Uniqueness of nodal radial solutions to nonlinear elliptic equations in the unit ball*

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Abstract

In this paper, we study the uniqueness of nodal radial solutions to nonlinear elliptic equations in the unit ball in $\mathbb{R}^3$. Under suitable conditions, we prove that, for any given positive integer k, the problem we considered has at most one solution possessing exactly $k-1$ nodes. Together with the results presented by Nagasaki [J. Fac. Sci. Univ. Tokyo Sect. IA Math. 36 (2): 211–232, 1989] and Tanaka [Proc. Roy. Soc. Edinburgh Sect. A. 138 (6): 1331–1343, 2008], we can prove that more types of nonlinear elliptic equations have the uniqueness of nodal radial solutions.

Keywords: Uniqueness; Nodal radial solution; Shooting method

1 Introduction

This paper investigates the uniqueness of radially symmetric solutions to nonlinear elliptic equations having exactly $k-1$ nodes in B:

$$\begin{cases} -\Delta u = K(|x|)|u|^{p-2}u, & x \in B, \\ u = 0, & x \in \partial B, \end{cases} \tag{1.1}$$

where $k \in \mathbb{N} := \{1, 2, \dots\}, B = \{x \in \mathbb{R}^3 : |x| < 1\}, p > 2, K \in C[0,1] \cap C^2(0,1],$ and $K(r) > 0$ for $r \in (0,1]$. The radial solutions to (1.1) having exactly $k-1$ nodes in B can be obtained by solving the following ordinary differential equation:

$$\begin{cases} u'' + \dfrac{2}{r}u' + K(r)|u|^{p-2}u = 0, & r \in (0,1), \\ u'(0) = u(1) = 0, \quad u(0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (0,1). \end{cases} \tag{1.2}$$

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Equation (1.1) with $K = 1$, known as the Lane–Emden equation in astrophysics, arises in the study of stellar structure [4]. In the case of $K = 1$ and $p = 6$, (1.1) is relevant to the problems in differential geometry such as the prescribed scalar curvature problem [3]. Since the beginning of last century, numerous contributions flourished within the topics of the existence and uniqueness of solution to problems related to (1.1). Here, we just mention, among many possible choices, the papers [1, 7, 8, 12, 13, 15, 16, 17]. Moreover, we refer to [5, 6, 19] for the existence and uniqueness results of problems related to (1.1) set on $\mathbb{R}^N$ and [2, 14] set on an annuli.

Let

$$V(r) = \frac{rK'(r)}{K(r)}, \quad r \in (0, 1).$$

In [10, Theorem 3], Naito showed that (1.2) admits at least one solution if $p > 2, K \in C[0, 1] \cap C^1(0, 1]$, $K(r) > 0$ for $r \in (0, 1]$, and

$$\liminf_{r \to 0^+} V(r) > \frac{1}{2}(p - 6). \tag{1.3}$$

We also know that, by the results in [7, 8], the solutions to the equation

$$\begin{cases} u'' + \dfrac{2}{r}u' + K(r)|u|^{p-2}u = 0, \quad r \in (0, 1), \\ u'(0) = 0, u(0) \neq 0 \end{cases}$$

have no zeros in $[0, 1]$ provided that

$$V(r) < \frac{1}{2}(p - 6), \quad r \in (0, 1).$$

As for the uniqueness results related to (1.1), we refer to [12, 13, 15, 16]. In [15], Tanaka showed that if $K \in C^2[0, 1]$, $K(r) > 0$ for $r \in [0, 1]$, and

$$(V(r) - p + 2)(V(r) - p + 4) - 2rV'(r) < 0, \quad r \in (0, 1), \tag{1.4}$$

then the solution to (1.2) is unique. As pointed out by [15, Remark 1.3], letting $r \to 0^+$, we have $p \leqslant 4$. Hence, roughly speaking, Tanaka [15] can only provide a method to judge the uniqueness for (1.2) in the case of $p \in (2, 4]$.

It seems natural to ask what happens when $p > 4$? To describe the idea of solving this problem, we will briefly review some previous results for the special case $K(r) = r^q$ with $q \geqslant 0$. In [11], Ni demonstrated the existence of positive radial solutions to (1.1) for $p \in (2, 6+2q)$. Nagasaki [9] showed that (1.2) has a unique solution for every $k \in \mathbb{N}$ and every $p \in (2, 6 + 2q)$, and has no solution if $p \geqslant 6 + 2q$. The above facts tell us that the potential function $|x|^q$ may expand the range of p for the uniqueness of solutions to (1.2). Motivated by these observations, we focus on the discussion of (1.2) in the case of $K(r) = r^q K_1(r)$ with $K_1(r) > 0$. We prove the unique result of (1.2) by using the shooting method and variable substitution.

