On the spectrum of Euler-Lagrange operator in the stability analysis of Bénard problem

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Abstract

In studying the stability of Bénard problem we usually have to solve a variational problem to determine the critical Rayleigh number for linear or nonlinear stability. To solve the variational problem one usually transform it to an eigenvalue problem which is called Euler-Lagrange equations. An operator related to the Euler-Lagrange equations is usually referred to as Euler-Lagrange operator whose spectrum is investigated in this paper. We have shown that the operator possesses only the point spectrum consisting of real number, which forms a countable set. Moreover, it is found that the spectrum of the Euler-Lagrange operator depends on the thickness of the fluid layer.

KEY WORDS: Bénard problem, Oberbeck-Boussinesq equations, variational problem, Euler-Lagrange operator, spectrum.

1 Introduction

Bénard problem refers to a fluid layer heated from below. It is not only a well known stability problem in fluid mechanics but also a standard situation in geophysical applications. From the perspective of mechanics, the main research object of the problem is the onset of convection and the transform from laminar fluid flow to turbulence. This problem has many extensions, such as: rotating Bénard problem[1-3], Bénard problem with magnetic field[4, 5] and Bénard problem for double diffusive convection in porous medium[6, 7], etc.

There are a lot of papers dealing with Bénard problem theoretically as well as experimentally[1]. The study of this problem is basically carried out under two kinds of boundary conditions. One is rigid boundaries, and the other is stress-free boundaries. The former is closer to practical applications, while the latter is more convenient for theoretical research. This paper has selected the case of the stress-free boundary conditions for our consideration. Most theoretical research considering this problem is to investigate its linear and nonlinear stability mathematically. For the linear stability one can refer to the celebrated monograph of Chandrasekhar[8], as for the nonlinear stability one can refer to [2-7] and the references

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therein. In the stability analysis of this problem we usually have to solve a variational problem by means of solving the corresponding Euler-Lagrange equations. In many situations, such as Bénard system with rotation or magnetic field, the Euler-Lagrange equation can be solved analytically for the stress boundaries, but not for the rigid boundaries.

In this paper we will study the spectrum of an operator derived from the Euler-Lagrange equations. For this reason, the operator is called Euler-Lagrange operator which is found to be connected with a compact operator. The compact operator is also a Hilbert Schmidt operator. The spectrum of Euler-Lagrange operator is closely related to the existence of solution of the variational problem. Unfortunately, this spectrum problem is rarely discussed in papers.

The paper is organized as follows: Section 2 contains the perturbation equations, boundary conditions, and the poloidal-toroidal decomposition of a solenoidal vector field. Section 3 deduces a variational problem. The calculation of the spectrum of the Euler-Lagrange operator and the related proofs are carried out in section 4.

2 Perturbation equations and boundary conditions

We consider an infinite horizontal layer $\mathbb{R}^2 \times (-1,1)$ which is filled with an incompressible fluid and heated from bottom. Then, the nondimensionalized perturbation equations for the motionless basic state are given by Oberbeck-Boussinesq equations as follows[9,10]:

$$(1) \qquad \begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} + \sqrt{R}\theta \boldsymbol{k} - \nabla p - \boldsymbol{u}\cdot\nabla\boldsymbol{u}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ p_r \partial_t \theta = \Delta\theta + \sqrt{R}\boldsymbol{u}\cdot\boldsymbol{k} - p_r \boldsymbol{u}\cdot\nabla\theta. \end{cases}$$

where R is Rayleigh number, p_r is Prandtl number, Δ is three-dimensional Laplacian and $\boldsymbol{k} = (0,0,1)^T$ is a unit vector being vertical to the fluid layer. $\boldsymbol{u} = (u, v, w)$, θ, p are, velocity, temperature and pressure field, respectively. $\boldsymbol{u}$, θ and p are assumed to be periodic in x, y with periodic cell $\mathcal{P} = (-\frac{\pi}{l_1}, \frac{\pi}{l_1}) \times (-\frac{\pi}{l_2}, \frac{\pi}{l_2})$.

The boundary conditions are stress free and given by

$$(2) \qquad \partial_z u = \partial_z v = w = \theta = 0 \quad at \quad z = \pm 1$$

In order to eliminate the pressure in the momentum conservation equation we apply the poloidal-toroidal-mean flow decomposition to an arbitrary solenoidal vector field[9]

$$(3) \qquad \begin{aligned} \boldsymbol{u} &= \text{curlcurl}\varphi\boldsymbol{k} + \text{curl}\psi\boldsymbol{k} + \boldsymbol{f} \\ &= \boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi + \boldsymbol{f}. \end{aligned}$$

where $\boldsymbol{\delta}\cdot = \text{curlcurl}.\boldsymbol{k} = (\partial_{xz}\cdot, \partial_{yz}\cdot, -\Delta_2\cdot)^T$, $\boldsymbol{\varepsilon}\cdot = \text{curl}.\boldsymbol{k} = (\partial_y\cdot, -\partial_x\cdot, 0)^T$, $\Delta_2 = \partial_x^2 + \partial_y^2$. $\boldsymbol{f} = (f_1, f_2, f_3)^T$ is called mean flow depending only on z and its third component is constant. With the boundary of (2) we obtain $f_3 = 0$. Here $.^T$ refers to transposition. For a given vector $\boldsymbol{V} = (V_1, V_2, V_3)^T$ we denote $\boldsymbol{\delta}\cdot\boldsymbol{V} = \partial_{xz}V_1 + \partial_{yz}V_2 + (-\Delta_2)V_3$.

