ON THE LOW MACH NUMBER LIMIT OF THE COMPRESSIBLE VISCOUS MICROPOLAR FLUID MODEL

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ABSTRACT. In this paper, we consider the low Mach number limit problem for the compressible viscous micropolar fluid model based on the concept of dissipative measure-valued solutions. We prove that the dissipative measure-valued solutions of the compressible micropolar fluid model converge to the smooth solution of the incompressible micropolar system in the case of well-prepared initial data when the Mach number tends to zero.

1. INTRODUCTION

The aim of this paper is to prove the low Mach number limit for a three-dimensional compressible viscous micropolar fluid model. We consider the (dimensionless) continuity, momentum and mirco-rotation equations for the density $\rho(t,x)$, the mean velocity $\mathbf{u}(t,x)$ and the microrotational velocity $\mathbf{w}(t,x)$ of a fluid in the three-dimensional smooth bounded domain $\Omega \subset \mathbb{R}^3$:

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 \ \text{ in } \Omega, \quad t > 0, \tag{1.1}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = (\mu+\xi)\Delta\mathbf{u} + (\mu+\lambda-\xi)\nabla\operatorname{div}\mathbf{u} + 2\xi\nabla\times\mathbf{w}, \tag{1.2}$$

$$(\rho\mathbf{w})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{w}) + 4\xi\mathbf{w} = \mu'\Delta\mathbf{w} + \left(\mu'+\lambda'\right)\nabla\operatorname{div}\mathbf{w} + 2\xi\nabla\times\mathbf{u}, \tag{1.3}$$

with the initial-boundary conditions

$$(\rho,\mathbf{u},\mathbf{w})|_{t=0}(t,x) = \left(\rho_0,\mathbf{u}_0,\mathbf{w}_0\right)(x) \text{ in } \Omega,$$
$$(\mathbf{u}^\epsilon,\mathbf{w}^\epsilon)|_{\partial\Omega}(t,x) = 0.$$

In the above equations, the coefficients of viscosity μ,λ and the coefficients of microviscosity μ',λ',ξ satisfy

$$\mu,\mu',\xi > 0, \quad 2\mu+3\lambda-4\xi \geq 0 \quad \text{and} \quad 2\mu'+3\lambda' \geq 0. \tag{1.4}$$

The pressure function $p(\rho)$ is given by the isentropic pressure-density state $p(\rho) = \frac{\rho^\gamma}{\gamma}$ with $\gamma > 1$, although more general cases can be treated as well.

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When the effect of the angular velocity field of the particle's rotation is neglected, i.e. $\mathbf{w} = 0$ and $\xi = 0$, the compressible micropolar fluid system (1.1)-(1.3) can be reduced to the classical Navier-Stokes equations. It is well-known that there are a lot of studies on the low Mach number limit of the compressible Navier-Stokes equations, for example, see [4–7, 19, 22, 28, 30] and references therein.

The theory of compressible micropolar fluids was introduced by Eringen [12, 13] for the first time to describe certain microscopic effects in some complex fluids. The micropolar fluid equations differ essentially from the Navier-Stokes equations through the sustenance of the couple stress and the non-symmetry of the stress tensor [16]. For the incompressible version ($\rho = constant$) of system (1.1)-(1.3), the existence and uniqueness of strong solutions have been studied in [24, 25, 34]. For the compressible case, we point out that there are a lot of research results on the existence of weak and strong solutions, the large time behavior, and asymptotic regime to compressible micropolar fluids [1, 10, 11, 17, 23, 27, 32, 33] and the references therein.

The main purpose of this paper is to rigorously prove the low Mach number limit in the framework of the measure-valued solutions to the compressible viscous micropolar system (1.1)-(1.3). To begin with, we introduce the following scalings for the unknowns

$$\rho(t,x) = \rho^\epsilon(\epsilon t, x), \quad \mathbf{u}(t,x) = \epsilon \mathbf{u}^\epsilon(t,x)(\epsilon t, x), \quad \mathbf{w}(t,x) = \epsilon \mathbf{w}^\epsilon(t,x)(\epsilon t, x) \tag{1.5}$$

and for the viscosity coefficients

$$\mu \longmapsto \epsilon\mu, \quad \lambda \longmapsto \epsilon\lambda, \quad \xi \longmapsto \epsilon\xi \quad \mu' \longmapsto \epsilon\mu', \quad \lambda' \longmapsto \epsilon\lambda', \tag{1.6}$$

where $\epsilon \in (0,1)$ is the scaled Mach number, which represents the norm of the velocity divided by the sound speed. With such scalings, the model (1.1)-(1.3) read as

$$\rho^\epsilon_t + \operatorname{div}(\rho^\epsilon \mathbf{u}^\epsilon) = 0, \tag{1.7}$$

$$(\rho^\epsilon \mathbf{u}^\epsilon)_t + \operatorname{div}(\rho^\epsilon \mathbf{u}^\epsilon \otimes \mathbf{u}^\epsilon) + \frac{\nabla(\rho^\epsilon)^\gamma}{\epsilon^2} = (\mu+\xi)\Delta \mathbf{u}^\epsilon + (\mu+\lambda-\xi)\nabla \operatorname{div} \mathbf{u}^\epsilon + 2\xi \nabla \times \mathbf{w}^\epsilon, \tag{1.8}$$

$$(\rho^\epsilon \mathbf{w}^\epsilon)_t + \operatorname{div}(\rho^\epsilon \mathbf{u}^\epsilon \otimes \mathbf{w}^\epsilon) + 4\xi \mathbf{w}^\epsilon = \mu' \Delta \mathbf{w}^\epsilon + \left(\mu' + \lambda'\right) \nabla \operatorname{div} \mathbf{w}^\epsilon + 2\xi \nabla \times \mathbf{u}^\epsilon. \tag{1.9}$$

The initial conditions and boundary conditions are prescribed by

$$(\rho^\epsilon, \mathbf{u}^\epsilon, \mathbf{w}^\epsilon)|_{t=0}(t,x) = \left(\rho^\epsilon_0, \mathbf{u}^\epsilon_0, \mathbf{w}^\epsilon_0\right)(x), \text{ for } x \in \Omega, \tag{1.10}$$

$$(\mathbf{u}^\epsilon, \mathbf{w}^\epsilon)|_{\partial\Omega}(t,x) = 0. \tag{1.11}$$

Here we use the superscript to emphasize the dependence of ϵ for each variables in (1.7)-(1.11).

In this paper, we shall consider the asymptotic limit of dissipative measure-valued solutions $(\rho^\epsilon, \mathbf{u}^\epsilon, \mathbf{w}^\epsilon)$ of the compressible viscous micropolar system (1.7)-(1.9) with the initial conditions and boundary conditions (1.10)-(1.11) for $\epsilon \to 0$, namely the low Mach number limit. This process represents a bridge between compressible and incompressible fluid flows. We expect that the limit system is the following incompressible micropolar fluid model

$$\operatorname{div} \mathbf{u}^0 = 0, \tag{1.12}$$

$$\partial_t \mathbf{u}^0 + (\mathbf{u}^0 \cdot \nabla)\mathbf{u}^0 + \nabla \pi^0 = (\mu + \xi)\Delta \mathbf{u}^0 + 2\xi \nabla \times \mathbf{w}^0, \tag{1.13}$$

$$\partial_t \mathbf{w}^0 + (\mathbf{u}^0 \cdot \nabla)\mathbf{w}^0 + 4\xi \mathbf{w}^0 = \mu' \Delta \mathbf{w}^0 + \left(\mu' + \lambda'\right) \nabla \operatorname{div} \mathbf{w}^0 + 2\xi \nabla \times \mathbf{u}^0, \tag{1.14}$$

if the limits $\mathbf{u}^\epsilon \to \mathbf{u}^0$ and $\mathbf{w}^\epsilon \to \mathbf{w}^0$ exist, where π^0 is the limit of $\frac{a(\rho^\epsilon)^\gamma}{\epsilon^2}$. The energy equality of incompressible micropolar system (1.12)-(1.14) is given by