Compared with [9], this paper has well established in the case where $K_1(r) \equiv c > 0$ to the case where $K_1(r)$ varies, where c is a constant. Yanagida in [18] proved that if $V(r)$ is non-increasing then uniqueness is ensured. We can also solve some cases where $V'(r) > 0$ for $r \in (0,1)$, as shown in the Remarks 1.3 and 1.5. Furthermore, we can not only prove the case where $K(0) = 0$, but also solve the case where $K(0) > 0$, presented in [15].

The main results of this paper are as follows.

Theorem 1.1. *Let $K(r) = r^q K_1(r) \in C[0,1]$ with $q > 0$ and $2 < p - q < 4$, and $\beta := (p-q-2)/2$. Assume that*

(i) $K_1 \in C^2(0,1]$ and $K_1(r) > 0$ for $r \in (0,1]$;

(ii) $\lim_{r\to 0^+} r^{\beta}[K_1(r)]^{-1/2} = 0$, $\lim_{r\to 0^+} r^{p-2-\beta}[K_1(r)]^{1/2} = 0$, $\lim_{r\to 0^+} r^{\beta+1}[K_1(r)]^{-3/2}K_1'(r) = 0$;

(iii) $4\beta(\beta-1)K_1^2(r) + 3r^2[K_1'(r)]^2 - 4\beta r K_1(r)K_1'(r) - 2r^2 K_1(r)K_1''(r) < 0$ for $r \in (0,1]$.

Then, for every $k \in \mathbb{N}$, the problem (1.2) has at most one solution that has exactly $k-1$ zeros in $(0,1)$.

In particular, if $K_1'(r) > 0$ for $r \in (0,1]$, then we have

Corollary 1.2. *Let $K(r) = r^q K_1(r) \in C[0,1]$ with $q > 0$ and $2 < p - q < 4$, and $\beta := (p-q-2)/2$. Assume that*

(i) $K_1 \in C^2(0,1]$, $K_1(r) > 0$ for $r \in (0,1]$, and $K_1'(r) > 0$ for $r \in (0,1)$;

(ii) $\lim_{r\to 0^+} r^{\beta}[K_1(r)]^{-1/2} = 0$, $\lim_{r\to 0^+} r^{p-2-\beta}[K_1(r)]^{1/2} = 0$, $\lim_{r\to 0^+} r^{\beta+1}[K_1(r)]^{-3/2}K_1'(r) = 0$;

(iii) $3[K_1'(r)]^2 < 2K_1(r)K_1''(r)$ for $r \in (0,1]$.

Then, for every $k \in \mathbb{N}$, the problem (1.2) has at most one solution that has exactly $k-1$ zeros in $(0,1)$.

Remark 1.3. In Theorem 1.1, the conditions $K_1 \in C^2(0,1]$ and $K_1(r) > 0$ for $r \in (0,1]$ allow us to consider the situation, $K(0) > 0$, presented in [15]. For example, let us consider the equation from [15, Example 1.5]:

$$\begin{cases} -\Delta u = \mathrm{e}^{3|x|}u^3, & x \in B, \\ u = 0, & x \in \partial B, \\ u(0) > 0. \end{cases} \tag{1.5}$$

We will show that (1.5) has a unique solution that has exactly $k-1$ nodes in B for every $k \in \mathbb{N}$. In fact, the existence is guaranteed by $V(r) = 3r$ satisfying (1.3). Letting $K_1(r) = \mathrm{e}^{3r}/r$, $p = 4$ and $q = 1$, because of $\beta = 1/2$, we have

$$\lim_{r\to 0^+} r^{\beta}[K_1(r)]^{-1/2} = \lim_{r\to 0^+} r\mathrm{e}^{-3r/2} = 0,$$

$$\lim_{r\to 0^+} r^{p-2-\beta}[K_1(r)]^{1/2} = \lim_{r\to 0^+} r\mathrm{e}^{3r/2} = 0,$$

$$\lim_{r\to 0^+} r^{\beta+1}[K_1(r)]^{-3/2}K_1'(r) = \lim_{r\to 0^+} r\mathrm{e}^{-3r/2}(3r-1) = 0,$$

and

$$
\begin{aligned}
& 4\beta(\beta-1)K_1^2(r)+3r^2[K_1'(r)]^2-4\beta rK_1(r)K_1'(r)-2r^2K_1(r)K_1''(r) \\
= \;& r^{-2}\mathrm{e}^{6r}[-1+3(3r-1)^2-2(3r-1)-2(3r-1)(3r-2)-6r] \\
= \;& r^{-2}\mathrm{e}^{6r}[(3r-1)^2-6r-1] \\
= \;& 3r^{-1}\mathrm{e}^{6r}(3r-4)<0, \quad r\in(0,1].
\end{aligned}
$$