Under the decomposition (3) the perturbation equations (1) can be equivalently transformed into the following ones[10]:

$$(4)\qquad \begin{cases} \partial_t(-\Delta)(-\Delta_2)\varphi = -(-\Delta)^2(-\Delta_2)\varphi + \sqrt{R}(-\Delta_2)\theta + \boldsymbol{\delta}\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) \\ \partial_t(-\Delta_2)\psi = -(-\Delta)(-\Delta_2)\psi + \boldsymbol{\varepsilon}\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) \\ p_r\partial_t\theta = -(-\Delta)\theta + \sqrt{R}(-\Delta_2)\varphi + p_r\boldsymbol{u}\cdot\nabla\theta \\ \partial_t f_1 = -(-\partial_z^2)f_1 - \frac{1}{|\mathcal{P}|}\int_{\mathcal{P}}\tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}}_1\,dxdy \\ \partial_t f_2 = -(-\partial_z^2)f_2 - \frac{1}{|\mathcal{P}|}\int_{\mathcal{P}}\tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}}_2\,dxdy \end{cases}$$

where $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ refer to the first and second component of $\tilde{\boldsymbol{u}} = \boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi$, respectively. φ, ψ are uniquely determined if we require them to be periodic in x, y (with the periodic cell $\mathcal{P}$) and to have vanishing mean value over $\mathcal{P}$, i.e. $\int_{\mathcal{P}}\varphi dxdy = \int_{\mathcal{P}}\psi dxdy = 0$.

The corresponding boundary conditions for the new dependent variables are

$$(5)\qquad \varphi = \partial_z^2\varphi = \partial_z\psi = \partial_z f_1 = \partial_z f_2 = \theta = 0 \quad \text{at} \quad z = \pm 1.$$

For the shortness, the system (4) can be written as

$$(6)\qquad J_1\Phi_t + J_2\Phi - \sqrt{R}J_3\Phi + M(\Phi) = 0$$

where $\Phi = (\varphi, \psi, \theta, f_1, f_2)^T$. The matrix operators J_1, J_2, J_3 and nonlinear term $M(\Phi)$ have the following form:

$$J_1 = \begin{pmatrix} (-\Delta)(-\Delta_2) & 0 & 0 & 0 & 0 \\ 0 & -\Delta_2 & 0 & 0 & 0 \\ 0 & 0 & p_r I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix},$$

$$J_2 = \begin{pmatrix} \Delta^2(-\Delta_2) & 0 & 0 & 0 & 0 \\ 0 & (-\Delta)(-\Delta_2) & 0 & 0 & 0 \\ 0 & 0 & -\Delta & 0 & 0 \\ 0 & 0 & 0 & -\partial_z^2 & 0 \\ 0 & 0 & 0 & 0 & -\partial_z^2 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 0 & 0 & -\Delta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\Delta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$M(\Phi) = \begin{pmatrix} -\boldsymbol{\delta}\cdot((\boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi + \boldsymbol{f})\cdot\nabla(\boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi + \boldsymbol{f})) \\ -\boldsymbol{\varepsilon}\cdot((\boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi + \boldsymbol{f})\cdot\nabla(\boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi + \boldsymbol{f})) \\ -p_r(\boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi + \boldsymbol{f})\cdot\nabla\theta \\ \frac{1}{|\mathcal{P}|}\int_{\mathcal{P}}(\boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi)\cdot\nabla(\boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi)_1\,dxdy \\ \frac{1}{|\mathcal{P}|}\int_{\mathcal{P}}(\boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi)\cdot\nabla(\boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi)_2\,dxdy \end{pmatrix}.$$

Where I is a unit operator. The Hilbert space H, in which we are going to study our problem, is given by

$$H = \mathcal{H}_M \times \mathcal{H}_M \times \mathcal{H} \times \mathcal{H}_M^1 \times \mathcal{H}_M^1$$

where $\mathcal{H}_M$, $\mathcal{H}$, $\mathcal{H}_M^1$ are defined as follows

$$\begin{cases} \mathcal{H}_M = W((-1,1), L_M^2(\mathcal{P})), \\ \mathcal{H} = W((-1,1), L^2(\mathcal{P})), \\ \mathcal{H}_M^1 = \{f | f \in L^2(-1,1), \int_{-1}^1 f(z)\, dz = 0\}. \end{cases}$$

Here, $L_M^2(\mathcal{P})$ contains the elements of $L^2(\mathcal{P})$ with vanishing mean value over $\mathcal{P}$. $W((-1,1), L^2(\mathcal{P}))$ consists of the mapping $h : (-1,1) \to L^2(\mathcal{P})$ with $h \in L^2((-1,1), L^2(\mathcal{P}))$. For both $\mathcal{H}$ and $\mathcal{H}_M^1$, the inner product are, respectively, given by

$$< f, g >= \int_\Omega f \cdot \bar{g}\, dxdydz, \Omega = \mathcal{P} \times (-1,1)$$

$$< f, g >= |\mathcal{P}| \int_{-1}^1 f \cdot \bar{g}\, dz.$$

With $\|\cdot\|$ we denote the corresponding norm to the inner product. It is easy to show that matrix operators J_1 and J_2 are strictly positive definite selfadjoint. Therefore, $J_1^{\frac{1}{2}}$ and $J_2^{\frac{1}{2}}$ are well defined.