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\left(|\mathbf{u}^0|^2 + |\mathbf{w}^0|^2\right) \mathrm{d}x + \mu \int_0^t \int_\Omega |\nabla \mathbf{u}^0|^2 \mathrm{d}x\mathrm{d}\tau + \mu' \int_0^t \int_\Omega |\nabla \mathbf{w}^0|^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad + (\mu' + \lambda') \int_0^t \int_\Omega |\operatorname{div} \mathbf{w}^0|^2 \mathrm{d}x\mathrm{d}\tau + 4\xi \int_0^t \int_\Omega |2\mathbf{w}^0 - \nabla \otimes \mathbf{u}^0|^2 \mathrm{d}x\mathrm{d}\tau \\
&= \int_\Omega \frac{1}{2}\left(|\mathbf{u}_0^0|^2 + |\mathbf{w}_0^0|^2\right) \mathrm{d}x.
\end{aligned} \tag{1.15}$$

Inspired by the works [15] on the low Mach number limit for the compressible Euler system and [2] on a singular limit for stratified compressible fluids, the main aim of this paper is to prove the low Mach number limit of the compressible micropolar model (1.7)-(1.9) based on the concept of dissipative measured-valued solutions. The concept of measure-valued solutions to partial differential equations was introduced by DiPerna [9] and recently developed by Feireisl, Gwiazda et al. [14, 18]. For more related topics, we refer to [3, 8, 21, 26, 29] and the references cited therein. Compared with [15] and [2], due to the adjunction of microrotational velocity, the proof of the low Mach number limit of the compressible viscous micropolar fluid model becomes more complex and difficult. Our analysis depends heavily on the special nonlinear structure of the compressible viscous micropolar fluid model and the refined estimation of relative energy.

Throughout this paper, we denote by χ the characteristics function and C the generic positive constants independent of ϵ.

The paper is organized as follows. In Sect. 2, we introduce the definition and the existence of dissipative measure-valued solutions to the compressible micropolar system (1.7)-(1.9), the existence of a smooth solution to the target system, and relative energy. In Sect. 3, we state our main theorem. Section 4 is devoted to the proof of the main theorem.

2. Preliminaries

We first state the precise definition of dissipative measure-valued solutions to the compressible micropolar system (1.7)-(1.9).

Let

$$\mathcal{Q} = \left\{ [s, \mathbf{v}, \omega] \mid s \geq 0, \mathbf{v} \in \mathbb{R}^3, \omega \in \mathbb{R}^3 \right\} \tag{2.1}$$

be the natural phase space. The symbol $\mathcal{P}(\mathcal{Q})$ denote the space of (Borel) probability measures, i.e., for $\nu \in \mathcal{P}(\mathcal{Q})$ we have $\nu(\mathcal{Q}) = 1$.

2.1. Dissipative measure-valued solutions to compressible micropolar system.

Definition 2.1. We say that a parameterized measure $\left\{ Y^{\epsilon}_{t,x} \right\}_{(t,x)\in[0,T]\times\Omega\}}$,

$$\begin{aligned} &Y^{\epsilon}_{t,x} \in L^{\infty}_{\text{weak- }(\star)}\left([0,T]\times\Omega; \mathcal{P}(\mathcal{Q})\right), \\ &\left\langle Y^{\epsilon}_{t,x}; s \right\rangle = \rho^{\epsilon}, \left\langle Y^{\epsilon}_{t,x}; \mathbf{v} \right\rangle = \mathbf{u}^{\epsilon}, \left\langle Y^{\epsilon}_{t,x}; \omega \right\rangle = \mathbf{w}^{\epsilon}, \end{aligned} \tag{2.2}$$

is a dissipative measure-valued solution of the compressible micropolar system (1.7)-(1.9) in $[0,T]\times\Omega$ with the initial conditions $Y^{\epsilon}_{0,x} \in \mathcal{P}(\mathcal{Q})$ and a non-negative function $\mathcal{D}^{\epsilon}(t) \in L^{\infty}(0,T)$ called dissipation defect of the total energy, if the following holds.

- The continuity equation:

$$\begin{aligned} &\int_0^t \int_{\Omega} \left[\left\langle Y^{\epsilon}_{\tau,x}; s \right\rangle \partial_t \phi + \left\langle Y^{\epsilon}_{\tau,x}; s\mathbf{v} \right\rangle \cdot \nabla\phi \right] \mathrm{d}x\mathrm{d}\tau \\ &\qquad = \int_{\Omega} \left\langle Y^{\epsilon}_{t,x}; s \right\rangle \phi(t,\cdot)\mathrm{d}x - \int_{\Omega} \left\langle Y^{\epsilon}_{0,x}; s \right\rangle \phi(0.\cdot)\mathrm{d}x, \end{aligned} \tag{2.3}$$

  for $t \in (0,T)$, and all $\phi \in C^1_c([0,T]\times\Omega)$.
- The momentum equation:

$$\begin{aligned} &\int_0^t \int_{\Omega} \left[\left\langle Y^{\epsilon}_{\tau,x}; s\mathbf{v} \right\rangle \partial_t \psi + \left\langle Y^{\epsilon}_{\tau,x}; s\mathbf{v}\otimes\mathbf{v} \right\rangle : \nabla\psi + \frac{1}{\epsilon^2} \left\langle Y^{\epsilon}_{\tau,x}; s^{\gamma} \right\rangle \operatorname{div}\psi \right] \mathrm{d}x\mathrm{d}\tau \\ &\quad - \int_0^t \int_{\Omega} [(\mu+\xi)\nabla\mathbf{u}^{\epsilon} : \nabla\psi + (\mu+\lambda-\xi)\operatorname{div}\mathbf{u}^{\epsilon}\operatorname{div}\psi]\mathrm{d}x\mathrm{d}\tau \\ &\quad + \int_0^t \int_{\Omega} 2\xi(\nabla\times\mathbf{w}^{\epsilon})\cdot\psi\mathrm{d}x\mathrm{d}\tau + \int_0^t \left\langle r^M_1; \nabla\psi \right\rangle \mathrm{d}\tau \\ &= \int_{\Omega} \left\langle Y^{\epsilon}_{t,x}; s\mathbf{v} \right\rangle \cdot \psi(t,\cdot)\mathrm{d}x - \int_{\Omega} \left\langle Y^{\epsilon}_{0,x}; s\mathbf{v} \right\rangle \cdot \psi(0,\cdot)\mathrm{d}x \end{aligned} \tag{2.4}$$

for $t \in [0,T]$ and all $\psi \in C_c^1([0,T) \times \Omega; \mathbb{R}^3)$, where $r_1^M \in L^\infty_{weak}([0,T]; \mathcal{M}(\Omega))$ is a tensor-valued measure satisfying

$$\left|\left\langle r_1^M(t); \nabla\psi \right\rangle\right| \leq C\mathcal{D}^\epsilon(t)\|\psi\|_{C^1(\Omega)}. \tag{2.5}$$

