On the stability of a time-varying single-species harvesting model with Allee effect

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Abstract

This work brings a one-dimensional logistic harvesting model with Allee effect to the time-varying framework. This new framework is more sober than the autonomous version of the system because it; the framework, permits all environment-dependent coefficients to depend on time. Based on these coefficients, we derive sets of conditions that drive population to "mathematical" extinction. More precisely, we investigate various local and global stability notions including uniform stability, attractivity, asymptotic stability and the (uniform) exponential stability.

Keywords: time-varying system, extinction, attractivity, asymptotic stability, Allee effect, harvest model

1 Introduction

Logistic growth models have been extensively used to describe mathematically many phenomena in a wide spectrum of applicability in biology, economy and ecology. To improve the realism of the aforementioned models, several issues should be subject to further research. To begin with, it is important to comprehend the long-term behavior and stability results of the population [12, 14, 15]. In addition, all coefficients; including the carrying capacity and the growth rate, have to be regarded as functions of time because they depend on environmental fluctuations. This idea has been initially explored by [5, 9]. Thereafter, many researchers have used oscillating carrying capacities to describe seasonal changes; e.g. [11, 19] and references therein. Lastly, special attention has to be paid to the harvesting strategies because they may sharply affect the long-term behavior of the population [3, 4, 10, 20]. The relative-rate harvesting occurs when the number of population caught per time is proportional to the total population size; whereas the constant-rate harvesting occurs when the number of population is caught at a constant rate per unit time independently of the size of the population [6, 13].

Among of the most influential dynamical behaviors of populations with logistic equations is the Allee effect which is phenomenon characterized by reduced per capita population growth rate at low population sizes [1, 7, 20, 21]. This phenomenon is nonlinear, caused by various biological factors and is associated with the increase of population extinction risk [7, 8]. In this paper, we generalize the typical autonomous population growth models in the presence of Allee effect and relative/constant-rate harvesting into the nonautonomous cases: $\dot{N}(t) = g(t, N(t)) - \lambda(t)N(t)$ and $\dot{N}(t) = g(t, N(t)) - \lambda(t)$; where $g(t, N(t)) = r(t)\, N(t) \left(1 - \frac{N(t)}{K(t)}\right)(N(t) - L(t))$, $t \geq t_0$, population density $N(t)$, time-varying capacity $K \in C^0(\mathbb{R}, \mathbb{R})$, Allee threshold $L \in C^0(\mathbb{R}, \mathbb{R})$, harvest term $\mu \in C^0(\mathbb{R}, \mathbb{R})$ and a function $r \in C^0(\mathbb{R}, \mathbb{R})$ that is related to the intrinsic growth rate.

The purpose of this study is to perform analysis concerning the stability of the two systems under study. In particular, we study several types of stabilities including positivity of solutions, (global) origin attractivity, (global) asymptotic stability and (global/uniform) exponential stability. The paper is organized as follows. Section 2 provides some background results that are used throughout the paper. Sets of sufficient conditions for the of the system in the presence of relative-rate harvesting and constant-rate harvesting are derived respectively in Sections 3 and 4.

2 Background results

We present the some results from the literature that are needed in the paper. The next theorem gives conditions for the attractivity of systems with vanishing perturbations.

Theorem 2.1. *[16, Theorem 2.1] Consider the differential equation*

$$\dot{u}(t) = e(t) - q(t)\, \beta(u(t)), \tag{1}$$

where $t \geq t_0$, state $u \in \mathbb{R}$, a strictly increasing function $\beta \in C^0(\mathbb{R}, \mathbb{R})$ that satisfies $\beta(0) = 0$, and continuous real-valued functions e, $q \in C^0(\mathbb{R}, \mathbb{R}_+)$. We assume that $q(t) > 0$ for all $t \geq t_0$, $\int_{t_0}^{\infty} q(t)\, dt = \infty$, $\lim_{t\to\infty} \frac{e(t)}{q(t)} = L \in [0, \infty)$ and $L \in$ Range$\{\beta\}$. Then for each nonnegative initial condition $u(t_0) \geq 0$ and each solution $u(t)$ with maximal interval of existence $[t_0, \omega)$ where $\omega \in (t_0, \infty]$, one has $\omega = \infty$, $\|u\|_\infty < \infty$ and $\lim_{t\to\infty} u(t) = \beta^{-1}(L)$.

In the next theorem, sufficient conditions are derived for the asymptotic stability of systems with unbounded perturbations.

Theorem 2.2. *[17, Theorem 6.2] Consider the class of systems (2)–(3):*

$$\dot{x}(t) = f\big(t, x(t)\big),\ t \geq t_0, \tag{2}$$

$$x(t_0) = x_0, \tag{3}$$

where $t \geq t_0$, solution $x(t)$ in $\mathbb{R}^m$, and a well-defined function $f : [t_0, \infty) \times \mathbb{R}^m \to \mathbb{R}^m$ such that $f(t, 0) = 0$ for all $t \geq t_0$. Assume that the system (2)–(3) satisfies Carathéodory conditions (and hence it has an absolutely continuous Carathéodory solution [2, Section 1.1]). Suppose that

1. *there exist positive constants $\alpha > 0$, $\beta > 0$ such that $\alpha < \beta$ and $(-1)^\alpha = -1$, continuous functions $R_1 \in C^0(\mathbb{R}, \mathbb{R})$, $R_2 \in C^0(\mathbb{R}, \mathbb{R})$ such that $R_1(t) > 0$, $R_2(t) > 0$ for all $t > t_0$, and a Lebesgue measurable function $h : \mathbb{R} \to \mathbb{R}$ satisfying $\int_{t_0}^{\infty} R_1(t)\, h(t)\, dt = \infty$ and $\lim_{t\to\infty} \frac{R_2(t)}{R_1(t)} = \infty$.*

2. $\lim_{t\to\infty} \Lambda(t) = 0$, *where the function Λ is defined almost everywhere on (t_0, ∞) as*

$$\Lambda(t) = \frac{R_1(t)\,\dot{R}_2(t) - \dot{R}_1(t)\,R_2(t)}{h(t)\,(R_1(t))^{\frac{2\beta-\alpha-1}{\beta-\alpha}}\,(R_2(t))^{\frac{\beta-2\alpha+1}{\beta-\alpha}}}.$$

3. *for every solution $x(t)$ of (2)–(3) with maximal interval of existence $[t_0, \omega)$, there exist positive constants c_1, c_2, δ, σ and a Lyapunov function $V \in C^1(\mathbb{R}\times\mathbb{R}^m, \mathbb{R}_+)$, satisfying*

$$c_1 |X|^\sigma \leq V(t, X) \leq c_2 |X|^\sigma, \forall t \in \mathbb{R}, \forall X \in \mathbb{R}^m, \tag{4}$$

$$\begin{aligned} \left.\frac{\partial V(t,X)}{\partial t}\right|_{X=x(t)} &+ \left.\frac{\partial V(t,X)}{\partial X}\right|_{X=x(t)} \cdot f(t, x(t)) \\ &\leq \Big(-R_1(t)\,V^\alpha(t, x(t)) + R_2(t)\,V^\beta(t, x(t))\Big)\,h(t), \end{aligned} \tag{5}$$

for almost all $t \in (t_0, \omega)$ that satisfy $V(t, x(t)) < \delta$.