3 A variational problem in linear stability analysis

By the method of linearized stability one neglects the nonlinear term $M(\Phi)$ and study the following equation

$$J_1\Phi_t + J_2\Phi - \sqrt{R}J_3\Phi = 0. \tag{7}$$

We investigate the solution of this equation of the form $\Phi(x, y, z, t) = \Psi e^{\sigma t}$ with $\Psi = \Psi(x, y, z)$ and obtain an eigenvalue problem:

$$\sigma J_1 \Psi = -J_2\Psi + \sqrt{R}J_3\Psi \tag{8}$$

where $\Psi = (\hat{\varphi}, \hat{\psi}, \hat{\theta}, \hat{f}_1, \hat{f}_2)^T$ which satisfy the following boundary conditions

$$\hat{\varphi} = \partial_z^2\hat{\varphi} = \partial_z\hat{\psi} = \partial_z\hat{f}_1 = \partial_z\hat{f}_2 = \hat{\theta} = 0 \quad \text{at} \quad z = \pm 1. \tag{9}$$

In related Hilbert space $\hat{\varphi}$, $\hat{\psi}$ and $\hat{\theta}$ can be expanded in Fourier series:

$$\begin{cases} \hat{\varphi}(x, y, z) = \frac{1}{\sqrt{|\mathcal{P}|}} \sum\limits_{\boldsymbol{k}\in\mathbb{Z}^2-\{0\}} a_{\boldsymbol{k}}(z) e^{i(l_1k_1x + l_2k_2y)} \\ \hat{\psi}(x, y, z) = \frac{1}{\sqrt{|\mathcal{P}|}} \sum\limits_{\boldsymbol{k}\in\mathbb{Z}^2-\{0\}} b_{\boldsymbol{k}}(z) e^{i(l_1k_1x + l_2k_2y)} \\ \hat{\theta}(x, y, z) = \frac{1}{\sqrt{|\mathcal{P}|}} \sum\limits_{\boldsymbol{k}\in\mathbb{Z}^2} c_{\boldsymbol{k}}(z) e^{i(l_1k_1x + l_2k_2y)}. \\ \text{with } \boldsymbol{k} = (k_1, k_2)^T \end{cases}$$

By denoting $\lambda_0 = sup\{Re\sigma\}$ we have

$$\begin{cases} \lambda_0 = sup\{Re\sigma\} > 0, \text{basic flow is linearly unstable ,} \\ \lambda_0 = sup\{Re\sigma\} < 0, \text{basic flow is linearly stable,} \\ \lambda_0 = sup\{Re\sigma\} = 0, \text{basic flow is marginally stable.} \end{cases}$$

The value λ_0 depends obviously on Rayleigh number R. The smallest Rayleigh number corresponding to $\lambda_0 = 0$ is critical Rayleigh number R_c which gives bound for linear stability.

Using boundary condition (9) and taking inner product to both sides of (8) with Ψ yield

$$\sigma \|J_1^{\frac{1}{2}}\Psi\|^2 = -\|J_2^{\frac{1}{2}}\Psi\|^2 + \sqrt{R} < J_3\Psi, \Psi > \tag{10}$$

where Ψ is eigenvector belonging to the eigenvalue σ, and

$$\|J_1^{\frac{1}{2}}\Psi\|^2 = \|\nabla_2\nabla\hat{\varphi}\|^2 + \|\nabla_2\hat{\psi}\|^2 + p_r\|\hat{\theta}\|^2 + \|\hat{f}_1\|^2 + \|\hat{f}_2\|^2,$$

$$\|J_2^{\frac{1}{2}}\Psi\|^2 = \|\nabla_2\Delta\hat{\varphi}\|^2 + \|\nabla_2\nabla\hat{\psi}\|^2 + \|\nabla\hat{\theta}\|^2 + \|\partial_z\hat{f}_1\|^2 + \|\partial_z\hat{f}_2\|^2,$$
$$< J_3\Psi, \Psi > = < -\Delta_2\hat{\theta}, \hat{\varphi} > + < -\Delta_2\hat{\varphi}, \hat{\theta} > = 2Re < -\Delta_2\hat{\theta}, \hat{\varphi} > .$$

Setting $\sigma = Re\sigma + i(Im\sigma)$ and putting it into (10), by comparing imaginary part of both sides of resulting equations we obtain $Im\sigma = 0$, suggesting that the eigenvalue problem (10) has only real eigenvalues.