- The micro-rotation equation:

$$\begin{aligned}
&\int_0^t \int_\Omega \left[\left\langle Y^\epsilon_{\tau,x}; s\omega \right\rangle \cdot \eta_t + \left\langle Y^\epsilon_{\tau,x}; s\omega \otimes \mathbf{v} \right\rangle : \nabla\eta + 2\xi(\nabla \times \mathbf{u}^\epsilon)\cdot\eta\right] \mathrm{d}x\mathrm{d}\tau \\
&\quad - \int_0^t \int_\Omega \left[4\xi\omega^\epsilon \cdot \eta + \mu'\nabla \mathbf{w}^\epsilon : \nabla\eta + \left(\mu' + \lambda'\right) \operatorname{div} \mathbf{w}^\epsilon \operatorname{div} \eta\right] \mathrm{d}x\mathrm{d}\tau + \int_0^t \left\langle r_2^M; \nabla\eta \right\rangle \mathrm{d}\tau \\
&= \int_\Omega \left\langle Y^\epsilon_{t,x}; s\omega \right\rangle \cdot \eta(t,\cdot)\mathrm{d}x - \int_\Omega \left\langle Y^\epsilon_{0,x}; s\omega \right\rangle \cdot \eta(0,\cdot)\mathrm{d}x
\end{aligned} \tag{2.6}$$

for $t \in [0,T]$ and all $\eta \in C_c^1([0,T] \times \Omega; \mathbb{R}^3)$, where $r_2^M \in L^\infty_{weak}([0,T]; \mathcal{M}(\Omega))$ is a tensor-valued measure satisfying

$$\left|\left\langle r_2^M(t); \nabla\psi \right\rangle\right| \leq C\mathcal{D}^\epsilon(t)\|\eta\|_{C^1(\Omega)}. \tag{2.7}$$

- The energy balance:

$$\begin{aligned}
&\int_\Omega \left\langle Y^\epsilon_{t,x}; \frac{1}{2}s|\mathbf{v}|^2 + \frac{1}{2}s|\omega|^2 + \frac{1}{\gamma(\gamma-1)\epsilon^2}\left(s^\gamma - \gamma(s-1) - 1\right) \right\rangle \mathrm{d}x + \mathcal{D}^\epsilon(t) \\
&\quad + \int_0^t \int_\Omega \left[\mu|\nabla \mathbf{u}^\epsilon|^2 + (\mu+\lambda)|\operatorname{div} \mathbf{u}^\epsilon|^2\right] \mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_0^t \int_\Omega \left[\mu'|\nabla \mathbf{w}^\epsilon|^2 + \lambda'|\operatorname{div} \mathbf{w}^\epsilon|^2 + \xi|2\mathbf{w}^\epsilon - \nabla \times \mathbf{u}^\epsilon|^2\right] \mathrm{d}x\mathrm{d}\tau \\
&\leq \int_\Omega \left\langle Y^\epsilon_{0,x}; \frac{1}{2}s|\mathbf{v}|^2 + \frac{1}{2}s|\omega|^2 + \frac{1}{\gamma(\gamma-1)\epsilon^2}\left(s^\gamma - \gamma(s-1) - 1\right) \right\rangle \mathrm{d}x
\end{aligned} \tag{2.8}$$

for $t \in [0,T]$.

- The generalized Poincaré's inequality

$$\int_0^t \int_\Omega \left\langle Y^\epsilon_{t,x}; |\mathbf{v} - \mathbf{u}^\epsilon|^2 + |\omega - \mathbf{w}^\epsilon|^2 \right\rangle \mathrm{d}x\mathrm{d}\tau \leq C\mathcal{D}^\epsilon(t) \tag{2.9}$$

for $t \in [0,T]$.

Remark 2.2. The dissipation defect $\mathcal{D}^\epsilon(t)$ controls the oscillations and concentrations of the gradient of velocity and rotation-velocity.

Remark 2.3. The parameterized family $Y^\epsilon_{t,x}$ is the Young measure generated by an oscillating sequence of approximate solutions while the measure $Y^\epsilon_{0,x}$ is determined by the initial data $(\rho_0^\lambda(x), \mathbf{u}_0^\lambda(x))$ of the compressible viscous micropolar fluid model (1.7)-(1.9) with $Y^\epsilon_{0,x} = \delta_{\rho_0^\epsilon(x), \mathbf{u}_0^\epsilon(x), \mathbf{w}_0^\epsilon(x)}$.

In [20], Huang established the existence of dissipative measure-valued solutions to the compressible viscous micropolar fluid model (1.7)-(1.9) with the initial-boundary conditions (1.10)-(1.11).

Proposition 2.4 (See [20]). Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. If $\left(\rho_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{w}_0^\epsilon\right)$ is the initial data with finite energy, then there exists a dissipative measure-valued solution in the sense specified in Definition 2.1 with initial data $Y_{0,x}^\epsilon = \delta_{\rho_0^\epsilon, \mathbf{u}_0^\epsilon, \mathbf{w}_0^\epsilon}$ to the micropolar system (1.7)-(1.9).

2.2. Solutions of the target system (1.12)-(1.14).

Now we recall the local existence of strong solutions to the incompressible micropolar system (1.12)-(1.14) in the bounded domain Ω. The proof can be found in [16, 31].

Proposition 2.5. If the initial data $\mathbf{u}_0^0$ of system (1.12)-(1.14) satisfies

$$(\mathbf{u}_0^0, \mathbf{w}_0^0) \in H^k\left(\Omega; \mathbb{R}^3\right), \quad k > \frac{3}{2} + 2, \quad \operatorname{div} \mathbf{u}_0^0 = 0, \tag{2.10}$$

we know that (1.12)-(1.14) admits a solution $(\mathbf{u}^0, \mathbf{w}^0)$ unique in the class

$$(\mathbf{u}^0, \mathbf{w}^0) \in C\left(\left[0, T_{\max}\right); H^k\left(\Omega; \mathbb{R}^N\right)\right), \tag{2.11}$$

$$\partial_t \mathbf{u}^0, \partial_t \mathbf{w}^0, \partial_t \pi^0, \nabla \pi^0 \in C\left(\left[0, T_{\max}\right); H^{k-2}\left(\Omega; \mathbb{R}^N\right)\right) \tag{2.12}$$

for some $T_{\max} > 0$.

2.3. Relative energy functional.

For a parameterized family $Y_{t,x}^\epsilon$ of probability measure defined on the phase space (2.1), we introduce the following form of the relative energy functional

$$\mathcal{H}^\epsilon(t) = \int_\Omega \left\langle Y_{t,x}^\epsilon; \frac{1}{2} s \left|\mathbf{v} - \mathbf{u}^0\right|^2 + \frac{1}{2} s \left|\omega - \mathbf{w}^0\right|^2 + \frac{1}{\gamma(\gamma-1)\epsilon^2}\left(s^\gamma - \gamma(s-1) - 1\right)\right\rangle \mathrm{d}x, \tag{2.13}$$

where $(\mathbf{u}^0, \mathbf{w}^0)$ is the smooth solution of the incompressible micropolar system (1.12)-(1.14). Our approach depends essentially on the fact that the limit field $(\mathbf{u}^0, \mathbf{w}^0)$ is a smooth function.

3. Main result

Now we state the main result of our paper.

