (Fig. 3), where branch-layer problems will be considered.

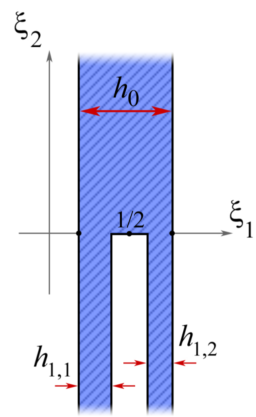

FIGURE 3 Domain Π_1

The ansatz for the inner asymptotic expansion in a neighborhood of $I_1 \cap \big(G_\varepsilon^{(0)} \cup G_\varepsilon^{(1)}\big)$ is as follows:

$$\begin{aligned} u^\varepsilon(x) \approx{}& v_0^{(0)}(x_1,-l_1) + \varepsilon\Big(Z_1^{(1)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1}{\varepsilon}\big)\partial_{x_1} v_0^{(0)}(x_1,-l_1) \\ &+ \Big\{\eta_1(x_1)\, \Xi_1^{(1)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1}{\varepsilon}\big) + (1-\eta_1(x_1))\, \Xi_2^{(1)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1}{\varepsilon}\big)\Big\}\partial_{x_2} v_0^{(0)}(x_1,-l_1)\Big) + \ldots \qquad (28) \end{aligned}$$

Substituting (28) in the corresponding differential equation of the problem (3) and boundary conditions, we arrive branch-layer problems for the functions $Z_1^{(1)}$, $\Xi_1^{(1)}$, $\Xi_2^{(1)}$. So, the functions $\Xi_1^{(1)}$ and $\Xi_2^{(1)}$ are solution to the following homogeneous problem

$$\begin{cases} -\Delta_\xi\, \Xi(\xi) = 0, & \xi \in \Pi_1, \\ \partial_\nu \Xi(\xi) = 0, & \xi \in \Pi_1. \end{cases} \tag{29}$$

Again using approach mentioned above (see also[13]), we conclude.

Proposition 2. There exist two solutions $\Xi_1, \Xi_2 \in H^1_{loc,\xi_2}(\Pi_1)$ of the problem (29), which have the following differentiable asymptotics:

$$\Xi_1(\xi) = \begin{cases} \xi_2 + \mathcal{O}\big(\exp(-\frac{\pi\xi_2}{h_0})\big), & \xi_2 \to +\infty,\ \xi \in \Pi^+_{h_0}, \\ \dfrac{h_0}{h_{1,1}}\,\xi_2 + C_1^{(1)} + \mathcal{O}\big(\exp(\frac{\pi\xi_2}{h_{1,1}})\big), & \xi_2 \to -\infty,\ \xi \in \Pi^-_{1,1}, \\ C_2^{(1)} + \mathcal{O}\big(\exp(\frac{\pi\xi_2}{h_{1,2}})\big), & \xi_2 \to -\infty,\ \xi \in \Pi^-_{1,2}, \end{cases} \tag{30}$$

$$\Xi_2(\xi) = \begin{cases} \xi_2 + \mathcal{O}\big(\exp(-\frac{\pi\xi_2}{h_0})\big), & \xi_2 \to +\infty,\ \xi \in \Pi^+_{h_0}, \\ C_1^{(2)} + \mathcal{O}\big(\exp(\frac{\pi\xi_2}{h_{1,1}})\big), & \xi_2 \to -\infty,\ \xi \in \Pi^-_{1,1}, \\ \dfrac{h_0}{h_{1,2}}\,\xi_2 + C_2^{(2)} + \mathcal{O}\big(\exp(\frac{\pi\xi_2}{h_{1,2}})\big), & \xi_2 \to -\infty,\ \xi \in \Pi^-_{1,2}, \end{cases} \tag{31}$$

where $C_1^{(1)}, C_2^{(1)}, C_1^{(2)}, C_2^{(2)}$ are some fixed constants.

Any another solution to the homogeneous problem (29), which has polynomial grow at infinity, can be presented as a linear combination $c_0 + c_1\Xi_1 + c_2\Xi_2$.

The function $Z_1^{(1)}$ is a solution to the following problem:

$$\begin{aligned} -\Delta_\xi\, Z(\xi) &= 0, && \xi \in \Pi_1, \\ \partial_{\xi_1} Z(\xi) &= -1, && \xi \in \partial_{\|}\Pi_1, \\ \partial_{\xi_2} Z(\xi_1, 0) &= 0, && (\xi_1, 0) \in \partial\Pi_1 \setminus \partial_{\|}\Pi_1, \end{aligned} \tag{32}$$

where $\partial_{\|}\Pi_1$ is the union of the vertical sides of $\partial\Pi_1$.

Proposition 3. There exists the unique solution $Z \in H^1_{loc,\xi_2}(\Pi_0)$ to problems (32), which has the following differentiable asymptotics:

$$Z(\xi) = \begin{cases} -\xi_1 + \frac{1}{2} + \mathcal{O}\big(\exp(-\frac{\pi\xi_2}{h_0})\big), & \xi_2 \to +\infty,\ \xi \in \Pi^+_{h_0}, \\ -\xi_1 + b_{1,1} + C_1 + \mathcal{O}\big(\exp(\frac{\pi\xi_2}{h_{1,1}})\big), & \xi_2 \to -\infty,\ \xi \in \Pi^-_{1,1}, \\ -\xi_1 + b_{1,2} + C_2 + \mathcal{O}\big(\exp(\frac{\pi\xi_2}{h_{1,2}})\big), & \xi_2 \to -\infty,\ \xi \in \Pi^-_{1,2}, \end{cases} \tag{33}$$

where C_1, C_2 are some fixed constants.

Thus, we set $\Xi_1^{(1)} = \Xi_1$, $\Xi_2^{(1)} = \Xi_2$ and $Z_1^{(1)} = Z$.

3.3.3 | Inner expansion in a neighborhood of the second branching zone I_2

In a neighborhood of I_2 we introduce the "rapid" coordinates $\xi = (\xi_1, \xi_2)$, where $\xi_1 = \varepsilon^{-1}x_1$ and $\xi_2 = \varepsilon^{-1}(x_2 + l_1 + l_2)$. Passing to $\varepsilon = 0$, we see that the rods $G_0^{(1,m)}(\varepsilon)$, $m \in \{1,2\}$, transform into the semi-infinite strips

$$\Pi^+_{1,m} = \Big(b_{1,m} - \frac{h_{1,m}}{2}, b_{1,m} + \frac{h_{1,m}}{2}\Big) \times (0, +\infty), \quad m \in \{1,2\},$$

respectively, and the rods $G_0^{(2,m)}(\varepsilon)$, $m \in \{1,2,3,4\}$, transform into the semi-infinite strips

$$\Pi^-_{2,m} = \Big(b_{2,m} - \frac{h_{2,m}}{2}, b_{2,m} + \frac{h_{2,m}}{2}\Big) \times (-\infty, 0], \quad m \in \{1,2,3,4\},$$

respectively.

In view of the periodic structure of Ω_ε in a neighborhood of I_2, we take the following two cells of periodicity

$$\Pi_2^{(1)} = \Pi^+_{1,1} \cup \Pi^-_{2,1} \cup \Pi^-_{2,2} \quad \text{and} \quad \Pi_2^{(2)} = \Pi^+_{1,2} \cup \Pi^-_{2,3} \cup \Pi^-_{2,4},$$

where branch-layer problems will be considered.

Now we propose the following two inner asymptotic expansions in a neighborhood of $I_2 \cap \big(G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}\big)$, namely the first one is as follows:

$$u^\varepsilon(x) \approx v_0^{(1,1)}(x_1, -l_1 - l_2) + \varepsilon\bigg(Z_1^{(2,1)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1+l_2}{\varepsilon}\big)\, \partial_{x_1} v_0^{(1,1)}(x_1, -l_1 - l_2) \\ + \Big\{ \eta_{2,1}(x_1)\, \Xi_1^{(2,1)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1+l_2}{\varepsilon}\big) + (1 - \eta_{2,1}(x_1))\, \Xi_2^{(2,1)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1+l_2}{\varepsilon}\big) \Big\} \partial_{x_2} v_0^{(1,1)}(x_1, -l_1 - l_2) \bigg) + \ldots \tag{34}$$

in a neighborhood of $I_2 \cap \Big(G_\varepsilon^{(1,1)} \bigcup \big(\bigcup_{m=1}^2 G_\varepsilon^{(2,m)}\big)\Big)$, and the second one

$$u^\varepsilon(x) \approx v_0^{(1,2)}(x_1, -l_1 - l_2) + \varepsilon\bigg(Z_1^{(2,2)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1+l_2}{\varepsilon}\big)\, \partial_{x_1} v_0^{(1,2)}(x_1, -l_1 - l_2) \\ + \Big\{ \eta_{2,2}(x_1)\, \Xi_1^{(2,2)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1+l_2}{\varepsilon}\big) + (1 - \eta_{2,2}(x_1))\, \Xi_2^{(2,2)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1+l_2}{\varepsilon}\big) \Big\} \partial_{x_2} v_0^{(1,2)}(x_1, -l_1 - l_2) \bigg) + \ldots \tag{35}$$

in a neighborhood of $I_2 \cap \Big(G_\varepsilon^{(1,2)} \bigcup \big(\bigcup_{m=3}^4 G_\varepsilon^{(2,m)}\big)\Big)$.

Coefficients $Z_1^{(2,1)}(\xi)$, $\Xi_1^{(2,1)}(\xi)$, $\Xi_2^{(2,1)}(\xi)$ $\big(\xi \in \Pi_2^{(1)}\big)$ in (34) and coefficients $Z_1^{(2,2)}(\xi)$, $\Xi_1^{(2,2)}(\xi)$, $\Xi_2^{(2,2)}(\xi)$ $\big(\xi \in \Pi_2^{(2)}\big)$ in (35) are solutions to branch-layer problems, which 1-periodic extended along the coordinate axis O_{ξ_1}; the functions $\eta_{2,1}$ and $\eta_{2,2}$ will be defined from matching conditions.

