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Group classification of the two-dimensional magnetogasdynamics equations in Lagrangian coordinates

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The present paper is devoted to the group classification of magnetogasdynamics equations in which dependent variables in Euler coordinates depend on time and two spatial coordinates. It is assumed that the continuum is inviscid and nonthermal polytropic gas with infinite electrical conductivity. The equations are considered in mass Lagrangian coordinates. Use of Lagrangian coordinates allows reducing number of dependent variables. The analysis presented in this article gives complete group classification of the studied equations. This analysis is necessary for constructing invariant solutions and conservation laws on the base of Noether's theorem.

Keywords: Magnetohydrodynamics, Lagrangian coordinates, Lie point symmetries

1. Introduction

The equations of magnetogasdynamics (MGD) describe motion of a gas under the action of the internal forces, which consist of the pressure and magnetic forces. These equations describe phenomena related to plasma flows, for example, in plasma confinement, as well as physical problems in astrophysics and fluid metals flows.

The present article considers MGD flows in which dependent variables in Euler coordinates depend on time and two spatial coordinates. It is assumed that the continuum is inviscid and non-thermal polytropic gas with infinite electrical conductivity. For the analysis of equations describing the behavior of such a continuum, the Lie group analysis method is applied.

Lie point symmetries are an effective tool for analyzing nonlinear differential equations [1–5]. They are related with the fundamental physical principles of the model under consideration and correspond to the important properties of the differential equations. Finding an admitted Lie group is one of the first and necessary steps in application of the group analysis method to partial differential equations. Using found symmetries one can construct a representation of invariant or partially invariant solution. The representation of a solution reduces the number of the independent variables. The group analysis method guarantees that the reduced system of equations for an invariant solution has fewer independent variables and is involutive. Admitted symmetry of variational partial differential equations is a necessary condition for application of Noether's theorem, which is used for deriving conservation laws.

Applications of the group analysis method for different versions of MGD equations have been considered in many publications. For example, the case of the finite conductivity was investigated in [6, 7]. The case of the infinite conductivity was examined in [8, 9]. Invariant solutions were considered in [10–16]. Comprehensive analysis of MGD equations in Eulerian and Lagrangian coordinates with plain and cylindrical symmetries were given in [17, 18].

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The present paper is devoted to the group classification of the MGD equations, where all dependent variables in Eulerian coordinates depend on time and two spatial coordinates[†]. The study is performed in mass Lagrangian coordinates. The transition to mass Lagrangian coordinates makes it possible to solve four MGD equations. As a result of this solving, four arbitrary functions of the mass Lagrangian coordinates are obtained. In group analysis, these functions are called arbitrary elements. The presence of arbitrary elements requires a group classification, which consists of finding all Lie groups admitted by a system of partial differential equations [2–4]. In practice, groups are represented by their generators. The generators admitted for all arbitrary elements form the kernel of the admitted Lie algebras. The group classification represents all non-equivalent extensions of the kernel and the corresponding concrete forms of arbitrary elements, where the equivalence is considered with respect to equivalence transformations that preserve the structure of the equations, but can change arbitrary elements.

The paper is organized as follows. The next section provides MGD equations in mass Lagrangian coordinates. Derivation of the equations in Lagrangian coordinates, when the dependent variables in Eulerian coordinates depend on time and two independent space variables. Section 4 provides equivalence transformations, which are used for simplification arbitrary elements. Sections 5 and 7 give the group classifications of nonisentropic and isentropic solutions when $b_{01}^2 + b_{02}^2 \neq 0$. Sections 6 and 8 are devoted to the group classifications of nonisentropic and isentropic solutions when $b_{01}^2 + b_{02}^2 = 0$. Conclusions are stated in Section 8.

2. Magnetogasdynamics equations in mass Lagrangian coordinates

The magnetogasdynamics equations of an ideal perfect polytropic gas can be written in the following form [15, 19]

$$
\begin{gathered}
D\rho + \rho\, div\, \mathbf{u} = 0, \\
D\mathbf{u} + \rho^{-1}\nabla(p + \tfrac{1}{2}\mathbf{H}^2) - \rho^{-1}(\mathbf{H}\cdot\nabla)\mathbf{H} = 0, \\
D\mathbf{H} + \mathbf{H}\, div\, \mathbf{u} - (\mathbf{H}\cdot\nabla)\mathbf{u} = 0, \;\; div\, \mathbf{H} = 0, \\
DS = 0,
\end{gathered}
\tag{1}
$$

where ρ, $\mathbf{u}$, p, S, and $\mathbf{H}$ correspond to the gas density, fluid velocity, pressure, entropy and magnetic induction, respectively, and γ is the polytropic exponent,

$$
D = \partial_t + \mathbf{u}\cdot\nabla, \;\; \mathbf{H} = (H_1, H_2, H_3), \;\; \mathbf{u} = (u_1, u_2, u_3), \;\; \boldsymbol{x} = (x_1, x_2, x_3).
$$

The magnetic field strength $\mathbf{H}$ and magnetic field induction $\mathbf{B}$ are related by the equation $\mathbf{B} = \sqrt{\mu_0}\mathbf{H}$, where μ_0 is the magnetic permeability. The pressure p, the density ρ and the entropy S are related by the state equation $p = A(S)\rho^\gamma$, where $A(S) = Re^{(S-S_0)/c_v}$, R is the gas constant, c_v is the dimensionless specific heat capacity at constant volume, and S_0 is constant.

In coordinate form equations (1) become

$$
\rho_t + u_i\rho_{x_i} + \rho u_{ix_i} = 0, \tag{2a}
$$
$$
\rho(u_{jt} + u_i u_{jx_i}) + H_i H_{ix_j} - H_i H_{jx_i} + p_{x_j} = 0, \;\; (j = 1, 2, 3), \tag{2b}
$$
$$
H_{jt} + u_i H_{jx_i} + H_j u_{ix_i} - H_i u_{jx_i} = 0, \;\; (j = 1, 2, 3), \tag{2c}
$$
$$
H_{ix_i} = 0, \tag{2d}
$$
$$
S_t + u_i S_{x_i} = 0, \tag{2e}
$$

where the energy equation is rewritten. Here summation with respect to a repeated index is assumed.

The mass Lagrangian coordinates are introduced by the relations

$$
\rho = J^{-1}, \;\; \varphi_{it}(t,\xi) = u_i(t, \varphi(t,\xi)), \tag{3}
$$

where

$$
\xi = (\xi_1, \xi_2, \xi_3), \;\; \varphi = (\varphi_1, \varphi_2, \varphi_3), \;\; J = \det\left(\frac{\partial\varphi}{\partial\xi}\right), \;\; T = \frac{\partial\varphi}{\partial\xi} = \begin{pmatrix} \varphi_{1,1} & \varphi_{2,1} & \varphi_{3,1} \\ \varphi_{1,2} & \varphi_{2,2} & \varphi_{3,2} \\ \varphi_{1,3} & \varphi_{2,3} & \varphi_{3,3} \end{pmatrix},
$$

and $\varphi_{i,j} = \dfrac{\partial\varphi_i}{\partial x_j}$.

In mass Lagrangian coordinates the conservation law of mass (2a) becomes identical and equation (2e) gives that $S = S_0(\xi)$, where $S_0(\xi)$ is an arbitrary function.

[†]Such solutions can also be three-dimensional.

For the sake of completeness we provide here the transition of equations (2) to mass Lagrangian coordinates [15]. Let $A = JT^{-1}$, then

$$A_{ik}T_{kl} = J\delta_{il}.$$

Noting that $\frac{\partial}{\partial \xi_j} = \varphi_{i,j}\frac{\partial}{\partial x_i}$, the operators

$$\frac{\partial}{\partial x} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}, \quad \frac{\partial}{\partial \xi} = \begin{pmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \\ \frac{\partial}{\partial \xi_3} \end{pmatrix},$$

can be represented as follows

$$\frac{\partial}{\partial \xi} = T\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x} = J^{-1}A\frac{\partial}{\partial \xi}.$$

Gauss' law (2d) gives

$$JH_{ix_i} = A_{ik}H_{i\xi_k} = 0. \tag{4}$$

Direct calculations show that

$$\frac{\partial}{\partial \xi_k}(A_{ik}) = 0, \ \ \forall i. \tag{5}$$

The latter leads to the relations

$$A_{ik}H_{j\xi_k} = (A_{ik}H_j)_{\xi_k}, \ \ A_{ik}p_{\xi_k} = (A_{ik}p)_{\xi_k}, \ \ \forall i,\, j.$$

Using these relations, the part of momentum equations (2b) in Lagrangian coordinates become

$$H_iH_{ix_j} - H_iH_{jx_i} + p_{x_j} = J^{-1}(H_iA_{jk}H_{i\xi_k} - H_iA_{ik}H_{j\xi_k} + A_{jk}p_{\xi_k}) =$$
$$= J^{-1}(\frac{1}{2}A_{jk}\frac{\partial H^2}{\partial \xi_k} - H_i(A_{ik}H_j)_{\xi_k} + A_{jk}p_{\xi_k}) =$$
$$= J^{-1}(\frac{\partial}{\partial \xi_k}(\frac{1}{2}A_{jk}H^2 + A_{jk}p) - H_i(A_{ik}H_j)_{\xi_k}).$$

By virtue of Gauss' law (4), one derives that

$$H_i(A_{ik}H_j)_{\xi_k} = (H_iA_{ik}H_j)_{\xi_k} - A_{ik}H_{i\xi_k}H_j = (H_iA_{ik}H_j)_{\xi_k}.$$

Hence,

$$H_iH_{ix_j} - H_iH_{jx_i} + p_{x_j} = J^{-1}\frac{\partial}{\partial \xi_k}\left(A_{jk}(\frac{1}{2}H^2 + p) - H_iA_{ik}H_j\right) =$$
$$= J^{-1}\frac{\partial}{\partial \xi_k}\left(\delta_{ij}A_{ik}(\frac{1}{2}H^2 + p) - H_iA_{ik}H_j\right) =$$
$$= J^{-1}\frac{\partial}{\partial \xi_k}\left(A_{ik}\left(\delta_{ij}(\frac{1}{2}H^2 + p) - H_iH_j\right)\right)$$

Then the momentum equations in Lagrangian coordinates have the form

$$\frac{\partial^2 \varphi_j}{\partial t^2} + \frac{\partial}{\partial \xi_k}\left(A_{ik}\left(\delta_{ij}(\frac{1}{2}H^2 + p) - H_iH_j\right)\right) = 0.$$

Faraday's equations (2c) in Lagrangian coordinates reduces as follows. Let $\boldsymbol{b} = \rho^{-1}\mathbf{H}$, then using the conservation law of mass and Faraday's equations, one obtains

$$\frac{db_j}{dt} = -\rho^{-2}\frac{d\rho}{dt}H_j + \rho^{-1}\frac{dH_j}{dt} = \rho^{-1}H_iu_{jx_i} = b_iu_{jx_i}.$$