Then, there exists a positive constant $c > 0$ such that for every $|x_0| < c$, we have $\omega = \infty$. Moreover, the origin $x = 0$ is uniformly stable and is asymptotically stable.

3 Stability of the system in the presence of relative-rate harvesting

We derive sufficient conditions for the positivity, the uniform stability, the asymptotic stability and the (uniform) exponential stability of the system with a relative-rate harvesting:

$$\dot{N}(t) = r(t)\,N(t)\left(1 - \frac{N(t)}{K(t)}\right)(N(t) - L(t)) - \lambda(t)\,N(t),\ t \geq t_0 \in \mathbb{R}. \tag{6}$$

Due to the continuity of the right-hand side of the differential equations (6), a continuously differentiable solution exists and is defined on a maximal intervals of the form $[t_0, \omega)$ where $t_0 < \omega \leq \infty$ [2, Section 1.1]. The origin $N = 0$ is an equilibrium point. Moreover, the system (6) can be rewritten as

$$\dot{N}(t) = -\left(r(t)\,L(t) + \lambda(t)\right)N(t) + r(t)\left(1 + \frac{L(t)}{K(t)}\right)N^2(t) - \frac{r(t)}{K(t)}N^3(t), \forall t \in [t_0, \omega). \tag{7}$$

In this section, we assume that $K(t) \neq 0$ for every $t \geq t_0$. The next lemma studies the positivity of solutions.

Lemma 3.1. *Assume that all time-varying coefficients of N, N^2 and N^3 in (7) are nonnegatives for all $t \geq t_0$. Then for each strictly positive initial value, the positivity of the solution $N(t)$ is guaranteed; i.e. $N(t) \geq 0$ for every $t \geq t_0$.*

Proof. Let $N(t_0) > 0$. To prove the positivity of solution, assume that there exists $t_1 \in (t_0, \omega)$ such that $N(t_1) < 0$. Thus, the continuity of the solution $N(t)$ and the Intermediate

Value Theorem ensure that the set $S = \{t \in (t_0, t_1) : N(t) = 0\}$ is nonempty. Let $t_2 = \sup S$. One can easily show that $t_2 \in S$ so that $N(t_2) = 0$ and that $N(t) < 0$ for all $t \in (t_2, t_1)$. Therefore, we deduce by (7) that $\dot{N}(t) \geq 0$ for all $t \in (t_2, t_1)$. This means that N is increasing on (t_2, t_1) which contradicts the facts that $N(t_1) < 0$ and $N(t_2) = 0$. This proves the positivity of solution. $\square$

Define the functions $Q \in C^0(\mathbb{R}, \mathbb{R})$ and $E \in C^0(\mathbb{R}, \mathbb{R})$ for every $t \geq t_0$ as

$$Q(t) = 2\left(L(t)\, r(t) + \lambda(t)\right),$$

$$E(t) = \frac{2\,|r(t)|\,|K(t) + L(t)|}{|K(t)|}.$$

One can consider the Lyapunov function $V = N^2$ and the differential equation (7) to obtain for all $t \in (t_0, \omega)$ that

$$\begin{aligned} \dot{V}(t) &= 2\dot{N}(t)\, N(t), \\ &= -Q(t)\, V(t) + 2r(t)\left(1 + \frac{L(t)}{K(t)}\right) N^3(t) - \frac{2r(t)}{K(t)} V^2(t). \end{aligned} \tag{8}$$

3.1 Stability conditions when $Q(\cdot) > 0$

The differential equation (8) gives

$$\begin{aligned} \dot{V}(t) &\leq -Q(t)\, V(t) + 2\left| r(t)\left(1 + \frac{L(t)}{K(t)}\right)\right| |N(t)|^3 \\ &\leq -Q(t)\, V(t) + E(t)\, V^{\frac{3}{2}}(t) \ \text{ for all } t \in (t_0, \omega) \text{ that satisfy } \frac{r(t)}{K(t)} \geq 0. \end{aligned} \tag{9}$$

Looking to the inequality (9), one can consider $E(t)$ as a perturbation when $\frac{r(\cdot)}{K(\cdot)} \geq 0$. Results (i) and (ii) of the Theorem 3.1 study respectively the stability of the system (6) for the cases $\lim_{t\to\infty} \frac{E(t)}{Q(t)} = \infty$ and $\lim_{t\to\infty} \frac{E(t)}{Q(t)} = 0$ in which the term $E(t)$ is treated as an unbounded and a vanishing perturbations.

Theorem 3.1. *If $Q(t) > 0$ for each $t \geq t_0$ and $\int_{t_0}^{\infty} Q(t)\, dt = \infty$, then we have the following detached results.*

(i) if $\lim_{t\to\infty}\left(\frac{E(t)}{Q(t)}\right) = \infty$, $\frac{r(t)}{K(t)} \geq 0$, $E(t) > 0, \forall t \geq t_0$ and both functions Q and E are locally absolutely continuous with

$$\lim_{t\to\infty} \frac{Q(t)\, \dot{E}(t) - \dot{Q}(t)\, E(t)}{Q^2(t)\ E(t)} = 0, \tag{10}$$

then $N = 0$ is uniformly stable and is asymptotically stable.

(ii) if $\lim_{t\to\infty}\left(\frac{E(t)}{Q(t)}\right) = 0$ and $\frac{r(t)}{K(t)} \geq 0$ for all $t \geq t_0$, then there exists some $c > 0$ such that for each initial value $N(t_0) \in \mathbb{R}$ with $|N(t_0)| < c$, the solution $N(t)$ of (6) is continuable on $[t_0, \infty)$ and $N = 0$ is locally attractive. If further we have $K(t) = -L(t)$ for all $t \geq t_0$, then the solution $N(t)$ is global and the origin is uniformly stable

and is globally asymptotically stable. Moreover, if there exist constants: $c_1 > 0$ *(that is independent of* t_0*) and* $c_2 \geq 0$ *(that may depend on* t_0*) satisfying* $\int_{t_0}^{t} Q(\tau)\, d\tau \geq c_1 (t - t_0) + c_2, \forall t \geq t_0$*, then the* $N = 0$ *is globally exponentially stable. Moreover, if* c_2 *is independent of* t_0*, then* $N = 0$ *is globally uniformly exponentially stable.*

(iii) if $\lim_{t\to\infty} \left(\frac{E(t)}{Q(t)}\right) = 0$ *and* $\lim_{t\to\infty} \left(\frac{r(t)}{Q(t)K(t)}\right) = 0$*, then for all initial condition* $N(t_0) \in \mathbb{R}$ *with* $|N(t_0)| < c$ *for some* $c > 0$*, the solution* $N(t)$ *of (6) is globally defined on* $[t_0, \infty)$ *and* $N = 0$ *is locally attractive.*

Proof. We prove each case separately.

