Ground-state solutions of Schrödinger-type equation with magnetic field *

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Abstract

In this paper, the nonlinear Schrödinger-type equation

$$-(\nabla + iA)^2 u + u + \lambda \left[I_\alpha * \left(K|u|^2\right)\right] Ku = a\frac{f(|u|)}{|u|}u \quad \text{in } \mathbb{R}^3$$

is considered in the presence of magnetic field, where $A \in C^1(\mathbb{R}^3, \mathbb{R}^3)$, $\alpha \in (0,3)$, I_α denotes the Riesz potential, $K \in L^p(\mathbb{R}^3, (0,\infty))$ for some $p \in (6/(1+\alpha), \infty]$, $a \in L^q(\mathbb{R}^3, [0,\infty)) \setminus \{0\}$ for some $q \in (3/2, \infty]$, and $f \in C(\mathbb{R}, [0,\infty))$ is assumed to be asymptotically linear at infinity. Under suitable assumptions regarding A, K, a, and f, variational methods are used to establish the existence of ground-state solutions of the above equation for sufficiently small values of the parameter λ.

Keywords: Schrödinger-type equation; Ground-state solution; Variational methods

1 Introduction

The aim of this paper is to study the existence of ground-state solutions of the nolinear Schrödinger-type equation

$$-(\nabla + iA)^2 u + u + \lambda \left[I_\alpha * \left(K|u|^2\right)\right] Ku = a\frac{f(|u|)}{|u|}u \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $A \in C^1(\mathbb{R}^3, \mathbb{R}^3)$, λ is a positive parameter, $\alpha \in (0,3)$, and I_α is the Riesz potential given by

$$I_\alpha(x) = \frac{\Gamma((3-\alpha)/2)}{2^\alpha \pi^{3/2}\Gamma(\alpha/2)|x|^{3-\alpha}}, \quad x \in \mathbb{R}^3,$$

where Γ denotes the Gamma function. Here, the operator $(\nabla + iA)^2$, known in the literature as the magnetic Schrödinger operator, is defined by

$$-(\nabla + iA)^2 u = -\Delta u - 2iA \cdot \nabla u - iu\,\mathrm{div}A + |A|^2 u.$$

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The main reason for studying Eq. (1.1) is the fact that if u is a solution of it with $\lambda = 0$, then $\Psi(x,t) = u(x)e^{-iEt/\hbar}$ is a solution of the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial \Psi}{\partial t} = -(\nabla + iA(x))^2\Psi + \Psi - g(x, |\Psi|^2)\Psi \tag{1.2}$$

with $g(x,t) = a(x)f(\sqrt{t})/\sqrt{t}$, where $\hbar$ is Planck's constant and i is the imaginary unit. Equation (1.2) appears in quantum mechanics to describe the dynamics of a particle in a nonrelativistic setting and also arises in different physical theories, e.g., the description of Bose-Einstein condensates and nonlinear optics.

In recent years, the nonlinear Schrödinger equation

$$-\Delta u + u = h(x,u) \;\text{ in } \mathbb{R}^3 \tag{1.3}$$

with no magnetic field has attracted increasing attention, and many results have been established regarding the existence, multiplicity, and qualitative properties of its solutions; for examples, see [3, 9, 10] and the references therein. Equation (1.1) with $A = 0, \alpha = 2$, and $K \in L^2(\mathbb{R}^3, [0,\infty))\backslash\{0\}$ is analogous to the nonautonomous Schrödinger-Poisson system. Recently asymptotically linear problems in the whole space were studied under various assumptions and potentials. Using results from critical-point theory, van Heerden in [13] proved the existence of multiple solutions of a Schrödinger equation with an asymptotically linear term. Using variational techniques, Wang et al. in [14] proved the existence of a positive solution of a nonlinear Schrödinger-type equation with asymptotically linear nonlinearity. For other interesting results related to problems with an asymptotically linear term, see [6, 7, 11, 12, 17].

Herein, we investigate the case of $A \neq 0$. To date, much consideration has been given to magnetic Schrödinger-type equation (for examples, see [1, 2, 4, 5, 16]), but to the best of our knowledge, scant consideration has been given to Schrödinger-type equations with a magnetic field and an asymptotically linear term. Inspired by the studies mentioned above, we focus on the existence of ground-state solutions for a magnetic Schrödinger equation with an asymptotically linear term and with K, a, f satisfying the following assumptions.

Assumpation (K): $K \in L^p(\mathbb{R}^3, (0,\infty))$ for some $p \in (6/(1+\alpha), \infty]$. If $p = \infty$, then also assume that

$$K_\infty := \lim_{r\to\infty} |K|_{\infty, B_r^c} = 0,$$

where $B_r = \{x \in \mathbb{R}^3 : |x| < r\}$ for $r > 0$.

Assumpation (a): $a \in L^q(\mathbb{R}^3, \mathbb{R}_+)$ and $a \neq 0$ for some $q \in (3/2, \infty]$, where $\mathbb{R}_+ = [0,\infty)$. If $q = \infty$, then also assume that

$$a_\infty := \lim_{r\to\infty} |a|_{\infty, B_r^c} = 0.$$

Let

$$S_a := \inf\left\{\|u\|_A^2 : u \in H_A^1(\mathbb{R}^3, \mathbb{C}), \int_{\mathbb{R}^3} a|u|^2 = 1\right\},$$

where $H^1_A(\mathbb{R}^3,\mathbb{C})$, defined in Section 2, is a Hilbert space and can be imbedded into $L^q(\mathbb{R}^3,\mathbb{C})$ for each $q\in[2,6]$. By the imbedding, we know that $S, S_a>0$.

Assumpation (f_1): $f\in C(\mathbb{R},\mathbb{R}_+), f(t)=0$ for $t\leqslant 0$, and $\lim_{t\to0^+} f(t)/t = f_0\in\mathbb{R}_+$ satisfying $f_0<S_a$.