Introducing the vector $\boldsymbol{b}_0$ such that $\boldsymbol{b} = T\boldsymbol{b}_0$, one derives

$$\frac{\partial b_j}{\partial t} - J^{-1}b_iA_{ik}u_{j\xi_k} = \frac{\partial b_{0\alpha}}{\partial t}\varphi_{j,\alpha} + b_{0\alpha}u_{j\xi_\alpha} - b_{0\alpha}(J^{-1}T_{\alpha i}A_{ik})u_{j\xi_k} = \frac{\partial b_{0\alpha}}{\partial t}\varphi_{j,\alpha} = 0.$$

The latter gives that

$$\frac{\partial b_{0\alpha}}{\partial t} = 0, \ \ \forall \alpha.$$

Hence, similar to the entropy, one integrates the Faraday's equation $\boldsymbol{b}_0 = \boldsymbol{b}_0(\xi)$, where $\boldsymbol{b}_0(\xi) = (b_{01}(\xi), b_{02}(\xi), b_{03}(\xi))$ is an arbitrary vector function of ξ. Gauss's equation (4) reduces as follows

$$\begin{aligned} H_{ix_i} = (\rho b_i)_{x_i} = (\rho b_{0\alpha}\varphi_{i,\alpha})_{x_i} = J^{-1}A_{ik}(\rho b_{0\alpha}\varphi_{i,\alpha})_{\xi_k} = J^{-1}(J^{-1}A_{ik}b_{0\alpha}\varphi_{i,\alpha})_{\xi_k} \\ = J^{-1}(J^{-1}T_{\alpha i}A_{ik}b_{0\alpha})_{\xi_k} = J^{-1}(b_{0k})_{\xi_k} = 0. \end{aligned}$$

Therefore, in mass Lagrangian coordinates equations (2) reduce to the equations

$$\frac{\partial^2\varphi_j}{\partial t^2} + \frac{\partial}{\partial \xi_k}\left(A_{ik}\left(\delta_{ij}(\frac{1}{2}H^2 + p) - H_iH_j\right)\right) = 0, \ \ (j = 1, 2, 3), \ \ \frac{\partial}{\partial \xi_k}b_{0k} = 0, \tag{6}$$

$$S = S(\xi), \ \ \boldsymbol{b}_0 = \boldsymbol{b}_0(\xi),$$

where

$$H_i = J^{-1}b_{0\alpha}\varphi_{i,\alpha}, \ \ H^2 = J^{-2}b_{0\alpha}b_{0\beta}\varphi_{i,\alpha}\varphi_{i,\beta}.$$

3. Equations (1) with two independent space variables in Lagrangian coordinates

We study the case, where all dependent functions in Eulerian coordinates only depend on two space variables x_1 and x_2. From equations (3) one obtains the Cauchy problem[‡]

$$\begin{aligned} (\varphi_{1,3})_t = u_{1x_1}\varphi_{1,3} + u_{1x_2}\varphi_{2,3}, \ \ \varphi_{1,3}(0, \xi_1, \xi_2, \xi_3) = 0, \\ (\varphi_{2,3})_t = u_{2x_1}\varphi_{1,3} + u_{2x_2}\varphi_{2,3}, \ \ \varphi_{2,3}(0, \xi_1, \xi_2, \xi_3) = 0. \end{aligned}$$

For sufficiently smooth functions $\boldsymbol{u}(t, \boldsymbol{x})$ the latter Cauchy problem has unique solution $\varphi_{i,3} = 0, \ (i = 1, 2)$ that means

$$\varphi_1 = \varphi(t, \xi_1, \xi_2), \ \ \varphi_2 = \zeta(t, \xi_1, \xi_2).$$

In this case the transition from Lagrangian coordinates to the mass Lagrangian coordinates can be done such that $\varphi_3(0, \xi_1, \xi_2, \xi_3) = \xi_3$. Hence, because of the uniqueness of a solution of the Cauchy problem

$$(\varphi_{3,3})_t = u_{3x_1}\varphi_{\xi_3} + u_{3x_2}\zeta_{\xi_3} = 0, \ \ \varphi_{3,3}(0, \xi_1, \xi_2, \xi_3) = 1,$$

on gets $\varphi_3 = \xi_3 + \chi(t, \xi_1, \xi_2)$. Further we use the notations $\xi_1 = \xi, \ \ \xi_2 = \eta$. Thus, one has

$$T = \frac{\partial\varphi}{\partial\xi} = \begin{pmatrix} \varphi_\xi & \zeta_\xi & \chi_\xi \\ \varphi_\eta & \zeta_\eta & \chi_\eta \\ 0 & 0 & 1 \end{pmatrix}, \ \ A = \begin{pmatrix} \zeta_\eta & -\zeta_\xi & \chi_\eta\zeta_\xi - \chi_\xi\zeta_\eta \\ -\ \varphi_\eta & \varphi_\xi & \chi_\xi\varphi_\eta - \chi_\eta\varphi_\xi \\ 0 & 0 & \varphi_\xi\zeta_\eta - \varphi_\eta\zeta_\xi \end{pmatrix}, \ \ J = \varphi_\xi\zeta_\eta - \varphi_\eta\zeta_\xi,$$

$$b_1 = b_{01}\varphi_\xi + b_{02}\varphi_\eta, \ \ b_2 = b_{01}\zeta_\xi + b_{02}\zeta_\eta,$$

$$b_3 = b_{01}\chi_\xi + b_{02}\chi_\eta + b_{03}.$$

The latter relations provide that $b_{0i} = b_{0i}(\xi, \eta), \ (i = 1, 2, 3)$. As all functions only depend on ξ and η, and the coefficients $A_{31} = 0$ and $A_{32} = 0$, then equations (6) become

$$\frac{\partial^2\varphi_j}{\partial t^2} + \sum_{k=1}^{2}\sum_{i=1}^{2}\frac{\partial}{\partial \xi_k}\left(A_{ik}\left(\delta_{ij}(\frac{1}{2}H^2 + p) - H_iH_j\right)\right) = 0, \ \ (j = 1, 2), \tag{7a}$$

$$\frac{\partial^2\chi}{\partial t^2} - \sum_{k=1}^{2}\sum_{i=1}^{2}\frac{\partial}{\partial \xi_k}\left(A_{ik}H_iH_3\right) = 0, \tag{7b}$$

where

$$\begin{aligned} H_1 = J^{-1}(b_{01}\varphi_\xi + b_{02}\varphi_\eta), \ \ H_2 = J^{-1}(b_{01}\zeta_\xi + b_{02}\zeta_\eta), \\ H_3 = J^{-1}(b_{01}\chi_\xi + b_{02}\chi_\eta + b_{03}), \ \ H^2 = H_1^2 + H_2^2 + H_3^2, \end{aligned}$$

and

$$S = S(\xi, \eta), \ \ \boldsymbol{b}_0 = (b_{01}(\xi, \eta), b_{02}(\xi, \eta), b_{03}(\xi, \eta))$$

are arbitrary functions such that

$$\frac{\partial}{\partial\xi}b_{01} + \frac{\partial}{\partial\eta}b_{02} = 0. \tag{8}$$

[‡]Here the Lagrangian space variables ξ_i are considered before the transition to the mass Lagrangian coordinates.

4. Equivalence transformations

The class of equations (7) is parameterized by arbitrary elements $S(\xi,\eta)$, $b_{0i}(\xi,\eta)$, $(i=1,2,3)$. The first step of the group classification of the class of equations of form (7) consists of describing the equivalence among the equations of this class. The group classification is considered with respect to these equivalence transformations.

Direct calculations show that the transformations corresponding to the generators

$$X_1^e=\frac{\partial}{\partial\xi},\ X_2^e=\frac{\partial}{\partial\eta},\ X_3^e=\frac{\partial}{\partial\varphi},\ X_4^e=\frac{\partial}{\partial\zeta},\ X_5^e=\frac{\partial}{\partial\chi},\ X_6^e=\frac{\partial}{\partial t},$$
$$X_7^e=t\frac{\partial}{\partial\varphi},\ X_8^e=t\frac{\partial}{\partial\zeta},\ X_9^e=t\frac{\partial}{\partial\chi},\ X_{10}^e=\zeta\frac{\partial}{\partial\varphi}-\varphi\frac{\partial}{\partial\zeta},$$
$$X_{11}^e=t\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}+\varphi\frac{\partial}{\partial\varphi}+\zeta\frac{\partial}{\partial\zeta}+\chi\frac{\partial}{\partial\chi},$$
$$X_{12}^e=t\frac{\partial}{\partial t}+2\xi\frac{\partial}{\partial\xi}+2\eta\frac{\partial}{\partial\eta}+4(1-\gamma)S\frac{\partial}{\partial S}-2b_{03}\frac{\partial}{\partial b_{03}},$$
$$X_{13}^e=-t\frac{\partial}{\partial t}+2S\frac{\partial}{\partial S}+b_{01}\frac{\partial}{\partial b_{01}}+b_{02}\frac{\partial}{\partial b_{02}}+b_{03}\frac{\partial}{\partial b_{03}},$$
$$X_f^e=f(\xi,\eta)\frac{\partial}{\partial\chi}.$$

do not change the structure of equations (7) and (8). Here the generators X_i^e, $(i=3,4,...11)$ are inherited by equations in Eulerian coordinates (2), where X_3^e, X_4^e, X_5^e correspond to the shifts with respect to x_i, $(i=1,2,3)$; X_6^e, X_7^e, X_8^e correspond to the Galilean boosts; X_{10}^e correspond to the rotation. The generator X_f^e allows adding a function $f(\xi,\eta)$ to χ. In particular, for given $b_{0i}(\xi,\eta)$, $(i=1,2,3)$ such that $b_{01}^2+b_{02}^2\neq 0$, choosing a function $f(\xi,\eta)$ satisfying the condition

$$b_{01}f_\xi+b_{02}f_\eta+b_{03}=0,$$

one can assume that after the transformation $b_{03}=0$. Indeed, for $\chi=\tilde\chi+f$ one derives that

$$b_3=b_{01}\tilde\chi_\xi+b_{02}\tilde\chi_\eta+b_{01}f_\xi+b_{02}f_\eta+b_{03}=b_{01}\tilde\chi_\xi+b_{02}\tilde\chi_\eta.$$

There are also two involutions

$$E_1:\ \ t\to -t,$$
$$E_2:\ \ (\xi,\eta,\varphi,\zeta,\chi)\to-(\xi,\eta,\varphi,\zeta,\chi),$$

where only changeable variables are presented.