Namely, $Z_1^{(2,1)}$ and $Z_1^{(2,2)}$ are solutions to the problem (32) in $\Pi_2^{(1)}$ and $\Pi_2^{(2)}$ respectively. Applying results of Proposition 3, we can state that there exist unique solutions with the following differentiable asymptotics:

$$Z_1^{(2,1)}(\xi) = \begin{cases} -\xi_1 + b_{1,1} + \mathcal{O}\big(\exp(-\frac{\pi\xi_2}{h_{1,1}})\big), & \xi_2 \to +\infty,\ \xi \in \Pi_{1,1}^+, \\ -\xi_1 + b_{2,1} + C_1^{(3)} + \mathcal{O}\big(\exp(\frac{\pi\xi_2}{h_{2,1}})\big), & \xi_2 \to -\infty,\ \xi \in \Pi_{2,1}^-, \\ -\xi_1 + b_{2,2} + C_2^{(3)} + \mathcal{O}\big(\exp(\frac{\pi\xi_2}{h_{2,2}})\big), & \xi_2 \to -\infty,\ \xi \in \Pi_{2,2}^-, \end{cases} \tag{36}$$

$$Z_1^{(2,2)}(\xi) = \begin{cases} -\xi_1 + b_{1,2} + \mathcal{O}\big(\exp(-\frac{\pi\xi_2}{h_{1,2}})\big), & \xi_2 \to +\infty,\ \xi \in \Pi_{1,2}^+, \\ -\xi_1 + b_{2,3} + C_1^{(4)} + \mathcal{O}\big(\exp(\frac{\pi\xi_2}{h_{2,3}})\big), & \xi_2 \to -\infty,\ \xi \in \Pi_{2,3}^-, \\ -\xi_1 + b_{2,4} + C_2^{(4)} + \mathcal{O}\big(\exp(\frac{\pi\xi_2}{h_{2,4}})\big), & \xi_2 \to -\infty,\ \xi \in \Pi_{2,4}^-. \end{cases} \tag{37}$$

Functions $\Xi_1^{(2,1)}, \Xi_2^{(2,1)}$ and $\Xi_1^{(2,2)}, \Xi_2^{(2,2)}$ are solutions to the problem (29) in $\Pi_2^{(1)}$ and $\Pi_2^{(2)}$ respectively. From Proposition 2 it follows that they have the corresponding differentiable asymptotics (30) and (31).

3.4 | Homogenized problem

To complete the formal asymptotic constructions and obtain transmission conditions for the coefficients of the outer asymptotic expansions

$$v_0^+,\ v_0^{(0)},\ v_0^{(1,1)},\ v_0^{(1,2)},\ v_0^{(2,1)},\ v_0^{(2,2)},\ v_0^{(2,3)},\ v_0^{(2,4)},$$

we should match the corresponding outer asymptotic expansions with the inner ones, namely, the asymptotics of the leading terms of outer expansions (12) and (13) as $x_2 \to \pm \sum_{p=0}^m l_p,\ m \in \{0,1,2\}$, have to coincide with the corresponding asymptotics of the inner expansions (24), (28), (34) and (35) as $\xi_2 \to \pm\infty$, respectively.

By making similar steps as in[4, Sec. 4], we come to the following transmission conditions:

$$v_0^+ = v_0^{(0)}, \quad \partial_{x_2} v_0^+ = h_0\, \partial_{x_2} v_0^{(0)} \quad \text{on } I_0; \tag{38}$$

$$v_0^{(0)} = v_0^{(1,1)} = v_0^{(1,1)}, \quad h_0\, \partial_{x_2} v_0^{(0)} = h_{1,1}\, \partial_{x_2} v_0^{(1,1)} + h_{1,2}\, \partial_{x_2} v_0^{(1,2)} \quad \text{on } I_1; \tag{39}$$

$$v_0^{(1,1)} = v_0^{(2,1)} = v_0^{(2,2)}, \quad h_{1,1}\, \partial_{x_2} v_0^{(1,1)} = h_{2,1}\, \partial_{x_2} v_0^{(2,1)} + h_{2,2}\, \partial_{x_2} v_0^{(2,2)} \quad \text{on } I_2; \tag{40}$$

$$v_0^{(1,2)} = v_0^{(2,3)} = v_0^{(2,4)}, \quad h_{1,2}\, \partial_{x_2} v_0^{(1,2)} = h_{2,3}\, \partial_{x_2} v_0^{(2,3)} + h_{2,4} \partial_{x_2} v_0^{(2,4)} \quad \text{on } I_2. \tag{41}$$

Relations (14), (20)-(23), (38)-(41) form *homogenized spectral problem* for problem (3) and an eigenfunction of this problem is a multi-sheeted function

$$\begin{array}{ccccccc}
 & & & & & & v_0^{(2,1)} \\
 & & & & & \nearrow & \\
 & & & & v_0^{(1,1)} & & \\
 & & & \nearrow & & \searrow & \\
 & & & & & & v_0^{(2,2)} \\
v_0^+ & \longrightarrow & v_0^{(0)} & & & & \\
 & & & & & & v_0^{(2,3)} \\
 & & & \searrow & & \nearrow & \\
 & & & & v_0^{(1,2)} & & \\
 & & & & & \searrow & \\
 & & & & & & v_0^{(2,4)} \\
 & I_0 & & I_1 & & I_2 & \\
\Omega_0 & & D_0 & & D_1 & & D_2
\end{array}$$

In addition, functions η_1 in (28), $\eta_{2,1}$ in (34), and $\eta_{2,2}$ in (35) are defined as follows:

$$\eta_1(x_1) = \frac{h_{1,1}\, \partial_{x_2} v_0^{(1,1)}(x_1, -l_1)}{h_{1,1}\, \partial_{x_2} v_0^{(1,1)}(x_1, -l_1) + h_{1,2}\, \partial_{x_2} v_0^{(1,2)}(x_1, -l_1)}, \tag{42}$$

$$\eta_{2,1}(x_1) = \frac{h_{2,1}\, \partial_{x_2} v_0^{(2,1)}(x_1, -(l_1 + l_2))}{h_{2,1}\, \partial_{x_2} v_0^{(2,1)}(x_1, -(l_1 + l_2)) + h_{2,2}\, \partial_{x_2} v_0^{(2,2)}(x_1, -(l_1 + l_2))}, \tag{43}$$

$$\eta_{2,2}(x_1) = \frac{h_{2,3}\, \partial_{x_2} v_0^{(2,3)}(x_1, -(l_1 + l_2))}{h_{2,3}\, \partial_{x_2} v_0^{(2,3)}(x_1, -(l_1 + l_2)) + h_{2,4}\, \partial_{x_2} v_0^{(2,4)}(x_1, -(l_1 + l_2))}. \tag{44}$$

for $x_1 \in (0, a)$.

4 | THE SPECTRUM STRUCTURE OF THE HOMOGENIZED PROBLEM

In order to avoid technical and huge calculations, we will consider the case when $i \in \{0, 1\}$ and $\alpha_i > 1$, while making remarks and comments for the other cases.

Let us first introduce an anisotropic Sobolev space $\mathbf{H}_0$ of multi-sheeted functions. A multi-sheeted function

$$\boldsymbol{\varphi} := \left(\varphi^+, \varphi^{(0)}, \left\{ \varphi^{(1,m)} \right\}_{m=1}^{2} \right) = \begin{cases} \varphi^+(x), & x \in \Omega_0, \\ \varphi^{(0)}(x), & x \in D_0, \\ \varphi^{(1,m)}(x), & x \in D_1, \quad m \in \{1, 2\}, \end{cases} \tag{45}$$

belongs to $\mathbf{H}_0$ if

$$\varphi^+ \in H^1_\sharp(\Omega_0; \Gamma_1) := \{ v \in H^1(\Omega_0) : \ v(0, x_2) = v(a, x_2), \ x_2 \in [0, \delta_0]; \ \ v|_{\Gamma_1} = 0 \},$$

$\varphi^{(0)} \in L^2(D_0)$, $\{\varphi^{(1,m)}\}_{m=1}^2 \subset L^2(D_1)$, there exist weak derivatives $\partial_{x_2} \varphi^{(0)} \in L^2(D_0)$, $\{\partial_{x_2} \varphi^{(1,m)}\}_{m=1}^2 \subset L^2(D_1)$, and

$$\varphi^+|_{I_0} = \varphi^{(0)}|_{I_0}, \qquad \varphi^{(0)}|_{I_1} = \varphi^{(1,1)}|_{I_1} = \varphi^{(1,2)}|_{I_1}.$$

The space $\mathbf{H}_0$ is continuously and densely embedded in a Hilbert space $\mathbf{V}_0$ of multi-sheeted functions whose components belong to L^2-spaces, i.e., $\boldsymbol{\varphi} \in \mathbf{V}_0$ if $\varphi^+ \in L^2(\Omega_0)$, $\varphi^{(0)} \in L^2(D_0)$, $\{\varphi^{(1,m)}\}_{m=1}^2 \subset L^2(D_1)$. Scalar products in these spaces are defined as follows:

$$(\boldsymbol{\varphi}, \boldsymbol{\psi})_{\mathbf{V}_0} := (\varphi^+, \psi^+)_{L^2(\Omega_0)} + h_0(\varphi^{(0)}, \psi^{(0)})_{L^2(D_0)} + \sum_{m=1}^{2} h_{1,m}(\varphi^{(1,m)}, \psi^{(1,m)})_{L^2(D_1)}, \tag{46}$$

$$(\boldsymbol{\varphi}, \boldsymbol{\psi})_{\mathbf{H}_0} := (\nabla\varphi^+, \nabla\psi^+)_{L^2(\Omega_0)} + h_0(\partial_{x_2}\varphi^{(0)}, \partial_{x_2}\psi^{(0)})_{L^2(D_0)} + \sum_{m=1}^{2} h_{1,m}(\partial_{x_2}\varphi^{(1,m)}, \partial_{x_2}\psi^{(1,m)})_{L^2(D_1)}. \tag{47}$$

If $i \in \{0, 1, 2\}$, then the corresponding summands appear in these scalar products, and if some $\alpha_{i_0} = 1$, then terms

$$2 \sum_{m=1}^{i_0+\left\lfloor \frac{i_0+2}{2} \right\rfloor} k_{i_0,m}(\varphi^{(i_0,m)}, \psi^{(i_0,m)})_{L^2(D_{i_0})}$$

appear in $(\cdot,\cdot)_{\mathbf{H}_0}$.

Definition 2. A number μ is called an eigenvalue of the homogenized spectral problem if there exists a function $\mathbf{v} \in \mathbf{H}_0 \setminus \{0\}$ such that

$$(\mathbf{v}, \boldsymbol{\psi})_{\mathbf{H}_0} = \mu\,(\mathbf{v}, \boldsymbol{\psi})_{\mathbf{V}_0} \quad \forall \boldsymbol{\psi} \in \mathbf{H}_0. \tag{48}$$

The function $\mathbf{v}$ is called an eigenfunction that corresponds to μ.