Assumpation (f_2): $\lim_{t\to\infty} f(t)/t = f_\infty \in (0,\infty)$ satisfies $f_\infty > S_a$.

Under the above assumptions, we have the following results.

Theorem 1.1. *If assumptions (K), (a), (f_1), and (f_2) hold, then for sufficient small $\lambda>0$, Eq.* (1.1) *has a ground-state solution in $H^1_A(\mathbb{R}^3,\mathbb{C})$.*

Remark 1.2. In Theorem 1.1, we say that u is a solution of Eq. (1.1) if

$$\int_{\mathbb{R}^3}\left(\nabla_A u\cdot\overline{\nabla_A v}+u\overline{v}\right)+\lambda\int_{\mathbb{R}^3}\left[I_\alpha*\left(K|u|^2\right)\right]Ku\overline{v}-\int_{\mathbb{R}^3}a\frac{f(|u|)}{|u|}u\overline{v}=0$$

for any $v\in H^1_A(\mathbb{R}^3,\mathbb{C})$.

Remark 1.3. It is easy to find some functions K, a, and f that satisfy the above assumpations. Letting $\mu>0$ and

$$f(t)=\begin{cases}\frac{\mu t^2}{1+t}, & t>0,\\ 0, & t\leqslant 0,\end{cases}$$

assumpations (f_1) and (f_2) hold for sufficiently large μ. Letting

$$K(x)=a(x)=\frac{1}{1+|x|^2},\quad x\in\mathbb{R}^3,$$

it is easy to see that $K_\infty=a_\infty=0$.

The outline of this paper is as follows. In Section 2, we introduce some notations and preliminary ideas. In Section 3, we establish the mountain-pass geometry structure and prove some useful lemmas and Theorem 1.1.

2 Variational setting and preliminaries

This section is devoted mainly to establishing some preliminary ideas. Let us denote $\nabla_A u:=\nabla u+iAu$ and consider the real Hilbert space

$$H^1_A(\mathbb{R}^3,\mathbb{C}):=\{u\in L^2(\mathbb{R}^3,\mathbb{C}):\nabla_A u\in L^2(\mathbb{R}^3,\mathbb{C}^3)\}$$

with the inner product and the corresponding norm defined by

$$(u,v)_A=\mathrm{Re}\int_{\mathbb{R}^3}\left(\nabla_A u\cdot\overline{\nabla_A v}+u\overline{v}\right),\quad \|u\|_A^2=\int_{\mathbb{R}^3}\left(|\nabla_A u|^2+|u|^2\right),\quad u,v\in H^1_A(\mathbb{R}^3,\mathbb{C}).$$

If $u\in H^1_A(\mathbb{R}^3,\mathbb{C})$, then $|u|\in H^1(\mathbb{R}^3)$ and

$$|\nabla|u|(x)|\leqslant|\nabla u(x)+iA(x)u(x)|,\quad \text{a.e. } x\in\mathbb{R}^3. \tag{2.1}$$

Inequality (2.1) is known as the diamagnetic inequality; see [8, Theorem 7.21]. The main idea behind proving Eq. (2.1) is given by

$$|\nabla |u|| = \left|\mathrm{Re}\left(\frac{\overline{u}}{|u|}\nabla u\right)\right| = \left|\mathrm{Re}\left[(\nabla u + iAu)\frac{\overline{u}}{|u|}\right]\right|.$$

The diamagnetic inequality leads to the continuous imbedding

$$H^1_A(\mathbb{R}^3, \mathbb{C}) \hookrightarrow L^q(\mathbb{R}^3, \mathbb{C})$$

for any given $q \in [2, 6]$. Hence, for each $q \in [2, 6]$, there exists $\gamma_q > 0$ such that

$$|u|_q \leqslant \gamma_q \|u\|_A, \quad u \in H^1_A(\mathbb{R}^3, \mathbb{C}),$$

where $|\cdot|_q$ denotes the norm on $L^q(\mathbb{R}^3, \mathbb{C})$.

Herein, we use the following notations:

- terms C_i for $i \in \mathbb{N} := \{1, 2, \dots\}$ denote various positive constants, which may vary from line to line;
- $B_r = \{x \in \mathbb{R}^3 : |x| < r\}$ denotes an open ball in $\mathbb{R}^3$ with its center at the origin and radius $r > 0$.

Lemma 2.1. *Suppose that assumpations (K), (a), (f_1), and (f_2) hold. The energy functional $I_A : H^1_A(\mathbb{R}^3, \mathbb{C}) \to \mathbb{R}$ associated with Eq. (1.1), namely*

$$I_A(u) = \frac{1}{2}\|u\|^2_A + \frac{\lambda}{4}\int_{\mathbb{R}^3}\left[I_\alpha * \left(K|u|^2\right)\right] K|u|^2 - \int_{\mathbb{R}^3} aF(|u|), \quad u \in H^1_A(\mathbb{R}^3, \mathbb{C}), \tag{2.2}$$

is of class C^1, where $F(t) = \int_0^t f(s)\mathrm{d}s$ for $t \in \mathbb{R}$ and its derivative is given by

$$\langle I'_A(u), v\rangle = (u, v)_A + \lambda \mathrm{Re}\int_{\mathbb{R}^3}\left[I_\alpha * \left(K|u|^2\right)\right] K u\overline{v} - \mathrm{Re}\int_{\mathbb{R}^3} a\frac{f(|u|)}{|u|}u\overline{v}, \quad u, v \in H^1_A(\mathbb{R}^3, \mathbb{C}),$$

where we define $zf(|z|)/|z| := 0$ as $z = 0$.