Proof of Result (i)***.***
We aim to use Theorem 2.2 to prove this case. Since $V = N^2$, the inequality (4) is satisfied with $c_1 = 1$, $c_2 = 1$ and $\sigma = 2$. On the other hand, by (9) we conclude that the inequality (5) is satisfied with $\alpha = 1$, $\beta = 1.5$, $R_1(t) = Q(t)$, $R_2(t) = E(t)$ and $h(t) = 1$ for all $t \geq t_0$. Observe that $\alpha < \beta$, $(-1)^\alpha = -1$ and $\int_{t_0}^{\infty} R_1(t)\, h(t)\, dt = \infty$ because $\int_{t_0}^{\infty} Q(t)\, dt = \infty$. Furthermore, we have $\lim_{t\to\infty} \frac{R_2(t)}{R_1(t)} = \infty$ because $\lim_{t\to\infty} \frac{E(t)}{Q(t)} = \infty$. Simple computations along with the fact that (10) yield $\lim_{t\to\infty} \Lambda(t) = 0$ where Λ in defined in Theorem 2.2. All conditions of Theorem 2.2 are satisfied. Thus, there exists some $c > 0$ such that for all $|N(t_0)| < c$, one has $\omega = \infty$. Furthermore, $N = 0$ is uniformly stable and is asymptotically stable.

Proof of Result (ii)***.***
Suppose that $\lim_{t\to\infty} \left(\frac{E(t)}{Q(t)}\right) = 0$ and $\frac{r(t)}{K(t)} \geq 0$ for all $t \geq t_0$. Without loss of generality, assume that E is not uniformly zero so that $0 < \left\|\frac{E}{Q}\right\|_\infty < \infty$. Pick a positive number δ with $\delta < \left\|\frac{E}{Q}\right\|_\infty^{-2}$. Thus, $\frac{1}{\sqrt{\delta}} > \left\|\frac{E}{Q}\right\|_\infty \geq \frac{E(t)}{Q(t)}$ for all $t > t_0$ so that

$$\frac{E(t)\delta^{\frac{3}{2}}}{Q(t)} < \delta, \forall t > t_0. \tag{11}$$

On the other hand, we obtain by (9) that

$$\dot{V}(t) \leq -Q(t)\, V(t) + E(t)\, \delta^{\frac{3}{2}} \text{ for all } t \in (t_0, \omega) \text{ that satisfy } V(t) < \delta. \tag{12}$$

Thus, we conclude by (11) that

$$\dot{V}(t) \leq 0 \text{ for all } t \in (t_0, \omega) \text{ that satisfy } \frac{E(t)\delta^{\frac{3}{2}}}{Q(t)} < V(t) < \delta. \tag{13}$$

We need to prove that for all $V(t_0) < \delta$, we have $V(t) < \delta$ for all $t \in [t_0, \omega)$. To this end, we use a proof technique similar to the proof of [18, Proposition 3.2]. We consider the cases $V(t_0) < \frac{E(t_0)\delta^{\frac{3}{2}}}{Q(t_0)}$ and $\frac{E(t_0)\delta^{\frac{3}{2}}}{Q(t_0)} \leq V(t_0) < \delta$. For the first case, assume that there exists $t_1 \in (t_0, \omega)$ such that $V(t_1) \geq \delta$. Thus, the continuity of the function E, Q and the Intermediate Value Theorem ensure that the set $S_1 = \left\{t \in [t_0, t_1] : V(t) = \frac{E(t)\delta^{\frac{3}{2}}}{Q(t)}\right\}$ is nonempty (this can be easily shown by contradiction). Let $t_2 = \sup S_1$. It can be shown that $t_2 \in S_1$ and $t_2 < t_1$. By the definition of t_2, we have

$$V\left(t\right) > \frac{E(t)\delta^{\frac{3}{2}}}{Q(t)} \text{ for all } t \in (t_2, t_1]. \tag{14}$$

The inequality (11) along with the assumption $V\left(t_1\right) \geq \delta$ imply that the set $S_2 = \{t \in [t_2, t_1] : V\left(t\right) = \delta\}$ is nonempty. Let $t_3 = \inf S_2$. We have $t_3 \in S_2$ and $t_2 < t_3 \leq t_1$. Furthermore, we deduce by (14) that

$$\frac{E(t)\delta^{\frac{3}{2}}}{Q(t)} < V\left(t\right) < \delta \text{ for all } t \in (t_2, t_3).$$

Thus, by the inequality (13), we get $\dot{V}\left(t\right) \leq 0$ on the interval $t \in (t_2, t_3)$ which means that the Lyapunov function $V(t)$ is nonincreasing on (t_2, t_3). This implies that $V(t_2) > V(t_3)$ which contradicts the facts that $t_2 \in S_1$, $t_3 \in S_2$ and (11). This proves the claim so that when $V(t_0) < \frac{E(t_0)\delta^{\frac{3}{2}}}{Q(t_0)}$, we have $V\left(t\right) < \delta$ for all $t > t_0$.

Similarly, for the second case $\frac{E(t_0)\delta^{\frac{3}{2}}}{Q(t_0)} \leq V(t_0) < \delta$, assume the existence of some $t_4 \in (t_0, \omega)$ such that $V(t_4) \geq \delta$. Consider the set $S_3 = \left\{t \in [t_0, t_4] : V\left(t\right) = \frac{E(t_0)\delta^{\frac{3}{2}}}{Q(t_0)}\right\}$. If S_3 is nonempty, then set $t_4 = \sup S_3$ so that $t_4 \in S_3$. Let $t_5 = \inf\{t \in [t_4, t_1] : V\left(t\right) = \delta\}$. We have $V\left(t_5\right) = \delta$ and $\frac{E(t_0)\delta^{\frac{3}{2}}}{Q(t_0)} < V\left(t\right) < \delta$ for each $t \in (t_4, t_5)$ and thus (13) implies that V is nonincreasing on (t_4, t_5) which contradicts the definitions of t_4 and t_5. If S_3 is empty, then we have $\frac{E(t_0)\delta^{\frac{3}{2}}}{Q(t_0)} < V\left(t\right) < \delta$ for all $t \in (t_0, t_6)$ where $t_6 = \inf\{t \in [t_0, t_1] : V\left(t\right) = \delta\}$. We have $V\left(t_6\right) = \delta$. Moreover, V is nonincreasing on (t_0, t_6) which contradicts the facts that $V\left(t_6\right) = \delta$ and $V(t_0) < \delta$. This ends the proof of the second case. As a result, the prior two cases along with (12) lead to

$$\text{if } V(t_0) < \delta \text{ we have } \dot{V}\left(t\right) \leq -Q\left(t\right)V\left(t\right) + E\left(t\right)\delta^{\frac{3}{2}} \text{ for all } t \in (t_0, \omega).$$

Thus, the facts that $\lim_{t\to\infty}\left(\frac{E(t)}{Q(t)}\right) = 0$, $Q\left(t\right) > 0, \forall t \geq t_0$ and $\int_{t_0}^{\infty} Q\left(t\right)dt = \infty$ ensure that by a comparison principle, one can well see all conditions of Theorem 2.1 are satisfied with β is the identity function, $L = 0$, $q\left(t\right) = Q\left(t\right)$ and $e\left(t\right) = \delta^{\frac{3}{2}}E\left(t\right)$ for all $t \geq t_0$. Thus, when $|N(t_0)| < c = \sqrt{\delta}$, we have $\omega = \infty$, $\|V\|_\infty < \infty$ and $\lim_{t\to\infty} V\left(t\right) = 0$ so that $\|N\|_\infty < \infty$ and $\lim_{t\to\infty} N\left(t\right) = 0$ which proves the local attractivity.