The admitted generator X is sought in the form

$$X=\xi^\xi\frac{\partial}{\partial\xi}+\xi^\eta\frac{\partial}{\partial\eta}+\xi^t\frac{\partial}{\partial t}+\zeta^\varphi\frac{\partial}{\partial\varphi}+\zeta^\zeta\frac{\partial}{\partial\zeta}+\zeta^\chi\frac{\partial}{\partial\chi},$$

where all coefficients of the generator X depend on $(t,\xi,\eta,\varphi,\zeta,\chi)$. The determining equations [2] are obtained by applying the prolongation of the generator X to the left-hand side of equations (7):

$$XF_{|(7)}=0,$$

where F is the left-hand side of equations (7), and $|(7)$ means to consider XF on the manifold defined by equations (7).

The analysis of the determining equations depend on the relations between the entropy $S(\xi,\eta)$ and the vector $\boldsymbol{b}_0(\xi,\eta)$. It breaks down into several cases. Globally, according to the equivalence transformations corresponding to the generator X_f^e, it decomposes into $b_{01}^2+b_{02}^2\neq 0$ and $b_{01}^2+b_{02}^2=0$, and each of these cases is divided into non-isentropic and isentropic solutions.

5. Nonisentropic case with $b_{01}^2+b_{02}^2\neq 0$

The general solution of Gauss' equation (8) can be written as

$$b_{01}=\psi_\eta,\ \ b_{02}=-\psi_\xi,$$

where $\psi=\psi(\xi,\eta)$. One also can assume that $\psi_\eta\neq 0$. By virtue of the equivalence transformation corresponding to the generator X_f^e it can be considered that $b_{03}=0$.

Partially solving the determining equations one derives that $\xi^\xi=\xi^\xi(\xi,\eta)$, $\xi^\eta=\xi^\eta(\xi,\eta)$, and

$$\zeta^{\varphi}=k_8\varphi+k_1\zeta+k_6t+k_7,\ \ \zeta^{\zeta}=k_8\zeta-k_1\varphi+k_{11}t+k_{12},$$
$$\zeta^{\chi}=k_8\chi+k_9t+k_{10},\ \ \xi^{t}=-2k_2t+2k_8t+k_5,$$

where k_i are constant. The remaining equations are

$$\begin{array}{c}\xi^{\xi}_{\eta}\psi_{\eta}Sj_1+\xi^{\xi}\left(gS(S_{\eta\eta}\psi_{\eta}-\psi_{\eta\eta}S_{\eta})-\frac{(2\gamma+3)}{2(\gamma-1)}S_{\eta}\psi_{\eta}(S_{\eta}g+j_1)+S\psi_{\eta}j_{1\eta}\right)\\ +\xi^{\eta}\left(S(S_{\eta\eta}\psi_{\eta}-\psi_{\eta\eta}S_{\eta})-\frac{(2\gamma+3)}{2(\gamma-1)}{S_{\eta}}^2\psi_{\eta}\right)+\frac{2k_2(\gamma+4)-5k_8}{\gamma-1}S\psi_{\eta}S_{\eta}=0,\end{array}\tag{9}$$

$$\begin{array}{c}\xi^{\xi}\left(Sgj_1\psi_{\eta\eta}+Sj_1\psi_{\eta}g_{\eta}+S\psi_{\eta}j_{1\xi}-\frac{2\gamma-5}{2(\gamma-1)}j_1\psi_{\eta}(S_{\eta}g+j_1)\right)\\ +\xi^{\eta}\left(S(\psi_{\eta\eta}j_1+j_{1\eta}\psi_{\eta})-\frac{2\gamma-5}{2(\gamma-1)}S_{\eta}\psi_{\eta}j_1\right)-\frac{2k_2(\gamma+2)-k_8(2\gamma+1)}{\gamma-1}Sj_1\psi_{\eta}=0,\end{array}\tag{10}$$

$$\begin{array}{c}\xi^{\eta}_{\xi}=-g^2\xi^{\xi}_{\eta}-\xi^{\xi}\left(2gg_{\eta}+2\frac{\psi_{\eta\eta}}{\psi_{\eta}}g^2+g_{\xi\eta}\psi_{\eta}+\frac{4g(gS_{\eta}+j_1)}{(\gamma-1)S}\right)\\ -\xi^{\eta}\left(g_{\eta}+2\frac{\psi_{\eta\eta}}{\psi_{\eta}}g+g_{\xi\eta}\psi_{\eta}+\frac{4gS_{\eta}}{(\gamma-1)S}\right)+\frac{2(\gamma+3)}{\gamma-1}(2k_2-k_8)g,\end{array}\tag{11}$$

$$\begin{array}{c}\xi^{\eta}_{\eta}=-g\xi^{\xi}_{\eta}-\xi^{\xi}\left(g_{\eta}+\frac{\psi_{\eta\eta}}{\psi_{\eta}}g+\frac{5}{2(\gamma-1)S}(S_{\eta}g+j_1)\right)\\ -\xi^{\eta}\left(\frac{\psi_{\eta\eta}}{\psi_{\eta}}+\frac{5}{2(\gamma-1)S}S_{\eta}\right)+\frac{2(\gamma+4)k_2-5k_8}{\gamma-1},\end{array}\tag{12}$$

$$\begin{array}{c}\xi^{\xi}_{\xi}=g\xi^{\xi}_{\eta}+\xi^{\xi}\left(g_{\eta}+\frac{\psi_{\eta\eta}}{\psi_{\eta}}g+\frac{3}{2(\gamma-1)S}(S_{\eta}g+j_1)\right)\\ +\xi^{\eta}\left(\frac{\psi_{\eta\eta}}{\psi_{\eta}}+\frac{3}{2(\gamma-1)S}S_{\eta}\right)-\frac{2(\gamma+2)k_2-(2\gamma+1)k_8}{\gamma-1}.\end{array}\tag{13}$$

where

$$j_1=S_{\xi}-gS_{\eta},\ \ g=\frac{\psi_{\xi}}{\psi_{\eta}}.\tag{14}$$

As a solution of equations (9)-(13) determines an admitted Lie group of equations (7), they are called the defining equations.

The generators admitted for any functions S, b_{01} and b_{02}, composes a Lie algebra, called the kernel of admitted Lie algebras. A basis of this Lie algebra consists of the generators

$$\begin{array}{c}X_1=\frac{\partial}{\partial\varphi},\ \ X_2=\frac{\partial}{\partial\zeta},\ \ X_3=\frac{\partial}{\partial\chi},\ \ X_4=\frac{\partial}{\partial t},\\ X_5=t\frac{\partial}{\partial\varphi},\ \ X_6=t\frac{\partial}{\partial\zeta},\ \ X_7=t\frac{\partial}{\partial\chi},\ \ X_8=\zeta\frac{\partial}{\partial\varphi}-\varphi\frac{\partial}{\partial\zeta}.\end{array}\tag{15}$$

The kernel extensions are discussed next.

5.1. *Case* $j_1\neq 0$

Introducing

$$h_1=\xi^{\xi}\psi_{\xi}+\xi^{\eta}\psi_{\eta},\ \ h_2=\xi^{\xi}S_{\xi}+\xi^{\eta}S_{\eta},$$

one finds

$$\xi^{\xi}=(\psi_{\eta}j_1)^{-1}(-S_{\eta}h_1+\psi_{\eta}h_2),\ \ \xi^{\eta}=(\psi_{\eta}j_1)^{-1}(S_{\xi}h_1-\psi_{\eta}gh_2).$$

From equation (12) one obtains

$$h_2=\frac{2S}{5}\left(\frac{h_{1\eta}}{\psi_{\eta}}(1-\gamma)+2k_2(\gamma+4)-5k_8\right).\tag{16}$$

Finding $h_{1\xi\eta}$ from equation (13), equation (11) becomes

$$h_{1\xi}-h_{1\eta}g=0.\tag{17}$$

Hence, $h_1=h_1(\psi)$, and equation (13) reduces to

$$\begin{array}{c}h_{1\eta}j_2+5h_1\left(\frac{(2\gamma+j_2-5)S_{\eta}}{2(\gamma-1)S}-\frac{j_{1\eta}}{j_1}-\frac{\psi_{\eta\eta}}{\psi_{\eta}}\right)\\ -\frac{\psi_{\eta}}{\gamma-1}\left(2k_2(j_2(\gamma+4)-5(\gamma+2))+5k_8(2\gamma-j_2+1)\right)=0,\end{array}\tag{18}$$

where

$$j_2=j_1^{-2}\left(2(\gamma-1)S(j_{1\xi}-gj_{1\eta}+j_1g_{\eta})-(2\gamma-5)j_1^2\right).\tag{19}$$

Notice that from the notation (19) one has

$$j_{1\xi}=gj_{1\eta}-j_1g_{\eta}+\frac{j_1^2}{2(\gamma-1)S}(j_2+(2\gamma-5)).\tag{20}$$

5.1.1. Case $j_2 \neq 0$ From equation (18) one finds $h_{1\eta}$. Introducing the function

$$j_3 = j_{2\xi} - g j_{2\eta}, \tag{21}$$

the compatibility condition $(h_{1\xi})_\eta = (h_{1\eta})_\xi$ becomes

$$h_1\mu - 4\psi_\eta^2 S j_1 j_3 \tilde{k}_2(\gamma+2) = 0, \tag{22}$$

where $\tilde{k}_2 = k_2 + k_8\dfrac{2\gamma+1}{2(\gamma+2)}$, and

$$\mu = j_3\left(2(\gamma-1)S(j_{1\eta}\psi_\eta + \psi_{\eta\eta}j_1) - S_\eta\psi_\eta j_1(2\gamma-5)\right) - j_{2\eta}\psi_\eta j_1^2 j_2. \tag{23}$$

Let $j_3\mu \neq 0$. Introducing the function

$$j_4 = 4(\gamma+2)\mu^{-1}\psi_\eta^2 S j_1 j_3, \tag{24}$$

equation (22) gives that $h_1 = j_4\tilde{k}_2$. As for $\tilde{k}_2 = 0$ there is no an extension of the kernel of admitted Lie algebras, and because $h_1 = h_1(\psi)$, then

$$j_4 = j_4(\psi).$$

From definition of j_4 one finds

$$\psi_{\eta\eta} = \frac{\psi_\eta}{(\gamma-1)}\left(\frac{(2\gamma-5)S_\eta}{2S} - \frac{(\gamma-1)j_{1\eta}}{j_1} + \frac{j_1 j_2 j_{2\eta}}{2Sj_3} - \frac{2(\gamma+2)\psi_\eta}{j_4}\right).$$

The compatibility condition $(\psi_{\eta\eta})_\xi = (\psi_\xi)_{\eta\eta}$ gives

$$j_{1\eta} = \frac{j_1}{j_3^2}(j_{3\eta}j_{2\xi} - j_{3\xi}j_{2\eta}) + \frac{j_1}{S}S_\eta + \frac{j_1 j_{2\eta}}{j_3}\left(\frac{j_1(j_2-3)}{2(\gamma-1)S} - g_\eta\right). \tag{25}$$

The relation $(j_{1\xi})_\eta = (j_{1\eta})_\xi$ provides the condition

$$\begin{gathered} j_3^2(j_{2\eta}g_{\xi\eta} - j_{2\xi}g_{\eta\eta}) + j_3(j_{5\xi}j_{2\eta} - j_{5\eta}j_{2\xi}) \\ + j_3 g_\eta(j_{3\eta}j_3 - j_{2\eta}j_5) + 2j_5(j_3 j_{3\eta} - j_{2\eta}j_5) = 0. \end{gathered} \tag{26}$$

Substituting h_1 into (18) one derives

$$k_8 = -\tilde{k}_2\frac{(\gamma+2)}{4(\gamma+3)(\gamma-1)}\left(2(\gamma-1)\frac{j_{4\eta}}{\psi_\eta} + 5Mj_4 + 4(\gamma+4)\right), \tag{27}$$

where

$$M = \frac{S_\eta j_3 - j_{2\eta}j_1}{\psi_\eta S j_3}. \tag{28}$$

Direct calculations show that M satisfies the relation

$$M_\xi - gM_\eta = 0,$$

which means that $M = M(\psi)$.