Proof. By the definition of $H^1_A(\mathbb{R}^3, \mathbb{C})$ and assumpations (K), (a), (f_1), and (f_2), it is easy to show that I_A is well defined in $H^1_A(\mathbb{R}^3, \mathbb{C})$.

Let $\phi(u) = \int_{\mathbb{R}^3} aF(|u|)$ for $u \in H^1_A(\mathbb{R}^3, \mathbb{C})$. Then for $u, v \in H^1_A(\mathbb{R}^3, \mathbb{C}), t \in \mathbb{R}\setminus\{0\}$, we have that

$$\frac{1}{t}[\phi(u + tv) - \phi(u)] = \int_{\mathbb{R}^3}\frac{1}{t}a[F(|u + tv|) - F(|u|)].$$

Using the mean-value theorem, there exists $\theta \in (0, 1)$ such that

$$\frac{1}{t}[F(|u + tv|) - F(|u|)] \;=\; \frac{1}{t}[f(|u| + \theta(|u + tv| - |u|))](|u + tv| - |u|).$$

For $t \in \mathbb{R}$ with $|t| \in (0, 1)$, it thus follows from assumpations (f_1) and (f_2) that

$$\left|\frac{1}{t}[F(|u + tv|) - F(|u|)]\right| \leqslant C(|u| + |v|)|v|.$$

By Lebesgue's dominated convergence theorem, we have that

$$\lim_{t\to 0}\frac{1}{t}[\phi(u+tv)-\phi(u)] = \mathrm{Re}\int_{\mathbb{R}^3} a\frac{f(|u|)}{|u|}u\overline{v}.$$

Similarly, let $\psi(u) = \int_{\mathbb{R}^3}[I_\alpha * (K|u|^2)]\,K|u|^2$ for $u \in H^1_A(\mathbb{R}^3,\mathbb{C})$. Then it follows that for $u, v \in H^1_A(\mathbb{R}^3,\mathbb{C}), t \in \mathbb{R}$ with $|t| \in (0,1)$,

$$\begin{aligned}
& \frac{1}{t}[\psi(u+tv)-\psi(u)] \\
= & \int_{\mathbb{R}^3}\frac{1}{t}\left\{\left[I_\alpha * \left(K|u+tv|^2\right)\right]K|u+tv|^2 - \left[I_\alpha * \left(K|u|^2\right)\right]K|u|^2\right\} \\
= & \int_{\mathbb{R}^3}\frac{1}{t}\left[I_\alpha * \left(K|u+tv|^2\right)\right]K(|u+tv|^2-|u|^2) + \int_{\mathbb{R}^3}\frac{1}{t}\left[I_\alpha * \left(K(|u+tv|^2-|u|^2)\right)\right]K|u|^2 \\
= & \int_{\mathbb{R}^3}\left[I_\alpha * \left(K|u+tv|^2\right)\right]K\left(2\mathrm{Re}(u\overline{v})+t|v|^2\right) + \int_{\mathbb{R}^3}\left[I_\alpha * \left(K\left(2\mathrm{Re}(u\overline{v})+t|v|^2\right)\right)\right]K|u|^2 \\
\leqslant & \ 2\left[I_\alpha * \left(K(|u|^2+|v|^2)\right)\right]K\left(2|u||v|+|v|^2\right) + \int_{\mathbb{R}^3}\left[I_\alpha * \left(K(2|u||v|+|v|^2)\right)\right]K|u|^2.
\end{aligned}$$

Since

$$\begin{aligned}
& \lim_{t\to 0}\left\{\left[I_\alpha * \left(K|u+tv|^2\right)\right]K\left(2\mathrm{Re}(u\overline{v})+t|v|^2\right) + \left[I_\alpha * \left(K\left(2\mathrm{Re}(u\overline{v})+t|v|^2\right)\right)\right]K|u|^2\right\} \\
= & \ 4\mathrm{Re}\left[I_\alpha * (K|u|^2)\right]Ku\overline{v},
\end{aligned}$$

by the Hardy-Littlewood-Sobolev (HLS) inequality, and Lebesgue's dominated convergence theorem, we have that

$$\lim_{t\to 0}\frac{1}{t}[\psi(u+tv)-\psi(u)] = 4\mathrm{Re}\int_{\mathbb{R}^3}\left[I_\alpha * (K|u|^2)\right]Ku\overline{v}.$$

Assuming $u_n \to u$ in $H^1_A(\mathbb{R}^3,\mathbb{C})$, since

$$\begin{aligned}
& \langle I'_A(u_n) - I'_A(u), v\rangle \\
= & \ (u_n-u,v)_A + \lambda\mathrm{Re}\int_{\mathbb{R}^3}\left\{\left[I_\alpha * (K|u_n|^2)\right]Ku_n\overline{v} - \left[I_\alpha * (K|u|^2)\right]Ku\overline{v}\right\} \\
& -\mathrm{Re}\int_{\mathbb{R}^3} a\left[\frac{f(|u_n|)}{|u_n|}u_n - \frac{f(|u|)}{|u|}u\right]\overline{v}, \quad v \in H^1_A(\mathbb{R}^3,\mathbb{C}),
\end{aligned}$$

and given the HLS inequality, the Hölder inequality and the continuity of Nemytskii operator, we have that $\|I'_A(u_n) - I_A(u)\| = o_n(1)$, and the proof is complete. □

Lemma 2.2. *Suppose that assumpations (K), (a), (f_1), and (f_2) hold. Then u is a solution of Eq.* (1.1) *if and only if u is a critical point of I_A.*

Proof. On one hand, if u is a weak solution of Eq. (1.1), then we have that

$$\int_{\mathbb{R}^3}\left(\nabla_A u\cdot\overline{\nabla_A v} + u\overline{v}\right) + \lambda\int_{\mathbb{R}^3}\left[I_\alpha * \left(K|u|^2\right)\right]Ku\overline{v} - \int_{\mathbb{R}^3} a\frac{f(|u|)}{|u|}u\overline{v} = 0, \quad v \in H^1_A(\mathbb{R}^3,\mathbb{C}).$$