Suppose that $\int_{t_0}^{\infty} Q\left(t\right)dt = \infty$ and $K\left(t\right) = -L\left(t\right)$ so that $E(t) = 0$ for all $t \geq t_0$. Then we deduce by (9) that $\dot{V}\left(t\right) \leq -Q\left(t\right)V\left(t\right)$ for all $t \in (t_0, \omega)$. Thus, by the comparison principle, all conditions of Theorem 2.1 are satisfied for each $N(t_0) \in \mathbb{R}$ with β is the identity function, $L = 0$, $q\left(t\right) = Q\left(t\right)$ and $e\left(t\right) = 0$ for each $t \geq t_0$. Therefore, $\omega = \infty$ and $N = 0$ is globally attractive. Furthermore, we have $\dot{V} \leq 0$ on (t_0, ∞) and hence $V\left(t\right) \leq V\left(t_0\right)$ so that $|N\left(t\right)| \leq |N\left(t_0\right)|$ for all $t \in (t_0, \infty)$. Hence, the origin is uniformly stable and is globally asymptotically stable. Additionally, one can use [22, Theorem 1] to conclude that if there exist constants: $c_1 > 0$ (that is independent of t_0) and $c_2 \geq 0$ (that may depend on t_0) satisfying $\int_{t_0}^{t} Q\left(\tau\right)d\tau \geq c_1\left(t - t_0\right) + c_2, \forall t \geq t_0$, then the $N = 0$ is globally exponentially stable. Moreover, if c_2 is independent of t_0, then the origin is globally uniformly exponentially stable.

Proof of Result (iii)**.**
Without loss of generality, assume that $\varepsilon := \left\|\frac{E(t)}{Q(t)} + 2\left|\frac{r(t)}{Q(t)K(t)}\right|\right\|_\infty > 0$. Let $0 < \delta < \min\left(1, \varepsilon^{-2}\right)$. We have

$$\left(\frac{E\left(t\right)}{Q\left(t\right)} + 2\left|\frac{r\left(t\right)}{Q\left(t\right)K(t)}\right|\right)\delta^{\frac{3}{2}} \leq \varepsilon\delta^{\frac{3}{2}} < \delta, \forall t \in [t_0, \infty). \tag{15}$$

We deduce by (8) that $\dot{V}\left(t\right) \leq -Q\left(t\right)V\left(t\right) + E\left(t\right)V^{\frac{3}{2}}\left(t\right) + 2\left|\frac{r(t)}{K(t)}\right|V^2\left(t\right)$ for all $t \in (t_0, \omega)$ so that

$$\dot{V}\left(t\right) \leq -Q\left(t\right)V\left(t\right) + E\left(t\right)\delta^{\frac{3}{2}} + 2\left|\frac{r\left(t\right)}{K(t)}\right|\delta^2, \text{ for all } t \in (t_0, \omega) \text{ that satisfy } V\left(t\right) < \delta,$$

and hence the fact that $\delta < 1$ gives

$$\dot{V}\left(t\right) \leq -Q\left(t\right)V\left(t\right) + \left(E\left(t\right) + 2\left|\frac{r\left(t\right)}{K(t)}\right|\right)\delta^{\frac{3}{2}}, \text{ for all } t \in (t_0, \omega) \text{ that satisfy } V\left(t\right) < \delta. \tag{16}$$

Thus, we get by (15) that

$$\dot{V}\left(t\right) \leq 0, \text{ for all } t \in (t_0, \omega) \text{ that satisfy } \left(\frac{E\left(t\right)}{Q\left(t\right)} + 2\left|\frac{r\left(t\right)}{Q\left(t\right)K(t)}\right|\right)\delta^{\frac{3}{2}} < V\left(t\right) < \delta.$$

As we have done in the proof of Result (ii), one can show that for all $V(t_0) < \delta$ we have $V(t) < \delta$ for each $t \in (t_0, \omega)$ so that $\|N\|_\infty < \infty$ and $\omega = \infty$. Thus, (16) gives $\dot{V}\left(t\right) \leq -Q\left(t\right)V\left(t\right) + \left(E\left(t\right) + 2\left|\frac{r(t)}{K(t)}\right|\right)\delta^{\frac{3}{2}}$ for all $t \geq t_0$. As a result, a comparison principle along with Theorem 2.1 give $\lim_{t\to\infty} V\left(t\right) = 0$ and $\lim_{t\to\infty} N\left(t\right) = 0$ whenever $|N(t_0)| < c = \sqrt{\delta}$ which ensures the local attractivity. $\square$

Simulations. Consider the initial time $t_0 = 0$ and the initial condition $N(0) = 1$. For every $t \geq t_0$, let $r\left(t\right) = \lambda\left(t\right) = L\left(t\right) = e^t$ and $K\left(t\right) = e^{-t}$. Note that $E\left(t\right) = 2e^t\left(1 + e^{2t}\right)$, $K(t) > 0$, $L(t) > 0$, $r(t) > 0$ and $\frac{r(t)}{K(t)} \geq 0$ for all $t \geq t_0$.

For the case $\lambda(t) = e^{2t}$, we have $Q(t) = 4e^{2t} > 0, \forall t \geq 0$ so that $\int_{t_0}^{\infty} Q\left(t\right)dt = \infty$ and $\lim_{t\to\infty}\left(\frac{E(t)}{Q(t)}\right) = \infty$. The solution $N(t)$ is nonnegative because of Lemma 3.1. Both functions Q and E are differentiable and thus locally absolutely continuous. Moreover, observe that $E\left(t\right) > 0, \forall t \geq 0$ and (10) is satisfied. Therefore, Result (i) of Theorem 3.1 implies that the origin is uniformly stable and is asymptotically stable.

For the case $\lambda(t) = e^{4t}$, we can simply conclude that $Q(t) = 2\left(e^{2t} + e^{4t}\right)$ so that $Q(t) > 0, \forall t \geq 0$. Thus, the positivity of the solution $N(t)$ of (6) is guaranteed Lemma 3.1. On the other hand, since $\int_{t_0}^{\infty} Q\left(t\right)dt = \infty$ and $\lim_{t\to\infty}\left(\frac{E(t)}{Q(t)}\right) = 0$, then we deduce by Result (ii) of Theorem 3.1 that for each initial value $N(t_0) \in \mathbb{R}$ that belong to some neighborhood about the origin, the solution $N(t)$ of (6) is global, bounded and the origin is locally attractive.

Both prior two cases are illustrated in Figure 1 where the attractivity is observed.