Theorem 3.1. *Let* $\left\{Y_{t,x}^\varepsilon\right\}_{(t,x)\in([0,T]\times\Omega)}$ *and* $\mathcal{D}^\epsilon(t)$ *be a family of dissipative measure-valued solutions of the compressible micropolar system* (1.7)-(1.9) *with the initial-boundary conditions*

(1.10)-(1.11) *obtained in Proposition [20] satisfying the compatibility condition* (2.5) *and* (2.7) *with ξ independent of λ. Assume that the initial data $Y^{\lambda}_{0,x}$ satisfy*

$$\int_\Omega \left\langle Y^{\epsilon}_{0,x}; \frac{1}{2}s\left|\mathbf{v}-\mathbf{u}^0\right|^2 + \frac{1}{2}s\left|\omega-\mathbf{w}^0\right|^2 + \frac{1}{\gamma(\gamma-1)\epsilon^2}\left(s^\gamma-\gamma(s-1)-1\right)\right\rangle \mathrm{d}x \to 0, \tag{3.1}$$

as $\epsilon \to 0$. Finally, suppose that $T < T_{max}$, where T_{max} denotes the maximal existence time of the solution to the incompressible micropolar system (1.12)-(1.14) *with the initial data* $(\mathbf{u}^0_0, \mathbf{w}^0_0)$. *Then as $\epsilon \to 0$, we have*

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega \left\langle Y^{\epsilon}_{t,x}; \frac{1}{2}s\left|\mathbf{v}-\mathbf{u}^0\right|^2 + \frac{1}{2}s\left|\omega-\mathbf{w}^0\right|^2 + \frac{1}{\gamma(\gamma-1)\epsilon^2}\left(s^\gamma-\gamma(s-1)-1\right)\right\rangle \mathrm{d}x \to 0$$

and

$$\mathcal{D}^\epsilon(t) \to 0,$$

where $(\mathbf{u}^0, \mathbf{w}^0)$ *is the solution of the incompressible micropolar system*(1.12)-(1.14) *with the initial data* $\mathbf{u}^0_0$.

Remark 3.2. Theorem 3.1 describes the low Mach number limit of the compressible micropolar system (1.7)-(1.9) with the initial-boundary conditions (1.10)-(1.11) with well-prepared initial data. For the general initial data, the fast singular oscillation appears. It is more difficult to prove the low Mach number limit in this situation, which will be studied in a forthcoming paper.

Remark 3.3. We remark that the estimate in Theorem 3.1 is uniform with respect to the coefficients of viscosity μ, λ and the coefficients of microviscosity μ', λ', ξ. Therefore, Theorem 3.1 is a stability result not only with respect to the scaling Mach number ϵ but also with respect to the coefficients of viscosity μ, λ and the coefficients of microviscosity μ', λ', ξ. In fact, we can show that the combined incompressible and vanishing viscosities limit of the compressible micropolar system (1.7)-(1.9) with the initial-boundary conditions (1.10)-(1.11) are the so-called ideal incompressible micropolar system

$$\operatorname{div}\mathbf{u}^0 = 0, \tag{3.2}$$

$$\partial_t\mathbf{u}^0 + (\mathbf{u}^0\cdot\nabla)\mathbf{u}^0 + \nabla\pi^0 = 0, \tag{3.3}$$

$$\partial_t\mathbf{w}^0 + (\mathbf{u}^0\cdot\nabla)\mathbf{w}^0 = 0. \tag{3.4}$$

4. The proof of Theorem 3.1

In this section, we give the proof of Theorem 3.1.

4.1. **Relative energy inequality.**

To derive the relative energy inequality, we use $\mathbf{u}^0$ and $\mathbf{w}^0$ —the smooth solution of the incompressible micropolar system (1.12)-(1.14) — as the test function in the momentum equation (1.8) and the micro-rotation equation (1.9) respectively, to yield the following equality

$$\begin{aligned}
-\int_\Omega \left\langle Y^\epsilon_{t,x}; s\mathbf{v}\cdot\mathbf{u}^0\right\rangle \mathrm{d}x = &-\int_\Omega \left\langle Y^\epsilon_{0,x}; s\mathbf{v}\cdot\mathbf{u}^0_0\right\rangle \mathrm{d}x - \int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s\mathbf{v}\right\rangle \partial_\tau \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
&-\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s\mathbf{v}\otimes\mathbf{v}\right\rangle : \nabla\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau + (\mu+\xi)\int_0^t\int_\Omega \nabla\mathbf{u}^\epsilon : \nabla\mathbf{u}^0\mathrm{d}x\mathrm{d}\tau \\
&-2\xi\int_0^t\int_\Omega (\nabla\times\mathbf{w}^\epsilon)\cdot\mathbf{u}^0\mathrm{d}x\mathrm{d}\tau - \int_0^t \left\langle r^M_1; \nabla\mathbf{u}^0\right\rangle \mathrm{d}\tau
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
-\int_\Omega \left\langle Y^\epsilon_{t,x}; s\omega\cdot\mathbf{w}^0\right\rangle \mathrm{d}x = &-\int_\Omega \left\langle Y^\epsilon_{0,x}; s\omega\cdot\mathbf{w}^0_0\right\rangle \mathrm{d}x - \int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s\omega\right\rangle \partial_\tau \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau \\
&-\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s\omega\otimes\upsilon\right\rangle : \nabla\mathbf{w}^0 \mathrm{d}x\mathrm{d} + 4\xi\int_0^t\int_\Omega \mathbf{w}^\epsilon\cdot\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau \\
&-2\xi\int_0^t\int_\Omega (\nabla\times\mathbf{u}^\epsilon)\cdot\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau + \mu'\int_0^t\int_\Omega \nabla\mathbf{w}^\epsilon : \nabla\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau \\
&+(\mu'+\lambda')\int_0^t\int_\Omega \operatorname{div}\mathbf{w}^\epsilon \operatorname{div}\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau - \int_0^t \left\langle r^M_2; \nabla\mathbf{u}^0\right\rangle \mathrm{d}\tau.
\end{aligned} \tag{4.2}$$

We use $|\mathbf{u}^0|^2$ and $|\mathbf{w}^0|^2$ as the test function in the continuity equation (1.7) respectively, to yield the following equalities:

$$\begin{aligned}
\int_\Omega \left\langle Y^\epsilon_{t,x}; \frac{s}{2}|\mathbf{u}^0|^2\right\rangle \mathrm{d}x = &\int_\Omega \left\langle Y^\epsilon_{0,x}; \frac{s}{2}|\mathbf{u}^0_0|^2\right\rangle \mathrm{d}x + \int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s\mathbf{u}^0\right\rangle \partial_t \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
&+\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s\mathbf{u}^0\otimes\mathbf{v}\right\rangle : \nabla\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
\int_\Omega \left\langle Y^\epsilon_{t,x}; \frac{s}{2}|\mathbf{w}^0|^2\right\rangle \mathrm{d}x = &\int_\Omega \left\langle Y^\epsilon_{0,x}; \frac{s}{2}|\mathbf{w}^0_0|^2\right\rangle \mathrm{d}x + \int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s\mathbf{w}^0\right\rangle \partial_t \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau \\
&+\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s\mathbf{w}^0\otimes\mathbf{v}\right\rangle : \nabla\mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{4.4}$$

In view of (1.15), (2.13) and (4.1)-(4.4), we have

$$\mathcal{H}^\epsilon(t) + \mathcal{D}^\epsilon(t) + \int_0^t\int_\Omega \left[\mu|\nabla\mathbf{u}^\epsilon|^2 + (\mu+\lambda)|\operatorname{div}\mathbf{u}^\epsilon|^2\right] \mathrm{d}x\mathrm{d}\tau$$