Therefore,

$$\mathrm{Re}\left[\int_{\mathbb{R}^3}\left(\nabla_A u\cdot\overline{\nabla_A v}+u\overline{v}\right)+\lambda\int_{\mathbb{R}^3}\left[I_\alpha*\left(K|u|^2\right)\right]Ku\overline{v}-\int_{\mathbb{R}^3}a\frac{f(|u|)}{|u|}u\overline{v}\right]=0,\quad v\in H_A^1(\mathbb{R}^3,\mathbb{C}),$$

that is, u is a critical point of I_A. On the other hand, if u is a critical point of I_A, then we have that $\langle I_A'(u),v\rangle=0$ for $v\in H_A^1(\mathbb{R}^3,\mathbb{C})$. Hence

$$\langle I_A'(u),iv\rangle=\mathrm{Re}\left[-i\left(\int_{\mathbb{R}^3}\left(\nabla_A u\cdot\overline{\nabla_A v}+u\overline{v}\right)+\lambda\int_{\mathbb{R}^3}\left[I_\alpha*\left(K|u|^2\right)\right]Ku\overline{v}-\int_{\mathbb{R}^3}a\frac{f(|u|)}{|u|}u\overline{v}\right)\right]=0.$$

Therefore, it follows that

$$\int_{\mathbb{R}^3}\left(\nabla_A u\cdot\overline{\nabla_A v}+u\overline{v}\right)+\lambda\int_{\mathbb{R}^3}\left[I_\alpha*\left(K|u|^2\right)\right]Ku\overline{v}-\int_{\mathbb{R}^3}a\frac{f(|u|)}{|u|}u\overline{v}=0,\quad v\in H_A^1(\mathbb{R}^3,\mathbb{C}).$$

Combining the above results, we see that the critical points of I_A are solutions of Eq. (1.1), and the proof is complete. □

Before concluding this section, we recall the celebrated mountain-pass theorem [15], which we use herein.

Theorem 2.3. *(Mountain-pass theorem) Let E be a real Banach space. Suppose that $I\in C^1(E,\mathbb{R})$ satisfies*

$$\max\{I(0),I(e)\}<\inf_{\|u\|=\rho}I(u)$$

for some $\rho>0$ and $e\in E$ with $\|e\|>\rho$. Let

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t)),\tag{2.3}$$

where $\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=0,\gamma(1)=e\}$ is the set of paths joining 0 and e. Then there exists a sequence $\{u_n\}\subset E$ such that as $n\to\infty$,

$$I(u_n)\to c,\quad I'(u_n)\to 0.$$

3 Proofs for Theorem 1.1

We begin this section by presenting Lemmas 3.1 and 3.2 that ensure that the functional I_A has the mountain-pass geometry.

Lemma 3.1. *Suppose that assumptions (K), (a), (f_1), and (f_2) hold. Then there exists $\rho>0$ such that* $\inf\{I_A(u):u\in H_A^1(\mathbb{R}^3,\mathbb{C}),\|u\|_A=\rho\}>0$.

Proof. Since $f_0<S_a$ from assumpation (f_1), there exists $\varepsilon\in(0,1)$ such that $f_0<(1-\varepsilon)S_a$. It follows from assumpations (f_1) and (f_2) that there exists $C_\varepsilon>0$ such that

$$f(t)\leqslant(1-\varepsilon)S_a|t|+C_\varepsilon|t|^{6/q'-1},\quad t\in\mathbb{R},\tag{3.1}$$

and then

$$F(t) \leqslant \frac{1}{2}(1-\varepsilon)S_a t^2 + \frac{1}{6}q' C_\varepsilon |t|^{6/q'}, \quad t \in \mathbb{R},$$

where $q' = q/(q-1)$ if $q \in (3/2, \infty)$; and $q' = 1$ if $q = \infty$. Furthermore, assumpation (a) gives $|a|_q < \infty$. Hence, from the Sobolev inequality, for all $u \in H^1_A(\mathbb{R}^3, \mathbb{C})$,

$$\int_{\mathbb{R}^3} aF(|u|) \leqslant \frac{1}{2}(1-\varepsilon)S_a \int_{\mathbb{R}^3} a|u|^2 + \frac{1}{6}q' C_\varepsilon \int_{\mathbb{R}^3} a|u|^{6/q'} \leqslant \frac{1}{2}(1-\varepsilon)\|u\|_A^2 + C_1 C_\varepsilon \|u\|_A^{6/q'},$$

where $C_1 = \frac{1}{6}q'|a|_q \gamma_6^{6/q'}$. It then follows from Eq. (2.2) that for all $u \in H^1_A(\mathbb{R}^3, \mathbb{C})$,

$$I_A(u) \geqslant \frac{1}{2}\|u\|_A^2 - \frac{1}{2}(1-\varepsilon)\|u\|_A^2 - C_1 C_\varepsilon \|u\|_A^{6/q'} = \frac{\varepsilon}{2}\|u\|_A^2 - C_1 C_\varepsilon \|u\|_A^{6/q'}. \tag{3.2}$$

Thus, choosing sufficiently small $\|u\|_A = \rho > 0$, we get the conclusion from the fact that $6/q' \in (2, 6]$, and the proof is complete. □

Lemma 3.2. *Suppose that assumpations (K), (a), (f$_1$), and (f$_2$) hold. Then for sufficiently small $\lambda > 0$, there exists $e \in H^1_A(\mathbb{R}^3, \mathbb{C})$ with $\|e\|_A > \rho$ such that $I_A(e) < 0$, where ρ is given by Lemma 3.1.*