Since the eigenvalue σ is real, suppose that Ψ is an eigenvector corresponding to σ, then its conjugate function $\overline{\Psi}$ is also an eigenvector to the same eigenvalue σ. Therefore, real functions $\frac{1}{2}(\Psi + \overline{\Psi})$ and $\frac{1}{2i}(\Psi - \overline{\Psi})$ is also eigenvectors to the same eigenvalue σ, if they are non-zero functions. Since both $\frac{1}{2}(\Psi+\overline{\Psi})$ and $\frac{1}{2i}(\Psi-\overline{\Psi})$ can not be vanishing simultaneously. It follows that it is enough to consider the equation (10) in real Hilbert space. This leads to

$$\sigma = Re\sigma = -\frac{\|J_2^{\frac{1}{2}}\Psi\|^2}{\|J_1^{\frac{1}{2}}\Psi\|^2}(1 - \frac{\sqrt{R} < J_3\Psi, \Psi >}{\|J_2^{\frac{1}{2}}\Psi\|^2}).$$

We denote

$$\begin{aligned} \frac{1}{\sqrt{R_c}} &= \sup_{\mathbb{H}-\{0\}} \frac{< J_3\Psi, \Psi >}{\|J_2^{\frac{1}{2}}\Psi\|^2} \\ &= \sup_{\mathbb{H}-\{0\}} \frac{2 < -\Delta_2\hat{\theta}, \hat{\varphi} >}{\|\nabla_2\Delta\hat{\varphi}\|^2 + \|\nabla_2\nabla\hat{\psi}\|^2 + \|\nabla\hat{\theta}\|^2 + \|\partial_z\hat{f}_1\|^2 + \|\partial_z\hat{f}_2\|^2}. \end{aligned} \tag{11}$$

where

$$\mathbb{H} = \Big\{\hat{\varphi},\, \hat{\psi},\, \hat{\theta},\, \hat{f}_1,\, \hat{f}_2 \Big| \{\hat{\varphi},\, \hat{\psi}\} \subset \mathcal{H}_M,\, \hat{\theta} \in \mathcal{H},\, \{\hat{f}_1,\, \hat{f}_2\} \subset \mathcal{H}_M^1,\ \hat{\varphi},\, \hat{\psi},\, \hat{\theta},\, \hat{f}_1,\, \hat{f}_2 \text{ satisfy}$$
$$\text{boundary conditions (9) with } l_1, l_2 \in [0, +\infty) \text{ and } l_1^2 + l_2^2 \neq 0\} .$$

It should be emphasized that a steady basic flow is said to be stable if it is stable with respect to all possible perturbations. This implies that the supremum in (11) is taken for any $l_1, l_2 \in [0, +\infty)$ and $l_1^2k_1^2 + l_2^2k_2^2 \neq 0$. The case of $l_1 \neq 0$ and $l_2 = 0$ indicates that the perturbation is two-dimensional, depending only on t, x and z. Similar understanding for the case of $l_1 = 0$ and $l_2 \neq 0$.

It is obvious that the variational problem (11) is actually equivalent to

$$\frac{1}{\sqrt{R_c}} = \sup_{\mathbb{H}-\{0\}} \frac{2 < -\Delta_2\hat{\theta}, \hat{\varphi} >}{\|\nabla_2\Delta\hat{\varphi}\|^2 + \|\nabla\hat{\theta}\|^2}. \tag{12}$$

Here we have set $\hat{\psi} = \hat{f}_1 = \hat{f}_2 = 0$. If $R < R_c$, we have $\lambda_0 < 0$, which implies that the steady basic flow is linearly stable. If $R > R_c$, we have $\lambda_0 > 0$, which implies that there is at least one positive eigenvalue. In this case, the steady basic flow is linearly unstable.

If the nonlinear stability of the motionless state of the Bénard problem is studied by considering the kinetic energy $E(t) = \|J_1^{\frac{1}{2}}\Psi\|^2$ as an energy functional. It is easy to find that the critical stability bound R_E is also determined by the variational problem (12)(i.e. $R_E = R_c$), and the motionless state is globally and exponentially stable for $R < E_E$.

4 Spectrum of Euler-Lagrange operator

Consider the variational problem (12). We shall replace $\frac{1}{\sqrt{R_c}}$ with parameter μ, then the Euler-Lagrange equation corresponding to the variational problem is given by

$$\begin{cases} \mu(-\Delta)^2(-\Delta_2)\hat{\varphi} = (-\Delta_2)\hat{\theta} \\ \mu(-\Delta)\hat{\theta} = (-\Delta_2)\hat{\varphi} \end{cases} \tag{13}$$

This is an eigenvalue problem, where $\mu \in \mathbb{C}$ is the eigenvalue parameter and the boundary conditions are given in (9). $\mathbb{C}$ denotes the set of complex numbers.