$$
\begin{aligned}
&+\int_0^t\int_\Omega\left[\mu'|\nabla\mathbf{w}^\epsilon|^2+\lambda'|\operatorname{div}\mathbf{w}^\epsilon|^2+\xi|2\mathbf{w}^\epsilon-\nabla\times\mathbf{u}^\epsilon|^2\right]\mathrm{d}x\mathrm{d}\tau\\
=&\int_\Omega\left\langle Y^\epsilon_{t,x};\frac{1}{2}s\,|\mathbf{v}|^2+\frac{1}{2}s\,|\omega|^2+\frac{1}{\gamma(\gamma-1)\epsilon^2}\left(s^\gamma-\gamma(s-1)-1\right)\right\rangle\mathrm{d}x\\
&+\int_0^t\int_\Omega\left[\mu|\nabla\mathbf{u}^\epsilon|^2+(\mu+\lambda)|\operatorname{div}\mathbf{u}^\epsilon|^2\right]\mathrm{d}x\mathrm{d}\tau\\
&+\int_0^t\int_\Omega\left[\mu'|\nabla\mathbf{w}^\epsilon|^2+\lambda'|\operatorname{div}\mathbf{w}^\epsilon|^2+\xi|2\mathbf{w}^\epsilon-\nabla\times\mathbf{u}^\epsilon|^2\right]\mathrm{d}x\mathrm{d}\tau\\
&-\int_\Omega\left\langle Y^\epsilon_{t,x};s\mathbf{v}\cdot\mathbf{u}^0\right\rangle\mathrm{d}x-\int_\Omega\left\langle Y^\epsilon_{t,x};s\omega\cdot\mathbf{w}^0\right\rangle\mathrm{d}x+\int_\Omega\left\langle Y^\epsilon_{t,x};\frac{1}{2}s\left|\mathbf{u}^0\right|^2+\frac{1}{2}s\left|\mathbf{w}^0\right|^2\right\rangle\mathrm{d}x\\
\le&\mathcal{H}^\epsilon(0)-\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x};s(\mathbf{v}-\mathbf{u}^0)\right\rangle\partial_\tau\mathbf{u}^0\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x};s(\mathbf{v}-\mathbf{u}^0)\otimes\mathbf{v}\right\rangle:\nabla\mathbf{u}^0\mathrm{d}x\mathrm{d}\tau\\
&-\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x};s(\omega-\mathbf{w}^0)\right\rangle\partial_\tau\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x};s(\omega-\mathbf{w}^0)\otimes\mathbf{v}\right\rangle:\nabla\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau\\
&+(\mu+\xi)\int_0^t\int_\Omega\nabla\mathbf{u}^\epsilon:\nabla\mathbf{u}^0\mathrm{d}x\mathrm{d}\tau-2\xi\int_0^t\int_\Omega(\nabla\times\mathbf{w}^\epsilon)\cdot\mathbf{u}^0\mathrm{d}x\mathrm{d}\tau\\
&+4\xi\int_0^t\int_\Omega\mathbf{w}^\epsilon\cdot\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau-2\xi\int_0^t\int_\Omega(\nabla\times\mathbf{u}^\epsilon)\cdot\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau+\mu'\int_0^t\int_\Omega\nabla\mathbf{w}^\epsilon:\nabla\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau\\
&+(\mu'+\lambda')\int_0^t\int_\Omega\operatorname{div}\mathbf{w}^\epsilon\operatorname{div}\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau-\int_0^t\left\langle r_1^M;\nabla\mathbf{u}^0\right\rangle\mathrm{d}\tau-\int_0^t\left\langle r_2^M;\nabla\mathbf{u}^0\right\rangle\mathrm{d}\tau.
\end{aligned}\tag{4.5}
$$

Inserting (1.13), (1.14) into (4.5) and using the following vector analysis formula

$$\operatorname{div}(\mathbf{a}\times\mathbf{b})=\nabla\times\mathbf{a}\cdot\mathbf{b}-\nabla\times\mathbf{b}\cdot\mathbf{a}$$

and the fact

$$\int_\Omega|\nabla\mathbf{u}^0|^2\mathrm{d}x=\int_\Omega|\nabla\times\mathbf{u}^0|^2\mathrm{d}x$$

due to $\operatorname{div}\mathbf{u}^0=0$, we get

$$
\begin{aligned}
&\mathcal{H}^\epsilon(t)+\mathcal{D}^\epsilon(t)+\mu\int_0^t\int_\Omega|\nabla\mathbf{u}^\epsilon-\nabla\mathbf{u}^0|^2\mathrm{d}x\mathrm{d}\tau+(\mu+\lambda)\int_0^t\int_\Omega|\operatorname{div}\mathbf{u}^\epsilon|^2\mathrm{d}x\mathrm{d}\tau\\
&+\mu'\int_0^t\int_\Omega|\nabla\mathbf{w}^\epsilon-\nabla\mathbf{w}^0|^2\mathrm{d}x\mathrm{d}\tau+(\mu'+\lambda')\int_0^t\int_\Omega|\operatorname{div}\mathbf{w}^\epsilon-\operatorname{div}\mathbf{w}^\epsilon|^2\mathrm{d}x\mathrm{d}\tau\\
&+\xi\int_0^t\int_\Omega|2(\mathbf{w}^\epsilon-\mathbf{w}^0)-\nabla\times(\mathbf{u}^\epsilon-\mathbf{u}^0)|^2\mathrm{d}x\mathrm{d}\tau\\
\le&\mathcal{H}^\epsilon(0)+\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x};s(\mathbf{v}-\mathbf{u}^0)\right\rangle\cdot(\mathbf{u}^0\cdot\nabla)\mathbf{u}^0\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x};s(\mathbf{v}-\mathbf{u}^0)\otimes\mathbf{v}\right\rangle:\nabla\mathbf{u}^0\mathrm{d}x\mathrm{d}\tau\\
&+\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x};s(\mathbf{v}-\mathbf{u}^0)\right\rangle\cdot\nabla\pi^0\mathrm{d}x\mathrm{d}\tau-\mu\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x};s(\mathbf{v}-\mathbf{u}^0)\right\rangle\cdot\Delta\mathbf{u}^0\mathrm{d}x\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&- \mu \int_0^t \int_\Omega (\nabla \mathbf{u}^\epsilon - \nabla \mathbf{u}^0) : \nabla \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau - \xi \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v} - \mathbf{u}^0) \right\rangle \cdot \Delta \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
&- \xi \int_0^t \int_\Omega (\nabla \mathbf{u}^\epsilon - \nabla \mathbf{u}^0) : \nabla \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau - 2\xi \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v} - \mathbf{u}^0) \right\rangle \cdot \nabla \times \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau \\
&+ 2\xi \int_0^t \int_\Omega (\mathbf{u}^\epsilon - \mathbf{u}^0) \cdot \nabla \times \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau + 2\xi \int_0^t \int_\Omega (\mathbf{w}^\epsilon - \mathbf{w}^0) \cdot \nabla \times \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
&- 2\xi \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \right\rangle \cdot \nabla \times \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
&+ \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \right\rangle (\mathbf{u}^0 \cdot \nabla) \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau - \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \otimes \mathbf{v} \right\rangle : \nabla \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau \\
&+ 4\xi \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \right\rangle \cdot \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau - 4\xi \int_0^t \int_\Omega (\mathbf{w}^\epsilon - \mathbf{w}^0) \cdot \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau \\
&- \mu' \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \right\rangle \cdot \Delta \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau - \mu' \int_0^t \int_\Omega (\nabla \mathbf{w}^\epsilon - \nabla \mathbf{w}^0) : \nabla \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau \\
&- (\mu' + \lambda') \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \right\rangle \cdot \nabla \operatorname{div} \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau \\
&- (\mu' + \lambda') \int_0^t \int_\Omega (\operatorname{div} \mathbf{w}^\epsilon - \operatorname{div} \mathbf{w}^0) \operatorname{div} \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau \\
&- \int_0^t \left\langle r_1^M; \nabla \mathbf{u}^0 \right\rangle \mathrm{d}\tau - \int_0^t \left\langle r_2^M; \nabla \mathbf{u}^0 \right\rangle \mathrm{d}\tau \\
=& \mathcal{H}^\epsilon(0) + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6 + \mathcal{I}_7 + \mathcal{I}_8 + \mathcal{I}_9 + \mathcal{I}_{10},
\end{aligned} \tag{4.6}
$$