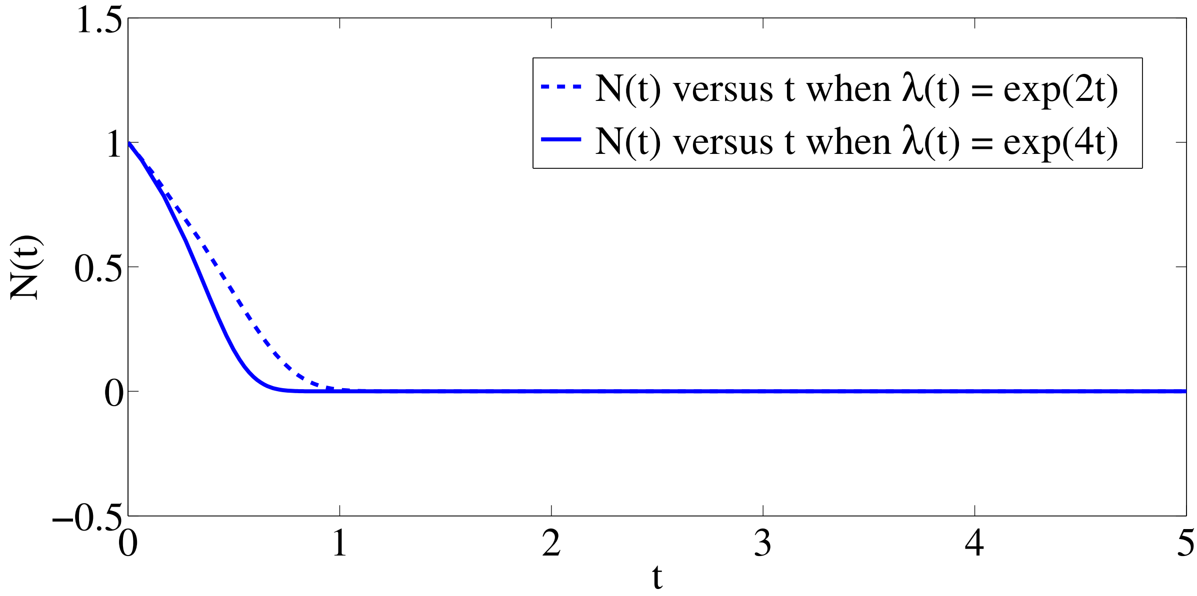


Figure 1: For the cases $\lambda(t) = e^{4t}$ and $\lambda(t) = e^{2t}$, the term $E(t)$ can be respectively considered as a vanishing and an unbounded perturbations of the model (6). Observe that the positivity of solutions and the origin attractivity are ensured for both cases.

3.2 Stability conditions when $\frac{r(\cdot)}{K(\cdot)} > 0$

One deduce by the differential equation (8) that

$$\dot{V}(t) \leq -\frac{2r(t)}{K(t)} V^2(t) + E(t) V^{\frac{3}{2}}(t) \text{ for all } t \in (t_0, \omega) \text{ that satisfy } Q(t) \geq 0. \tag{17}$$

Theorem 3.2. *Assume that* $\frac{r(t)}{K(t)} > 0$ *for each* $t \geq t_0$ *and* $\int_{t_0}^{\infty} \frac{r(t)}{K(t)} dt = \infty$. *We have the following separate results.*

(i) if $\lim_{t\to\infty}(K(t) + L(t)) = 0$ *and* $Q(t) \geq 0$ *for all* $t \geq t_0$*, then for any initial condition* $N(t_0) \in \mathbb{R}$*,* $N(t)$ *is global, bounded and* $N = 0$ *is globally attractive. If further we have* $K(t) = -L(t)$ *for all* $t \geq t_0$*, then* $N = 0$ *is uniformly stable and is globally asymptotically stable.*

(ii) if $\lim_{t\to\infty}(K(t) + L(t)) = 0$ *and* $\lim_{t\to\infty}\left(\frac{K(t)Q(t)}{r(t)}\right) = 0$*, then* $N(t)$ *is global, bounded and* $N = 0$ *is globally attractive.*

Proof. We consider each case separately.
Proof of Result (i)**.**
Let $\delta > \max\left(V(t_0), \|L + K\|_{\infty}^2\right)$ so that

$$\delta^{\frac{3}{4}} \sqrt{|L(t) + K(t)|} \leq \delta^{\frac{3}{4}} \|L + K\|_{\infty}^{\frac{1}{2}} < \delta, \forall t \geq t_0. \tag{18}$$

We get by (17) that

$$\dot{V}(t) \leq \frac{2r(t)}{K(t)}\left(-V^2(t) + |K(t) + L(t)|\,\delta^{\frac{3}{2}}\right) \text{ for all } t \in (t_0, \omega) \text{ that satisfy } V(t) < \delta, \tag{19}$$

and hence (18) gives

$$\dot{V}(t) \leq 0, \text{ for all } t \in (t_0, \omega) \text{ that satisfy } \delta^{\frac{3}{4}}\sqrt{|L(t) + K(t)|} < V(t) < \delta.$$

Observe that δ depends on the choice of the initial condition $V(t_0)$ with $V(t_0) < \delta$ for any initial condition $V(t_0) \in \mathbb{R}$. Moreover, using the same technique of the proof of Result (i) of Theorem 3.1, one can use the contradiction technique to prove that $V(t) < \delta$ for all $t \in (t_0, \omega)$ and thus (19) gives

$$\dot{V}(t) \leq -Q_0(t)\,V^2(t) + E_0(t) \text{ for all } t \in (t_0, \omega),$$

where $Q_0(t) = \frac{2r(t)}{K(t)}$ and $E_0(t) = \frac{2\delta^{\frac{3}{2}}\,r(t)}{K(t)}|K(t) + L(t)|$. Since $\int_{t_0}^{\infty} \frac{r(t)}{K(t)} dt = \infty$ and $\lim_{t\to\infty}(K(t) + L(t)) = 0$, we have $\lim_{t\to\infty}\left(\frac{E_0(t)}{Q_0(t)}\right) = 0$ and $\int_{t_0}^{\infty} Q_0(t)\,dt = \infty$. Therefore, all conditions of Theorem 2.1 come true with $\beta(\tau) = \tau^2, \forall \tau \in \mathbb{R}$, $L = 0$, $q(t) = Q_0(t)$ and $e(t) = E_0(t)$ for all $t \geq t_0$. Thus, both $V(t)$ and $N(t)$ are globally defined on $[t_0, \infty)$ and are bounded with $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} N(t) = 0$ which proves the global attractivity. On the other hand, if $K(t) = -L(t)$ for every $t \geq t_0$, then we have $\dot{V}(t) \leq -Q_0(t)\,V^2(t)$ for each $t > t_0$. Thus, the global attractivity is guaranteed by a comparison principle along with Theorem 2.1. The uniform stability and the global asymptotic stability is also guaranteed by the fact that $\dot{V} \leq 0$ on (t_0, ∞).