Proof. It follows from assumpation (f$_2$) that there exists $\psi_0 \in H^1_A(\mathbb{R}^3, \mathbb{C})$ with $\|\psi_0\|_A = 1$ such that $f_\infty \int_{\mathbb{R}^3} a|\psi_0|^2 > 1$. Using assumpations (a), (f$_1$), and (f$_2$), we obtain

$$\lim_{t\to\infty} \int_{\mathbb{R}^3} \frac{1}{t^2} aF(t|\psi_0|) = \int_{\mathbb{R}^3} \frac{1}{2} f_\infty a |\psi_0|^2.$$

Then there exists $t_0 > \rho$ such that

$$\int_{\mathbb{R}^3} \frac{1}{t_0^2} aF(t_0|\psi_0|) > \frac{1}{2}.$$

Finally, choosing sufficiently small $\lambda_0 > 0$ such that

$$\frac{1}{2} + \frac{1}{4}\lambda_0 t_0^2 \int_{\mathbb{R}^3} \left[I_\alpha * (K|\psi_0|^2)\right] K|\psi_0|^2 - \int_{\mathbb{R}^3} \frac{1}{t_0^2} aF(t_0|\psi_0|) < 0,$$

it follows from Eq. (2.2) that for $\lambda \in (0, \lambda_0]$,

$$I_A(t_0\psi_0) = t_0^2 \left(\frac{1}{2} + \frac{1}{4}\lambda t_0^2 \int_{\mathbb{R}^3} \left[I_\alpha * \left(K|\psi_0|^2\right)\right] K|\psi_0|^2 - \int_{\mathbb{R}^3} \frac{1}{t_0^2} aF(t_0|\psi_0|) \right) < 0.$$

Let $e = t_0\psi_0$, and the proof is complete. □

By Lemma 3.1, 3.2, and Theorem 2.3, there is a sequence $\{u_n\} \subset H^1_A(\mathbb{R}^3, \mathbb{C})$ such that as $n \to \infty$,

$$I_A(u_n) \to c, \quad I'_A(u_n) \to 0, \tag{3.3}$$

where c is defined in Eq. (2.3) and $c > 0$.

For $r > 0$, we define

$$\begin{aligned} S_{a,r} &= \inf\left\{\|u\|_A^2 : u \in H_A^1(\mathbb{R}^3,\mathbb{C}), \int_{B_r^c} a|u|^2 = 1\right\} \\ &= \inf\left\{\|u\|_A^2 / \int_{B_r^c} a|u|^2 : u \in H_A^1(\mathbb{R}^3,\mathbb{C}), \int_{B_r^c} a|u|^2 \neq 0\right\}. \end{aligned}$$

Particularly, for the case of $a \in L^q(\mathbb{R}^3,\mathbb{R}_+)\backslash\{0\}$ and $a(x) = 0, x \in B_r^c$, we say that $S_{a,r} = \infty$. Since $S_{a,r}$ is nondecreasing for $r \in (0,\infty)$, it makes sense that

$$S_{a,\infty} := \lim_{r\to\infty} S_{a,r}.$$

Remark 3.3. If $q \in (3/2,\infty)$, then we can show that $S_{a,\infty} = \infty$. Indeed, for each $u \in H_A^1(\mathbb{R}^3,\mathbb{C})\backslash\{0\}$, by $a \in L^q(\mathbb{R}^3,\mathbb{R}_+)$ and $a \neq 0$ for $q \in (3/2,\infty)$, we have that $\int_{B_r^c} a|u|^2 \leqslant \gamma_{2q'}^2 |a|_{q,B_r^c}\|u\|_A^2$ for $r > 0$. It follows that $S_{a,r} \geqslant \gamma_{2q'}^{-2}|a|_{q,B_r^c}^{-1}$ for $|a|_{q,B_r^c} \neq 0$, or $S_{a,r} = \infty$. Thus $\lim_{r\to\infty} S_{a,r} = \infty$, that is, $S_{a,\infty} = \infty$. Here, $q' = q/(q-1), \gamma_{2q'}$ is the imbedding constant from $H_A^1(\mathbb{R}^3,\mathbb{C})$ to $L^{2q'}(\mathbb{R}^3,\mathbb{C})$. Moreover, if $a \in L^\infty(\mathbb{R}^3,\mathbb{R}_+)$ and $a_\infty = 0$, then for any $u \in H_A^1(\mathbb{R}^3,\mathbb{C})\backslash\{0\}$, we have that $\int_{B_r^c} a|u|^2 \leqslant |a|_{\infty,B_r^c}\gamma_2^2\|u\|_A^2$. Hence $S_{a,\infty} = \infty$.

Lemma 3.4. *Suppose that assumpations (K), (a), (f$_1$), and (f$_2$) hold. Let $\{\mu_n\} \subset (0,\infty)$ with $\{\mu_n\}$ having a positive lower bound, $v_n \rightharpoonup v$ in $H_A^1(\mathbb{R}^3,\mathbb{C})$, and $\liminf_{n\to\infty}\|v_n\|_A > 0$. If $\langle I_A'(\mu_n v_n), v_n\rangle \to 0$, then the weak limit v of $\{v_n\}$ is nonzero.*

Proof. Since $v_n \rightharpoonup v$ in $H_A^1(\mathbb{R}^3,\mathbb{C})$, there exists a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, such that

$$\begin{aligned} &v_n \rightharpoonup v \ \text{ in } H_A^1(\mathbb{R}^3,\mathbb{C}), \\ &v_n \to v \ \text{ in } L_{\text{loc}}^{2q'}(\mathbb{R}^3,\mathbb{C}), \\ &v_n \to v \ \text{ a.e. on } \mathbb{R}^3. \end{aligned}$$