Define the operator $\mathsf{A}_0 : \mathbf{H}_0 \mapsto \mathbf{H}_0$ by the equality

$$(\mathsf{A}_0\boldsymbol{\varphi}, \boldsymbol{\psi})_{\mathbf{H}_0} = (\boldsymbol{\varphi}, \boldsymbol{\psi})_{\mathbf{V}_0} \quad \forall\, \boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathbf{H}_0. \tag{49}$$

Then the homogenized spectral problem is equivalent to the spectral problem

$$\mathsf{A}_0\mathbf{v} = \mu^{-1}\,\mathbf{v} \quad \text{in } \mathbf{H}_0.$$

It is easy to verify that A_0 is self-adjoint, positive, continuous, but non-compact and the spectrum $\sigma(\mathsf{A}_0)$ belongs to $(c_0, +\infty)$, where $c_0 > 0$.

4.1 | Reducing to a spectral problem for an operator-function

Solving ordinary differential equations (20), (21), and taking into account the Neumann condition at $x_2 = -l_1 - l_2$, we get

$$v_0^{(0)}(x) = B^{(0)} \cos\left(\sqrt{\mu_0}\,(x_2 + l_1)\right) + C^{(0)} \sin\left(\sqrt{\mu_0}\,(x_2 + l_1)\right), \tag{50}$$

$$v_0^{(1,m)}(x) = \frac{B^{1,m}}{\cos\left(\sqrt{\mu_0}\,l_1\right)\cos\left(\sqrt{\mu_0}\,l_2\right)} \cos\left(\sqrt{\mu_0}\,(x_2 + l_1 + l_2)\right), \quad m = 1, 2. \tag{51}$$

Gradually substituting representations (50) and (51) in the transmission conditions (38) and (39), we find constants $B^{(0)}, C^{(0)}, B^{1,1}, B^{1,2}$ and arrive at the relation

$$\partial_{x_2} v_0^+(x_1, 0) = -h_0\sqrt{\mu_0}\, \frac{\sin\sqrt{\mu_0}l_1 \cos\sqrt{\mu_0}l_2 + \theta\cos\sqrt{\mu_0}l_1 \sin\sqrt{\mu_0}l_2}{\cos\sqrt{\mu_0}l_1 \cos\sqrt{\mu_0}l_2 - \theta\sin\sqrt{\mu_0}l_1 \sin\sqrt{\mu_0}l_2}\, v_0^+(x_1, 0) \tag{52}$$

for any $x_1 \in (0, a)$. Here $\theta = \frac{h_{1,1}+h_{1,2}}{h_0} < 1$.

Relations (14) and (52) form the *resulting spectral problem* in the junction's body Ω_0 with the spectral parameter μ_0 occurring both in the differential equation and in the boundary condition on I_0.

Remark 3. In the case $i \in \{0, 1, 2\}$ the condition on I_0 is more complicated, namely,

$$\partial_{x_2} v_0^+(x_1, 0) = -\varphi(\sqrt{\mu})v_0^+(x_1, 0),$$

where

$$\varphi(\mu) = \mu h_0 \frac{\tan\left(\mu\, l_1\right) + \theta\tan\left(\mu\, l_2\right) + \frac{h_{2,1}+\ldots+h_{2,4}}{h_0}\tan\left(\mu\, l_3\right) - \frac{h_{2,1}+h_{2,2}}{h_{1,1}}\tan\left(\mu\, l_1\right)\tan\left(\mu\, l_2\right)\tan\left(\mu\, l_3\right)}{1 - \theta\tan\left(\mu\, l_1\right)\tan\left(\mu\, l_2\right) - \frac{h_{2,1}+h_{2,2}}{h_{1,1}}\tan\left(\mu\, l_3\right)\tan\left(\mu\, l_2\right) - \frac{h_{2,1}+\ldots+h_{2,4}}{h_0}\tan\left(\mu\, l_1\right)\tan\left(\mu\, l_3\right)},$$

for the case $\frac{h_{2,1}+h_{2,2}}{h_{1,1}} = \frac{h_{2,3}+h_{2,4}}{h_{1,2}}$.

Remark 4. In the case $i \in \{0, 1\}$ and $\alpha_0 = 1,\ \alpha_1 > 1$ the condition on I_0 is as follows

$$\partial_{x_2} v_0^+(x_1, 0) = -g(\mu)v_0^+(x_1, 0),$$

where

$$g(\mu) = h_0 \frac{\sqrt{\mu_0 - \frac{2k_0}{h_0}} \tan\left(\sqrt{\mu_0 - \frac{2k_0}{h_0}}\, l_1\right) + \theta \sqrt{\mu_0} \tan\left(\sqrt{\mu_0}\, l_2\right)}{1 - \theta \tan\left(\sqrt{\mu_0 - \frac{2k_0}{h_0}}\, l_1\right) \tan\left(\sqrt{\mu_0}\, l_2\right)}.$$

Multiplying the differential equation in (14) with an arbitrary function $\psi \in H^1_\sharp(\Omega_0;\Gamma_1)$ and then integrating by parts over Ω_0 with regards to (52), the resulting spectral problem is reduced to the spectral problem

$$\mathbf{L}(\mu) v_0^+ = 0 \quad \text{in } H^1_\sharp(\Omega_0;\Gamma_1), \qquad \mu \in [c_0, +\infty),$$

for the following operator-function

$$\mathbf{L}(\mu) := \mu\, A_1 + h_0 \sqrt{\mu}\, f(\sqrt{\mu})\, A_2 - \mathbb{I}, \tag{53}$$

where $\theta = \frac{h_{1,1}+h_{1,2}}{h_0}$, $\mathbb{I}$ is the identity operator in $H^1_\sharp(\Omega_0;\Gamma_1)$,

$$f(\mu) := \frac{\sin \mu l_1 \cos \mu l_2 + \theta \cos \mu l_1 \sin \mu l_2}{\cos \mu l_1 \cos \mu l_2 - \theta \sin \mu l_1 \sin \mu l_2}, \tag{54}$$

$A_1,\ A_2$ are self-adjoint, compact operators in $H^1_\sharp(\Omega_0;\Gamma_1)$ such that

$$(A_1\varphi, \psi)_{H^1_\sharp(\Omega_0;\Gamma_1)} = \int\limits_{\Omega_0} \varphi(x)\, \psi(x)\, dx, \qquad (A_2\varphi, \psi)_{H^1_\sharp(\Omega_0;\Gamma_1)} = \int\limits_{I_0} \varphi(x_1, 0)\, \psi(x_1, 0)\, dx_1 \quad \forall \varphi, \psi \in H^1_\sharp(\Omega_0;\Gamma_1).$$

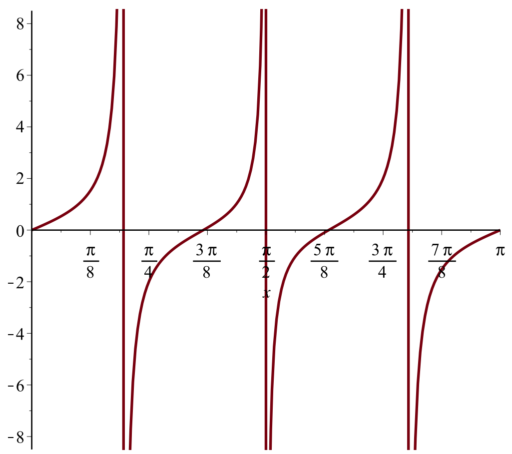

FIGURE 4 The graph of the function f for $\theta = \frac{1}{2},\ l_1 = 2,\ l_2 = 1$

It is easy to verify that $f'(\mu)$

$$f'(\mu) = \frac{(l_1 + \theta l_2)(\cos^2 \mu l_1 \cos^2 \mu l_2 + \sin^2 \mu l_1 \cos^2 \mu l_2) + \theta(l_2 + \theta l_1)(\sin^2 \mu l_1 \sin^2 \mu l_2 + \cos^2 \mu l_1 \sin^2 \mu l_2)}{(\cos \mu l_1 \cos \mu l_2 - \theta \sin \mu l_1 \sin \mu l_2)^2} > 0$$

for all $\mu \in \mathbb{R} \setminus \{\mu :\ \tan(\mu l_1)\tan(\mu l_2) = \theta^{-1}\}$. The graph of this function resembles the graph of the tangent (see Fig. 4 made by Maple).

Theorems on existence and concentration of the spectrum for self-adjoint discontinuous operator functions like (53) and the minimax principles for eigenvalues were proved in [14]. These results yield the following theorem.

Theorem 1. The spectrum of the operator-function $\mathbf{L}$ contains normal eigenvalues and also the left accumulation points $\{p_k\}_{k\in\mathbb{N}}$ that are roots of the transcendental equation

$$\tan\left(\sqrt{\mu}\, l_1\right) \tan\left(\sqrt{\mu}\, l_2\right) = \frac{h_0}{h_{1,1} + h_{1,2}}.$$

These points divide the eigenvalues into the sequences

$$0 < \mu_1^{(1)} \le \ldots \le \mu_n^{(1)} \le \ldots \to p_1 \quad \text{as} \quad n \to +\infty, \tag{55}$$

$$p_k < \mu_1^{(k+1)} \le \ldots \le \mu_n^{(k+1)} \le \ldots \to p_{k+1} \quad \text{as} \quad n \to +\infty. \tag{56}$$

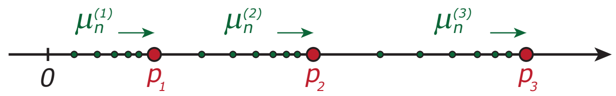

FIGURE 5 The spectrum structure of the operator-function $\mathbf{L}$

Recall that an eigenvalue is called normal if it has finite multiplicity and the corresponding eigenvectors have no Jordan chains. In the cases considered in Remarks 3 and 4 the spectrum structure of the corresponding operator-function is the same.