where

$$
\begin{aligned}
\mathcal{I}_1 &= \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v} - \mathbf{u}^0) \right\rangle \cdot (\mathbf{u}^0 \cdot \nabla) \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau - \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v} - \mathbf{u}^0) \otimes \mathbf{v} \right\rangle : \nabla \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
\mathcal{I}_2 &= \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v} - \mathbf{u}^0) \right\rangle \cdot \nabla \pi^0 \mathrm{d}x\mathrm{d}\tau, \\
\mathcal{I}_3 &= -(\mu + \xi) \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v} - \mathbf{u}^0) \right\rangle \cdot \Delta \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau - (\mu + \xi) \int_0^t \int_\Omega (\nabla \mathbf{u}^\epsilon - \nabla \mathbf{u}^0) : \nabla \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau, \\
\mathcal{I}_4 &= -2\xi \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v} - \mathbf{u}^0) \right\rangle \cdot \nabla \times \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau + 2\xi \int_0^t \int_\Omega (\mathbf{u}^\epsilon - \mathbf{u}^0) \cdot \nabla \times \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau, \\
\mathcal{I}_5 &= -2\xi \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \right\rangle \cdot \nabla \times \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau + 2\xi \int_0^t \int_\Omega (\mathbf{w}^\epsilon - \mathbf{w}^0) \cdot \nabla \times \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau, \\
\mathcal{I}_6 &= 4\xi \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \right\rangle \cdot \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau - 4\xi \int_0^t \int_\Omega (\mathbf{w}^\epsilon - \mathbf{w}^0) \cdot \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau,
\end{aligned}
$$

$$\mathcal{I}_7 = -\mu' \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \right\rangle \cdot \Delta \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau - \mu' \int_0^t \int_\Omega (\nabla \mathbf{w}^\epsilon - \nabla \mathbf{w}^0) : \nabla \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau,$$

$$\mathcal{I}_8 = -(\mu' + \lambda') \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \right\rangle \cdot \nabla \operatorname{div} \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau$$
$$- (\mu' + \lambda') \int_0^t \int_\Omega (\operatorname{div} \mathbf{w}^\epsilon - \operatorname{div} \mathbf{w}^0) \operatorname{div} \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau,$$

$$\mathcal{I}_9 = \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \right\rangle (\mathbf{u}^0 \cdot \nabla)\mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau - \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega - \mathbf{w}^0) \otimes \mathbf{v} \right\rangle : \nabla \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau,$$

$$\mathcal{I}_{10} = -\int_0^t \left\langle r_1^M; \nabla \mathbf{u}^0 \right\rangle \mathrm{d}\tau - \int_0^t \left\langle r_2^M; \nabla \mathbf{u}^0 \right\rangle \mathrm{d}\tau.$$

4.2. **The estimates of $\mathcal{I}_k (k = 1, 2, \cdots, 10)$.**

In order to obtain the estimates of $\mathcal{I}_k (k = 1, 2, \cdots, 10)$, we need to the following Lemma.

Lemma 4.1. *Let $\left\{ Y^\epsilon_{t,x} \right\}_{(t,x)\in([0,T]\times\Omega)}$ be a family of dissipative measure-valued solutions of the compressible microploar system (1.7)-(1.9) obtained in Theorem 3.1 on $[0, T]$ with $T < T_{\max}$. Then there exists a constant $C > 0$ such that for all $\epsilon \in (0, 1)$ and $\gamma > 1$,*

$$\int_\Omega \left\langle Y^\epsilon_{t,x}; |s-1|^\alpha \right\rangle \mathrm{d}x \le C\epsilon^\alpha, \quad a.e.\ t \in [0, T], \tag{4.7}$$

where, $\alpha = \min\{2, \gamma\}$. Moreover, we have

$$\int_\Omega \left\langle Y^\epsilon_{t,x}; \left| \sqrt{s} - 1 \right|^2 \right\rangle \mathrm{d}x \le C\epsilon^2. \quad a.e.\ t \in [0, T] \tag{4.8}$$

Proof. Noting that the measure-valued solutions satisfy the energy inequality (2.8), we deduce from (3.1) that

$$\int_\Omega \left\langle Y^\epsilon_{t,x}; \frac{1}{\gamma(\gamma-1)\epsilon^2} (s^\gamma - \gamma(s-1) - 1) \right\rangle \mathrm{d}x \le C. \tag{4.9}$$

(1) If $\gamma \ge 2$, we know that

$$|s-1|^\gamma \le s^\gamma - \gamma(s-1) - 1$$

for all $s \ge 0$. Then, we have

$$\int_\Omega \left\langle Y^\epsilon_{t,x}; |s-1|^\gamma \right\rangle \mathrm{d}x \le \int_\Omega \left\langle Y^\epsilon_{t,x}; \frac{1}{\gamma(\gamma-1)} (s^\gamma - \gamma(s-1) - 1) \right\rangle \mathrm{d}x \le C\epsilon^2. \tag{4.10}$$

(2) If $1 < \gamma < 2$, by [22], we claim that

$$|s-1|^2 \le M\left(s^\gamma - \gamma(s-1) - 1\right), \quad \text{when } s \le a,$$

$$|s-1|^\gamma \le M\left(s^\gamma - \gamma(s-1) - 1\right), \quad \text{when } s > a,$$

for some positive constants M and a. Hence, using Hölder's inequality and the following Jensen's inequality

$$\left\langle Y_{t,x}^{\lambda}; |\mathbf{U}| \right\rangle^{q} \leq \left\langle Y_{t,x}^{\lambda}; |\mathbf{U}|^{q} \right\rangle \text{ with } q \geq 1, \tag{4.11}$$

we have

$$\begin{aligned}
\int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; |s-1|^{\gamma} \right\rangle \mathrm{d}x \leq & C \left(\int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; |s-1|^{\gamma} \right\rangle^{\frac{2}{\gamma}} \chi_{(s\leq a)} \mathrm{d}x \right)^{\frac{\gamma}{2}} + \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; |s-1|^{\gamma} \right\rangle \chi_{(s>a)} \mathrm{d}x \\
\leq & C \left(\int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; |s-1|^{2} \right\rangle \chi_{(s\leq a)} \mathrm{d}x \right)^{\frac{\gamma}{2}} + \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; |s-1|^{\gamma} \right\rangle \chi_{(s>a)} \mathrm{d}x \\
\leq & C \left(\int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; \frac{1}{\gamma(\gamma-1)} \left(s^{\gamma} - \gamma(s-1) - 1\right) \right\rangle \chi_{(s\leq a)} \mathrm{d}x \right)^{\frac{\gamma}{2}} \\
& + C \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; \frac{1}{\gamma(\gamma-1)} \left(s^{\gamma} - \gamma(s-1) - 1\right) \right\rangle \chi_{(s>a)} \mathrm{d}x \\
\leq & C \left(\varepsilon^{\gamma} + \varepsilon^{2} \right) \\
\leq & C \epsilon^{\gamma}.
\end{aligned} \tag{4.12}$$