Remark 1.4. It follows from [15, Theorem 1.6] that, for any $p\in(2,6)$ and $k\in\mathbb{N}$, there exists $K\in C^\infty[0,1]$ with $K(r)>0$ for $r\in[0,1]$ such that (1.2) has at least three solutions. Therefore, we can not expect to prove the uniqueness of solutions to (1.2) for all K satisfying the existence conditions of solution; we can only increase our understanding of the uniqueness of solution to (1.2) by looking for more judgment conditions of the uniqueness of solutions. Theorem 1.1 provides a new condition for the uniqueness of solutions to (1.2). Let us consider the following equation:

$$
\begin{cases}
-\Delta u=|x|^q\mathrm{e}^{-\delta|x|}|u|^{p-2}u, & x\in B,\\
u=0, & x\in\partial B,\\
u(0)>0,
\end{cases}
\tag{1.6}
$$

where $q>0$ with $p-q\in(2,4)$, and $\delta\in(0,\sqrt{2(p-q-2)}-(p-q-2))$. We will show that, for $p\in(2+q,4+q)$, (1.6) has a unique solution that has exactly $k-1$ nodes in B for every $k\in\mathbb{N}$. In fact, the existence is guaranteed by $V(r)=q-\delta r$ satisfying (1.3) for $p\in(2+q,4+q)$. Since p may be greater than 4, the uniqueness of solution cannot be guaranteed by (1.4). When $p\in(2+q,4+q)$, letting $K_1(r)=\mathrm{e}^{-\delta r}$, we have

$$
\lim_{r\to0^+}r^\beta[K_1(r)]^{-1/2}=\lim_{r\to0^+}r^\beta\mathrm{e}^{\delta r/2}=0,
$$

$$
\lim_{r\to0^+}r^{p-2-\beta}[K_1(r)]^{1/2}=\lim_{r\to0^+}r^{p-2-\beta}\mathrm{e}^{-\delta r/2}=0,
$$

$$
\lim_{r\to0^+}r^{\beta+1}[K_1(r)]^{-3/2}K_1'(r)=\lim_{r\to0^+}-\delta r^{\beta+1}\mathrm{e}^{\delta r/2}=0,
$$

and

$$
\begin{aligned}
& 4\beta(\beta-1)K_1^2(r)+3r^2[K_1'(r)]^2-4\beta rK_1(r)K_1'(r)-2r^2K_1(r)K_1''(r) \\
= \;& \mathrm{e}^{-2\delta r}[(p-q-2)(p-q-4)+r^2\delta^2+2\delta r(p-q-2)] \\
= \;& \mathrm{e}^{-2\delta r}[(\delta r+p-q-2)^2-2(p-q-2)]<0, \quad r\in(0,1].
\end{aligned}
$$

The desired uniqueness result is obtained by Theorem 1.1.

Remark 1.5. Let us consider the following equation:

$$
\begin{cases}
-\Delta u=|x|^q(|x|^2+3)|u|^{p-2}u, & x\in B,\\
u=0, & x\in\partial B,\\
u(0)>0,
\end{cases}
\tag{1.7}
$$

where $q > 0$ with $p - q \in (2, 4)$. We will prove that (1.7) has a unique solution which has exactly $k - 1$ nodes in B for every $k \in \mathbb{N}$. In fact, the existence is guaranteed by $V(r) = [(q+2)r^2 + 3q]/(r^2+3)$ satisfying (1.3). Because $K(0) = 0$ and $V'(r) > 0$, the uniqueness of solution cannot be guaranteed by [15] and [18]. Letting $K_1(r) = r^2 + 3$, we obtain

$$\lim_{r\to 0^+} r^\beta [K_1(r)]^{-1/2} = \lim_{r\to 0^+} r^\beta (r^2+3)^{-1/2} = 0,$$

$$\lim_{r\to 0^+} r^{p-2-\beta} [K_1(r)]^{1/2} = \lim_{r\to 0^+} r^{p-2-\beta} (r^2+3)^{1/2} = 0,$$

$$\lim_{r\to 0^+} r^{\beta+1} [K_1(r)]^{-3/2} K_1'(r) = \lim_{r\to 0^+} 2r^{\beta+2} (r^2+3)^{-3/2} = 0.$$

Meanwhile, $K_1''(r) = 2 > 0$ and

$$3[K_1'(r)]^2 - 2K_1(r)K_1''(r) = 12r^2 - 4(r^2+3) = 8r^2 - 12 < 0, \quad r \in (0, 1].$$

The desired uniqueness result is obtained by Corollary 1.2.

Remark 1.6. In this paper, we only discuss the case of B in $\mathbb{R}^3$. In fact, a similar conclusion to Theorem 1.1 can be obtained for the case of B in $\mathbb{R}^N$ with $N > 3$. To prove this, we need only make appropriate modifications to the relevant formulas in all sections.