5 | ASYMPTOTIC APPROXIMATIONS FOR THE EIGENFUNCTIONS

Let μ_0 be an eigenvalue of the homogenized problem. Consider the corresponding multi-sheeted eigenfunction

$$\mathbf{v}_0(x) = \begin{cases} v_0^+(x), & x \in \Omega_0, \\ v_0^{(0)}(x), & x \in D_0, \\ v_0^{(1,m)}(x), & x \in D_1,\ m \in \{1,2\}, \end{cases}$$

from $\mathbf{H}_0$. With the help of $\mathbf{v}_0$, the junction-layer solutions $Z_1^{(0)}$ and $Z_2^{(0)}$ (see Proposition 1), the branch-layer solutions $\{Z_1^{(1)}, \Xi_1^{(1)}, \Xi_2^{(1)}\}$ (see Propositions 2, 3) in a neighborhood of the first branching zone I_1 we construct the approximation function

$$R_\varepsilon(x) = v_0^+(x) + \varepsilon \chi_0(x_2)\, \mathcal{N}_+^{(0)}\big(\tfrac{x}{\varepsilon}, x_1\big), \quad x \in \Omega_0; \tag{57}$$

$$R_\varepsilon = v_0^{(0)}(x) + \varepsilon\Big(Y_0(\tfrac{x_1}{\varepsilon})\, \partial_{x_1} v_0^{(0)}(x) + \chi_0(x_2)\, \mathcal{N}_-^{(0)}\big(\tfrac{x}{\varepsilon}, x_1\big) + \chi_1(x_2)\mathcal{N}^{(1)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1}{\varepsilon}, x_1\big)\Big), \quad x \in G_\varepsilon^{(0)}; \tag{58}$$

$$R_\varepsilon = v_0^{(1,m)}(x) + \varepsilon\Big(Y_{1,m}(\tfrac{x_1}{\varepsilon})\, \partial_{x_1} v_0^{(1,m)}(x) + \chi_1(x_2)\, \mathcal{N}_{1,m}^{(1)}\big(\tfrac{x_1}{\varepsilon}, \tfrac{x_2+l_1}{\varepsilon}, x_1\big)\Big), \quad x \in G_\varepsilon^{(1,m)},\ m \in \{1,2\}; \tag{59}$$

where χ_0 is a smooth cutoff function such that $\chi_0(x_2) = 1$ for $|x_2| \le \tau_0/2$, and $\chi_0(x_2) = 0$ for $|x_2| \ge \tau_0$ (τ_0 is sufficiently small number); $\chi_1(x_2) := \chi_0(x_2 + l_1)$;

$$\mathcal{N}_+^{(0)}(\xi, x_1) = \sum_{i=1}^{2} \big(Z_i^{(0)}(\xi) - \delta_{i,2}\, \xi_2\big) \partial_{x_i} v_0^+(x_1, 0), \quad \xi = \tfrac{x}{\varepsilon},$$

where $\delta_{i,2}$ is the Kronecker delta; $Y_0(\xi_1) = -\xi_1 + \frac{1}{2} + \lfloor \xi_1 \rfloor$;

$$\mathcal{N}_-^{(0)}(\xi, x_1) = \Big(Z_1^{(0)}(\xi) - Y_0(\xi_1)\Big)\partial_{x_1} v_0^+(x_1, 0) + \Big(Z_2^{(0)}(\xi) - \tfrac{\xi_2}{h_0}\Big)\partial_{x_2} v_0^+(x_1, 0), \quad \xi = \tfrac{x}{\varepsilon};$$

$$\begin{aligned}\mathcal{N}^{(1)}(\xi, x_1) = {} & \Big(Z_1^{(1)}(\xi) - Y_0(\xi_1)\Big)\partial_{x_1} v_0^{(0)}(x_1, -l_1) \\ & + \Big(\eta_1(x_1)\Xi_1^{(1)}(\xi) + (1 - \eta_1(x_1))(\Xi_2^{(1)}(\xi) - \xi_2)\Big)\partial_{x_2} v_0^{(0)}(x_1, -l_1), \quad \xi_1 = \tfrac{x_1}{\varepsilon},\ \xi_2 = \tfrac{x_2+l_1}{\varepsilon};\end{aligned}$$

$Y_{1,m}(\xi_1) = -\xi_1 + b_{1,m} + \lfloor \xi_1 \rfloor$, $m \in \{1,2\}$, and

$$\begin{aligned}\mathcal{N}_{1,m}^{(1)}(\xi, x_1) = {} & \Big(Z_1^{(1)}(\xi) - Y_{1,m}(\xi_1)\Big)\partial_{x_1} v_0^{(0)}(x_1, -l_1) \\ & + \Big(\eta_1(x_1)\big(\Xi_1^{(1)}(\xi) - \delta_{1,m}\tfrac{h_0}{h_{1,1}}\xi_2\big) + (1 - \eta_1(x_1))\big(\Xi_2^{(1)}(\xi) - \delta_{2,m}\tfrac{h_0}{h_{1,2}}\xi_2\big)\Big)\partial_{x_2} v_0^{(0)}(x_1, -l_1), \quad \xi_1 = \tfrac{x_1}{\varepsilon},\ \xi_2 = \tfrac{x_2+l_1}{\varepsilon}.\end{aligned}$$

Due to (38) and (39), the jumps $\big[R_\varepsilon\big]\,|_{Q_\varepsilon^{(i)}} = 0$, $i = 0, 1$. This means that the approximation function R_ε belongs to $\mathcal{H}_\varepsilon$.

Substituting R_ε and μ_0 in problem (3) instead of u^ε and λ^ε, respectively, calculating residuals with regard to Propositions 1, 2, 3 and relations (14), (20), (21), (23) (38), (39), we obtain

$$\langle R_\varepsilon, \varphi \rangle_\varepsilon - \mu_0 \left(R_\varepsilon, \varphi \right)_{L^2(\Omega_\varepsilon)} = \ell_\varepsilon(\varphi) \quad \forall \varphi \in \mathcal{H}_\varepsilon, \tag{60}$$

where the linear functional ℓ_ε on the space $\mathcal{H}_\varepsilon$ and ℓ_ε is the sum of the integrals of the residues that the approximation function R_ε leaves after substitution into the integral identity. Its norm is estimated similar as in[4, Sec. 6]. As a result, with the help of the operator $A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$ defined in (6), we get from (60) the inequality

$$\| R_\varepsilon - \mu_0 A_\varepsilon R_\varepsilon \|_\varepsilon \le C(\varrho) \Big(\varepsilon^{1-\varrho} + \sum_{i=0}^{1} \varepsilon^{\alpha_i - 1 + \delta_{\alpha_i,1}} \Big), \tag{61}$$

where ϱ is an arbitrary positive number and $\delta_{\alpha_i,1}$ is the Kronecker delta.

From Vishik-Lyusternik lemma[15] and inequalities (61) and (10) it follows that for any $n \in \mathbb{N}$ there exist positive constants τ and ε_0 such that for all values of $\varepsilon \in (0, \varepsilon_0)$

$$0 < C_1^2 \le \lambda_n(\varepsilon) \le p_1 - \tau, \tag{62}$$

where p_1 is the first point of the essential spectrum of the operator-function $\mathbf{L}$ (see Theorem 1).

6 | JUSTIFICATION OF THE ASYMPTOTICS AND ASYMPTOTIC ESTIMATES

To justify the asymptotic expansions constructed above, we use the scheme suggested in[16] to study the asymptotic behaviour of eigenvalues and eigenvectors of a family of abstract operators. For the convenience of readers, the conditions of this scheme are written here specifically for the problems and homogenized spectral problem.

In our case, the family of operators is $\{A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon\}_{\varepsilon>0}$, where the operator A_ε defined by the formula (6) and corresponds to the problem (3).

Let $N(\frac{1}{\mu}, \mathsf{A}_0)$ denote the proper subspace corresponding to the eigenvalue $\frac{1}{\mu}$ of the operator $\mathsf{A}_0 : \mathbf{H}_0 \mapsto \mathbf{H}_0$ defined in (49) and let $\left\{ \left(u_n^\varepsilon, \lambda_n^\varepsilon \right) \right\}_{\varepsilon>0}$ denote a sequence whose components are the eigenfunction u_n^ε ($\|u_n^\varepsilon\|_{\mathcal{V}_\varepsilon} = 1$) and the corresponding characteristic number of A_ε. Here, $\mathcal{V}_\varepsilon := L^2(\Omega_\varepsilon)$.

To clarify these conditions, the following diagram is proposed:

$$\begin{tikzcd} \mathcal{H}_\varepsilon \arrow[r, "I_\varepsilon"] \arrow[d, shift right, "\mathbf{P}_\varepsilon"'] & \mathcal{V}_\varepsilon \\ \mathbf{Z}_0 \arrow[r, "\mathbf{I}_0"] \arrow[u, shift right, "\mathrm{S}_\varepsilon"'] & \mathbf{V}_0 \end{tikzcd}$$

Here $\mathbf{Z}_0$ is a subspace of $\mathbf{H}_0$, in which each component of a multi-sheeted function $\boldsymbol{\varphi} \in \mathbf{Z}_0$ belongs to the corresponding H^1-Sobolev space; the operators $I_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{V}_\varepsilon$ and $\mathbf{I}_0 : \mathbf{Z}_0 \mapsto \mathbf{V}_0$ are identical imbedding operators. Obviously, these operators are compact.

There are five conditions $\mathbf{C_1} - \mathbf{C_5}$ in this scheme. The first one is verified in Sec. 5 and it means that for each eigenvalue of the operator A_0 one can construct an approximation function such that this pair is, respectively, an almost eigenvalue and an almost eigenfunction of the operator A_ε, i.e., the inequality (61) holds.

Condition $\mathbf{C_2}$. There exists a linear operator $\mathrm{S}_\varepsilon : \mathbf{Z}_0 \mapsto \mathcal{H}_\varepsilon$ such that

$$\|\mathrm{S}_\varepsilon \mathbf{u}\|_{\mathcal{H}_\varepsilon} \le c_1 \|\mathbf{u}\|_{\mathbf{Z}_0}, \quad \forall \mathbf{u} \in \mathbf{Z}_0,$$

where the constant c_1 is independent of ε and u.

Condition $\mathbf{C_3}$. There exists a linear operator $\mathbf{P}_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathbf{Z}_0$ such that

$$\forall n \in \mathbb{N} \;\; \exists c_2 > 0 \;\; \exists \varepsilon_0 > 0 \;\; \forall \varepsilon \in (0, \varepsilon_0) : \quad \|\mathbf{P}_\varepsilon u_n^\varepsilon\|_{\mathbf{H}_0} \le c_2 \, \|u_n^\varepsilon\|_{\mathcal{H}_\varepsilon}.$$

Conditions $\mathbf{C_2}$ and $\mathbf{C_3}$ are connecting conditions between the spaces $\mathcal{H}_\varepsilon$ and $\mathbf{Z}_0$. The operator $\mathrm{S}_\varepsilon : \mathbf{Z}_0 \mapsto \mathcal{H}_\varepsilon$ associates any multi-sheeted function $\boldsymbol{\varphi} \in \mathbf{Z}_0$ (see (45)) with a function $\mathrm{S}_\varepsilon \boldsymbol{\varphi}$ that is equal to φ^+ in Ω_0, to $\varphi^{(0)}|_{G_\varepsilon^{(0)}}$, to $\varphi^{(1,1)}|_{G_\varepsilon^{(1,1)}}$ and to $\varphi^{(1,2)}|_{G_\varepsilon^{(1,2)}}$, where $\varphi^{(i,m)}|_{G_\varepsilon^{(i,m)}}$ is the restriction of $\varphi^{(i,m)}$ on $G_\varepsilon^{(i,m)}$. It is easy to see that the operator S_ε is uniformly bounded with respect to ε. Thus, condition $\mathbf{C_2}$ is satisfied.