From (4.10) and (4.12), one obtains the estimate (4.7). Using the following two elementary inequalities

$$|\sqrt{x}-1|^{2} \leq M|x-1|^{\gamma}, |x-1| \geq b, \gamma \geq 1,$$
$$|\sqrt{x}-1|^{2} \leq M|x-1|^{2}, x \geq 0$$

for some positive constant M and $0 < b < 1$, we obtain

$$\begin{aligned}
& \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; \left|\sqrt{s}-1\right|^{2} \right\rangle \mathrm{d}x \\
= & \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; \left|\sqrt{s}-1\right|^{2} \right\rangle \chi_{(|x-1|<b)} \mathrm{d}x + \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; \left|\sqrt{s}-1\right|^{2} \right\rangle \chi_{(|x-1|\geq b)} \mathrm{d}x \\
\leq & C \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; |s-1|^{2} \right\rangle \chi_{(|x-1|<b)} \mathrm{d}x + C \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; |s-1|^{\gamma} \right\rangle \chi_{(|x-1|\geq b)} \mathrm{d}x \\
\leq & C \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; |s-1|^{2} \right\rangle \chi_{(|x-1|<b)} \mathrm{d}x + C \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; |s-1|^{\gamma} \right\rangle \chi_{(|x-1|\geq b)} \mathrm{d}x \\
\leq & C \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; \frac{1}{\gamma(\gamma-1)} \left(s^{\gamma} - \gamma(s-1) - 1\right) \right\rangle \chi_{(|x-1|<b)} \mathrm{d}x \\
& + C \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; \frac{1}{\gamma(\gamma-1)} \left(s^{\gamma} - \gamma(s-1) - 1\right) \right\rangle \chi_{(|x-1|\geq b)} \mathrm{d}x \\
\leq & C \int_{\Omega} \left\langle Y_{t,x}^{\epsilon}; \frac{1}{\gamma(\gamma-1)} \left(s^{\gamma} - \gamma(s-1) - 1\right) \right\rangle \mathrm{d}x
\end{aligned}$$

$$\leq C\epsilon^2.$$

The proof of Lemma 4.1 is completed. □

Now we begin to estimate $\mathcal{I}_k(1 \leq k \leq 11)$. For the term $\mathcal{I}_1$, we have

$$\begin{aligned}
\mathcal{I}_1 =& \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v}-\mathbf{u}^0)\right\rangle \cdot (\mathbf{u}^0 \cdot \nabla)\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau - \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v}-\mathbf{u}^0)\otimes \mathbf{v}\right\rangle : \nabla \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
=& \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v}-\mathbf{u}^0)\otimes \mathbf{u}^0\right\rangle : \nabla \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau - \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v}-\mathbf{u}^0)\otimes \mathbf{v}\right\rangle : \nabla \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
=& - \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v}-\mathbf{u}^0)\otimes (\mathbf{v}-\mathbf{u}^0)\right\rangle : \nabla \mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
\leq& C\int_0^t \mathcal{H}^\epsilon(\tau)\mathrm{d}\tau.
\end{aligned} \tag{4.13}$$

Using the Jensen's inequality (4.11) and Lemma 4.1, the term $\mathcal{I}_2$ can be estimated

$$\begin{aligned}
\mathcal{I}_2 =& \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v}-\mathbf{u}^0)\right\rangle \cdot \nabla \pi^0 \mathrm{d}x\mathrm{d}\tau \\
=& \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s\mathbf{v}\right\rangle \cdot \nabla \pi^0 \mathrm{d}x\mathrm{d}\tau - \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s\mathbf{u}^0\right\rangle \cdot \nabla \pi^0 \mathrm{d}x\mathrm{d}\tau \\
=& \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; s-1\right\rangle \partial_t \pi^0 \mathrm{d}x\mathrm{d}\tau + \left[\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s-1\right\rangle \pi^0 \mathrm{d}x\right]_{t=0}^{\tau=t} \\
&- \int_0^t \int_\Omega \left\langle Y^\epsilon_{\tau,x}; (s-1)\mathbf{u}^0\right\rangle \cdot \nabla \pi^0 \mathrm{d}x\mathrm{d}\tau \\
\leq& \int_0^t \left[\left(\int_\Omega \left\langle Y^\epsilon_{\tau,x}; |s-1|\right\rangle^\alpha \mathrm{d}x\right)^{\frac{1}{\alpha}} \left(\int_\Omega |\partial_\tau \pi^0|^{\frac{\alpha}{\alpha-1}} \mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} \mathrm{d}x\right] \mathrm{d}\tau \\
&+ \left[\left(\int_\Omega \left\langle Y^\epsilon_{\tau,x}; |s-1|\right\rangle^\alpha \mathrm{d}x\right)^{\frac{1}{\alpha}} \left(\int_\Omega |\pi^0|^{\frac{\alpha}{\alpha-1}} \mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} \mathrm{d}x\right]_{t=0}^{\tau=t} \\
&+ \int_0^t \left[\left(\int_\Omega \left\langle Y^\epsilon_{\tau,x}; |s-1|\right\rangle^\alpha \mathrm{d}x\right)^{\frac{1}{\alpha}} \left(\int_\Omega |\mathbf{u}^0 \cdot \nabla \pi^0|^{\frac{\alpha}{\alpha-1}} \mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} \mathrm{d}x\right] \mathrm{d}\tau \\
\leq& C\int_0^t \left[\left(\int_\Omega \left\langle Y^\epsilon_{\tau,x}; |s-1|^\alpha\right\rangle \mathrm{d}x\right)^{\frac{1}{\alpha}} \mathrm{d}x\right] \mathrm{d}\tau \\
\leq& C\epsilon,
\end{aligned} \tag{4.14}$$

where we have used the incompressibility condition $\operatorname{div} \mathbf{u}^0 = 0$.

For the term $\mathcal{I}_3$, by integration by parts and using Lemma 4.1, we obtain

$$
\begin{aligned}
\mathcal{I}_3 =& -(\mu+\xi)\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v}-\mathbf{u}^0)\right\rangle \cdot \Delta\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau - \mu\int_0^t\int_\Omega (\nabla\mathbf{u}^\epsilon - \nabla\mathbf{u}^0) : \nabla\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
=& -(\mu+\xi)\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\mathbf{v}-\mathbf{u}^0)\right\rangle \cdot \Delta\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau + \mu\int_0^t\int_\Omega (\mathbf{u}^\epsilon - \mathbf{u}^0) : \Delta\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
=& -(\mu+\xi)\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; (s-1)(\mathbf{v}-\mathbf{u}^0)\right\rangle \cdot \Delta\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
=& -(\mu+\xi)\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; (\sqrt{s}-1)\cdot\sqrt{s}(\mathbf{v}-\mathbf{u}^0)\right\rangle \cdot \Delta\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
& -(\mu+\xi)\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; (\sqrt{s}-1)(\mathbf{v}-\mathbf{u}^0)\right\rangle \cdot \Delta\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
\le& C(\mu+\xi)\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; |\sqrt{s}-1|^2\right\rangle \mathrm{d}x\mathrm{d}\tau + C(\mu+\xi)\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s|\mathbf{v}-\mathbf{u}^0|^2\right\rangle \mathrm{d}x\mathrm{d}\tau \\
& + C(\mu+\xi)\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; \left|\frac{\sqrt{s}-1}{\sqrt{\epsilon}}\right|^2\right\rangle \mathrm{d}x\mathrm{d}\tau + C(\mu+\xi)\epsilon\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; |\mathbf{v}-\mathbf{u}^0|^2\right\rangle \mathrm{d}x\mathrm{d}\tau \\
\le& C(\mu+\xi)\epsilon + C(\mu+\xi)\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s|\mathbf{v}-\mathbf{u}^0|^2\right\rangle \mathrm{d}x\mathrm{d}\tau + C(\mu+\xi)\epsilon\int_0^t\int_\Omega |\mathbf{u}^\epsilon-\mathbf{u}^0|^2 \mathrm{d}x\mathrm{d}\tau \\
\le& C(\mu+\xi)\epsilon + C(\mu+\xi)\int_0^t \mathcal{H}^\epsilon(\tau)\mathrm{d}\tau + C(\mu+\xi)\epsilon\int_0^t\int_\Omega |\nabla\mathbf{u}^\epsilon-\nabla\mathbf{u}^0|^2 \mathrm{d}x\mathrm{d}\tau,
\end{aligned}
\tag{4.15}
$$

where we have used $\left\langle Y^\epsilon_{t,x}; \mathbf{v}\right\rangle = \mathbf{u}^\epsilon$.