Proof of Result (ii)**.**
We obtain by (8) for all $t \in (t_0, \omega)$ that

$$\dot{V}(t) \leq -\frac{2r(t)}{K(t)}V^2(t) + \frac{2r(t)}{K(t)}|K(t) + L(t)|\,V^{\frac{3}{2}}(t) + |Q(t)|\,V(t). \tag{20}$$

Pick $\delta > \max\left(1, V(t_0), \left\| |L + K| + \frac{K|Q|}{2r} \right\|_{\infty}^{2}\right)$ which gives

$$\sqrt{\delta^{\frac{3}{2}}\left(|L(t) + K(t)| + \frac{K(t)\,|Q(t)|}{2r(t)}\right)} \leq \delta^{\frac{3}{4}}\sqrt{\left\| |L + K| + \frac{K\,|Q|}{2r} \right\|_{\infty}} < \delta, \forall t \geq t_0. \tag{21}$$

On the other hand, (20) leads to

$$\begin{aligned} \dot{V}(t) \quad &\leq \quad -\frac{2r(t)}{K(t)}V^2(t) + \frac{2r(t)}{K(t)}|K(t) + L(t)|\,\delta^{\frac{3}{2}} + |Q(t)|\,\delta, \\ &\qquad \text{for all } t \in (t_0, \omega) \text{ that satisfy } V(t) < \delta, \end{aligned} \tag{22}$$

and hence (18) gives

$$\begin{aligned} \dot{V}(t) \quad &\leq \quad 0, \text{ for all } t \in (t_0, \omega) \text{ that satisfy} \\ &\qquad \sqrt{\delta^{\frac{3}{2}}\left(|L(t) + K(t)| + \frac{K(t)\,|Q(t)|}{2r(t)}\right)} \leq V(t) < \delta, \end{aligned}$$

Similar to Result (i), δ depends on the initial value $V(t_0)$ and $V(t_0) < \delta$ for each $V(t_0) \in \mathbb{R}$. As result, one can use the contradiction technique to show that $V(t) < \delta$ for all $t \in (t_0, \omega)$ and hence (22) yields

$$\dot{V}(t) \leq -Q_1(t) V^2(t) + E_1(t) \text{ for all } t \in (t_0, \omega),$$

where $Q_1(t) = \frac{2r(t)}{K(t)}$ and $E_1(t) = \frac{2\delta^{\frac{3}{2}} r(t)}{K(t)} |K(t) + L(t)| + |Q(t)| \delta$. Therefore, one can easily see by a comparison principle that all conditions of all conditions of Theorem 2.1 are satisfied $\beta(\tau) = \tau^2, \forall \tau \in \mathbb{R}$ and $L = 0$. Hence, the functions $V(t)$ and $N(t)$ are defined on $[t_0, \infty)$ and are bounded with $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} N(t) = 0$. This proves the global attractivity because the above results valid uniformly in $N(t_0) \in \mathbb{R}$. □

4 Stability results for the system in the presence of constant-rate harvesting

The next theorem studies the positivity, the uniform stability and the asymptotic stability of the system with a constant-rate harvesting:

$$\dot{N}(t) = r(t) N(t) \left(1 - \frac{N(t)}{K(t)}\right) (N(t) - L(t)) - \lambda(t), \; t \geq t_0. \tag{23}$$

We assume that $K(t) \neq 0$ for all $t \geq t_0$. Let $[t_0, \omega)$ be the maximal interval of existence. One can rewrite (23) into the form

$$\dot{N}(t) = -r(t) L(t) N(t) + r(t) \left(1 + \frac{L(t)}{K(t)}\right) N^2(t) - \frac{r(t)}{K(t)} N^3(t) - \lambda(t), \forall t \in (t_0, \omega). \tag{24}$$

4.1 Stability conditions when $r(\cdot) L(\cdot) > 0$

The next theorem studies the case when $r(\cdot) L(\cdot) > 0$ and $\int_{t_0}^{\infty} r(t) L(t) \, dt = \infty$.

Theorem 4.1. *Suppose that $r(t) L(t) > 0$ for every $t \geq t_0$ and $\int_{t_0}^{\infty} r(t) L(t) \, dt = \infty$. We have the following detached results.*

(i) assume that $\frac{r(t)}{K(t)} \geq 0$ for all $t \geq t_0$ and

$$\lim_{t\to\infty} \left(\frac{1}{L(t)} + \frac{1}{K(t)}\right) = \lim_{t\to\infty} \left(\frac{\lambda(t)}{r(t) L(t)}\right) = 0. \tag{25}$$

Moreover, suppose that the initial time t_0 is large enough to satisfy

$$\left\| \left| \frac{1}{L} + \frac{1}{K} \right| + \frac{|\lambda|}{rL} \right\|_{[t_0, \infty)} < 1. \tag{26}$$

Then for each initial value $N(t_0) \in \mathbb{R}$ that belong to some neighborhood about the origin, the solution $N(t)$ of (6) is global and bounded and $N = 0$ is locally attractive.

(ii) assume that

$$\lim_{t\to\infty}\left(\frac{1}{L(t)}+\frac{1}{K(t)}\right)=\lim_{t\to\infty}\left(\frac{1}{K(t)\,L(t)}\right)=\lim_{t\to\infty}\left(\frac{\lambda(t)}{r(t)\,L(t)}\right)=0. \tag{27}$$

We further assume that the initial time t_0 is large enough to satisfy

$$\left\|\left|\frac{1}{L}+\frac{1}{K}\right|+\frac{|\lambda|}{rL}+\frac{1}{|KL|}\right\|_{[t_0,\infty)}<1. \tag{28}$$

Then for all initial condition $N(t_0)\in\mathbb{R}$ that belong to some neighborhood about the origin, $N(t)$ is continuable on $[t_0,\infty)$ and $N=0$ is locally attractive.

Proof. Let $Q(t)=2r(t)L(t)$ for all $t\geq t_0$. Let $V=N^2$. For all $t\in(t_0,\omega)$, we get by (24) that

$$\begin{aligned}\dot V(t) &= 2\dot N(t)\,N(t)\\ &= -2r(t)L(t)V(t)+2r(t)\left(1+\frac{L(t)}{K(t)}\right)N^3(t)-\frac{2r(t)}{K(t)}N^4(t)+2\lambda(t)\,N(t)\\ &\leq -2r(t)L(t)V(t)+2\left|r(t)\left(\frac{K(t)+L(t)}{K(t)}\right)\right||N(t)|^3-\frac{2r(t)}{K(t)}N^4(t)\\ &\quad +2|\lambda(t)|\,|N(t)|. \end{aligned} \tag{29}$$

***Proof of Result** (i).*
Since $\frac{r(t)}{K(t)}\geq 0$ for all $t\geq t_0$, we deduce by (29) that

$$\dot V(t)\leq -2r(t)L(t)V(t)+2\left|r(t)\left(1+\frac{L(t)}{K(t)}\right)\right|V^{\frac{3}{2}}(t)+2|\lambda(t)|\sqrt{V(t)}. \tag{30}$$

As a result when $V(t)<1$ we get $\dot V(t)\leq -Q(t)\,V(t)+E(t)$ where (for all $t\geq t_0$):

$$E(t)=2\left|\frac{r(t)\,(K(t)+L(t))}{K(t)}\right|+2|\lambda(t)|.$$

Observe that it is assumed that $Q(t)>0$ for all $t\geq t_0$. Moreover, simple computations and (26) give

$$\frac{E(t)}{Q(t)}\leq\left\|\left|\frac{1}{L}+\frac{1}{K}\right|+\frac{|\lambda|}{rL}\right\|_{[t_0,\infty)}<1, \forall t\geq t_0.$$

Thus, we have

$$\dot V(t)\leq 0,\ \text{for all } t>t_0 \text{ that satisfy } \frac{E(t)}{Q(t)}<V(t)<1.$$

One can use the contradiction technique to conclude that if $V(t_0)<1$ we have $V(t)<1$ for all $t>t_0$. This gives $\dot V(t)\leq -Q(t)\,V(t)+E(t)$. We have $\lim_{t\to\infty}\frac{E(t)}{Q(t)}=0$ because of (25). Thus, all conditions of Theorem 2.1 are satisfied with $e(\cdot)=E(\cdot)$, $q(\cdot)=Q(\cdot)$, β is the identity function and $L=0$. Note that $\int_{t_0}^{\infty}q(t)\,dt=\infty$ because it is assumed that $\int_{t_0}^{\infty}r(t)\,L(t)\,dt=\infty$. Therefore, each initial value $V(t_0)<1$, the both $V(t)$ and $N(t)$ are globally defined on $[t_0,\infty)$, $\|V\|_\infty<\infty$ (and hence $\|N\|_\infty<\infty$) and

$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} N(t) = 0$. This proves that $N = 0$ is locally attractive.