For functions defined in thick junctions, there is no extension operator uniformly bounded with respect to ε in H^1-Sobolev spaces (see [12]). Nevertheless, it was shown in [12] that for eigenfunctions of spectral problems in thick junctions one can construct a special extension which is bounded uniformly in ε on every eigenfunction. In our case, we construct a special multi-sheeted extension and prove the following theorem.

Theorem 2. There exists a linear operator $\mathbf{P}_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathbf{Z}_0$, where $\mathbf{P}_\varepsilon = \left(P^{(0)}_\varepsilon, P^{(1,1)}_\varepsilon, P^{(1,2)}_\varepsilon\right)$ and

$$P^{(0)}_\varepsilon : H^1(G^{(0)}_\varepsilon) \mapsto H^1(D_0), \quad P^{(1,m)}_\varepsilon : H^1(G^{(1,m)}_\varepsilon) \mapsto H^1(D_1), \quad m \in \{1,2\},$$

such that for any eigenfunction u^ε_n of problem (3) condition $\mathbf{C_3}$ is satisfied.

Proof. **1.** Let χ_{δ_0} be a smooth cut-off function such that $\chi_{\delta_0}(x_2) = 0$ for $x_2 \geq \delta_0$, and $\chi_{\delta_0}(x_2) = 1$ for $x_2 \leq \frac{\delta_0}{2}$, where δ_0 is defined at the beginning of Section 2.

Let u^ε_n $(\|u^\varepsilon_n\|_{\mathcal{V}_\varepsilon} = 1)$ be an eigenfunction of problem (3). Then the function $v^\varepsilon_n = \chi_{\delta_0} u^\varepsilon_n$ is the solution to the following problem:

$$\begin{cases} -\Delta_x v^\varepsilon_n = f^\varepsilon_n(x) + \lambda^\varepsilon_n v^\varepsilon_n & \text{in } \Omega_{0,\delta_0}, \\ -\Delta_x v^\varepsilon_n = \lambda^\varepsilon_n v^\varepsilon_n & \text{in } G^{(0)}_\varepsilon \cup G^{(1)}_\varepsilon, \\ -\partial_\nu v^\varepsilon_n = \varepsilon^{\alpha_i} k_{i,m} v^\varepsilon_n & \text{on } \Upsilon^{(i,m)}_\varepsilon, \ i = 0,1, \\ \partial^p_{x_1} v^\varepsilon_n|_{x_1=0} = \partial^p_{x_1} v^\varepsilon_n|_{x_1=a}, & x_2 \in [0,\delta_0], \ p = 0,1, \\ v^\varepsilon_n = 0 & \text{on } \Gamma_{\delta_0}, \\ \partial_\nu v^\varepsilon_n = 0 & \text{on } \partial\Omega_{\varepsilon,\delta_0} \setminus (\Upsilon_\varepsilon \cup \Gamma_0 \cup \Gamma_{\delta_0}). \end{cases} \tag{63}$$

Here $\Omega_{0,\delta_0} = (0,a) \times (0,\delta_0)$, $\Gamma_{\delta_0} = \{x : x_1 \in (0,a), \ x_2 = \delta_0\}$, $f^\varepsilon_n = 2\chi'_{\delta_0} \partial_{x_2} u^\varepsilon_n + \chi''_{\delta_0} u^\varepsilon_n$, $\operatorname{supp}(\chi'_{\delta_0}) \subset [0,a] \times (\frac{\delta_0}{2}, \delta_0)$. To simplify formulas, the conjugation conditions on $Q^{(i)}_\varepsilon$, $i \in \{0,1\}$, are not written hereinafter.

Since the problem (63) is invariant with respect to shifts by ε along the axis Ox_1, the function (the index n is omitted)

$$V^\varepsilon(x) = \varepsilon^{-1}(v^\varepsilon(x + \varepsilon \bar{e}_1) - v^\varepsilon(x)), \quad (\bar{e}_1 = (1,0)), \tag{64}$$

that is a−periodic in x_1, satisfies the following relations

$$\begin{aligned} -\Delta_x V^\varepsilon &= \varepsilon^{-1}\left(f^\varepsilon(x + \varepsilon\bar{e}_1) - f^\varepsilon(x)\right) + \lambda^\varepsilon V^\varepsilon && \text{in } \Omega_{0,\delta_0}, \\ -\Delta_x V^\varepsilon &= \lambda^\varepsilon V^\varepsilon && \text{in } G^{(0)}_\varepsilon \cup G^{(1)}_\varepsilon, \\ -\partial_\nu V^\varepsilon &= \varepsilon^{\alpha_i} k_{i,m} V^\varepsilon && \text{on } \Upsilon^{(i,m)}_\varepsilon, \ i = 0,1, \\ \partial^p_{x_1} V^\varepsilon|_{x_1=0} &= \partial^p_{x_1} V^\varepsilon|_{x_1=a}, && x_2 \in [0,\delta_0], \ p = 0,1, \\ V^\varepsilon &= 0 && \text{on } \Gamma_{\delta_0}, \\ \partial_\nu V^\varepsilon &= 0 && \text{on } \partial\Omega_{\varepsilon,\delta_0} \setminus (\Upsilon_\varepsilon \cup \Gamma_0 \cup \Gamma_{\delta_0}), \end{aligned}$$

whence, multiplying the differential equations by V^ε and integrating by parts, we get

$$\|\nabla V^\varepsilon\|^2_{L^2(\Omega_{\varepsilon,\delta_0})} \leq \lambda^\varepsilon \|V^\varepsilon\|^2_{L^2(\Omega_{\varepsilon,\delta_0})} + \varepsilon^{-1} \int\limits_{\Omega_{0,\delta_0}} (f^\varepsilon(x + \varepsilon\bar{e}_1) - f^\varepsilon(x)) V^\varepsilon \, dx, \tag{65}$$

where $\Omega_{\varepsilon,\delta_0}$ is the interior of the union $\overline{\Omega}_{0,\delta_0} \cup \overline{G^{(0)}_\varepsilon} \cup \overline{G^{(1)}_\varepsilon}$.

Let us estimate the right-hand side of (65). Using the Cauchy-Bunyakovsky inequality and changing the order of integration, we derive the inequality

$$\int\limits_{\Omega_{0,\delta_0}} (V^\varepsilon)^2\, dx = \varepsilon^{-2} \int\limits_0^{\delta_0} \int\limits_0^a \left| \int\limits_{x_1}^{x_1+\varepsilon} \partial_t v^\varepsilon(t,x_2)\, dt \right|^2 dx_1 dx_2$$

$$\le \varepsilon^{-1} \int\limits_0^{\delta_0} \int\limits_0^a \int\limits_{x_1}^{x_1+\varepsilon} \left|\partial_t v^\varepsilon(t,x_2)\right|^2 dt dx_1 dx_2 = \int\limits_0^{\delta_0} \int\limits_0^a \left|\partial_t v^\varepsilon(t,x_2)\right|^2 dt dx_2 \le \|\partial_{x_1} u_n^\varepsilon\|^2_{L^2(\Omega_0)}. \quad (66)$$

Due to (9) the second summand the right-hand side of (65) is estimated with the value

$$\|\varepsilon^{-1}(f^\varepsilon(x+\varepsilon\bar{e}_1) - f^\varepsilon(x))\|_{L^2(\Omega_{0,\delta_0})} \cdot \|V_\varepsilon\|_{L^2(\Omega_{0,\delta_0})} \le \|\partial_{x_1} f^\varepsilon\|_{L^2(\Omega_{0,\delta_0})} \|\partial_{x_1} u_n^\varepsilon\|_{L^2(\Omega_0)}$$

$$\le c_1 \left(\|u_n^\varepsilon\|_{H^1(\Omega_0)} + \|(\chi_{\delta_0})' \partial^2_{x_1 x_2} u_n^\varepsilon\|_{L^2(\Omega_{0,\delta_0})\|} \right) \|\partial_{x_1} u_n^\varepsilon\|_{L^2(\Omega_0)} \le c_2 \|u_n^\varepsilon\|^2_{H^1(\Omega_0)}. \quad (67)$$

Here, in order to estimate the mixed second-order derivative, we have used so-called the second energy inequality for elliptic operators in the domain $(0,a)\times(\frac{\delta_0}{2}, \delta_0)$, i.e., the a-priori estimate $\|u\|^2_{H^2(\Omega)} \le c(\|\Delta u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})$ (see e.g. [17]) with a suitable cut-off function.

Now it remains to estimate L^2-norm of V^ε in each thin rod of the thick fractal junction Ω_ε. For this we represent V^ε on the rod $G_j^{(i,m)}(\varepsilon)$ in the following form:

$$V^\varepsilon(x) = \varphi_j^{(i,m)}(x_2) + U_j^{(i,m)}(x), \quad x \in G_j^{(i,m)}(\varepsilon), \quad (68)$$

$$\int\limits_{\varkappa_j^{(i,m)}(\varepsilon)} U_j^{(i,m)}(x)\, dx_1 = 0,$$

where $\varkappa_j^{(i,m)}(\varepsilon)$ is the cross-section of the rod $G_j^{(i,m)}(\varepsilon)$. We will regard that $U_j^{(i,m)}$ vanishes outside of $G_j^{(i,m)}(\varepsilon)$.