With the denotation of

$$A = \begin{pmatrix} \Delta^2(-\Delta_2) & 0 \\ 0 & -\Delta \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\Delta_2 \\ -\Delta_2 & 0 \end{pmatrix}.$$

the equations (13) are written in the form of $(\mu A - B)\hat{\Phi} = 0$, with $\hat{\Phi} = (\hat{\varphi}, \hat{\theta})^T$. The matrix operator $\mu A - B$ is generally referred to as Euler-Lagrange operator and will be represented by $\hat{E}(\hat{H}, \mu)$, that is $\hat{E}(\hat{H}, \mu) = \mu A - B$ with $\hat{H} = \mathcal{H}_M \times \mathcal{H}_M$. The set

$$\rho(\hat{E}(\hat{H}, \mu)) = \{\mu \in \mathbb{C} |\, \mu A - B \,\text{is injective and}\, (\mu A - B)^{-1}\, \text{a is bounded operator in}\, \hat{H}\}$$

is called resolvent set of the operator $\hat{E}(\hat{H}, \mu)$. The complementary set $\sigma(\hat{E}(\hat{H}, \mu)) = \mathbb{C} \setminus \rho(\hat{E}(\hat{H}, \mu))$ is the spectrum of $\hat{E}(\hat{H}, \mu)$.

For the eigenvalue problem we have the following lemma.

Lemma 4.1 Assuming l_1 and l_2 is given with $l_1^2 + l_2^2 \neq 0$. Then the eigenvalues of the eigenvalue problem (13) form a countable set whose elements are given by

$$\mu_\pm = \pm \frac{({l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)^{\frac{1}{2}}}{(\frac{n^2\pi^2}{4} + {l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)^{\frac{3}{2}}}, \ n = 1, 2, \cdots, \ (k_1, k_2)^T \in \mathbb{Z}^2 - \{0\},$$

where ${l_1}^2{k_1}^2+{l_2}^2{k_2}^2\neq 0$, i.e. $\{\mu_\pm\}\subset\sigma(\hat{E}(\hat{H},\mu))$. The corresponding eigenvectors are then given by

$$\hat{\Phi}_\pm=\begin{pmatrix}\hat{\varphi}\\ \hat{\theta}\end{pmatrix}=\begin{pmatrix}C\\ D_\pm\end{pmatrix}\sin\frac{n\pi}{2}(z+1)\,e^{i(k_1l_1x+k_2l_2y)} \tag{14}$$

where $D_\pm=\frac{C}{(\frac{n^2\pi^2}{4}+{l_1}^2{k_1}^2+{l_2}^2{k_2}^2)\mu_\pm}$, C is any non-zero constant.

Proof. Since $\mu=0$ can not be an eigenvalue. By taking operator $(-\Delta_2)^{-1}$ to first equation of (13), then applying $(-\Delta)$ to the resulting equation, after using second equation of (13) to eliminate $\hat{\theta}$ we get

$$\mu^2(-\Delta)^3\hat{\varphi}=(-\Delta_2)\hat{\varphi}. \tag{15}$$

With boundary conditions $\hat{\varphi}|_{z=\pm1}=\partial_z^2\hat{\varphi}|_{z=\pm1}=\hat{\theta}|_{z=\pm1}=0$ and equation (13) we obtain $\partial_z^2\hat{\theta}|_{z=\pm1}=\partial_z^4\hat{\varphi}|_{z=\pm1}=0$. Combining equation (15) it follows that $\partial_z^6\hat{\varphi}|_{z=\pm1}=0$. Differentiating equation (15) twice with respect to z we have $\partial_z^8\hat{\varphi}|_{z=\pm1}=0$. By further repeating this process successively we conclude that $\partial_z^{2n}\hat{\varphi}|_{z=\pm1}=0$ for $n=0,1,2,\cdots$. Therefore, we need only consider $\hat{\varphi}$ of the form:

$$\hat{\varphi}=C\sin\frac{n\pi}{2}(z+1)\,e^{i(k_1l_1x+k_2l_2y)} \tag{16}$$

where C is non-zero constant. Putting (16) into (15), we obtain

$$\mu=\pm\frac{({l_1}^2{k_1}^2+{l_2}^2{k_2}^2)^{\frac{1}{2}}}{(\frac{n^2\pi^2}{4}+{l_1}^2{k_1}^2+{l_2}^2{k_2}^2)^{\frac{3}{2}}}.$$

Putting (16) into the first equation of (13) we conclude that

$$\hat{\theta}=D_\pm\sin\frac{n\pi}{2}(z+1)\,e^{i(k_1l_1x+k_2l_2y)},\ \text{with } D_\pm=\frac{C}{(\frac{n^2\pi^2}{4}+{l_1}^2{k_1}^2+{l_2}^2{k_2}^2)\mu}.$$

From this we have completed the proof of this lemma.

Lemma 4.2 Suppose $L^2(-1,1)$ is given in section 2, then the following conclusions hold.

(1) The set of functions $\{\cos\frac{(2n-1)\pi}{2}z\}_{n=1}^\infty$ is complete in $L^2(0,1)$, i.e. for any even function $f(z)\in L^2(-1,1)$, $f(z)$ can be expanded in Fourier series of $\{\cos\frac{(2n-1)\pi}{2}z\}_{n=1}^\infty$.