2 Preliminaries

In this section, the conditions of Theorem 1.1 are satisfied. The proof of Theorem 1.1 is based on the shooting method. Assume that $u(\cdot, \alpha)$ is the solution to the following problem:

$$\begin{cases} u'' + \dfrac{2}{r}u' + K(r)|u|^{p-2}u = 0, & r \in (0, 1), \\ u(0) = \alpha > 0, \quad u'(0) = 0, \end{cases} \tag{2.1}$$

where $\alpha > 0$ is a parameter. Because $K \in C[0, 1]$, we have that $u(\cdot, \alpha)$ exists on $[0, 1]$ and is unique, $u, u' \in C^1([0, 1] \times (0, \infty))$, and $w(\cdot, \alpha) := u_\alpha(\cdot, \alpha) := \partial u(\cdot, \alpha)/\partial\alpha$ is a solution to the following problem:

$$\begin{cases} v'' + \dfrac{2}{r}v' + (p-1)K(r)|u|^{p-2}v = 0, & r \in (0, 1], \\ v(0) = 1, \quad v'(0) = 0. \end{cases} \tag{2.2}$$

Henceforth we assume that $u(\cdot, \alpha)$ has exactly k zeros in $(0, 1]$, denoted as $0 < z_1 < z_2 < \cdots < z_k := 1$. Note that $u(\cdot, \alpha)$ and $u'(\cdot, \alpha)$ cannot vanish at the same time. Then, we have

$$(-1)^i u'(z_i, \alpha) = (-1)^i \frac{\mathrm{d}}{\mathrm{d}r} u(z_i, \alpha) > 0, \quad i = 1, 2, \ldots, k. \tag{2.3}$$

In order to obtain the uniqueness of problem (1.2), it is not enough to study the properties of the derivative of u at the zero points, we also need to consider the properties of w at the zero points of u. Thus, we make the following variable substitution:

$$U(t,\alpha) = tu(t,\alpha), \quad W(t) = tw(t,\alpha), \quad t \in [0,1].$$

Then, U and W satisfy

$$U'' + M(t)|U|^{p-2}U = 0, \quad t \in (0,1], \tag{2.4}$$

$$U(0,\alpha) = 0, \quad U'(0,\alpha) = \alpha, \tag{2.5}$$

$$W'' + (p-1)M(t)|U|^{p-2}W = 0, \quad t \in (0,1], \tag{2.6}$$

$$W(0) = 0, \quad W'(0) = 1. \tag{2.7}$$

Here $' = \mathrm{d}/\mathrm{d}t$, and

$$M(t) = t^{-(p-2)}K(t) = t^{-p+q+2}K_1(t) > 0, \quad t \in (0,1],$$

$$[M(t)]^{-1/2} = t^{(p-q-2)/2}[K_1(t)]^{-1/2} := t^{\beta}[K_1(t)]^{-1/2} > 0, \quad t \in (0,1], \tag{2.8}$$

where $\beta = (p-q-2)/2$. Because $p-q \in (2,4)$, we have $\beta \in (0,1)$.

We denote $z_0 := 0$. Then,

$$U(z_i,\alpha) = z_i u(z_i,\alpha) = 0, \quad i = 0,1,2,\ldots,k,$$

$$(-1)^{i-1}U(t,\alpha) > 0, \quad t \in (z_{i-1},z_i), i = 1,2,\ldots,k.$$

It follows from (2.4) that there exist the unique s_i such that

$$\begin{cases} s_i \in (z_{i-1},z_i), \quad U'(s_i,\alpha) = 0, \quad i = 1,2,\ldots,k, \\ U'(t,\alpha) > 0, \quad t \in (0,s_1), \\ (-1)^i U'(t,\alpha) > 0, \quad t \in (s_i,s_{i+1}), i = 1,2,\ldots,k-1, \\ (-1)^k U'(t,\alpha) > 0, \quad t \in (s_k,z_k]. \end{cases} \tag{2.9}$$

Obviously, we can find that $(0,s_1),(s_1,s_2),(s_2,s_3),\ldots,(s_{k-1},s_k),(s_k,z_k)$ are the monotonic intervals of U.

In Lemmas 2.1–2.5 below, we always assume that U and W are solutions to (2.4)–(2.5) and (2.6)–(2.7), respectively. Then, U satisfies (2.9).

Lemma 2.1. *For any given $i \in \{1,2,\ldots,k\}$, W has at least one zero in (z_{i-1},z_i).*

Proof. Assume to the contrary that $W(t) \neq 0$ for $t \in (z_{i-1},z_i)$. Without loss of generality, we may suppose that $W(t) > 0$ and $U(t) > 0$ for $t \in (z_{i-1},z_i)$. Other cases can be treated similarly. Therefore, we have that $U'(z_i) < 0, U'(z_{i-1}) > 0$, and then

$$W(z_i)U'(z_i) - W(z_{i-1})U'(z_{i-1}) \leqslant 0. \tag{2.10}$$

A simple calculation gives

$$(WU' - W'U)' = (p-2)M(t)|U|^{p-2}UW, \quad t \in (0,1]. \tag{2.11}$$

It follows from $K \in C[0,1]$ that

$$\begin{aligned}\lim_{t\to 0^+} M(t)|U|^{p-2}UW &= \lim_{t\to 0^+} t^{-(p-2)}t^q K_1(t)|U|^{p-2}UW \\ &= \lim_{t\to 0^+} K(t)\left|\frac{U}{t}\right|^{p-2} UW = 0.\end{aligned}$$