Integrating the differential equation for V^ε in $G_j^{(i,m)}(\varepsilon)$ over the cross-section $\varkappa_j^{(i,m)}(\varepsilon)$, we get

$$\partial^2_{x_2 x_2} \varphi_j^{(i,m)}(x_2) + \lambda_n^\varepsilon\, \varphi_j^{(i,m)}(x_2) = 0. \quad (69)$$

Substituting the representation (68) in the Neumann conditions on the lower bases of the rods $G_\varepsilon^{(1)}$ and integrating over their cross-sections, we find that for each $j \in \{0,1,\ldots,N-1\}$

$$\partial_{x_2} \varphi_j^{(1,m)} \Big|_{x_2=-l_2-l_1} = 0, \quad m \in \{1,2\}; \quad (70)$$

similarly, we get

$$\varphi_j^{(0)}(-l_1) = \varphi_j^{(1,1)}(-l_1) = \varphi_j^{(1,2)}(-l_1). \quad (71)$$

Substituting (68) in the second conjugation conditions on $Q_\varepsilon^{(1)}$ and in the Neumann conditions on the other parts of the lower bases of the rods $G_\varepsilon^{(0)}$ and integrating over their cross-sections, we find that for each $j \in \{0,1,\ldots,N-1\}$

$$h_{1,1}\partial_{x_2}\varphi_j^{(0)}(-l_1) + \frac{\partial}{\partial x_2}\Big(\int\limits_{\varkappa_j^{(1,1)}(\varepsilon)} U_j^{(0)}(x)\, dx_1 \Big)\Big|_{x_2=-l_1} = h_{1,1}\partial_{x_2}\varphi_j^{(1,1)}(-l_1),$$

$$h_{1,2}\partial_{x_2}\varphi_j^{(0)}(-l_1) + \frac{\partial}{\partial x_2}\Big(\int\limits_{\varkappa_j^{(1,2)}(\varepsilon)} U_j^{(0)}(x)\, dx_1 \Big)\Big|_{x_2=-l_1} = h_{1,2}\partial_{x_2}\varphi_j^{(1,2)}(-l_1),$$

$$(h_0 - h_{1,1} - h_{1,2})\partial_{x_2}\varphi_j^{(0)}(-l_1) + \frac{\partial}{\partial x_2}\Big(\int\limits_{\varkappa_j^{(0)}(\varepsilon)\setminus(\varkappa_j^{(1,1)}(\varepsilon)\cup\varkappa_j^{(1,2)}(\varepsilon))} U_j^{(0)}(x)\, dx_1 \Big)\Big|_{x_2=-l_1} = 0.$$

Summing these equalities, we obtain

$$h_0\, \partial_{x_2}\varphi_j^{(0)}\Big|_{x_2=-l_1} = h_{1,1}\, \partial_{x_2}\varphi_j^{(1,1)}\Big|_{x_2=-l_1} + h_{1,2}\, \partial_{x_2}\varphi_j^{(1,2)}\Big|_{x_2=-l_1}. \quad (72)$$

Taking (62) into account, it follows from (69) and (70) that

$$\varphi_j^{(1,m)}(x_2) = \frac{B_j^{1,m}}{\cos\big(\sqrt{\lambda_n^\varepsilon}\, l_1\big)\cos\big(\sqrt{\lambda_n^\varepsilon}\, l_2\big)} \cos\Big(\sqrt{\lambda_n^\varepsilon}\,(x_2 + l_1 + l_2)\Big) \tag{73}$$

for $m \in \{1,2\}$; here $x_2 \in (-l_2 - l_1, -l_1)$. A general solution of (69) for $i = 0$ is given by the representation

$$\varphi_j^{(0)}(x_2) = B_j^{(0)} \cos\Big(\sqrt{\lambda_n^\varepsilon}\,(x_2 + l_1)\Big) + C_j^{(0)} \sin\Big(\sqrt{\lambda_n^\varepsilon}\,(x_2 + l_1)\Big), \quad x_2 \in (-l_1, 0). \tag{74}$$

Gradually substituting (73) and (74) in (71) and (72), we find

$$B_j^{1,m} = \cos\big(\sqrt{\lambda_n^\varepsilon}\, l_1\big)\, B_j^{(0)}, \quad m \in \{1,2\}, \quad C_j^{(0)} = -\frac{h_{1,1} + h_{1,2}}{h_0}\, \tan\big(\sqrt{\lambda_n^\varepsilon}\, l_2\big)\, B_j^{(0)}. \tag{75}$$

From the equality $V^\varepsilon(x_1, 0+) = V^\varepsilon(x_1, 0-)$ and (68) it follows that

$$B_j^{(0)} \cos\big(\sqrt{\lambda_n^\varepsilon}\, l_1\big)\Big(1 - \frac{h_{1,1} + h_{1,2}}{h_0}\, \tan\big(\sqrt{\lambda_n^\varepsilon}\, l_2\big)\tan\big(\sqrt{\lambda_n^\varepsilon}\, l_1\big)\Big) = \frac{1}{\varepsilon h_0} \int\limits_{\varkappa_j^{(0)}(\varepsilon)} V^\varepsilon(x_1, 0)\, dx_1. \tag{76}$$

Taking into account (62), (75) - (76), we deduce that for each $j \in \{0, 1, \ldots, N-1\}$ and $m \in \{1,2\}$

$$\|\varphi_j^{(i,m)}\|^2_{L^2(G_j^{(i,m)}(\varepsilon))} \le c(n) \int\limits_{\varkappa_j^{(0)}(\varepsilon)} \big(V^\varepsilon(x_1,0)\big)^2\, dx_1.$$

Using the Poincaré inequality, we get

$$\|U_j^{(i,m)}\|^2_{L^2(G_j^{(i,m)}(\varepsilon))} \le c_1 \varepsilon^2 \|\partial_{x_1} U_j^{(i,m)}\|^2_{L^2(G_j^{(i,m)}(\varepsilon))} = c_1 \varepsilon^2 \|\partial_{x_1} V^\varepsilon\|^2_{L^2(G_j^{(i,m)}(\varepsilon))}.$$

Therefore,

$$\begin{aligned}\|V^\varepsilon\|^2_{L^2(G_\varepsilon^{(0)} \cup G_\varepsilon^{(1)})} &\le c_2(n) \int\limits_{I_0} \big(V^\varepsilon(x_1,0)\big)^2\, dx_1 + 2c_1\varepsilon^2 \|\partial_{x_1} V^\varepsilon\|^2_{L^2(G_\varepsilon^{(0)} \cup G_\varepsilon^{(1)})} \\ &\le c_3(n)\Big(\tau \|\nabla V^\varepsilon\|_{L^2(\Omega_{0,\delta_0})} + \frac{2}{\tau}\|V^\varepsilon\|_{L^2(\Omega_{0,\delta_0})} + \varepsilon^2 \|\partial_{x_1} V^\varepsilon\|^2_{L^2(G_\varepsilon^{(0)} \cup G_\varepsilon^{(1)})}\Big) \quad (\tau > 0).\end{aligned} \tag{77}$$

Choosing τ and ε small enough, we obtain from (65), (66), (67) and (77)

$$\|\nabla V^\varepsilon\|_{L^2(\Omega_{\varepsilon,\delta_0})} \le c_4(n) \|u_n^\varepsilon\|_{H^1(\Omega_0)} \le c_5(n). \tag{78}$$

This inequality means that the eigenfunctions have no strong variation of values on the corresponding branches of neighbouring trees.

2. The construction of a multi-sheeted extension will be carried out in several steps. First, similarly as in [12, §4], we construct the extension $P_\varepsilon^{(0)} :\ H^1(G_\varepsilon^{(0)}) \mapsto H^1(D_0)$ such that

$$\|P_\varepsilon^{(0)} u_n^\varepsilon\|_{H^1(D_0)} \le c_5(n) \|u_n^\varepsilon\|_{H^1(G_\varepsilon^{(0)})} \quad \text{and} \quad P_\varepsilon^{(0)} u_n^\varepsilon\Big|_{x_2=0} = u_n^\varepsilon\Big|_{x_2=0}.$$

Now we fix the index $m \in \{1,2\}$. Using the "linear matching", we extent the eigenfunction u_n^ε from the class $G_\varepsilon^{(1,m)}$ into the rectangle $D_1^\varepsilon = (0,a) \times \big(-l_2 - l_1, -l_1 - \varepsilon\big)$, i.e.,

$$E_\varepsilon^{(m)} u_n^\varepsilon = \begin{cases} u_n^\varepsilon, & x \in G_\varepsilon^{(1,m)} \\ \alpha_j(x_2;\varepsilon) + \beta_j(x_2;\varepsilon)\cdot\big(x_1 - \varepsilon(j + b_{1,m} + \frac{h_{1,m}}{2})\big), & x \in \widehat{G}_j^{(1,m)}(\varepsilon), \ \ j \in \{-1, 0, 1, \ldots, N\}, \end{cases}$$

where

$$\alpha_j(x_2;\varepsilon) = u_n^\varepsilon\big(\varepsilon(j + b_{1,m} + \tfrac{h_{1,m}}{2}), x_2\big),$$

$$\beta_j(x_2;\varepsilon) = \frac{1}{1 - h_{1,m}}\Big(u_n^\varepsilon\big(\varepsilon(j + 1 + b_{1,m} - \tfrac{h_{1,m}}{2}), x_2\big) - \alpha_j(x_2;\varepsilon)\Big),$$

$$\widehat{G}_j^{(1,m)}(\varepsilon) = \big(\varepsilon(j + b_{1,m} + \tfrac{h_{1,m}}{2}), \varepsilon(j + 1 + b_{1,m} - \tfrac{h_{1,m}}{2})\big) \times \big(-l_2 - l_1, -l_1 - \varepsilon\big).$$

Recall that the problem (63) is invariant with respect to shifts by ε along the axis Ox_1; therefore, in the cases of extreme rods we take $j=-1$ or $j=N$, respectively. Direct calculations give (for more detail see [12, §4])

$$\|E_\varepsilon^{(m)}u_n^\varepsilon\|_{H^1(D_1^\varepsilon)} \le c_6(n)\|u_n^\varepsilon\|_{H^1(G_\varepsilon^{(1,m)})}.$$

Further extension of $E_\varepsilon^{(m)}u_n^\varepsilon$ to the rectangle D_1 is performed in the same way as for ε-perforated domains [18,19]. The theorem is proved. □

The next two conditions of this scheme are the conditions for the convergence of some quantities.