(2) The set of functions $\{\sin\frac{n\pi}{2}(z+1)\}_{n=1}^\infty$ is complete in $L^2(-1,1)$.

Proof. It is well known that the set of functions $\{1,\,\cos n\pi z,\,\sin n\pi z\}_{n=1}^\infty$ is complete in $L^2(-1,1)$. If $f(z)\in L^2(-1,1)$ is even, then the Fourier extension of $f(z)$ reduces to a cose series:

$$f(z)=\frac{a_0}{2}+\sum_{n=1}^\infty a_n\cos n\pi z$$

with $a_n=2\int_0^1 f(z)\cos n\pi z dz$. Since a function defined in $(0,1)$ can be extended to the interval $(-1,1)$ by defining it to be an even function. This implies that the function set

$\{1, \cos n\pi z\}_{n=1}^{\infty}$ is complete in $L^2(0,1)$. If $f(z) \in L^2(0,1)$, by setting $z = 2t$, we obtain a function $F(t) = f(2t), 0 < t < \frac{1}{2}$. If $F(t)$ is extended to the interval $(0,1)$ in the way of $F(t) = -F(1-t)$, $\frac{1}{2} < t < 1$, then from foregoing discussion we find that $F(t)$ has Fourier extension:

$$F(t) = \frac{\hat{a}_0}{2} + \sum_{n=1}^{\infty} \hat{a}_n \cos n\pi t,$$

where $\hat{a}_n = 2\int_0^1 F(t)\cos n\pi t dt$. Note that

$$\hat{a}_n = 2\{\int_0^{\frac{1}{2}} F(t)\cos n\pi t dt + \int_{\frac{1}{2}}^1 F(t)\cos n\pi t dt\}. \tag{17}$$

Applying the transformation $\tau = 1 - t$ to the second integral we get

$$\int_{\frac{1}{2}}^1 F(t)\cos n\pi t dt = -\int_{\frac{1}{2}}^1 F(1-t)\cos n\pi t dt = \int_{\frac{1}{2}}^0 F(\tau)\cos n\pi(1-\tau)d\tau$$

$$= -(-1)^n \int_0^{\frac{1}{2}} F(\tau)\cos n\pi\tau d\tau.$$

Combining with (17) we see that $\hat{a}_n = 0$ for even number of n. Putting $t = \dfrac{z}{2}$ back to $F(t) = f(2t)$ we finally obtain the Fourier extension for $f(z)$ as follows

$$f(z) = \sum_{n=1}^{\infty} \hat{a}_{2n-1} \cos \frac{(2n-1)\pi}{2} z.$$

Therefore, the completeness of $\{\cos \frac{(2n-1)\pi}{2} z\}_{n=1}^{\infty}$ in $L^2(0,1)$ is proved.

By letting $n = 2k-1$ and $n = 2k$ for $k = 1, 2, \cdots$, separately, it follows that

$$\mathrm{span}\{\sin \frac{n\pi}{2}(z+1)\}_{n=1}^{\infty} = \mathrm{span}\{\sin n\pi z, \cos \frac{(2n-1)\pi}{2} z\}_{n=1}^{\infty}.$$

Note that for $f(z) \in L^2(-1,1)$ we have

$$f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}.$$

Observing that $\frac{f(z)+f(-z)}{2}$ is an even function. From just proved results we know that $\frac{f(z)+f(-z)}{2}$ can be extended in Fourier series with respect to $\{\cos \frac{(2n-1)\pi}{2} z\}_{n=1}^{\infty}$. Similarly, $\frac{f(z)-f(-z)}{2}$ is an odd function and can be extended in Fourier series with respect to $\{\sin n\pi z\}_{n=1}^{\infty}$. Therefore, $f(z) \in L^2(-1,1)$ can be expended in Fourier series with respect to $\{\sin \frac{n\pi}{2}(z+1)\}_{n=1}^{\infty}$. So that the second assertion is also proved.

Since A is a strictly positive definite selfadjoint operator, it allows us to study the spectrum of operator $L = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and to prove the following theorem.

Theorem 4.1 Assuming l_1 and l_2 is given in lemma 4.1. Then the operator $L = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has $\mu_\pm$ as its eigenvalues and the corresponding eigenvectors are given by

$$\{\eta_\pm\} = \{\frac{A^{\frac{1}{2}}\hat{\Phi}_+}{\|A^{\frac{1}{2}}\hat{\Phi}_+\|}, \frac{A^{\frac{1}{2}}\hat{\Phi}_-}{\|A^{\frac{1}{2}}\hat{\Phi}_-\|}\}.$$

The eigenvectors $\{\eta_\pm\}$ form a complete orthonormal system in $\hat{H}$.