Similarly to the estimate of term $\mathcal{I}_3$, the terms $\mathcal{I}_k (k = 4, 5, 6, 7, 8)$ can be estimated as follows:

$$
\begin{aligned}
\mathcal{I}_4 =& -2\xi\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; (s-1)(\mathbf{v}-\mathbf{u}^0)\right\rangle \cdot \nabla\times\mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau \\
\le& C\xi\epsilon + C\xi\int_0^t \mathcal{H}^\epsilon(\tau)\mathrm{d}\tau + C\xi\epsilon\int_0^t\int_\Omega |\nabla\mathbf{u}^\epsilon-\nabla\mathbf{u}^0|^2 \mathrm{d}x\mathrm{d}\tau, \\
\mathcal{I}_5 =& -2\xi\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega-\mathbf{w}^0)\right\rangle \cdot \nabla\times\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau + 2\xi\int_0^t\int_\Omega (\mathbf{w}^\epsilon-\mathbf{w}^0)\cdot\nabla\times\mathbf{u}^0 \mathrm{d}x\mathrm{d}\tau \\
\le& C\epsilon + C\int_0^t \mathcal{H}^\epsilon(\tau)\mathrm{d}\tau + C\xi\epsilon\int_0^t\int_\Omega |\nabla\mathbf{w}^\epsilon-\nabla\mathbf{w}^0|^2 \mathrm{d}x\mathrm{d}\tau, \\
\mathcal{I}_6 =& 4\xi\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega-\mathbf{w}^0)\right\rangle \cdot \mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau - 4\xi\int_0^t\int_\Omega (\mathbf{w}^\epsilon-\mathbf{w}^0)\cdot\mathbf{w}^0 \mathrm{d}x\mathrm{d}\tau \\
\le& C\xi\epsilon + C\xi\int_0^t \mathcal{H}^\epsilon(\tau)\mathrm{d}\tau + C\xi\epsilon\int_0^t\int_\Omega |\nabla\mathbf{w}^\epsilon-\nabla\mathbf{w}^0|^2 \mathrm{d}x\mathrm{d}\tau,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_7 =& -\mu'\int_0^t\int_\Omega \left\langle Y^\epsilon_{\tau,x}; s(\omega-\mathbf{w}^0)\right\rangle\cdot\Delta\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau - \mu'\int_0^t\int_\Omega(\nabla\mathbf{w}^\epsilon-\nabla\mathbf{w}^0):\nabla\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau,\\
\le& C\mu'\epsilon + C\mu'\int_0^t\mathcal{H}^\epsilon(\tau)\mathrm{d}\tau + C\mu'\epsilon\int_0^t\int_\Omega|\nabla\mathbf{w}^\epsilon-\nabla\mathbf{w}^0|^2\mathrm{d}x\mathrm{d}\tau,\\
\mathcal{I}_8 =& -(\mu'+\lambda')\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x}; s(\omega-\mathbf{w}^0)\right\rangle\cdot\nabla\operatorname{div}\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau\\
&-(\mu'+\lambda')\int_0^t\int_\Omega(\operatorname{div}\mathbf{w}^\epsilon-\operatorname{div}\mathbf{w}^0)\operatorname{div}\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau\\
\le& C(\mu'+\lambda')\epsilon + C(\mu'+\lambda')\int_0^t\mathcal{H}^\epsilon(\tau)\mathrm{d}\tau + C(\mu'+\lambda')\epsilon\int_0^t\int_\Omega|\nabla\mathbf{w}^\epsilon-\nabla\mathbf{w}^0|^2\mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

For the term $\mathcal{I}_9$, using the Cauchy inequality, we have

$$
\begin{aligned}
\mathcal{I}_9 =& \int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x}; s(\omega-\mathbf{w}^0)\right\rangle(\mathbf{u}^0\cdot\nabla)\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau - \int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x}; s(\omega-\mathbf{w}^0)\otimes\mathbf{v}\right\rangle:\nabla\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau\\
=& \int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x}; s(\omega-\mathbf{w}^0)\otimes\mathbf{u}^0\right\rangle:\nabla\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau - \int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x}; s(\omega-\mathbf{w}^0)\otimes\mathbf{v}\right\rangle:\nabla\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau\\
=& -\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x}; \sqrt{s}(\omega-\mathbf{w}^0)\otimes\sqrt{s}(\mathbf{v}-\mathbf{u}^0)\right\rangle:\nabla\mathbf{w}^0\mathrm{d}x\mathrm{d}\tau\\
\le& C\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x}; s|\omega-\mathbf{w}^0|^2\right\rangle\mathrm{d}x\mathrm{d}\tau + C\int_0^t\int_\Omega\left\langle Y^\epsilon_{\tau,x}; s|\mathbf{v}-\mathbf{u}^0|^2\right\rangle\mathrm{d}x\mathrm{d}\tau\\
\le& C\int_0^t\mathcal{H}^\epsilon(\tau)\mathrm{d}\tau.
\end{aligned}\tag{4.16}
$$

In addition, since the compatibility conditions (2.5) and (2.7) are satisfied uniformly with respect to ϵ, we can estimate the term $\mathcal{I}_{11}$ in the following:

$$
\mathcal{I}_{10} = -\int_0^t\left\langle r_1^M;\nabla\mathbf{u}^0\right\rangle\mathrm{d}\tau - \int_0^t\left\langle r_2^M;\nabla\mathbf{u}^0\right\rangle\mathrm{d}\tau \le C\int_0^t\mathcal{D}^\epsilon(\tau)\mathrm{d}\tau. \tag{4.17}
$$

Noting that the initial data are well-prepared, i.e.

$$
\mathcal{H}^\epsilon(0) = \int_\Omega\left\langle Y^\epsilon_{0,x}; \frac{1}{2}s\left|\mathbf{v}-\mathbf{u}^0\right|^2 + \frac{1}{2}s\left|\omega-\mathbf{w}^0\right|^2 + \frac{a}{(\gamma-1)\epsilon^2}\left(s^\gamma-\gamma(s-1)-1\right)\right\rangle\mathrm{d}x \to 0
$$

and combing all above estimates, by using the smallness of ϵ, one gets

$$
\begin{aligned}
&\mathcal{H}^\epsilon(t) + \mathcal{D}^\epsilon(t) + \frac{\mu}{2}\int_0^t\int_\Omega|\nabla\mathbf{u}^\epsilon-\nabla\mathbf{u}^0|^2\mathrm{d}x\mathrm{d}\tau + (\mu+\lambda)\int_0^t\int_\Omega|\operatorname{div}\mathbf{u}^\epsilon|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad+\frac{\mu'}{2}\int_0^t\int_\Omega|\nabla\mathbf{w}^\epsilon-\nabla\mathbf{w}^0|^2\mathrm{d}x\mathrm{d}\tau + (\mu'+\lambda')\int_0^t\int_\Omega|\operatorname{div}\mathbf{w}^\epsilon-\operatorname{div}\mathbf{w}^\epsilon|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad+\xi\int_0^t\int_\Omega|2(\mathbf{w}^\epsilon-\mathbf{w}^0)-\nabla\times(\mathbf{u}^\epsilon-\mathbf{u}^0)|^2\mathrm{d}x\mathrm{d}\tau
\end{aligned}
$$

$$\leq \beta(\epsilon) + C \int_0^t [\mathcal{H}^\epsilon(\tau) + \mathcal{D}^\epsilon(\tau)]\, \mathrm{d}\tau, \ \beta(\epsilon) \to 0 \text{ as } \epsilon \to 0. \tag{4.18}$$

Applying the Grönwall inequality to (4.18), we conclude the proof of Theorem 3.1.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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