Lemma 1 (Condition $\mathbf{C_4}$). If for certain functions w^ε and v^ε from $\mathcal{H}_\varepsilon$ we have

$$\mathbf{P}_\varepsilon w^\varepsilon \to \mathbf{w} \ \text{ and } \ \mathbf{P}_\varepsilon v^\varepsilon \to \mathbf{v} \ \text{ weakly in } \ \mathbf{Z}_0 \ \text{ as } \ \varepsilon \to 0, \tag{79}$$

then

$$\lim_{\varepsilon\to 0}\big(w^\varepsilon, v^\varepsilon\big)_{\mathcal{V}_\varepsilon} = \big(\mathbf{w},\mathbf{v}\big)_{\mathbf{V}_0}.$$

Proof. By χ_S we denote the characteristic function of a set S in $\mathbb{R}^2$ (resp. $\mathbb{R}$). It easy to verify that for each $i\in\{0,1\}$ and $m\in\{1,2\}$ (recall that if $i=0$, then the index m is absent)

$$\chi_{G_\varepsilon^{(i,m)}} \rightharpoonup h_{i,m} \ \text{ weakly-star in } \ L^\infty(D_i) \quad \text{as } \ \varepsilon\to 0. \tag{80}$$

Since the space $\mathbf{Z}_0$ is compactly imbedded in $\mathbf{V}_0$, we have that $\mathbf{P}_\varepsilon w^\varepsilon \to \mathbf{w}$ and $\mathbf{P}_\varepsilon v^\varepsilon \to \mathbf{v}$ in $\mathbf{V}_0$ as $\varepsilon\to 0$. Therefore,

$$\begin{aligned}\big(w^\varepsilon, v^\varepsilon\big)_{\mathcal{V}_\varepsilon} &= (w^\varepsilon, v^\varepsilon)_{L^2(\Omega_0)} + \int_{D_0} \chi_{G_\varepsilon^{(0)}} P_\varepsilon^{(0)}\big(w^\varepsilon\big) P_\varepsilon^{(0)}\big(v^\varepsilon\big)\,dx + \sum_{m=1}^{2}\int_{D_1} \chi_{G_\varepsilon^{(1,m)}} P_\varepsilon^{(1,m)}\big(w^\varepsilon\big) P_\varepsilon^{(1,m)}\big(v^\varepsilon\big)\,dx \\ &\longrightarrow \ (w^+, v^+)_{L^2(\Omega_0)} + h_0(w^{(0)}, v^{(0)})_{L^2(D_0)} + \sum_{m=1}^{2} h_{1,m}(w^{(1,m)}, v^{(1,m)})_{L^2(D_1)} = \big(\mathbf{w},\mathbf{v}\big)_{\mathbf{V}_0} \quad \text{as } \ \varepsilon\to 0.\end{aligned}$$

□

Lemma 2 (Condition $\mathbf{C_5}$). Let $\{\lambda(\varepsilon)\}_{\varepsilon>0}$ be a sequence of eigenvalues of the problem (3) such that

$$\lim_{\varepsilon\to 0}\lambda(\varepsilon) = \Lambda \ \text{ and } \ \frac{1}{\Lambda} \notin \sigma_{ess}(\mathsf{A}_0); \tag{81}$$

$\{u^\varepsilon\}_{\varepsilon>0}$ is the corresponding sequence of eigenfunctions such that

$$\|u^\varepsilon\|_{\mathcal{V}_\varepsilon} = 1 \ \text{ and } \ \mathbf{P}_\varepsilon u^\varepsilon \to \mathbf{u} \ \text{ weakly in } \ \mathbf{Z}_0 \ \text{ as } \ \varepsilon\to 0. \tag{82}$$

Then Λ is an eigenvalue and $\mathbf{u} = \big(u^+, u^{(0)}, \big\{u^{(1,m)}\big\}_{m=1}^{2}\big)$ is the corresponding multi-sheeted eigenfunction of the homogenized problem.

Proof. First we note that from Lemma 1 it follows than $\|\mathbf{u}\|_{\mathbf{V}_0} = 1$.

The characteristic function $\chi_{G_\varepsilon^{(i,m)}}$ can be represented as follows

$$\chi_{G_\varepsilon^{(i,m)}}(x) = \chi_{h_{i,m}^{(i)}}\Big(\frac{x_1}{\varepsilon}\Big), \qquad x\in D_i,$$

where $\chi_{h_{i,m}^{(i)}}(\xi), \xi\in\mathbb{R}$, is 1-periodic function that equals 1 on the interval $\big(b_{i,m}^{(i)} - \frac{h_{i,m}^{(i)}}{2}, b_{i,m}^{(i)} + \frac{h_{i,m}^{(i)}}{2}\big)$ and vanishing on the rest of the segment $[0,1]$; here, if $i=0$, then $b_{0,m}^{(0)} = \frac{1}{2}$ and $h_{0,m}^{(0)} = h_0$, and if $i=1$, then $b_{1,m}^{(1)} = b_{1,m}$ and $h_{1,m}^{(1)} = h_{1,m}$; the same for other notation.

For each index i and m we define the function

$$Y_{i,m}^{(i)}(\xi) = \begin{cases} -\xi + b_{i,m}^{(i)}, & \xi\in\big[b_{i,m}^{(i)} - \frac{h_{i,m}^{(i)}}{2}, b_{i,m}^{(i)} + \frac{h_{i,m}^{(i)}}{2}\big] \\ 0, & \xi\in\big[0,1\big]\setminus\big[b_{i,m}^{(i)} - \frac{h_{i,m}^{(i)}}{2}, b_{i,m}^{(i)} + \frac{h_{i,m}^{(i)}}{2}\big], \end{cases}$$

and then periodically extend it to $\mathbb{R}$. Integrating by parts the integral

$$\varepsilon\int_{G_\varepsilon^{(i,m)}} Y_{i,m}^{(i)}\Big(\frac{x_1}{\varepsilon}\Big)\,\partial_{x_1}v(x)\,dx,$$

we get the identity

$$\varepsilon \frac{h^{(i)}_{i,m}}{2} \int\limits_{\Upsilon^{(i,m)}_\varepsilon} v\, dx_2 = \int\limits_{G^{(i,m)}_\varepsilon} v\, dx - \varepsilon \int\limits_{G^{(i,m)}_\varepsilon} Y^{(i)}_{i,m}\Big(\frac{x_1}{\varepsilon}\Big)\, \partial_{x_1} v\, dx \quad \forall\, v \in \mathcal{H}_\varepsilon. \tag{83}$$

Using this identity, we get

$$\varepsilon^{\alpha_i} k_{i,m} \int\limits_{\Upsilon^{(i,m)}_\varepsilon} u^\varepsilon\, \phi\, dx_2 = \varepsilon^{\alpha_i - 1} \frac{2k_{i,m}}{h^{(i)}_{i,m}} \Bigg(\int\limits_{D_i} \chi_{h^{(i)}_{i,m}}\Big(\frac{x_1}{\varepsilon}\Big) P^{(i,m)}_\varepsilon\big(u^\varepsilon\big) \phi\, dx - \varepsilon \int\limits_{G^{(i,m)}_\varepsilon} Y^{(i)}_{i,m}\Big(\frac{x_1}{\varepsilon}\Big)\, \partial_{x_1}\big(u^\varepsilon\, \phi\big)\, dx \Bigg) \quad \forall\, \phi \in C^1(\overline{D_i}),$$

whence, taking the boundedness of $Y^{(i)}_{i,m}$ and (82) into account, it follows that

$$\lim_{\varepsilon \to 0} \varepsilon^{\alpha_i} k_{i,m} \int\limits_{\Upsilon^{(i,m)}_\varepsilon} u^\varepsilon\, \phi\, dx_2 = \delta_{\alpha_i,1} 2k_{i,m} \int\limits_{D_i} u^{(i,m)}\, \phi\, dx. \tag{84}$$

Here $\delta_{\alpha_i,1}$ is the Kronecker delta.

Now take any function $\phi \in C^\infty_0(D_i)$ and consider the function

$$\psi(x) = \begin{cases} 0, & x \in \Omega_\varepsilon \setminus G^{(i,m)}_\varepsilon, \\ \varepsilon\, Y^{(i)}_{i,m}\big(\frac{x_1}{\varepsilon}\big)\, \phi(x), & x \in G^{(i,m)}_\varepsilon. \end{cases}$$

It is obvious that $\psi \in \mathcal{H}_\varepsilon$. Considering the corresponding integral identity (5) with the test-function ψ, we obtain

$$\int\limits_{G^{(i,m)}_\varepsilon} \partial_{x_1} u^\varepsilon\, \phi\, dx = \mathcal{O}(\varepsilon) \quad \text{as} \;\; \varepsilon \to 0. \tag{85}$$

Direct calculations show that

$$\int\limits_{G^{(i,m)}_\varepsilon} \partial_{x_2} u^\varepsilon\, \phi\, dx = \int\limits_{D_i} \chi_{h^{(i)}_{i,m}}\Big(\frac{x_1}{\varepsilon}\Big)\, \partial_{x_2}\Big(P^{(i,m)}_\varepsilon u^\varepsilon\Big)\, \phi\, dx = -\int\limits_{D_i} \chi_{h^{(i)}_{i,m}}\Big(\frac{x_1}{\varepsilon}\Big) P^{(i,m)}_\varepsilon\big(u^\varepsilon\big)\, \partial_{x_2}\phi\, dx$$

$$\longrightarrow \;\; -h^{(i)}_{i,m} \int\limits_{D_i} u^{(i,m)}\, \partial_{x_2}\phi\, dx = h^{(i)}_{i,m} \int\limits_{D_i} \partial_{x_2} u^{(i,m)}\, \phi\, dx \quad \text{as} \;\; \varepsilon \to 0.$$

Thus

$$\chi_{h^{(i)}_{i,m}}\Big(\frac{x_1}{\varepsilon}\Big) \partial_{x_2}\Big(P^{(i,m)}_\varepsilon u^\varepsilon\Big) \rightharpoonup h^{(i)}_{i,m}\, \partial_{x_2} u^{(i,m)} \;\; \text{weakly in } L^2(D_i) \;\; \text{as } \varepsilon \to 0. \tag{86}$$

Writing down the integral identity (5) with the test function $\mathrm{S}_\varepsilon \boldsymbol{\varphi}$, where $\boldsymbol{\varphi}$ is an arbitrary function from the space $\mathbf{Z}_0$ and $\mathrm{S}_\varepsilon : \mathbf{Z}_0 \mapsto \mathcal{H}_\varepsilon$ is the operator from the condition $\mathbf{C_2}$, and taking the limits (81) - (86) into account, we find

$$\big(\mathbf{u}, \boldsymbol{\varphi}\big)_{\mathbf{H}_0} = \Lambda\big(\mathbf{u}, \boldsymbol{\varphi}\big)_{\mathbf{V}_0} \quad \forall\, \boldsymbol{\varphi} \in \mathbf{Z}_0. \tag{87}$$

Since the subspace $\mathbf{Z}_0$ is dense in $\mathbf{H}_0$, the identity (87) means (see Definition 2) that Λ is an eigenvalue and $\mathbf{u}$ is the corresponding multi-sheeted eigenfunction of the homogenized problem. □

6.1 | The main results

Thus, all conditions $\mathbf{C_1}$ - $\mathbf{C_5}$ of the scheme from [16] are satisfied. Applying this scheme, we obtain the following theorems.

Theorem 3 (the Hausdorff convergence)**.** Only points of the spectrum of the homogenized problem are accumulation points for the spectrum of the problem (3) as $\varepsilon \to 0$.