For the existence of an extension of the kernel of admitted Lie algebras one needs to assume that $k_8/\tilde{k}_2$ is constant. Thus,

$$2(\gamma-1)\frac{j_{4\eta}}{\psi_\eta} + 5Mj_4 = k, \tag{29}$$

where k is some constant.

Equation (9) becomes

$$M_\eta = \frac{\psi_\eta M}{2(\gamma-1)}\left(5M - \frac{k}{j_4}\right). \tag{30}$$

The extension of the kernel of admitted Lie algebras is defined by the generator

$$\begin{aligned} X_9^{(1)} = {} & \tfrac{j_4}{\psi_\eta j_3}\left(-j_{2\eta}\tfrac{\partial}{\partial\xi} + j_{2\xi}\tfrac{\partial}{\partial\eta}\right) + \tfrac{3k+4(2\gamma+3)(\gamma+2)}{4(\gamma-1)(\gamma+3)}t\tfrac{\partial}{\partial t} \\ & + \tfrac{(k+4(\gamma+4))(\gamma+2)}{4(\gamma-1)(\gamma+3)}\left(\varphi\tfrac{\partial}{\partial\varphi} + \zeta\tfrac{\partial}{\partial\zeta} + \chi\tfrac{\partial}{\partial\chi}\right). \end{aligned} \tag{31}$$

Summarizing, one can state that if the functions $\psi(\xi,\eta)$ and $S(\xi,\eta)$ satisfy the conditions (25), (29) and (30), where j_i, $(i = 1,2,3,4)$ are defined by the formulas (14), (19), (21) and (24), then the extension of the kernel of admitted Lie algebras is defined by the generator (31). Here condition (26) guarantees the existence of the functions $\psi(\xi,\eta)$, $S(\xi,\eta)$ satisfying conditions (25) and (29).

Case $j_i \neq 0,\ \ (i=1,2,3)$ and $\mu=0$. Equation (22) provides that $\tilde{k}_2=0$. From $\mu=0$ one finds that

$$\psi_{\eta\eta} = -\frac{j_{1\eta}}{j_1}\psi_\eta + \frac{1}{2(\gamma-1)S}\left((2\gamma-5)S_\eta\psi_\eta + j_{2\eta}\psi_\eta j_1 j_2 j_3^{-1}\right) \tag{32}$$

The compatibility relation $(\psi_{\eta\eta})_\xi = (\psi_\xi)_{\eta\eta}$ is

$$j_{1\eta} = \frac{j_1}{j_3^2}\left(j_{2\eta}(-g_\eta j_3 - j_5 + \frac{j_1 j_3(j_2-3)}{2(\gamma-1)S}) + j_3(\frac{S_\eta}{S}j_3 + j_{3\eta})\right)\ , \tag{33}$$

where $j_5 = j_{3\xi} - g j_{3\eta}$. The compatibility condition $(j_{1\eta})_\xi = (j_{1\xi})_\eta$ also coincides with (26).

Equation (18) becomes

$$h_{1\eta} + \frac{5}{2(\gamma-1)}h_1 M\psi_\eta - k_8\frac{2(\gamma+3)}{\gamma+2}\psi_\eta = 0. \tag{34}$$

Equation (33) provides that

$$M_\xi - gM_\eta = 0, \tag{35}$$

which also means that $M = M(\psi)$. Equation (9) reduces to

$$h_1\nu + 4k_8\frac{(\gamma+3)(\gamma-1)}{\gamma+2}M\psi_\eta = 0,$$

where $\nu = 2(\gamma-1)M_\eta - 5M^2\psi_\eta$.

Assuming that $\nu \neq 0$, one obtains

$$h_1 = k_8\lambda,$$

where

$$\lambda = -\frac{4(\gamma+3)(\gamma-1)M\psi_\eta}{(\gamma+2)\nu}.$$

For an existence of the extension of the kernel of admitted Lie algebras it is necessary that λ is constant, say $\lambda = k$:

$$h_1 = kk_8.$$

Substituting the latter into (34),

$$M = \frac{4(\gamma-1)(\gamma+3)}{5k(\gamma+2)} \tag{36}$$

or

$$\frac{S_\eta j_{2\xi} - j_{2\eta}S_\xi}{\psi_\eta S j_3} = \frac{4(\gamma-1)(\gamma+3)}{5k(\gamma+2)}, \tag{37}$$

and the extension of the kernel of admitted Lie algebras is defined by the generator

$$X_9^{(2)} = \frac{k}{\psi_\eta j_3}\left(-j_{2\eta}\frac{\partial}{\partial\xi} + j_{2\xi}\frac{\partial}{\partial\eta}\right) + \frac{3}{\gamma+2}t\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi}. \tag{38}$$

Let $\nu = 0$, then $k_8 M = 0$.

Consider $M = 0$ or

$$j_1 = S_\xi - gS_\eta,\ \ g = \frac{\psi_\xi}{\psi_\eta},\ \ j_3 = j_{2\xi} - gj_{2\eta},$$

$$S_\eta\left(j_{2\xi} - gj_{2\eta}\right) - j_{2\eta}\left(S_\xi - gS_\eta\right) = S_\eta j_{2\xi} - j_{2\eta}S_\xi = 0$$

$$S_\eta j_{2\xi} - j_{2\eta}S_\xi = 0.$$

The latter means that $j_2 = j_2(S)$. Integrating (34), one obtains

$$h_1 = k_8\frac{2(\gamma+3)}{\gamma+2}\psi + k_{12}. \tag{39}$$

The extension of the kernel of admitted Lie algebras is defined by the generators

$$X_9^{(3)} = \frac{2(\gamma+3)\psi}{(\gamma+2)\psi_\eta j_3}\left(-j_{2\eta}\frac{\partial}{\partial\xi} + j_{2\xi}\frac{\partial}{\partial\eta}\right) + \frac{3}{\gamma+2}t\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi}. \tag{40}$$

$$X_{10}^{(3)} = \frac{1}{\psi_\eta j_3}\left(-j_{2\eta}\frac{\partial}{\partial\xi} + j_{2\xi}\frac{\partial}{\partial\eta}\right). \tag{41}$$

If $M \neq 0$, then $k_8 = 0$,

$$M = \left(q - \frac{5}{2(\gamma-1)}\psi\right)^{-1},$$

with some constant q, and equation (34) reduces to

$$h_{1\eta} + \frac{5}{2(\gamma-1)} h_1 M \psi_\eta = 0. \tag{42}$$

Hence, $h_1 = k/M$ and the extension of the kernel of admitted Lie algebras is defined by the generator

$$X_9^{(4)} = \frac{1}{M\psi_\eta j_3}\left(-j_{2\eta}\frac{\partial}{\partial\xi} + j_{2\xi}\frac{\partial}{\partial\eta}\right). \tag{43}$$

Case $j_1 j_2 \neq 0$ and $j_3 = 0$. The assumption $j_3 = 0$ gives that $j_2 = j_2(\psi)$, and equation (22) becomes $h_1 j_{2\eta} = 0$. If $j_{2\eta} \neq 0$, then $h_1 = 0$ and equation (18) leads to the condition

$$(\tilde{k}_2(\gamma-1)(\gamma+3) + k_8(\gamma+2)(\gamma+4))j_2 - 5(\gamma+2)^2\tilde{k}_2 = 0.$$

As $j_{2\eta} \neq 0$, then the latter equation provides that $\tilde{k}_2 = 0$ and $k_8 = 0$. Hence, for $j_{2\eta} \neq 0$ there is no an extension of the kernel of admitted Lie algebras. Thus, one should assume that $j_{2\eta} = 0$, which gives that $j_2 = k$, where $k \neq 0$ is constant. Equation (18) reduces to

$$h_{1\eta} - h_1\lambda\psi_\eta + \beta\psi_\eta = 0, \tag{44}$$

where

$$\lambda = \frac{5}{k\psi_\eta}\left(\frac{\psi_{\eta\eta}}{\psi_\eta} + \frac{j_{1\eta}}{j_1} - \frac{(2\gamma-5+k)S_\eta}{2(\gamma-1)S}\right), \quad \beta = 2\tilde{k}_2\frac{5(\gamma+2)-k(\gamma+4)}{k(\gamma-1)} - k_8\frac{2(\gamma+3)}{\gamma+2}.$$

Finding $\psi_{\eta\eta}$ from the latter notation of λ, the condition $(\psi_{\eta\eta})_\xi = (\psi_\xi)_{\eta\eta}$ provides that $\lambda = \lambda(\psi)$.

Equation (9) becomes

$$h_1 j_5 - \lambda\beta = 0, \tag{45}$$

where

$$j_5 = \frac{\lambda_\eta}{\psi_\eta} + \lambda^2.$$

As $\lambda = \lambda(\psi)$, then $j_5 = j_5(\psi)$.