Proof. Take operator $A^{-\frac{1}{2}}$ to $(\mu A - B)\hat{\Phi} = 0$ we have $\mu A^{\frac{1}{2}}\hat{\Phi} - A^{-\frac{1}{2}}B\hat{\Phi} = 0$. Setting $\hat{\Psi} = A^{\frac{1}{2}}\hat{\Phi}$ yields $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\hat{\Psi} = \mu\hat{\Psi}$. This indicates that μ is the eigenvalue of the operator L with corresponding eigenvector $\hat{\Psi} = A^{\frac{1}{2}}\hat{\Phi}$, if μ is the eigenvalue of the Euler-Lagrange operator $\hat{E}(\hat{H}, \mu)$ with eigenvector $\hat{\Phi}$. Thus, it is proved that $\mu_\pm$ are the eigenvalue of the operator L with corresponding eigenvector $\hat{\Psi} = A^{\frac{1}{2}}\hat{\Phi}_\pm$. Unitizing them we obtain an orthonormal system consisting of eigenvectors of L as follows

$$\{\frac{A^{\frac{1}{2}}\hat{\Phi}_+}{\|A^{\frac{1}{2}}\hat{\Phi}_+\|}, \quad \frac{A^{\frac{1}{2}}\hat{\Phi}_-}{\|A^{\frac{1}{2}}\hat{\Phi}_-\|}\}.$$

In what follows we denote this system by $\{\eta_\pm\}$, with

$$\eta_+ = \frac{A^{\frac{1}{2}}\hat{\Phi}_+}{\|A^{\frac{1}{2}}\hat{\Phi}_+\|}, \quad \eta_- = \frac{A^{\frac{1}{2}}\hat{\Phi}_-}{\|A^{\frac{1}{2}}\hat{\Phi}_-\|}$$

we shall prove that $\{\eta_\pm\}$ is complete in $\hat{H}$.

Suppose that $\{\eta_\pm\}$ is not complete in $\hat{H}$, then there exists a vector $\hat{\Phi}_0 \in \hat{H}(\hat{\Phi}_0 \neq 0)$ satisfying $\hat{\Phi}_0 \in \overline{\text{span}\{\eta_\pm\}}^\perp$. From this it follows that

$$< \hat{\Phi}_0, \eta_\pm > = < A^{\frac{1}{2}}A^{-\frac{1}{2}}\hat{\Phi}_0, \eta_\pm > = < A^{-\frac{1}{2}}\hat{\Phi}_0, A^{\frac{1}{2}}\eta_\pm > = 0.$$

If we can prove that

$$\overline{\text{span}\{A^{\frac{1}{2}}\eta_\pm\}} = \hat{H}. \tag{18}$$

then it turns out that $A^{-\frac{1}{2}}\hat{\Phi}_0 = 0$, this leads to $\hat{\Phi}_0 = 0$. This contradicts the hypothesis, so that $\{\eta_\pm\}$ is complete in $\hat{H}$.

Now we prove (18). From lemma 4.2 and the completeness of system

$$\{e^{i(k_1 l_1 x + k_2 l_2 y)}\}_{(k_1,k_2)^T \in \mathbb{Z}^2 - \{0\}}$$

in $L^2_M(\mathcal{P})$ we know that $\hat{e}(n, \boldsymbol{k}) = \frac{\sqrt{l_1 l_2}}{2\pi} \sin\frac{n\pi}{2}(z+1)\, e^{i(k_1 l_1 x + k_2 l_2 y)}$ is a complete orthonormal system in $\mathcal{H}_M$. From this it follows that

$$\left\{ \begin{pmatrix} \hat{e}(n, \boldsymbol{k}) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \hat{e}(m, \boldsymbol{k}') \end{pmatrix} \right\} \begin{array}{l} \boldsymbol{k}, \boldsymbol{k}' \in \mathbb{Z}^2 - \{0\} \\ n, m \in \mathbb{N} \end{array} \tag{19}$$

form a complete orthonormal system in $\hat{H}$.

With

$$\begin{aligned} A^{\frac{1}{2}}\eta_\pm &= \frac{A\hat{\Phi}_\pm}{\|A^{\frac{1}{2}}\hat{\Phi}_\pm\|} = \frac{1}{\|A^{\frac{1}{2}}\hat{\Phi}_\pm\|} \begin{pmatrix} \Delta^2(-\Delta_2)C \\ (-\Delta)D_\pm \end{pmatrix} \sin\frac{n\pi}{2}(z+1)\, e^{i(k_1 l_1 x + k_2 l_2 y)} \\ &= \frac{1}{\|A^{\frac{1}{2}}\hat{\Phi}_\pm\|} \begin{pmatrix} (\frac{n^2\pi^2}{4} + {l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)^2({l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)C \\ (\frac{n^2\pi^2}{4} + {l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)D_\pm \end{pmatrix} \sin\frac{n\pi}{2}(z+1)\, e^{i(k_1 l_1 x + k_2 l_2 y)} \end{aligned}$$