Proof of Result (ii)**.**
We obtain by (29) that

$$\begin{aligned}\dot{V}(t) &\leq -2r(t)L(t)V(t) + 2\left|r(t)\left(\frac{K(t)+L(t)}{K(t)}\right)\right|V^{\frac{3}{2}}(t) + 2\left|\frac{r(t)}{K(t)}\right|V^2(t) \\ &\quad +2|\lambda(t)|\sqrt{V(t)}, \text{ for all } t \in (t_0, \omega).\end{aligned}$$

Thus, for the case $V(t) < 1$ we have $\dot{V}(t) \leq -Q(t)V(t) + E_*(t)$ where

$$E_*(t) = 2\left|\frac{r(t)(K(t)+L(t))}{K(t)}\right| + 2|\lambda(t)| + \frac{r(t)}{|K(t)|L(t)}, \forall t \geq t_0.$$

We get by (28) that

$$\frac{E_*(t)}{Q(t)} \leq \left\|\left|\frac{1}{L} + \frac{1}{K}\right| + \frac{|\lambda|}{rL} + \frac{1}{|KL|}\right\|_{[t_0,\infty)} < 1, \forall t \geq t_0.$$

Therefore, we have $\dot{V}(t) \leq 0$, for all $t > t_0$ that satisfy $\frac{E_*(t)}{Q(t)} < V(t) < 1$. A contradiction argument gives that if $V(t_0) < 1$ we get $V(t) < 1$ for every $t > t_0$ so that $\dot{V}(t) \leq -Q(t)V(t) + E_*(t)$. Note that $\lim_{t\to\infty} \frac{E_*(t)}{Q(t)} = 0$ because of (27) and that $\int_{t_0}^{\infty} Q(t)\,dt = \infty$ because $\int_{t_0}^{\infty} r(t)L(t)\,dt = \infty$. All conditions of Theorem 2.1 are satisfied. Thus, for every initial value $V(t_0) < 1$, both $V(t)$ and $N(t)$ are globally defined on $[t_0, \infty)$, $\|V\|_\infty < \infty$, $\|N\|_\infty < \infty$ and $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} N(t) = 0$. Thus, $N = 0$ is locally attractive. □

Simulations. Consider the initial condition $N(t_0) = 0.9$. For every $t \geq t_0$, let $r(t) = e^t$, $L(t) = 30e^t$ and $K(t) = t$. We have $\int_{t_0}^{\infty} r(t)L(t)\,dt = \infty$, $K(t) \neq 0$ and $r(t)L(t) > 0$ for all $t \geq t_0 > 0$. Moreover, (25) is satisfied.

For the initial time $t_0 = 0.8$, the assumption (26) is not satisfied and thus Theorem 4.1 can not guarantee the boundedness and/or the attractivity of the solution. In fact, MATLAB simulations carried out in Figure 2a have clarified that; for the current case, the solution $N(t)$ escapes to infinity in a finite time.

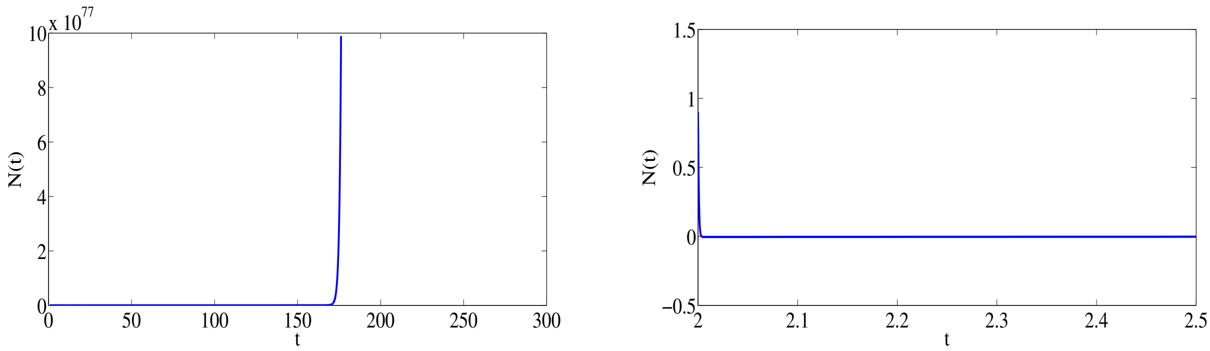
For the initial time $t_0 = 2$, the assumption (26) is satisfied and thus all Theorem 4.1 ensures the solution is globally defined, bounded and the origin is locally attractivity when $N(t_0) \in \mathbb{R}$ is small enough as shown in Figure 2b.

4.2 Stability conditions when $\frac{r(\cdot)}{K(\cdot)} > 0$

The case when $\frac{r(\cdot)}{K(\cdot)} > 0$ and $\int_{t_0}^{\infty} \frac{r(t)}{K(t)}dt = \infty$ is considered in Theorem 4.2.

Theorem 4.2. *Assume that* $\frac{r(t)}{K(t)} > 0$ *for each* $t \geq t_0$ *and* $\int_{t_0}^{\infty} \frac{r(t)}{K(t)}dt = \infty$. *We have the following separate results.*

(i) if $\lim_{t\to\infty}(K(t) + L(t)) = \lim_{t\to\infty} \frac{\lambda(t)K(t)}{r(t)} = 0$ *and* $r(t)L(t) \geq 0$ *for every* $t \geq t_0$*, then for each initial condition* $N(t_0) \in \mathbb{R}$*,* $N(t)$ *is continuable on* $[t_0, \infty)$*, bounded and* $N = 0$ *is globally attractive.*



(a) $N(t)$ versus t when $t_0 = 0.8$ (b) $N(t)$ versus t when $t_0 = 2$

Figure 2: For the figure on the left, the initial time $t_0 = 0.8$ is not large enough to satisfy the sufficient condition (26) of Theorem 4.1. As a result, no wonder that $N(t)$ escapes to infinity in a finite time. Unlike the prior case, the initial time $t_0 = 2$ used in the figure on the right satisfies (26) and thus Theorem 4.1 guarantees the boundedness of $N(t)$ and the attractivity of the origin.