Consider $j_5 \neq 0$. Substituting $h_1 = \beta\dfrac{\lambda}{j_5}$ into (44), one gets

$$\beta(\lambda j_{5\eta} + 2j_5(\lambda^2 - j_5)\psi_\eta) = 0.$$

If $\lambda j_{5\eta} + 2j_5(\lambda^2 - j_5)\psi_\eta \neq 0$, then $\beta = 0$ or

$$k_8 = \tilde{k}_2\frac{(\gamma+2)(5(\gamma+2)-k(\gamma+4))}{k(\gamma-1)(\gamma+3)}.$$

The extension of the kernel of admitted Lie algebras is defined by the generator

$$\begin{array}{c} X_9^{(5)} = \frac{4S}{kj_1}\left(-\frac{\partial}{\partial\xi} + g\frac{\partial}{\partial\eta}\right) + \frac{k(2\gamma+3)-15}{k(\gamma-1)(\gamma+3)}t\frac{\partial}{\partial t} \\ + \frac{k(\gamma+4)-5(\gamma+2)}{k(\gamma-1)(\gamma+3)}\left(\varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi}\right). \end{array} \tag{46}$$

If $\lambda j_{5\eta} + 2j_5(\lambda^2 - j_5)\psi_\eta = 0$, then the extension of the kernel of admitted Lie algebras is defined by the generator $X_9^{(6)} = X_9^{(5)}$ and one more generator

$$\begin{array}{c} X_{10}^{(6)} = \frac{2(\gamma+3)\lambda}{5\psi_\eta j_1 j_5}\left((5S_\eta + 2(\gamma-1)\lambda S\psi_\eta)\frac{\partial}{\partial\xi} - (5S_\xi + 2(\gamma-1)\lambda S\psi_\xi)\frac{\partial}{\partial\eta}\right) \\ + \frac{3}{\gamma+2}t\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi}. \end{array} \tag{47}$$

Considering $j_5 = 0$, one obtains that $\lambda\beta = 0$.

If $\lambda \neq 0$, then $\beta = 0$ and the extension of the kernel of admitted Lie algebras is defined by the generator $X_9^{(7)} = X_9^{(5)}$ and by one more generator

$$X_{10}^{(7)} = \frac{h_{11}}{5\psi_\eta j_1}\left(-(5S_\eta + 2(\gamma-1)\lambda S\psi_\eta)\frac{\partial}{\partial\xi} + (5S_\xi + 2(\gamma-1)\lambda S\psi_\xi)\frac{\partial}{\partial\eta}\right), \tag{48}$$

where $h_{11}(\psi)$ is the general solution of equation (44):

$$h'_{11} = h_{11}\lambda. \tag{49}$$

If $\lambda = 0$, then solving equation (44), one derives

$$h_1 = -\beta\psi + k_{20}, \tag{50}$$

where k_{20} is an arbitrary constant. The extension of the kernel of admitted Lie algebras is defined by the generators

$$X_9^{(8)} = \frac{2(\gamma+3)\psi}{(\gamma+2)\psi_\eta j_1}\left(-S_\eta\frac{\partial}{\partial\xi} + S_\xi\frac{\partial}{\partial\eta}\right) + \frac{3}{\gamma+2}t\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi}. \tag{51}$$

$$\begin{array}{c} X_{10}^{(8)} = \frac{2}{\psi_\eta j_1}\left(\left(\psi S_\eta(k(\gamma+4) - 5(\gamma+2)) - 2(\gamma-1)(\gamma+2)S\psi_\eta\right)\frac{\partial}{\partial\xi}\right. \\ \left. -\left(\psi S_\xi(k(\gamma+4) - 5(\gamma+2)) - 2(\gamma-1)(\gamma+2)S\psi_\xi\right)\frac{\partial}{\partial\eta}\right) + 2k(\gamma-1)t\frac{\partial}{\partial t}, \end{array} \tag{52}$$

$$X_{11}^{(8)} = \frac{1}{\psi_\eta j_1}\left(-S_\eta\frac{\partial}{\partial\xi} + S_\xi\frac{\partial}{\partial\eta}\right),$$

5.1.2. Case $j_1 \neq 0$ and $j_2 = 0$. Equation (18) becomes

$$h_1 N + \frac{2}{\gamma-1}\left(2k_2(\gamma+2) - k_8(2\gamma+1)\right) = 0, \tag{53}$$

where

$$N = \frac{1}{\psi_\eta}\left(\frac{(2\gamma-5)S_\eta}{2(\gamma-1)S} - \frac{j_{1\eta}}{j_1} - \frac{\psi_{\eta\eta}}{\psi_\eta}\right).$$

Conditions (20) provide that $N = N(\psi)$.

Assume that $N = 0$. Finding $\psi_{\eta\eta}$ from the condition $N = 0$:

$$\psi_{\eta\eta} = \psi_\eta\left(\frac{(2\gamma-5)S_\eta}{2(\gamma-1)S} - \frac{j_{1\eta}}{j_1}\right), \tag{54}$$

one checks that $(\psi_\xi)_{\eta\eta} = (\psi_{\eta\eta})_\xi$. Equation (53) reduces to the equation

$$k_2 = k_8\frac{2\gamma+1}{2(\gamma+2)},$$

and equation (9) becomes

$$\left(\frac{h_{1\eta}}{\psi_\eta}\right)_\eta = 0.$$

As $h_1 = h_1(\psi)$, one finds that

$$h_1 = k_{21}\psi + k_{20},$$

where k_{21} and k_{20} are arbitrary constants. The extension of the kernel of admitted Lie algebras (15) is defined by the generators

$$X_9^{(9)} = \frac{4(\gamma-1)(\gamma+3)S}{5j_1(\gamma+2)}\left(\frac{\partial}{\partial\xi} - g\frac{\partial}{\partial\eta}\right) + \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi} + \frac{3t}{(\gamma+2)}\frac{\partial}{\partial t}, \tag{55}$$

$$X_{10}^{(9)} = -(5\psi S_\eta + 2(\gamma-1)S\psi_\eta)\frac{\partial}{\partial\xi} + (5\psi S_\xi + 2(\gamma-1)S\psi_\xi)\frac{\partial}{\partial\eta}, \tag{56}$$

$$X_{11}^{(9)} = \frac{1}{\psi_\eta j_1}\left(-S_\eta\frac{\partial}{\partial\xi} + S_\xi\frac{\partial}{\partial\eta}\right) \tag{57}$$

Assuming that $N \neq 0$, one can introduce the function $P(\psi)$ instead of the function $N(\psi)$ by the formula

$$P = \frac{5}{(\gamma-1)N}$$

or the function P is introduced by the formula

$$\psi_{\eta\eta} = \psi_\eta\left(\frac{(2\gamma-5)S_\eta}{2(\gamma-1)S} - \frac{j_{1\eta}}{j_1} - \frac{2\psi_\eta}{(\gamma-1)P}\right). \tag{58}$$

As in the previous case the compatibility condition $(\psi_\xi)_{\eta\eta} = (\psi_{\eta\eta})_\xi$ is also satisfied. Equations (53) and (9) become

$$h_1 = 5\frac{2k_2(\gamma+2) - k_8(2\gamma+1)}{2}P,$$
$$P''(2k_2(\gamma+2) - k_8(2\gamma+1)) = 0,$$

where it is used the dependence $P = P(\psi)$ that leads to the equality $\left(\frac{P_\eta}{\psi_\eta}\right)_\eta = P''\psi_\eta$.

If $P'' \neq 0$, then the extension of the kernel of admitted Lie algebras (15) is defined by the generator $X_9^{(10)} = X_9^{(9)}$, and if $P'' = 0$, then by two generators $X_9^{(11)} = X_9^{(9)}$ and $X_{10}^{(11)} = X_{10}^{(9)}$.

5.2. Case $j_1 = 0$

The condition $j_1 = 0$ provides that $S = S(\psi)$, equation (10) is satisfied, and equation (9) becomes

$$(\xi^\eta + \xi^\xi g)S'\psi_\eta g_1 + 2S(2k_2(\gamma+4) - 5k_8) = 0, \tag{59}$$

where $g_1 = 2(\gamma-1)SS''/S'^2 - (2\gamma+3)$

Assume that $g_1 \neq 0$. From the latter equation one finds

$$\xi^\eta = -\xi^\xi g - \frac{2S}{g_1\psi_\eta S'}(2(\gamma+4)k_2 - 5k_8).$$

Substituting ξ^η into (11) and (12), they reduce to the single equation

$$g_1'(2(\gamma+4)k_2 - 5k_8) = 0.$$

Let $g_1' \neq 0$, then $k_8 = 2(\gamma+4)k_2/5$, and equation (13) reduces to the quasilinear first-order partial differential equation for the single function ξ^ξ:

$$\xi^\xi_\xi - g\xi^\xi_\eta = g_\eta\xi^\xi + k_2\frac{4(\gamma+3)}{5}.$$

The general solution of the latter equation can be written as follows

$$\xi^\xi = \psi_\eta\left(h_{11} + k_2\frac{4(\gamma+3)}{5}h_{12}\right),$$

where $h_{11} = h_{11}(\psi)$ is an arbitrary function and $h_{12}(\xi,\eta)$ is an arbitrary solution of the linear equation

$$h_{12\xi} - gh_{12\eta} = \psi_\eta^{-1}.$$

The extension of the kernel of admitted Lie algebras (15) is defined by the generators

$$X_9^{(12)} = h_{11}\psi_\eta\left(\frac{\partial}{\partial\xi} - g\frac{\partial}{\partial\eta}\right), \tag{60}$$

$$X_{10}^{(12)} = \frac{2(\gamma+3)}{\gamma+4}h_{12}\left(\psi_\eta\frac{\partial}{\partial\xi} - \psi_\xi\frac{\partial}{\partial\eta}\right) + \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi} + \frac{2\gamma+3}{\gamma+4}t\frac{\partial}{\partial t}. \tag{61}$$

Let $g_1' = 0$, say $g_1 = k$, where $k \neq 0$ is constant. Equation (13) becomes

$$\begin{array}{c}\xi^\xi_\xi - g\xi^\xi_\eta = g_\eta\xi^\xi + k_2\left(-\frac{4(\gamma+4)S}{kS'^2\psi_\eta^2}S_{\eta\eta} + \frac{4(\gamma(\gamma+4)+k)}{k(\gamma-1)}\right)\\ +k_8\left(\frac{10S}{kS'^2\psi_\eta^2}S_{\eta\eta} + \frac{2(k(\gamma-2)-5\gamma)}{k(\gamma-1)}\right).\end{array}$$

The general solution of the latter equation is written in the form

$$\xi^\xi = \psi_\eta(h_{11} + k_2h_{12} + k_8h_{13}),$$

where $h_{11} = h_{11}(\psi)$ is an arbitrary function, $h_{12}(\xi,\eta)$ and $h_{13}(\xi,\eta)$ are arbitrary solutions of the linear equations

$$h_{12\xi} - gh_{12\eta} = -\frac{4(\gamma+4)S}{kS'^2\psi_\eta^3}S_{\eta\eta} + \frac{4(\gamma(\gamma+4)+k)}{k(\gamma-1)\psi_\eta},$$

$$h_{13\xi} - gh_{13\eta} = \frac{10S}{kS'^2\psi_\eta^3}S_{\eta\eta} + \frac{2(k(\gamma-2)-5\gamma)}{k(\gamma-1)\psi_\eta}.$$