By contradiction, suppose that $v = 0$. It follows from assumpations (f$_1$) and (f$_2$) that $f(t)/t$ is bounded on $(0,\infty)$. Let

$$l := \sup\{f(t)/t : t > 0\}.$$

So there exist $\theta \in (0,1)$ and sufficiently large $r > 0$ such that

$$l/\theta < S_{a,r}.$$

This yields that for all n,

$$\int_{B_r^c} \frac{1}{\mu_n} af(|\mu_n v_n|)|v_n| \leqslant l\int_{B_r^c} a|v_n|^2 \leqslant \theta\|v_n\|_A^2. \tag{3.4}$$

According to the compactness of the embedding $H_A^1(B_r,\mathbb{C}) \hookrightarrow L^{2q'}(B_r,\mathbb{C})$, it holds that $v_n \to 0$ in $L^{2q'}(B_r,\mathbb{C})$. It follows from assumpations (f$_1$) and (f$_2$) that for all n,

$$0 \leqslant \int_{B_r} \frac{1}{\mu_n} af(|\mu_n v_n|)|v_n| \leqslant l\int_{B_r} a|v_n|^2 \leqslant l|a|_q|v_n|_{2q',B_r}^2,$$

which implies that

$$\lim_{n\to\infty}\int_{B_r}\frac{1}{\mu_n}af(|\mu_n v_n|)|v_n|=0. \tag{3.5}$$

Thus, we obtain from (3.4) and (3.5) that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}\frac{1}{\mu_n}af(|\mu_n v_n|)|v_n|\leqslant\theta\limsup_{n\to\infty}\|v_n\|_A^2. \tag{3.6}$$

On the other hand, it follows that

$$\langle I_A'(\mu_n v_n),v_n\rangle/\mu_n=o(1),$$

that is,

$$\begin{aligned} o(1) &= \|v_n\|_A^2+\lambda\int_{\mathbb{R}^3}\left[I_\alpha*\left(K|\mu_n v_n|^2\right)\right]K|v_n|^2-\int_{\mathbb{R}^3}\frac{1}{\mu_n}af(|\mu_n v_n|)|v_n| \\ &\geqslant \|v_n\|_A^2-\int_{\mathbb{R}^3}\frac{1}{\mu_n}af(|\mu_n v_n|)|v_n|, \end{aligned}$$

where here and in what follows $o(1)$ denotes a quantity that goes to zero as $n\to\infty$. Therefore,

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}\frac{1}{\mu_n}af(|\mu_n v_n|)|v_n|\geqslant\limsup_{n\to\infty}\|v_n\|_A^2,$$

which contradicts Eq. (3.6). Thus, $v\neq 0$, and the proof is complete. □

In the following, for $u\in H_A^1(\mathbb{R}^3,\mathbb{C})$, let $\hat{u}=\begin{cases} u/\|u\|_A, & u\neq 0,\\ 0, & u=0,\end{cases}$

Lemma 3.5. *Suppose that assumpations (K), (a), (f₁), and (f₂) hold. If* $\{u_n\}\subset H_A^1(\mathbb{R}^3,\mathbb{C})\setminus\{0\}$ *satisfies* $\langle I_A'(u_n),\hat{u}_n\rangle\to 0$*, then* $\{u_n\}$ *is bounded in* $H_A^1(\mathbb{R}^3,\mathbb{C})$*.*

Proof. By contradiction, suppose that $\|u_n\|_A\to\infty$ as $n\to\infty$. Let us denote $v_n=\hat{u}_n$ for all n. Then $\{v_n\}$ is bounded in $H_A^1(\mathbb{R}^3,\mathbb{C})$ and there exists $v\in H_A^1(\mathbb{R}^3,\mathbb{C})$ such that, up to a subsequence,

$$\begin{aligned} &v_n\rightharpoonup v \ \text{ in } H_A^1(\mathbb{R}^3,\mathbb{C}),\\ &v_n\to v \ \text{ in } L_{\text{loc}}^{2q'}(\mathbb{R}^3,\mathbb{C}),\\ &v_n\to v \ \text{ a.e. on } \mathbb{R}^3. \end{aligned}$$

Applying Lemma 3.4 with $\mu_n=\|u_n\|_A$, we obtain that $v\neq 0$.

Since $\|u_n\|_A\to\infty$ as $n\to\infty$, it follows that

$$\langle I_A'(u_n),u_n\rangle/\|u_n\|_A^4=o(1),$$

that is,

$$o(1)=\frac{1}{\|u_n\|_A^2}+\lambda\int_{\mathbb{R}^3}\left[I_\alpha*\left(K|v_n|^2\right)\right]K|v_n|^2-\frac{1}{\|u_n\|_A^2}\int_{\mathbb{R}^3}a\frac{f(|u_n|)}{|u_n|}|v_n|^2.$$

Hence,

$$\int_{\mathbb{R}^3}\left[I_\alpha*\left(K|v_n|^2\right)\right]K|v_n|^2=o(1).$$

From Fatou's lemma, we obtain that $\int_{\mathbb{R}^3}[I_\alpha*(K|v|^2)]\,K|v|^2=0$, which implies from assumption (K) that $v=0$. That is a contradiction, and the proof is complete. □

Lemma 3.6. *Suppose that assumptions (K), (a), (f_1), and (f_2) hold. If $u_n\rightharpoonup u$ in $H^1_A(\mathbb{R}^3,\mathbb{C})$ with $u\neq 0$, and $I'_A(u_n)\to 0$, then u is a nonzero critical point of I_A and $u_n\to u$.*