Therefore, $(WU'-W'U)'$ is integrable on $[0,1]$. Integrating (2.11) over the interval (z_{i-1}, z_i), we have that

$$W(z_i)U'(z_i) - W(z_{i-1})U'(z_{i-1}) > 0.$$

This contradicts (2.10). Hence, W has at least one zero in (z_{i-1}, z_i). □

Lemma 2.2. *For U and W, we have*

$$\left[[M(t)]^{-1/2}[W'U' - WU''] - ([M(t)]^{-1/2})'WU'\right]' = -([M(t)]^{-1/2})''WU', \quad t \in (0,1]. \tag{2.12}$$

Proof. From (2.4), we note that

$$U''' = -M'(t)|U|^{p-2}U - (p-1)M(t)|U|^{p-2}U', \quad t \in (0,1].$$

By a direct calculation, we have that

$$\begin{aligned} & \left[[M(t)]^{-1/2}[W'U' - WU''] - ([M(t)]^{-1/2})'WU'\right]' \\ = \ & ([M(t)]^{-1/2})'[W'U' - WU''] + [M(t)]^{-1/2}[W''U' - WU'''] \\ & -([M(t)]^{-1/2})''WU' - ([M(t)]^{-1/2})'[W'U' + WU''] \\ = \ & -2([M(t)]^{-1/2})'WU'' - ([M(t)]^{-1/2})''WU' \\ & +[M(t)]^{-1/2}[-(p-1)M(t)|U|^{p-2}WU' + M'(t)|U|^{p-2}UW + (p-1)M(t)|U|^{p-2}WU'] \\ = \ & [M(t)]^{-3/2}M'(t)W[-M(t)|U|^{p-2}U] - ([M(t)]^{-1/2})''WU' + [M(t)]^{-1/2}WM'(t)|U|^{p-2}U \\ = \ & -([M(t)]^{-1/2})''WU', \quad t \in (0,1]. \end{aligned}$$

The proof was completed. □

Lemma 2.3. *For U and W, we also have*

$$\lim_{t\to 0^+} \left[[M(t)]^{-1/2}[W'U' - WU''] - ([M(t)]^{-1/2})'WU'\right] = 0. \tag{2.13}$$

Proof. According to (2.4), (2.5), (2.7), (2.8), and the conditions of Theorem 1.1, we get that

$$\lim_{t\to 0^+} [M(t)]^{-1/2}W'U' = \lim_{t\to 0^+} t^{\beta}[K_1(t)]^{-1/2}W'U' = 0,$$

and

$$\begin{aligned}
\lim_{t\to 0^+}[M(t)]^{-1/2}WU'' &= -\lim_{t\to 0^+}[M(t)]^{1/2}|U|^{p-2}UW \\
&= -\lim_{t\to 0^+} t^{p-2-\beta}[K_1(t)]^{1/2}\left|\frac{U}{t}\right|^{p-2}UW = 0.
\end{aligned}$$

Meanwhile, we observe that

$$([M(t)]^{-1/2})' = \beta t^{\beta-1}[K_1(t)]^{-1/2} - \frac{1}{2}t^{\beta}[K_1(t)]^{-3/2}K_1'(t), \quad t \in (0,1].$$

Hence,

$$\lim_{t\to 0^+}([M(t)]^{-1/2})'W = \lim_{t\to 0^+}\left[\beta t^{\beta}[K_1(t)]^{-1/2} - \frac{1}{2}t^{\beta+1}[K_1(t)]^{-3/2}K_1'(t)\right]\frac{W}{t} = 0.$$

We conclude that (2.13) holds. This completes the proof. □

Lemma 2.4. $\left([M(t)]^{-1/2}\right)'' < 0$ *for* $t \in (0,1]$.

Proof. Note that $M(t) = t^{-2\beta}K_1(t) > 0$ for $t \in (0,1]$. Thus, for $t \in (0,1]$,