The extension of the kernel of admitted Lie algebras (15) is defined by the generators $X_9^{(13)} = X_9^{(12)}$ and

$$X_{10}^{(13)} = \quad h_{12}\psi_\eta(\frac{\partial}{\partial\xi} - g\frac{\partial}{\partial\eta}) - \frac{4(\gamma+4)S}{kS'\psi_\eta}\frac{\partial}{\partial\eta} - 2t\frac{\partial}{\partial t},$$

$$X_{11}^{(13)} = \quad h_{13}\psi_\eta(\frac{\partial}{\partial\xi} - g\frac{\partial}{\partial\eta}) + \frac{10S}{kS'\psi_\eta}\frac{\partial}{\partial\eta} + \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi} + 2t\frac{\partial}{\partial t}.$$

Case $g_1 = 0$. Equation (59) gives that $k_8 = 2k_2(\gamma+4)/5$, and equation (13) takes the form

$$\xi^\xi_\xi - g\xi^\xi_\eta = (g_2 + g_\eta)\xi^\xi + g_2\xi^\eta + k_2\frac{4(\gamma+3)}{5}, \tag{62}$$

where

$$g_2 = \frac{\psi_{\eta\eta}}{\psi_\eta} + \frac{3S_\eta}{2(\gamma-1)S}. \tag{63}$$

Case $g_2 \neq 0$. Finding ξ^η from equation (62), and substituting it into equations (11) and (12), one obtains two second-order equations for ξ^ξ. These equations can be simplified by the substitution

$$\xi^\xi_\xi = g\xi^\xi_\eta + g_\eta\xi^\xi + 4k_2\frac{\gamma+3}{5} + S^{-1/(\gamma-1)}g_2 h,$$

where $h(\xi,\eta)$ is some unknown function. Equations (11) and (12) become, respectively,

$$h_\xi = -h(g_\eta + g g_2), \;\; h_\eta = -h g_2. \tag{64}$$

For compatibility of these equations one needs to satisfy the condition $(h_\xi)_\eta = (h_\eta)_\xi$:

$$h g_3 = 0, \tag{65}$$

where $g_3 = g_{2\xi} - g g_{2\eta} - g_2 g_\eta - g_{\eta\eta}$.

Case $g_3 \neq 0$. Hence, $h = 0$ and then

$$\xi^\xi_\xi = g\xi^\xi_\eta + g_\eta\xi^\xi + 4k_2\frac{\gamma+3}{5}.$$

The general solution of the latter equation is presented in the form

$$\xi^\xi = \psi_\eta(h_{11} + k_2\frac{4(\gamma+3)}{5}h_{12}),$$

where $h_{11} = h_{11}(\psi)$ is arbitrary function, and $h_{12}(\xi,\eta)$ is an arbitrary solution of the linear equation

$$h_{12\xi} - g h_{12\eta} = \psi_\eta^{-1}. \tag{66}$$

The extension of the kernel of admitted Lie algebras (15) is defined by the generators (60) and (61):

$$X_9^{(14)} = h_{11}\psi_\eta\left(\frac{\partial}{\partial\xi} - g\frac{\partial}{\partial\eta}\right),$$

$$X_{10}^{(14)} = \quad \frac{2(\gamma+3)}{\gamma+4}h_{12}\psi_\eta(\frac{\partial}{\partial\xi} - g\frac{\partial}{\partial\eta}) + \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi} + \frac{2\gamma+3}{\gamma+4}t\frac{\partial}{\partial t}.$$

Case $g_3 = 0$. From equations (64) and the representation (63), one derives that

$$h = k_{21}\psi_\eta^{-1}S^{-3/(2(\gamma-1))},$$

where k_{21} is constant.

The extension of the kernel of admitted Lie algebras (15) is defined by the generators (60) and (61):

$$X_9^{(15)} = h_{11}\psi_\eta\left(\frac{\partial}{\partial\xi} - g\frac{\partial}{\partial\eta}\right), \;\; X_{11}^{(15)} = \frac{e^{-5/(2(\gamma-1))}}{\psi_\eta}\frac{\partial}{\partial\eta},$$

$$X_{10}^{(15)} = \quad \frac{2(\gamma+3)}{\gamma+4}h_{12}\psi_\eta(\frac{\partial}{\partial\xi} - g\frac{\partial}{\partial\eta}) + \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi} + \frac{2\gamma+3}{\gamma+4}t\frac{\partial}{\partial t}.$$

Let $g_2 = 0$.

$$\frac{\psi_{\eta\eta}}{\psi_\eta} + \frac{3S_\eta}{2(\gamma-1)S} = 0 \Rightarrow \left(\psi_\eta S^{3/(2(\gamma-1))}\right)_\eta = 0.$$

The compatibility condition $(\psi_{\eta\eta})_\xi = (\psi_\xi)_{\eta\eta}$ gives that $g(\xi,\eta)$ is a linear function with respect to η, say

$$g = -\frac{\mu_1''}{\mu_1'}\eta + \mu_2'\mu_1',$$

where $\mu_1(\xi)$ and $\mu_2(\xi)$ are some functions such that $\mu_1' \neq 0$. Here the representation for g is chosen for convenience of further integration. In particular, solving the equation $\psi_\xi = g\psi_\eta$, one finds

$$\psi = \psi(z), \;\; z = \frac{\eta}{\mu_1'} + \mu_2.$$

The relation $g_2 = 0$ provides that

$$\psi' = qe^{-\frac{S}{2(\gamma-1)}},$$

where q is constant.

Introducing $h_1 = \xi^\eta + g\xi^\xi$, one derives

$$\xi^\eta = h_1 - g\xi^\xi.$$

Then equation (11) reduces to

$$\left(h_1 S^{1/(\gamma-1)}\right)_\eta = 0,$$

which gives

$$h_1 = \mu_3 S^{-1/(\gamma-1)},$$

where $\mu_3(\xi)$ is an arbitrary function. Substituting h_1 into equation (12), one obtains that $\mu_3 = k_{20}\mu_1'$ with constant k_{20}. Equation (13) takes the form

$$\xi^\xi_\xi + \left(\frac{\mu_1''}{\mu_1'}\eta - \mu_2'\mu_1'\right)\xi^\xi_\eta = -\frac{\mu_1''}{\mu_1'}\xi^\xi + k_2\frac{4(\gamma+3)}{5}.$$

The general solution of the latter equation is

$$\xi^\xi = k_2\frac{4(\gamma+3)\mu_1}{5\mu_1'} + \frac{1}{\mu_1'}F(z),$$

where $F(z)$ is an arbitrary function.

The extension of the kernel of admitted Lie algebras (15) is defined by the generators

$$X_9^{(16)} = F\left(z_\eta\frac{\partial}{\partial\xi} - z_\xi\frac{\partial}{\partial\eta}\right),\;\; X_{10}^{(16)} = \mu_1' e^{-1/(\gamma-1)}\frac{\partial}{\partial\eta},$$

$$X_{11}^{(16)} = \frac{2(\gamma+3)}{\gamma+4}\mu_1\left(z_\eta\frac{\partial}{\partial\xi} - z_\xi\frac{\partial}{\partial\eta}\right) + \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + \chi\frac{\partial}{\partial\chi} + \frac{2\gamma+3}{\gamma+4}t\frac{\partial}{\partial t}.$$

6. Nonisentropic case with $b_{01}^2 + b_{02}^2 = 0$

In this case $H_1 = 0$ and $H_2 = 0$, and equation (7b) is integrated

$$\chi = t\chi_1 + \chi_0,$$

where $\chi_0(\xi,\eta)$ and $\chi_1(\xi,\eta)$ are arbitrary functions. Then the variable $\chi(t,\xi,\eta)$ is excluded from the consideration. It is also assumed that $S_\eta \neq 0$. Partially solving the determining equations, one derives that $\xi^\xi = \xi^\xi(\xi,\eta)$, $\xi^\eta = \xi^\eta(\xi,\eta)$, and

$$\zeta^\varphi = k_1\zeta + 2k_2\varphi + \frac{3\tilde{k}_4}{\gamma+2}\varphi + tk_{10} + k_{11},$$

$$\zeta^\zeta = 2k_2\zeta + \frac{3\tilde{k}_4}{\gamma+2}\zeta - k_1\varphi + tk_5 + k_6,$$

$$\xi^t = 2k_2 t + \frac{6\tilde{k}_4}{\gamma+2}t\tilde{k}_4 + k_9.$$

The remaining equations are

$$\begin{gathered} S(\xi^\xi_\eta S_\xi - \xi^\xi_\xi S_\eta) + \xi^\xi\left(SS_{\xi\eta} - \frac{\gamma}{\gamma-1}S_\xi S_\eta\right) + \xi^\eta\left(SS_{\eta\eta} - \frac{\gamma}{\gamma-1}S_\eta^2\right) \\ +2SS_\eta\left(2k_2 + \tfrac{3(\gamma-2)}{(\gamma-1)(\gamma+2)}\tilde{k}_4\right) = 0, \end{gathered} \tag{67}$$

$$
\begin{array}{c}
S^2(\xi^\xi_\eta b_{03\xi}-\xi^\xi_\xi b_{03\eta})+\xi^\eta\left(b_{03\eta\eta}S^2-\frac{3}{4(\gamma-1)^2}b_{03}S_\eta^2\right)\\
+\xi^\xi\left(b_{03\xi\eta}S^2+\frac{1}{2(\gamma-1)}S(B_{03\eta}S_\xi-B_{03\eta}S_\eta)-\frac{3}{4(\gamma-1)^2}b_{03}S_\xi S_\eta\right)\\
+4k_2S^2b_{03\eta}+3\tilde{k}_4\frac{S}{(\gamma-1)^2}\left(\frac{Sb_{03\eta}(3\gamma-2)(\gamma-1)}{\gamma+2}-\frac{1}{2}b_{03}S_\eta\right)=0,
\end{array}
\tag{68}
$$

$$
\xi^\eta f_{1\eta}+\xi^\xi f_{1\xi}+2\tilde{k}_4 f_1=0, \tag{69}
$$

$$
S(\xi^\eta S_\eta+\xi^\xi S_\xi)_\xi-S_\xi(\xi^\eta S_\eta+\xi^\xi S_\xi)=0, \tag{70}
$$

$$
\xi^\eta_\eta+\xi^\xi_\xi=\frac{2}{3b_{03}}(\xi^\eta+\xi^\xi)+4k_2+\frac{8}{\gamma+2}\tilde{k}_4, \tag{71}
$$

where

$$
f_1=b_{03}^{2(\gamma-1)/3}S.
$$

Let $h_1=\xi^\xi S_\xi+\xi^\eta S_\eta$, $h_2=\xi^\xi f_{1\xi}+\xi^\eta f_{1\eta}$, and $f_2=b_{03\xi}S_\eta-b_{03\eta}S_\xi$.