Proof. It is sufficient to prove that $\|u_n\|_A\to\|u\|_A$ as $n\to\infty$. From $I'_A(u_n)\to 0$ and $u_n\rightharpoonup u$, we have that

$$\langle I'_A(u_n),u_n\rangle=\int_{\mathbb{R}^3}\left(|\nabla_A u_n|^2+|u_n|^2\right)+\lambda\int_{\mathbb{R}^3}\left[I_\alpha*\left(K|u_n|^2\right)\right]K|u_n|^2-\int_{\mathbb{R}^3}af(|u_n|)|u_n|=o(1),$$

$$\begin{aligned}&\langle I'_A(u_n),u\rangle\\ =\;&\mathrm{Re}\int_{\mathbb{R}^3}\left(\nabla_A u_n\cdot\overline{\nabla_A u}+u_n\overline{u}\right)+\lambda\mathrm{Re}\int_{\mathbb{R}^3}\left[I_\alpha*\left(K|u_n|^2\right)\right]Ku_n\overline{u}-\mathrm{Re}\int_{\mathbb{R}^3}a\frac{f(|u_n|)}{|u_n|}u_n\overline{u}\\ =\;&o(1),\end{aligned}$$

and

$$\mathrm{Re}\int_{\mathbb{R}^3}\left(\nabla_A u_n\cdot\overline{\nabla_A u}+u_n\overline{u}\right)=\int_{\mathbb{R}^3}\left(|\nabla_A u|^2+|u|^2\right)+o(1).$$

It thus follows that showing that $\|u_n\|_A\to\|u\|_A$ is equivalent to proving that

$$\int_{\mathbb{R}^3}[I_\alpha*\left(K|u_n|^2\right)]K|u_n|^2-\mathrm{Re}\int_{\mathbb{R}^3}\left[I_\alpha*\left(K|u_n|^2\right)\right]Ku_n\overline{u}=o(1),\tag{3.7}$$

$$\int_{\mathbb{R}^3}af(|u_n|)|u_n|-\mathrm{Re}\int_{\mathbb{R}^3}a\frac{f(|u_n|)}{|u_n|}u_n\overline{u}=o(1).\tag{3.8}$$

First, we prove equality (3.8). In fact, by assumpations (f_1) and (f_2), we have that for $r>0$,

$$\begin{aligned}&\left|\int_{B_r^c}af(|u_n|)|u_n|-\mathrm{Re}\int_{B_r^c}a\frac{f(|u_n|)}{|u_n|}u_n\overline{u}\right|\\ \leqslant\;&\int_{B_r^c}\left|af(|u_n|)|u_n|-a\frac{f(|u_n|)}{|u_n|}u_n\overline{u}\right|\\ =\;&\int_{B_r^c}af(|u_n|)|u_n-u|\\ \leqslant\;&l\int_{B_r^c}a|u_n||u_n-u|\\ \leqslant\;&lS^{-1}_{a,r}\|u_n\|_A\|u_n-u\|_A\leqslant CS^{-1}_{a,r}.\end{aligned}$$

This and the compactness of embedding $H^1(\mathbb{R}^3,\mathbb{C})\hookrightarrow L^{2q'}_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{C})$ imply Eq. (3.8).

Now we verify that equality (3.7) holds. First, consider the case that $p \in (6/(1+\alpha), \infty)$. Since $u_n \rightharpoonup u$ in $H^1_A(\mathbb{R}^3, \mathbb{C})$, it follows that $u_n \to u$ in $L^{12p/[p(3+\alpha)-6]}_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C})$ and then for any given $\varepsilon > 0$, there exist $r > 0$ and n_0 such that for all $n > n_0$,

$$|K|_{p,B_r^c} < \varepsilon, \quad |u_n - u|_{12p/[p(3+\alpha)-6],B_r} < \varepsilon.$$

Hence, by the HLS inequality, we have that for all $n > n_0$,

$$\begin{aligned}
& \left| \int_{\mathbb{R}^3} \left[I_\alpha * \left(K|u_n|^2 \right) \right] K|u_n|^2 - \mathrm{Re} \int_{\mathbb{R}^3} \left[I_\alpha * \left(K|u_n|^2 \right) \right] K u_n \overline{u} \right| \\
= & \left| \mathrm{Re} \int_{\mathbb{R}^3} \left[I_\alpha * \left(K|u_n|^2 \right) \right] K \left(|u_n|^2 - u_n \overline{u} \right) \right| \\
\leqslant & \int_{\mathbb{R}^3} \left[I_\alpha * \left(K|u_n|^2 \right) \right] K|u_n||\overline{u}_n - \overline{u}| \\
\leqslant & \; C_1 \left| K|u_n|^2 \right|_{6/(3+\alpha)} \left(\int_{B_r} (K|u_n||\overline{u}_n - \overline{u}|)^{6/(3+\alpha)} + \int_{B_r^c} (K|u_n||\overline{u}_n - \overline{u}|)^{6/(3+\alpha)} \right)^{(3+\alpha)/6} \\
\leqslant & \; C_1 |K|_p |u_n|^2_{12p/[p(3+\alpha)-6]} \left(\int_{B_r} (K|u_n||\overline{u}_n - \overline{u}|)^{6/(3+\alpha)} + \int_{B_r^c} (K|u_n||\overline{u}_n - \overline{u}|)^{6/(3+\alpha)} \right)^{(3+\alpha)/6} \\
\leqslant & \; C_2 \Big(|K|_p^{6/(3+\alpha)} |u_n|^{6/(3+\alpha)}_{12p/[p(3+\alpha)-6]} |u_n - u|^{6/(3+\alpha)}_{12p/[p(3+\alpha)-6],B_r} \\
& + |K|^{6/(3+\alpha)}_{p,B_r^c} |u_n|^{6/(3+\alpha)}_{12p/[p(3+\alpha)-6]} |u_n - u|^{6/(3+\alpha)}_{12p/[p(3+\alpha)-6]} \Big)^{(3+\alpha)/6} \\
\leqslant & \; C_3 \varepsilon.
\end{aligned}$$