$$([M(t)]^{-1/2})' = -\frac{1}{2}[M(t)]^{-3/2}M'(t),$$

and

$$\begin{aligned}
& ([M(t)]^{-1/2})'' \\
= & \frac{3}{4}[M(t)]^{-5/2}[M'(t)]^2 - \frac{1}{2}[M(t)]^{-3/2}M''(t) \\
= & \frac{1}{4}[M(t)]^{-5/2}\left[3[M'(t)]^2 - 2M(t)M''(t)\right] \\
= & \frac{1}{4}[M(t)]^{-5/2}\left\{3(-2\beta t^{-2\beta-1}K_1(t) + t^{-2\beta}K_1'(t))^2\right. \\
& \left.-2t^{-2\beta}K_1(t)[-2\beta(-2\beta-1)t^{-2\beta-2}K_1(t) - 4\beta t^{-2\beta-1}K_1'(t) + t^{-2\beta}K_1''(t)]\right\} \\
= & \frac{1}{4}[M(t)]^{-5/2}t^{-4\beta-2}\left[12\beta^2K_1^2(t) + 3t^2[K_1'(t)]^2 - 12\beta tK_1(t)K_1'(t)\right. \\
& \left.-4\beta(2\beta+1)K_1^2(t) + 8\beta tK_1(t)K_1'(t) - 2t^2K_1(t)K_1''(t)\right] \\
= & \frac{1}{4}[M(t)]^{-5/2}t^{-4\beta-2}\left[4\beta(\beta-1)K_1^2(t) + 3t^2[K_1'(t)]^2 - 4\beta tK_1(t)K_1'(t) - 2t^2K_1(t)K_1''(t)\right].
\end{aligned}$$

It follows from the conditions of Theorem 1.1, we have $\left([M(t)]^{-1/2}\right)'' < 0$ for $t \in (0,1]$. This completes the proof. □

Now, we study the number of zeros of W in the monotonic intervals of U.

Lemma 2.5. *(i)* $W(t) > 0$ *for* $t \in (0, s_1]$.

(ii) For every $i \in \{1, 2, \ldots, k-1\}$, W *has at most one zero in* $(s_i, s_{i+1}]$.

(iii) W *has at most one zero in* $(s_k, z_k]$.

Proof. (i) Suppose that there exists some $t_2 \in (0, s_1]$ such that $W(t_2) = 0$ and $W(t) > 0$ for $t \in (0, t_2)$. Then, $W'(t_2) < 0$. Because $t_2 \in (0, s_1]$, we find that $U'(t_2) \geqslant 0$. Therefore, $W'(t_2)U'(t_2) \leqslant 0$. Integrating (2.12) over $(0, t_2]$ and applying Lemmas 2.3 and 2.4, we obtain $W'(t_2)U'(t_2) > 0$, which contradicts $W'(t_2)U'(t_2) \leqslant 0$. Hence, conclusion (i) holds.

We now show part (ii) only, as part (iii) can be obtained in the same way.

(ii) Suppose that there exist $t_1, t_2 \in (s_i, s_{i+1}]$ with $t_1 < t_2$ such that $W(t_1) = W(t_2) = 0$ and $W(t) \neq 0$ for $t \in (t_1, t_2)$. We may assume that $W(t) > 0$ for $t \in (t_1, t_2)$. The case where $W(t) < 0$ for $t \in (t_1, t_2)$ can be treated in a similar way. Therefore, we have that $W'(t_1) > 0$ and $W'(t_2) < 0$. Integrate (2.12) over $[t_1, t_2]$ and then multiply the result by $(-1)^i$. By Lemma 2.4 and (2.9), we have that

$$[M(t_2)]^{-1/2} W'(t_2)(-1)^i U'(t_2) - [M(t_1)]^{-1/2} W'(t_1)(-1)^i U'(t_1) > 0,$$

which contradicts (2.9), $W'(t_1) > 0$, and $W'(t_2) < 0$. This completes the proof. □

Before finishing this section, we prove an important lemma, which describes the properties of w at the zero points of u.

Lemma 2.6. $(-1)^i w(z_i) > 0$ *for* $i = 1, 2, \ldots, k$.

Proof. According to Lemmas 2.1 and 2.5, there exists some $c_1 \in (s_1, z_1)$ such that $W(c_1) = 0$, $W(t) > 0$ for $t \in (0, c_1)$, and $W(t) < 0$ for $t \in (c_1, s_2]$. Thus, we have $W(z_1) < 0$. From Lemmas 2.1 and 2.5, we also imply that there exists some $c_2 \in (s_2, z_2)$ such that $W(c_2) = 0$, $W(t) < 0$ for $t \in (s_2, c_2)$, and $W(t) > 0$ for $t \in (c_2, s_3]$. Therefore, $W(z_2) > 0$. By continuing this process, we obtain $(-1)^i W(z_i) = (-1)^i z_i w(z_i) > 0$ for $i = 1, 2, \ldots, k$. This means that $(-1)^i w(z_i) > 0$ for $i = 1, 2, \ldots, k$. □