6.1. Case $f_2\neq 0$

One can derive

$$
\xi^\xi=\Delta^{-1}(h_1f_{1\eta}-h_2S_\eta),\;\; \xi^\eta=\Delta^{-1}(h_2S_\xi-h_1f_{1\xi}),\;\; \Delta=-\frac{2(\gamma-1)}{3b_{03}}f_1f_2.
$$

Equation (69) gives

$$
h_2=-2\tilde{k}_4f_1.
$$

From equation (70) one finds

$$
h_1=h_{10}S,
$$

where $h_{10}=h_{10}(\eta)$ is an arbitrary function.

The linear combination of equations (67) and (71) gives that h_{10} is constant, say $h_{10}=k_{20}$.

Equation (67) provides

$$
k_2=f_3\tilde{k}_4+bk_{20} \tag{72}
$$

where

$$
f_3=\frac{3}{4(\gamma-1)}\left(\frac{b_{03}}{f_2^2}(S_\eta f_{2\xi}-S_\xi f_{2\eta})-\frac{3\gamma+2}{\gamma+2}\right),\;\; b=f_4+\frac{1}{2}f_3+\frac{\gamma+6}{4(\gamma+2)},
$$

and

$$
f_4=\frac{Sb_{03\eta}}{S_\eta b_{03}}\left(\frac{\gamma-1}{3}f_3+\frac{3\gamma-2}{4(\gamma+2)}\right)-\frac{Sf_{2\eta}}{4S_\eta f_2}.
$$

Differentiating k_2 with respect to ξ and η, one derives that it is necessary to study the cases (a) $f_3\neq const$ and (b) $f_3=const$.

If $f_3\neq const$, then one can assume that $f_3\neq 0$. Hence, from equation (72) one obtains that there exist constants k and q such that $b=kf_3+q$ and $\tilde{k}_4=-kk_{20}$.

Thus, the extension of the kernel of admitted Lie algebras is defined by the generator

$$
\begin{array}{c}
X_9^{(17)}=-\left(\frac{3k}{\gamma+2}-2q\right)(\varphi\frac{\partial}{\partial\varphi}+\zeta\frac{\partial}{\partial\zeta})-2\left(\frac{3k}{\gamma+2}-q\right)t\frac{\partial}{\partial t}\\
+\frac{3b_{03}S}{2(\gamma-1)f_2f_1}\left(-f_{1\eta}\frac{\partial}{\partial\xi}+f_{1\xi}\frac{\partial}{\partial\eta}\right)+\frac{3kb_{03}}{f_2(\gamma-1)}\left(S_\eta\frac{\partial}{\partial\xi}-S_\xi\frac{\partial}{\partial\eta}\right).
\end{array}
$$

Notice that as $S_\eta\neq 0$, then from the definition of f_3 one can find $f_{2\xi}$. Finding $f_{2\eta}$ from the equation $b=kf_3+q$, the compatibility condition $(f_{2\xi})_\eta-(f_{2\eta})_\xi=0$ gives

$$
S(f_{1\eta}f_{3\xi}-f_{1\xi}f_{3\eta})-2kf_1(S_\eta f_{3\xi}-S_\xi f_{3\eta})=0.
$$

Let f_4 be constant, say $f_4=m$. In this case $f_2=q/(B_{03}^{q_2}S^{q_3})$, where q is an arbitrary constant and

$$
q_2=-\frac{4}{3}f_3(\gamma-1)-\frac{3\gamma-2}{\gamma+2},\;\; q_3=4m.
$$

Thus,

$$
b_{03\xi}=(b_{03\eta}S_\xi+f_2)/S_\eta,
$$

and the extension of admitted Lie algebras occurs by the generators

$$X_9^{(18)} = \left(f_3 + \frac{\gamma+6}{2(\gamma+2)} + 2m\right)\left(\varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + t\frac{\partial}{\partial t}\right) + \frac{3b_{03}S}{2(\gamma-1)f_2f_1}\left(-f_{1\eta}\frac{\partial}{\partial\xi} + f_{1\xi}\frac{\partial}{\partial\eta}\right),$$

$$X_{10}^{(18)} = (2f_3 + \frac{3}{\gamma+2})\left(\varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + 2(f_3 + \frac{3}{\gamma+2})t\frac{\partial}{\partial t}\right) + \frac{3}{q(\gamma-1)}S^{4m}b_{03}^{q_4}\left(-S_\eta\frac{\partial}{\partial\xi} + S_\xi\frac{\partial}{\partial\eta}\right),$$

where

$$q_4 = -\frac{4}{3}f_3(\gamma-1) - 2\frac{\gamma-2}{\gamma+2}.$$

If f_4 is not constant, then $k_{20} = 0$, $f_4 = f_4(S)$, and

$$b_{03\xi} = (b_{03\eta}S_\xi + f_2)/S_\eta.$$

The extension of the kernel of admitted Lie algebras consists of the generator

$$X_9^{(19)} = (2f_3 + \frac{3}{\gamma+2})\left(\varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + t\frac{\partial}{\partial t}\right) + \frac{3b_{03}}{f_2(\gamma-1)}\left(-S_\eta\frac{\partial}{\partial\xi} + S_\xi\frac{\partial}{\partial\eta}\right).$$

6.2. Case $f_2 = 0$

In this case $b_{03} = b_{03}(S)$. From (70) one finds that $h_1 = h_{10}S$, where $h_{10} = h_{10}(\eta)$ is an arbitrary function. Equation (69) becomes

$$f_5h_{10} + 6\tilde{k}_4 = 0,$$

where

$$f_5 = 2(\gamma-1)\frac{b_{03\eta}}{S_\eta b_{03}} + \frac{3}{S}.$$

Notice that $f_5 = f_5(S)$.

If $f_5 = 0$, then $\tilde{k}_4 = 0$, and excluding ξ^ξ_ξ (71) by taking a linear combination with (67), one finds $h_{10} = k_{20}$, where k_{20} is constant. The general solution of equation (67) can be presented in the form

$$\xi^\xi = S_\eta(\psi_1 + \psi_2k_2 + \psi_3k_{20}).$$

Substituting the latter into (67), one finds that

$$\psi_{1\xi}S_\eta - \psi_{1\eta}S_\xi = 0,\ \ \psi_{2\xi}S_\eta - \psi_{2\eta}S_\xi = 4,$$

$$\psi_{3\xi}S_\eta - \psi_{3\eta}S_\xi - SS_\eta^{-2}S_{\eta\eta} = -\frac{\gamma}{\gamma-1}.$$

The extension of the kernel of admitted Lie algebras occurs by the generators

$$X_9^{(20)} = \psi_1\left(S_\eta\frac{\partial}{\partial\xi} - S_\xi\frac{\partial}{\partial\eta}\right),\ \ X_{10}^{(20)} = \psi_3\left(S_\eta\frac{\partial}{\partial\xi} - S_\xi\frac{\partial}{\partial\eta}\right) + \frac{S}{S_\eta}\frac{\partial}{\partial\eta},$$

$$X_{11}^{(20)} = 2\left(\varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + t\frac{\partial}{\partial t}\right) + \psi_2\left(S_\eta\frac{\partial}{\partial\xi} - S_\xi\frac{\partial}{\partial\eta}\right).$$

If $f_5 \neq 0$, then $h_{10} = -6f_5^{-1}\tilde{k}_4$, equations (67) and (68) give

$$f_5'\tilde{k}_4 = 0, \tag{73}$$

and equation (71) become

$$\xi^\xi_\eta S_\xi - \xi^\xi_\xi S_\eta + \xi^\xi\left(S_{\xi\eta} - \frac{S_\xi S_{\eta\eta}}{S_\eta}\right) + 4k_2S_\eta - 6\tilde{k}_4\left(\frac{SS_{\eta\eta}}{S_\eta f_5} - \frac{\gamma S_\eta}{(\gamma-1)f_5} - \frac{(\gamma-2)S_\eta}{(\gamma-1)(\gamma+2)}\right) = 0.$$

Substituting in the latter equation the representation

$$\xi^\xi = S_\eta(\psi_1 + \psi_2k_2 + \psi_3\tilde{k}_4),$$

one finds

$$S_\eta\psi_{1\xi}-S_\xi\psi_{1\eta}=0,\quad S_\eta\psi_{2\xi}-S_\xi\psi_{2\eta}=4,$$
$$S_\eta\psi_{3\xi}-S_\xi\psi_{3\eta}=-6S_\eta^2\left(\frac{SS_{\eta\eta}}{f_5}-\frac{\gamma}{(\gamma-1)f_5}-\frac{\gamma-2}{(\gamma-1)(\gamma+2)}\right).$$

The extension of the kernel of admitted Lie algebras is defined by the generators

$$X_9^{(21)}=\psi_1\left(S_\eta\frac{\partial}{\partial\xi}-S_\xi\frac{\partial}{\partial\eta}\right),$$

$$X_{10}^{(21)}=2\left(\varphi\frac{\partial}{\partial\varphi}+\zeta\frac{\partial}{\partial\zeta}+t\frac{\partial}{\partial\varphi_t}\right)+\psi_2\left(S_\eta\frac{\partial}{\partial\xi}-S_\xi\frac{\partial}{\partial\eta}\right),$$
$$X_{11}^{(21)}=\frac{3}{\gamma+2}\left(\varphi\frac{\partial}{\partial\varphi}+\zeta\frac{\partial}{\partial\zeta}+2t\frac{\partial}{\partial t}\right)+\psi_3\left(S_\eta\frac{\partial}{\partial\xi}-S_\xi\frac{\partial}{\partial\eta}\right)-6\frac{S}{S_\eta f_5}\frac{\partial}{\partial\eta},$$

where if $f_5'\neq 0$, then the generator $X_{11}^{(21)}$ is not admitted.