Second, consider the case that $p = \infty$. By the HLS inequality, assumption (K), and local compact imbedding theorems, we have that for sufficiently large $r > 0$,

$$\begin{aligned}
& \left| \int_{\mathbb{R}^3} \left[I_\alpha * \left(K|u_n|^2 \right) \right] K|u_n|^2 - \mathrm{Re} \int_{\mathbb{R}^3} \left[I_\alpha * \left(K|u_n|^2 \right) \right] K u_n \overline{u} \right| \\
= & \left| \mathrm{Re} \int_{\mathbb{R}^3} \left[I_\alpha * \left(K|u_n|^2 \right) \right] K \left(|u_n|^2 - u_n \overline{u} \right) \right| \\
\leqslant & \; C_1 \left| K|u_n|^2 \right|_{6/(3+\alpha)} \left(\int_{B_r} (K|u_n||\overline{u}_n - \overline{u}|)^{6/(3+\alpha)} + \int_{B_r^c} (K|u_n||\overline{u}_n - \overline{u}|)^{6/(3+\alpha)} \right)^{(3+\alpha)/6} \\
\leqslant & \; C_1 |K|_\infty |u_n|^2_{12/(3+\alpha)} \left(\int_{B_r} (K|u_n||\overline{u}_n - \overline{u}|)^{6/(3+\alpha)} + \int_{B_r^c} (K|u_n||\overline{u}_n - \overline{u}|)^{6/(3+\alpha)} \right)^{(3+\alpha)/6} \\
\leqslant & \; C_2 \Big(|K|_\infty^{6/(3+\alpha)} |u_n|^{6/(3+\alpha)}_{12/(3+\alpha)} |u_n - u|^{6/(3+\alpha)}_{12/(3+\alpha),B_r} \\
& + |K|^{6/(3+\alpha)}_{\infty,B_r^c} |u_n|^{6/(3+\alpha)}_{12/(3+\alpha)} |u_n - u|^{6/(3+\alpha)}_{12/(3+\alpha)} \Big)^{(3+\alpha)/6} \\
\leqslant & \; C_3 \varepsilon,
\end{aligned}$$

and the proof is complete. □

Finally, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Assume that $\{u_n\} \subset H^1_A(\mathbb{R}^3, \mathbb{C})$ is a sequence satisfying Eq. (3.3). Then by Lemma 3.5, the sequence $\{u_n\}$ is bounded in $H^1_A(\mathbb{R}^3, \mathbb{C})$. We then may assume that, up to a subsequence, $u_n \rightharpoonup u$ in $H^1_A(\mathbb{R}^3, \mathbb{C})$ for some $u \in H^1_A(\mathbb{R}^3, \mathbb{C})$. Since $I_A(u_n) \to c$ and $c > 0$, it follows that $\liminf_{n\to\infty} \|u_n\|_A > 0$. Applying Lemma 3.4 with $\mu_n = 1$ for all n, we obtain that $u \neq 0$. It follows from Lemma 3.6 that u is a nonzero critical point of I_A.

Let us define the set of nonzero critical points of I_A, namely

$$\mathcal{K}_A = \{u \in H^1_A(\mathbb{R}^3, \mathbb{C}) \setminus \{0\} : I'_A(u) = 0\},$$

and

$$m = \inf\{I_A(u) : u \in \mathcal{K}_A\}.$$

Then $\mathcal{K}_A$ is nonempty and $m \geqslant -\infty$. Furthermore, we have that for all $u \in \mathcal{K}_A$,

$$0 = \|u\|_A^2 + \lambda \int_{\mathbb{R}^3} \left[I_\alpha * \left(K|u|^2\right)\right] K|u|^2 - \int_{\mathbb{R}^3} af(|u|)|u| \geqslant \|u\|_A^2 - \int_{\mathbb{R}^3} af(|u|)|u|.$$

Now, choosing $\varepsilon \in (0, 1)$ such that $f_0 < (1-\varepsilon)S_a$, it follows from Eq. (3.1) that for $u \in H^1_A(\mathbb{R}^3, \mathbb{C})$,

$$\int_{\mathbb{R}^3} af(|u|)|u| \leqslant \int_{\mathbb{R}^3} a\left[(1-\varepsilon)S_a|u|^2 + C_\varepsilon|u|^{6/q'}\right] \leqslant (1-\varepsilon)\|u\|_A^2 + |a|_q C_\varepsilon \gamma_6^{6/q'} \|u\|_A^{6/q'}.$$

Therefore, we have that for all $u \in \mathcal{K}_A$,

$$0 \geqslant \|u\|_A^2 - (1-\varepsilon)\|u\|_A^2 - |a|_q C_\varepsilon \gamma_6^{6/q'} \|u\|_A^{6/q'}. \tag{3.9}$$

Since $u \neq 0$ for $u \in \mathcal{K}_A$, it follows from Eq. (3.9) that

$$\|u\|_A \geqslant \left(\frac{\varepsilon}{|a|_q C_\varepsilon \gamma_6^{6/q'}}\right)^{q'/(6-2q')} > 0, \quad u \in \mathcal{K}_A.$$

Hence any limit point of a sequence in $\mathcal{K}_A$ is different from zero.

We claim that I_A is bounded from below on $\mathcal{K}_A$. In fact, assume that $\{u_n\} \subset \mathcal{K}_A$ is such that $I_A(u_n) \to m$ as $n \to \infty$. Then it follows from Lemmas 3.5 and 3.4 that $\{u_n\}$ is bounded in $H^1_A(\mathbb{R}^3, \mathbb{C})$ and has a weakly convergent subsequence, still denoted by $\{u_n\}$, that converges weakly to $u_0 \in H^1_A(\mathbb{R}^3, \mathbb{C})$ and $u_0 \neq 0$. Thus, I_A is bounded from below on $\mathcal{K}_A$ and $m > -\infty$. Furthermore, it follows from Lemma 3.6 that $u_n \to u_0$ and then $I'_A(u_0) = 0, I_A(u_0) = m$. □

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