3 Proof of main theorem

In this section we apply the Prüfer transformation for the solution $u(\cdot, \alpha)$ to problem (2.1). For the definition and properties of the Prüfer transformation, we can refer to [17, p.270]. For the solution $u(r, \alpha)$ with $\alpha > 0$, $\rho(r, \alpha)$ and $\theta(r, \alpha)$ are defined as

$$\begin{cases} u(r, \alpha) = \rho(r, \alpha) \sin\theta(r, \alpha), \\ r^2 u'(r, \alpha) = \rho(r, \alpha) \cos\theta(r, \alpha), \end{cases} \tag{3.1}$$

where $' = \mathrm{d}/\mathrm{d}r$. Let us consider the implicit function equation

$$\begin{cases} F(r, \rho, \theta) := u(r, \alpha) - \rho \sin\theta = 0, \\ G(r, \rho, \theta) := r^2 u'(r, \alpha) - \rho \cos\theta = 0 \end{cases} \tag{3.2}$$

with initial conditions

$$\begin{cases} F\left(0, \alpha, \frac{\pi}{2}\right) = 0, \\ G\left(0, \alpha, \frac{\pi}{2}\right) = 0. \end{cases} \tag{3.3}$$

Because $u(\cdot,\alpha)$ and $u'(\cdot,\alpha)$ cannot disappear at the same time, we note that

$$\rho(r,\alpha) = \left([u(r,\alpha)]^2 + r^4[u'(r,\alpha)]^2\right)^{1/2} > 0.$$

At any point $(\rho,\theta) \in (0,\infty) \times \mathbb{R}$, the Jacobi determinant is

$$J(\rho,\theta) = \frac{\partial(F,G)}{\partial(\rho,\theta)} = \begin{vmatrix} F_\rho & F_\theta \\ G_\rho & G_\theta \end{vmatrix} = \begin{vmatrix} -\sin\theta & -\rho\cos\theta \\ -\cos\theta & \rho\sin\theta \end{vmatrix} = -\rho\sin^2\theta - \rho\cos^2\theta = -\rho < 0.$$

Thus, by (3.2) and (3.3), $\rho = f(r)$ and $\theta = g(r)$ such that $F(r,f(r),g(r)) \equiv 0$ and $G(r,f(r),g(r)) \equiv 0$ for $r \in [0,1]$ are uniquely determined. It follows from $u, u' \in C^1([0,1] \times (0,\infty))$ that $\rho, \theta \in C^1[0,1]$.

We denote $H(r,\rho,\theta) = (F(r,\rho,\theta), G(r,\rho,\theta))$. Then, we obtain

$$\begin{aligned}
\begin{bmatrix} \frac{\mathrm{d}\rho}{\mathrm{d}r} \\ \frac{\mathrm{d}\theta}{\mathrm{d}r} \end{bmatrix} &:= \begin{bmatrix} \rho' \\ \theta' \end{bmatrix} \\
&= -[H'_{(\rho,\theta)}]^{-1} H'_r \\
&= -\begin{bmatrix} -\sin\theta & -\rho\cos\theta \\ -\cos\theta & \rho\sin\theta \end{bmatrix}^{-1} \begin{bmatrix} u'(r,\alpha) \\ 2ru'(r,\alpha) + r^2u''(r,\alpha) \end{bmatrix} \\
&= -\frac{1}{\rho} \begin{bmatrix} -\rho\sin\theta & -\rho\cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} u'(r,\alpha) \\ 2ru'(r,\alpha) + r^2u''(r,\alpha) \end{bmatrix} \\
&= \begin{bmatrix} u'(r,\alpha)\sin\theta + \cos\theta[2ru'(r,\alpha) + r^2u''(r,\alpha)] \\ \frac{1}{\rho}u'(r,\alpha)\cos\theta - \frac{1}{\rho}\sin\theta[2ru'(r,\alpha) + r^2u''(r,\alpha)] \end{bmatrix} \\
&= \begin{bmatrix} u'(r,\alpha)\sin\theta - r^2\cos\theta[K(r)|u(r,\alpha)|^{p-2}u(r,\alpha)] \\ \frac{1}{\rho^2}r^2[u'(r,\alpha)]^2 + \frac{1}{\rho^2}u(r,\alpha)r^2[K(r)|u(r,\alpha)|^{p-2}u(r,\alpha)] \end{bmatrix}.
\end{aligned}$$

Thus, we find that

$$\theta' = \frac{1}{\rho^2}r^2[u'(r,\alpha)]^2 + \frac{1}{\rho^2}u(r,\alpha)r^2[K(r)|u(r,\alpha)|^{p-2}u(r,\alpha)] > 0, \quad r \in (0,1].$$

This shows that, for any given $\alpha > 0$, $\theta(\cdot,\alpha)$ is increasing in $(0,1]$. By (3.3), it is easy to see that $u(\cdot,\alpha)$ is a solution to problem (1.2) if and only if

$$\theta(1,\alpha) = k\pi. \tag{3.4}$$

Therefore, the number of solutions to (1.2) is the same as the number of roots of (3.4).