it follows that

$$
(20)\quad \begin{cases}
A^{\frac{1}{2}}\eta_+\|A^{\frac{1}{2}}\hat{\Phi}_+\| + A^{\frac{1}{2}}\eta_-\|A^{\frac{1}{2}}\hat{\Phi}_-\| \\
= \begin{pmatrix} 2(\frac{n^2\pi^2}{4} + {l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)^2({l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)C \\ 0 \end{pmatrix} \sin\frac{n\pi}{2}(z+1)\, e^{i(k_1 l_1 x + k_2 l_2 y)}, \\
= 4(\frac{n^2\pi^2}{4} + {l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)^2({l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)\frac{C\pi}{\sqrt{l_1 l_2}} \begin{pmatrix} \hat{e}(n,\boldsymbol{k}) \\ 0 \end{pmatrix}. \\
A^{\frac{1}{2}}\eta_+\|A^{\frac{1}{2}}\hat{\Phi}_+\| - A^{\frac{1}{2}}\eta_-\|A^{\frac{1}{2}}\hat{\Phi}_-\| \\
= \begin{pmatrix} 0 \\ 2(\frac{n^2\pi^2}{4} + {l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)D_+ \end{pmatrix} \sin\frac{n\pi}{2}(z+1)\, e^{i(k_1 l_1 x + k_2 l_2 y)}. \\
= 4(\frac{n^2\pi^2}{4} + {l_1}^2{k_1}^2 + {l_2}^2{k_2}^2)\frac{\pi D_+}{\sqrt{l_1 l_2}} \begin{pmatrix} 0 \\ \hat{e}(n,\boldsymbol{k}) \end{pmatrix}.
\end{cases}
$$

These two equations in (20) imply that the system (19) is equivalent to $\{A^{\frac{1}{2}}\eta_+,\, A^{\frac{1}{2}}\eta_-\}$, so (18) is proved.

Theorem 4.2 Assuming l_1 and l_2 is given in lemma 4.1. Then $L = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ is not only a compact but also a Hilbert-Schmidt operator and its spectrum is given by

$$\sigma(L) = \{0\} \cup \{\mu_\pm\}$$

where $\{\mu_\pm\}$ is given in lemma 4.1. Moreover, we have $\sigma(\hat{E}(\hat{H},\mu)) = \{\mu_\pm\}$.

Proof. From the proof of the theorem 4.1 we know that the eigenvectors of operator L form a complete orthonormal system in $\hat{H}$. Combining with the fact that $\{\mu_\pm\}$ constitutes a null-sequence regarding n^2 and $\boldsymbol{k}$, it follows immediately that L is compact.

Observing that

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^2-\{0\} \\ n=1,2,\cdots}} \|L(\eta_\pm)\|^2 = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^2-\{0\} \\ n=1,2,\cdots}} \|L(\frac{A^{\frac{1}{2}}\hat{\Phi}_\pm}{\|A^{\frac{1}{2}}\hat{\Phi}_\pm\|})\|^2 = \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^2-\{0\} \\ n=1,2,\cdots}} |\mu_\pm|^2 < +\infty.$$

With this we conclude that L is a Hilbert-Schmidt operator.

As for the last assertion we need only to pay attention to a fact that

$$\hat{E}(\hat{H},\mu) = A^{\frac{1}{2}}(\mu - L)A^{\frac{1}{2}}.$$

For $\mu \in \rho(L)$, $(\mu - L)^{-1}$ is bounded in $\hat{H}$, together with

$$[\hat{E}(\hat{H},\mu)]^{-1} = A^{-\frac{1}{2}}(\mu - L)^{-1}A^{-\frac{1}{2}}$$

we obtain $\mu \in \rho(\hat{E}(\hat{H},\mu))$. Thus $\sigma(L) \supset \sigma(\hat{E}(\hat{H},\mu))$. Note that the operator $(-\Delta_2)$ is only nonnegative selfadjoint in $\mathcal{H}$, but strictly positive definite selfadjoint in $\mathcal{H}_M$. From this it follows that B^{-1} exists and is bounded in $\hat{H}$. Hence we have $0 \in \rho(\hat{E}(\hat{H},\mu))$. Combine with the results in lemma 4.1 we have proved the assertion.

Remarks: The Euler-Lagrange operator studied in this paper is derived under the background of considering the fluid layer $\mathbb{R}^2 \times (-1,1)$. If the fluid layer $\mathbb{R}^2 \times (-\frac{1}{2},\frac{1}{2})$ is considered, one can follow the same process of this paper to show that the spectrum of the Euler-Lagrange operator of the case $\mathbb{R}^2 \times (-\frac{1}{2},\frac{1}{2})$ is a true subset of the spectrum for $\mathbb{R}^2 \times (-1,1)$. In fact, one can show that the spectrum for the Euler-Lagrange operator for the fluid layer $\mathbb{R}^2 \times (-\frac{1}{2},\frac{1}{2})$ is

$$\{\pm\frac{({l_1}^2{k_1}^2+{l_2}^2{k_2}^2)^{\frac{1}{2}}}{(n^2\pi^2+{l_1}^2{k_1}^2+{l_2}^2{k_2}^2)^{\frac{3}{2}}} \mid n=1,2,\cdots,\ (k_1,k_2)^T\in\mathbb{Z}^2-\{0\}\}.$$

The results in this manuscript show that the eigenvectors of the eigenvalue problem (13) constitute a complete orthonormal system of related Hilbert space, this suggests that the supremum of variational problem can be attained, it is the maximum of the eigenvalues given in lemma 4.1. Therefore, the supremum of the variational problem (12) is also a maximum.

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