(ii) if $\lim_{t\to\infty}(K(t)+L(t)) = \lim_{t\to\infty}\frac{\lambda(t)K(t)}{r(t)} = \lim_{t\to\infty} L(t)K(t) = 0$, *then for any* $N(t_0)\in\mathbb{R}$, $N(t)$ *is continuable on* $[t_0,\infty)$ *and* $N=0$ *is globally attractive.*

Proof. Let $Q(t) = \frac{2r(t)}{K(t)}$ for all $t \geq t_0$. We have $\int_{t_0}^{\infty} Q(t)\,dt = \infty$ because it is assumed that $\int_{t_0}^{\infty}\frac{r(t)}{K(t)}dt = \infty$.

Proof of Result (i)***.***

Since $r(\cdot)L(\cdot) \geq 0$, (24) gives (for all $t \in (t_0,\omega)$)

$$\dot{V}(t) \leq -\frac{2r(t)}{K(t)}V^2(t) + \frac{2r(t)}{K(t)}\left|L(t)+K(t)\right|V^{\frac{3}{2}}(t) + 2\left|\lambda(t)\right|\sqrt{V(t)}. \tag{31}$$

Pick $\delta > \max\left(1, V(t_0), \left\| |K+L| + \frac{|\lambda|K}{r}\right\|_\infty^2\right)$. We have

$$\sqrt{\left|K(t)+L(t)\right|\delta^{\frac{3}{2}} + \frac{\left|\lambda(t)\right|K(t)\sqrt{\delta}}{r(t)}} \leq \delta^{\frac{3}{4}}\sqrt{\left\| |K+L| + \frac{|\lambda|K}{r}\right\|_\infty} < \delta, \forall t \geq t_0. \tag{32}$$

We have by (31) that

$$\begin{aligned} \dot{V}(t) \quad &\leq \quad -\frac{2r(t)}{K(t)}V^2(t) + \frac{2r(t)}{K(t)}\left|L(t)+K(t)\right|\delta^{\frac{3}{2}} + 2\left|\lambda(t)\right|\sqrt{\delta} \\ &\text{for all } t\in(t_0,\omega) \text{ that satisfy } V(t)<\delta. \end{aligned} \tag{33}$$

The prior inequality along with (32) yield

$$\begin{aligned} \dot{V}(t) \quad &\leq \quad 0, \text{ for all } t\in(t_0,\omega) \text{ that satisfy} \\ &\sqrt{\left|K(t)+L(t)\right|\delta^{\frac{3}{2}} + \frac{\left|\lambda(t)\right|K(t)\sqrt{\delta}}{r(t)}} < V(t) < \delta. \end{aligned}$$

Since for all $N(t_0) \in \mathbb{R}$, we have $\delta > V(t_0)$, one can prove that $V(t) < \delta$ for all $t \in (t_0, \omega)$ for any $N(t_0) \in \mathbb{R}$ and hence (33) yields $\dot{V}(t) \leq -Q(t)V^2(t) + E_1(t)$ for all $t \in (t_0, \omega)$, where $E_1(t) = |K(t) + L(t)|\delta^{\frac{3}{2}} + \frac{|\lambda(t)|K(t)\sqrt{\delta}}{r(t)}$. We have $\lim_{t\to\infty}\left(\frac{E_1(t)}{Q(t)}\right) = 0$ because it is assumed that $\lim_{t\to\infty}(K(t) + L(t)) = \lim_{t\to\infty}\frac{\lambda(t)K(t)}{r(t)} = 0$. Therefore, a comparison principle along with Theorem 2.1 ensure that $\omega = \infty$ and $N = 0$ is globally attractive.

Proof of Result (ii)***.***
The system (24) leads to

$$\begin{aligned} \dot{V}(t) &\leq -\frac{2r(t)}{K(t)}V^2(t) + 2|r(t)L(t)|V(t) + \frac{2r(t)}{K(t)}|L(t) + K(t)|V^{\frac{3}{2}}(t) \\ &\quad +2|\lambda(t)|\sqrt{V(t)} \text{ for all } t \in (t_0, \omega). \end{aligned} \tag{34}$$

Let $\delta > \max\left(1, V(t_0), \left\||LK| + |K+L| + \frac{|\lambda|K}{r}\right\|_\infty^2\right)$. We obtain

$$\begin{aligned} &\sqrt{|L(t)K(t)|\delta + |K(t) + L(t)|\delta^{\frac{3}{2}} + \frac{|\lambda(t)|K(t)\sqrt{\delta}}{r(t)}} \leq \\ &\qquad \delta^{\frac{3}{4}}\sqrt{\left\||LK| + |K+L| + \frac{|\lambda|K}{r}\right\|_\infty} < \delta \text{ for all } t \geq t_0. \end{aligned} \tag{35}$$

The prior inequality along with (34) imply

$$\begin{aligned} \dot{V}(t) &\leq -\frac{2r(t)}{K(t)}V^2(t) + \frac{2r(t)}{K(t)}|L(t) + K(t)|\delta^{\frac{3}{2}} + 2|\lambda(t)|\sqrt{\delta} \\ &\quad \text{for all } t \in (t_0, \omega) \text{ that satisfy } V(t) < \delta, \end{aligned}$$

and

$$\begin{aligned} \dot{V}(t) &\leq 0, \text{ for all } t \in (t_0, \omega) \text{ that satisfy} \\ &\quad \sqrt{|L(t)K(t)|\delta + |K(t) + L(t)|\delta^{\frac{3}{2}} + \frac{|\lambda(t)|K(t)\sqrt{\delta}}{r(t)}} < V(t) < \delta. \end{aligned}$$

Similar to Case (i), we note that for every $N(t_0) \in \mathbb{R}$: $\delta > V(t_0)$. Therefore, it can be shown that for any $N(t_0) \in \mathbb{R}$: $V(t) < \delta$ for each $t \in (t_0, \omega)$ and $\dot{V}(t) \leq -Q(t)V^2(t) + E_2(t)$ for all $t \in (t_0, \omega)$, where $E_2(t) = |L(t)K(t)|\delta + |K(t) + L(t)|\delta^{\frac{3}{2}} + \frac{|\lambda(t)|K(t)\sqrt{\delta}}{r(t)}$. We get $\lim_{t\to\infty}\left(\frac{E_2(t)}{Q(t)}\right) = 0$ because all limits $\lim_{t\to\infty}(K(t) + L(t))$, $\lim_{t\to\infty}\frac{\lambda(t)K(t)}{r(t)}$ and $\lim_{t\to\infty}L(t)K(t)$ are equal to zero. Thus, a comparison principle and Theorem 2.1 completes the proof of the theorem. □

5 Conclusion

The stability of a time-varying single-species model with Allee effect and relative/constant-rate harvesting has been considered. In the presence of relative-rate harvesting, several sets of sufficient conditions have been derived in Theorems 3.1 and 3.2 for many types of local and global stabilities including attractivity, uniform stability, asymptotic stability and exponential stability. Similarly, Theorems 4.1 and 4.2 have given conditions for the constant-rate harvesting case to ensure the local/global attractivity. The proposed results have been confirmed by numerical simulations.

Declaration Statements

Funding

No funding sources to be declared.

Availability of data and material

The data are available to academic researchers upon request.

Acknowledgements

The author is grateful to the responsible editor and the anonymous referees for their valuable efforts.

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