Lemma 3.1. *Let $k \in \mathbb{N}$ and $u(\cdot,\alpha_0)$ be a solution to* (1.2) *for some $\alpha_0 > 0$. Then, $\theta_\alpha(1,\alpha_0) > 0$.*

Proof. By (3.1), we find that

$$\theta_\alpha(r,\alpha) = \frac{u_\alpha(r,\alpha)r^2u'(r,\alpha) - u(r,\alpha)r^2u'_\alpha(r,\alpha)}{[u(r,\alpha)]^2 + [r^2u'(r,\alpha)]^2}.$$

Because $z_k = 1$ and $u(1, \alpha_0) = 0$, we have that

$$\theta_\alpha(1, \alpha_0) = \frac{u_\alpha(z_k, \alpha_0)}{u'(z_k, \alpha_0)}.$$

By (2.3), we observe that $(-1)^k u'(z_k, \alpha_0) > 0$. It follows from Lemma 2.6 that

$$(-1)^k u_\alpha(z_k, \alpha_0) > 0.$$

Therefore, $\theta_\alpha(1, \alpha_0) > 0$. $\square$

Proof of Theorem 1.1. Because $\theta(1, \cdot)$ is a continuous differentiable function in $(0, \infty)$, and when α_0 satisfies $\theta(1, \alpha_0) - k\pi = 0$, by Lemma 3.1, we have $\theta_\alpha(1, \alpha_0) > 0$. Therefore, $\theta(1, \cdot) - k\pi$ has at most one zero in $(0, \infty)$. This completes the proof of the theorem. $\square$

References

[1] Gianni Arioli and Hans Koch. Non-radial solutions for some semilinear elliptic equations on the disk. *Nonlinear Anal.*, 179:294–308, 2019.

[2] Mohamed Ben Chrouda and Kods Hassine. Uniqueness of positive radial solutions for elliptic equations in an annulus. *Proc. Amer. Math. Soc.*, 149(2):649–660, 2021.

[3] Luis A. Caffarelli, Basilis Gidas, and Joel Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.*, 42(3):271–297, 1989.

[4] S. Chandrasekhar. *An introduction to the study of stellar structure*. Dover Publications, Inc., New York, 1957.

[5] Jann-Long Chern, Zhi-You Chen, and Yong-Li Tang. Uniqueness of finite total curvatures and the structure of radial solutions for nonlinear elliptic equations. *Trans. Amer. Math. Soc.*, 363(6):3211–3231, 2011.

[6] Nichiro Kawano, Eiji Yanagida, and Shoji Yotsutani. Structure theorems for positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbf{R}^n$. *Funkcial. Ekvac.*, 36(3):557–579, 1993.

[7] Takaŝi Kusano and Manabu Naito. Positive entire solutions of superlinear elliptic equations. *Hiroshima Math. J.*, 16(2):361–366, 1986.

[8] Takaŝi Kusano and Manabu Naito. Oscillation theory of entire solutions of second order superlinear elliptic equations. *Funkcial. Ekvac.*, 30(2-3):269–282, 1987.

[9] Ken'ichi Nagasaki. Radial solutions for $\Delta u + |x|^l |u|^{p-1} u = 0$ on the unit ball in $\mathbf{R}^n$. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 36(2):211–232, 1989.

[10] Yūki Naito. Bounded solutions with prescribed numbers of zeros for the Emden-Fowler differential equation. *Hiroshima Math. J.*, 24(1):177–220, 1994.

[11] Wei-Ming Ni. A nonlinear Dirichlet problem on the unit ball and its applications. *Indiana Univ. Math. J.*, 31(6):801–807, 1982.

[12] Wei-Ming Ni. Uniqueness of solutions of nonlinear Dirichlet problems. *J. Differential Equations*, 50(2):289–304, 1983.

[13] Wei-Ming Ni and Roger D. Nussbaum. Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u,r) = 0$. *Comm. Pure Appl. Math.*, 38(1):67–108, 1985.

[14] Naoki Shioji, Satoshi Tanaka, and Kohtaro Watanabe. Uniqueness of positive radial solutions of superlinear elliptic equations in annuli. *J. Differential Equations*, 284:522–545, 2021.

[15] Satoshi Tanaka. Uniqueness of nodal radial solutions of superlinear elliptic equations in a ball. *Proc. Roy. Soc. Edinburgh Sect. A*, 138(6):1331–1343, 2008.

[16] Satoshi Tanaka. Uniqueness and nonuniqueness of nodal radial solutions of sublinear elliptic equations in a ball. *Nonlinear Anal.*, 71(11):5256–5267, 2009.

[17] Wolfgang Walter. *Ordinary differential equations*, volume 182 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998. Translated from the sixth German (1996) edition by Russell Thompson, Readings in Mathematics.

[18] Eiji Yanagida. Structure of radial solutions to $\Delta u + K(|x|)|u|^{p-1}u = 0$ in $\mathbf{R}^n$. *SIAM J. Math. Anal.*, 27(4):997–1014, 1996.

[19] Eiji Yanagida and Shoji Yotsutani. Classification of the structure of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.*, 124(3):239–259, 1993.