The convergence of the eigenvalue λ^ε_n at a fixed index n as ε tends to zero is usually called *low -frequency convergence of the spectrum.*

Theorem 4 (low-frequency convergence of the spectrum)**.** For any $n \in \mathbb{N}$

$$\lambda^\varepsilon_n \to \mu^{(1)}_n \quad \text{as} \;\; \varepsilon \to 0,$$

where $\mu^{(1)}_n$ is the eigenvalue of the homogenized spectral problem from the first sequence (55).

There exists a subsequence of $\{\varepsilon\}$ (again denoted by $\{\varepsilon\}$) such that

$$\mathbf{P}_\varepsilon u_n^\varepsilon \to \mathbf{v}_n^{(1)} \quad \text{weakly in } \mathbf{Z}_0 \quad \text{as } \varepsilon \to 0,$$

$\{\mathbf{v}_n^{(1)}\}_{n\in\mathbb{N}}$ are the corresponding eigenfunctions of the homogenized problem such that

$$\left(\mathbf{v}_n^{(1)}, \mathbf{v}_m^{(1)}\right)_{\mathbf{V}_0} = \delta_{n,m}, \quad n, m \in \mathbb{N}.$$

Theorem 5. Let $\mu_n^{(1)} = \mu_{n+1}^{(1)} = \ldots = \mu_{n+r-1}^{(1)}$ be an r-multiple eigenvalue of the homogenized problem from the first sequence (55). Then for any positive number ϱ and sufficiently small ε, we have

$$|\lambda_n(\varepsilon) - \mu_n^{(1)}| \ \leq \ C(n,\varrho)\,\zeta(\varepsilon),$$

where

$$\zeta(\varepsilon) := \left(\varepsilon^{1-\varrho} + \sum_{i=0}^{1} \varepsilon^{\alpha_i - 1 + \delta_{\alpha_i,1}}\right).$$

For the approximation function $R_\varepsilon^{(n+j)}$ ($j \in \{0,1,\ldots,r-1\}$) constructed with the help of the eigenfunction $\mathbf{v}_{n+j}^{(1)}$ by the formulas (57)-(59) the inequality

$$\left\| R_\varepsilon^{(n+j)} - \sum_{s=0}^{r-1} \beta_{js}(\varepsilon) u_{n+k}(\varepsilon,\cdot) \right\|_{H^1(\Omega_\varepsilon)} \leq C_j(n,\varrho)\,\zeta(\varepsilon)$$

is satisfied for ε small enough, where $0 < c_1 < \sum_{k=0}^{r-1}(\beta_{js}(\varepsilon))^2 < c_2$.

It follows from Theorem 3 that there exist other converging sequences of eigenvalues $\lambda_{n(\varepsilon)}^\varepsilon \to \mu_n^{(k)}$ as $\varepsilon \to 0$, where $\mu_n^{(k)}$ is an eigenvalue of the homogenized problem from the kth sequence (56) (obviously, in this case $n(\varepsilon) \to +\infty$ as $\varepsilon \to 0$); these limits are usually called *high-frequency convergence of the spectrum.*

Theorem 6. Let $\mu_n^{(k)} = \mu_{n+1}^{(k)} = \ldots = \mu_{n+r-1}^{(k)}$ be an r-multiple eigenvalue of the homogenized problem from the kth sequence (56). Then

$$\forall \varrho > 0 \ \exists \varepsilon_{n,k} > 0, c > 0 \text{ such that } \forall \varepsilon \in (0, \varepsilon_{n,k}) \text{ in the interval}$$

$$I_n^{(k)}(\varepsilon) := \left(\mu_n^{(k)} - c\,\zeta(\varepsilon)\,,\ \mu_n^{(k)} + c\,\zeta(\varepsilon)\right)$$

there are exactly r eigenvalues of the problem (3).

For the approximation function $R_\varepsilon^{(n+j,k)}$ ($i \in \{0,1,\ldots,r-1\}$) constructed with the help of the eigenfunction $\mathbf{v}_{n+j}^{(k)}$ by the formulas (57)-(59) we have

$$\left\| \frac{R_\varepsilon^{(n+j,k)}}{\|R_\varepsilon^{(n+j,k)}\|_{\mathcal{H}_\varepsilon}} - \widetilde{U}_\varepsilon \right\|_{\mathcal{H}_\varepsilon} \leq c(n,k,\varrho)\,\zeta(\varepsilon), \qquad \|\widetilde{U}_\varepsilon\|_{\mathcal{H}_\varepsilon} = 1,$$

for ε small enough; here $\widetilde{U}_\varepsilon$ is a linear combination of eigenfunctions of the problem (3), which correspond to the eigenvalues from the interval $I_n^{(k)}(\varepsilon)$.

7 | CONCLUSION

1. In this paper the asymptotic analysis of the spectral problem (3) is presented. The asymptotic behavior of the spectrum is determined by the spectrum of the corresponding non-standard homogenized spectral problem consisting of relations (14), (20)-(23), (38)-(41). The spectrum of the homogenized problem has a complex structure, namely, there is a countable set of gaps in the spectrum (see Theorem 1). This structure of the spectrum is the main argument for the mathematical justification of the well-known "loss reduction" phenomenon in comb-like waveguides (for more detail see [12]). Moreover, the question of existence of spectral gaps has been actively investigated in last time since it is very important for the description of wave propagations in different mediums (see [3] for a lot of examples and references on this topic). The left accumulation points $\{p_k\}_{k\in\mathbb{N}}$ form the essential spectrum of the homogenized problem. In some cases, these points can stretch into segments. Based on my previous results (see the review on spectral problems in thick junctions [5, Chap. 1], this can happen when thin branches (rods) forming trees smoothly change their length. Thus, choosing appropriately lengths of the thin branches one can build a thick fractal junction so that the spectrum of a problem under consideration has the given number of gaps.

2. The differential equations (20)-(23) contain extra zero-order terms that catch the effect of the parameters $\{\alpha_i\}$ $(\alpha_i \geq 1)$. A natural question arises, what happens when $\alpha_i < 1$ for some $i \in \{0,1,2\}$; to be specific, we put $\alpha_0 < 1$. Then from the integral identity (5), (8) and (9) it follows that for a fixed index $n \in \mathbb{N}$

$$\varepsilon^{\alpha_0} k_0 \int\limits_{\Upsilon_\varepsilon^{(0)}} (u_n^\varepsilon)^2 \, dx_2 \leq C_1.$$

Now, with the help of (83), we get

$$\int\limits_{G_\varepsilon^{(0)}} (u_n^\varepsilon)^2 \, dx \leq C_2 \varepsilon^{\vartheta} \to 0 \quad \text{as} \ \ \varepsilon \to 0,$$

where $\vartheta := \min\{1, \ 1-\alpha_0\}$. So, we can state that the spectral problem (3) splits into three independent problems as $\varepsilon \to 0$, namely

$$\begin{cases} -\Delta_x \, v(x) = \mu \, v(x), & x \in \Omega_0 \\ \partial_{x_1}^p v|_{x_1=0} = \partial_{x_1}^p v|_{x_1=a}, & x_2 \in [0,\delta_0], \ \ p = 0,1 \\ v = 0, & \text{on } \Gamma_1 \cup I_0, \\ \partial_\nu v = 0, & \text{on } \partial\Omega_0 \setminus (\Gamma_0 \cup \Gamma_1 \cup I_0), \end{cases}$$

$$\begin{cases} -h_{1,1} \, \partial^2_{x_2x_2} v^{(1,1)} + 2\delta_{\alpha_1,1} k_{1,1} v^{(1,1)} = \mu \, v^{(1,1)} \quad \text{in } D_1, \\ -h_{2,m} \, \partial^2_{x_2x_2} v^{(2,m)} + 2\delta_{\alpha_2,1} k_{2,m} v^{(2,m)} = \mu \, h_{2,m} v^{(2,m)} \quad \text{in } D_2, \ \ m \in \{1,2\}, \\ v^{(1,1)} = 0 \quad \text{on } I_1, \\ v^{(1,1)} = v^{(2,1)} = v^{(2,2)}, \quad h_{1,1}\partial_{x_2} v^{(1,1)} = h_{2,1}\partial_{x_2} v^{(2,1)} + h_{2,2}\partial_{x_2} v^{(2,2)} \quad \text{on } I_2, \\ \partial_{x_2} v^{(2,m)}\Big|_{x_2=-(l_1+l_2+l_3)} = 0, \quad m \in \{1,2\}; \end{cases}$$

and

$$\begin{cases} -h_{1,2} \, \partial^2_{x_2x_2} v^{(1,2)} + 2\delta_{\alpha_1,1} k_{1,2} v^{(1,2)} = \mu \, v^{(1,2)} \quad \text{in } D_1, \\ -h_{2,m} \, \partial^2_{x_2x_2} v^{(2,m)} + 2\delta_{\alpha_2,1} k_{2,m} v^{(2,m)} = \mu \, h_{2,m} v^{(2,m)} \quad \text{in } D_2, \ \ m \in \{3,4\}, \\ v^{(1,2)} = 0 \quad \text{on } I_1, \\ v^{(1,2)} = v^{(2,3)} = v^{(2,4)}, \quad h_{1,2}\partial_{x_2} v^{(1,2)} = h_{2,3}\partial_{x_2} v^{(2,3)} + h_{2,4}\partial_{x_2} v^{(2,4)} \quad \text{on } I_2, \\ \partial_{x_2} v^{(2,m)}\Big|_{x_2=-(l_1+l_2+l_3)} = 0, \quad m \in \{3,4\}. \end{cases}$$

Thus, in this case, a new essential feature appears, which is different from the cases studied here and which cannot be studied by any simple modifications of the approaches of this article; the cases when $\alpha_i < 1$ for some $i \in \{0,1,2\}$ are postponed by the author to a planned forthcoming paper.

3. It should be noted that for each new proposed asymptotic method, it is very important to justify its accuracy. Thus, the proof of the residual error estimate for discrepancy between the constructed approximations and solutions is a general principle applied to the analysis of the efficiency of the proposed asymptotic method. With the help of special branch-layer solutions, whose properties were studied in [13], the method of matched asymptotic expansions [7] and the approach within the conceptual framework of [12,16,4,5], the approximation for the eigenfunctions are constructed and the corresponding asymptotic error estimates in the Sobolev space $H^1(\Omega_\varepsilon)$ are proved in Theorems 4 and 5. In addition, the Hausdorff convergence of the spectrum and the rate of this convergence (asymptotic estimates for eigenvalues) depending on the parameters ε and $\{\alpha_i\}$ are proved.

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