7. Isentropic case with $b_{01}^2+b_{02}^2\neq 0$

For the isentropic case, equations (9)-(13) reduce to the following

$$\begin{array}{c}\xi^\eta_\xi=-g^2\xi^\xi_\eta-\xi^\xi\left(2gg_\eta+2\frac{\psi_{\eta\eta}}{\psi_\eta}g^2+g_{\xi\eta}\psi_\eta\right)\\-\xi^\eta\left(g_\eta+2\frac{\psi_{\eta\eta}}{\psi_\eta}g+g_{\xi\eta}\psi_\eta\right)+\frac{2(\gamma+3)}{\gamma-1}(2k_2-k_8)g,\end{array}\tag{74}$$

$$\xi^\eta_\eta+\xi^\xi_\xi=2\frac{2k_2+(\gamma-2)k_8}{\gamma-1},\tag{75}$$

$$\xi^\xi_\xi=g\xi^\xi_\eta+\xi^\xi\left(g_\eta+\frac{\psi_{\eta\eta}}{\psi_\eta}g\right)+\xi^\eta\frac{\psi_{\eta\eta}}{\psi_\eta}-\frac{2(\gamma+2)k_2-(2\gamma+1)k_8}{\gamma-1}.\tag{76}$$

Assume that $\psi_{\eta\eta}\neq 0$. Substituting ξ^η, found from equation (76), into (74) and (75), one can integrate them

$$\xi^\xi_\xi=g\xi^\xi_\eta+g_\eta\xi^\xi+\hat{k}_2\frac{\psi\psi_{\eta\eta}}{\psi_\eta^2}+\hat{k}_8+k_{21}\frac{\psi_{\eta\eta}}{\psi_\eta^2},\tag{77}$$

where k_{21} is the constant of integration, and

$$k_2=\frac{1}{4(\gamma+3)}(\hat{k}_2(2\gamma+1)+3\hat{k}_8),\quad k_8=\frac{1}{2(\gamma+3)}(\hat{k}_2(\gamma+2)+3\hat{k}_8(\gamma+4)).$$

As the latter equation is linear with respect to ξ^ξ, then one can look for a solution in the form

$$\xi^\xi=\psi_\eta(\psi_1+\hat{k}_2\psi_2+\hat{k}_8\psi_3+k_{21}\psi_4).$$

Substituting this representation into (77) and splitting it, one derives

$$\psi_\eta\psi_{1\xi}=\psi_\xi\psi_{1\eta},\quad \psi_\eta\psi_{2\xi}-\psi_\xi\psi_{2\eta}=\frac{\psi\psi_{\eta\eta}}{\psi_\eta^2},$$

$$\psi_\eta\psi_{3\xi}-\psi_\xi\psi_{3\eta}=1,\quad \psi_\eta\psi_{4\xi}=\psi_\xi\psi_{4\eta}+\frac{\psi_{\eta\eta}}{\psi_\eta^2}.$$

The extension of the kernel of admitted Lie algebras (15) is defined by the generators

$$X_9^{(22)}=\frac{\gamma+2}{2(\gamma+3)}\left(\varphi\frac{\partial}{\partial\varphi}+\zeta\frac{\partial}{\partial\zeta}+\chi\frac{\partial}{\partial\chi}+\frac{3}{\gamma+2}t\frac{\partial}{\partial t}\right)+\psi_2\left(\psi_\eta\frac{\partial}{\partial\xi}-\psi_\xi\frac{\partial}{\partial\eta}\right)+\frac{\psi}{\psi_\eta}\frac{\partial}{\partial\eta},\tag{78}$$

$$X_{10}^{(22)}=\frac{\gamma+2}{2(\gamma+3)}\left(\varphi\frac{\partial}{\partial\varphi}+\zeta\frac{\partial}{\partial\zeta}+\chi\frac{\partial}{\partial\chi}+\frac{2\gamma+3}{\gamma+2}t\frac{\partial}{\partial t}\right)+\psi_3\left(\psi_\eta\frac{\partial}{\partial\xi}-\psi_\xi\frac{\partial}{\partial\eta}\right),\tag{79}$$

$$X_{11}^{(22)}=\psi_4\left(\psi_\eta\frac{\partial}{\partial\xi}-\psi_\xi\frac{\partial}{\partial\eta}\right)+\frac{1}{\psi_\eta}\frac{\partial}{\partial\eta},\quad X_{12}^{(16)}=\psi_1\left(\psi_\eta\frac{\partial}{\partial\xi}-\psi_\xi\frac{\partial}{\partial\eta}\right).\tag{80}$$

Case $\psi_{\eta\eta}=0$ or $\psi=\eta g_1+g_0$, where $g_1(\xi)$ and $g_0(\xi)$. Hence $g=(g_0'+g_1'\eta)/g_1$.

Let $\xi^\eta = -g\xi^\xi + \xi_0$, where $\xi_0(\xi,\eta)$ some function. Then equation (75) can be integrated

$$\xi_0 = \frac{2k_2(\gamma+4)-5k_8}{\gamma-1}\eta + \xi_{00},$$

where $\xi_{00}(\xi)$ is a function of integration. Equation (74) becomes

$$\left(\xi_{00}g_1 - \frac{2k_2(\gamma+4)-5k_8}{\gamma-1}g_0\right)' = 0$$

or

$$\xi_{00}g_1 - \frac{2k_2(\gamma+4)-5k_8}{\gamma-1}g_0 = k_{22},$$

where k_{22} is an arbitrary constant. The remaining equation (76) is

$$\xi^\xi_\xi = \left(\frac{g_0'+g_1'\eta}{g_1}\xi^\xi\right)_\eta - k_2\frac{2(\gamma+2)}{\gamma-1} + k_8\frac{2\gamma+1}{\gamma-1}.$$

Seeking for a solution of the latter equation in the form

$$\xi^\xi = g_1(\psi_1 + k_2\psi_2 + k_8\psi_3),$$

one derives that

$$\psi_{1\xi} = \left(\frac{g_0'+g_1'\eta}{g_1}\psi_1\right)_\eta, \quad \psi_{2\xi} = \left(\frac{g_0'+g_1'\eta}{g_1}\psi_2\right)_\eta - \frac{2(\gamma+2)}{\gamma-1},$$
$$\psi_{3\xi} = \left(\frac{g_0'+g_1'\eta}{g_1}\psi_3\right)_\eta + \frac{2\gamma+1}{\gamma-1},$$

and an extension of the kernel of admitted Lie algebras (15) is defined by the generators

$$X_9^{(23)} = \psi_1\left(g_1\frac{\partial}{\partial\xi} - (g_0'+g_1'\eta)\frac{\partial}{\partial\eta}\right), \quad X_{12}^{(23)} = \frac{1}{g_1}\frac{\partial}{\partial\eta},$$

$$X_{10}^{(23)} = -2t\frac{\partial}{\partial t} + \psi_2\left(g_1\frac{\partial}{\partial\xi} - (g_0'+g_1'\eta)\frac{\partial}{\partial\eta}\right) + 2\frac{(\gamma+4)(g_1\eta+g_0)}{(\gamma-1)g_1}\frac{\partial}{\partial\eta},$$
$$X_{11}^{(23)} = \frac{\partial}{\partial\varphi}\varphi + \frac{\partial}{\partial\zeta}\zeta + \frac{\partial}{\partial\chi}\chi + 2t\frac{\partial}{\partial t} + \psi_3\left(g_1\frac{\partial}{\partial\xi} - (g_0'+g_1'\eta)\frac{\partial}{\partial\eta}\right) - \frac{5(g_1\eta+g_0)}{(\gamma-1)g_1}\frac{\partial}{\partial\eta}.$$

8. Isentropic case with $b_{01}^2 + b_{02}^2 = 0$

The defining equations (67)-(71) reduce to the equations

$$\xi^\eta b_{03\eta} + \xi^\xi b_{03\xi} + \frac{3b_{03}}{\gamma-1}\tilde{k}_4 = 0, \tag{81}$$

$$\xi^\eta_\eta + \xi^\xi_\xi = \frac{2}{3b_{03}}(\xi^\eta b_{03\eta} + \xi^\xi b_{03\xi}) + 4k_2 + \frac{8}{\gamma+2}\tilde{k}_4. \tag{82}$$

Assume that b_{03} is not constant, for example, $b_{03\eta} \neq 0$. Finding ξ^η from (81) and substituting it into (82), one obtains a linear first-order partial differential equation for the function ξ^ξ. Representing the general solution of this equation in the form

$$\xi^\xi = b_{03\eta}(\psi_1 + \psi_2 k_2 + \psi_3\tilde{k}_4),$$

one derives

$$\psi_{1\xi}b_{03\eta} - \psi_{1\eta}b_{03\xi} = 0, \quad \psi_{2\xi}b_{03\eta} - \psi_{2\eta}b_{03\xi} = 4,$$
$$\psi_{3\xi}b_{03\eta} - \psi_{3\eta}b_{03\xi} = \frac{3}{\gamma-1}\left(\frac{3\gamma-2}{\gamma+2} - \frac{b_{03\eta\eta}b_{03}}{b_{03\eta}^2}\right).$$

The extension of admitted Lie algebras occurs by the generators

$$X_9^{(24)} = \psi_1\left(b_{03\eta}\frac{\partial}{\partial\xi} - b_{03\xi}\frac{\partial}{\partial\eta}\right),$$

$$X_{10}^{(24)} = 2\left(\varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + t\frac{\partial}{\partial t}\right) + \psi_2\left(b_{03\eta}\frac{\partial}{\partial\xi} - b_{03\xi}\frac{\partial}{\partial\eta}\right),$$

$$X_{11}^{(24)} = \psi_3\left(b_{03\eta}\frac{\partial}{\partial\xi} - b_{03\xi}\frac{\partial}{\partial\eta}\right) - \frac{3b_{03}}{(\gamma-1)b_{03\eta}}\frac{\partial}{\partial\eta} + \frac{3}{\gamma+2}\left(\varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + 2t\frac{\partial}{\partial t}\right).$$

If b_{03} is constant, then $\tilde{k}_4 = 0$, $\xi^\eta = \psi_{1\xi}$, $\xi^\xi = -\psi_{1\eta} + 4k_2\xi$, where $\psi_1(\xi,\eta)$ is an arbitrary function, and the extension of admitted Lie algebras occurs by the generator

$$X_9^{(25)} = \psi_{1\eta}\frac{\partial}{\partial\xi} - \psi_{1\xi}\frac{\partial}{\partial\eta},\;\; X_{10}^{(25)} = \varphi\frac{\partial}{\partial\varphi} + \zeta\frac{\partial}{\partial\zeta} + t\frac{\partial}{\partial t} + 2\xi\frac{\partial}{\partial\xi}.$$

Conclusions

The transition to Lagrangian coordinates allows integrating four equations of magnetogasdynamics of an ideal perfect polytropic gas: the entropy $S(\xi,\eta)$ and the functions associated with the magnetic field $(b_{01}(\xi,\eta), b_{02}(\xi,\eta), b_{03}(\xi,\eta))$ are arbitrary functions of the integration. This leads to complications in the study of group classification: consideration of the many possibilities of these functions. The analysis presented in this article gives a complete investigation of all these possibilities. Figures 1-3 provide the trees of the study of nonisentropic cases, where (i,j) means the following: i is the number of the extension of the kernel of admitted Lie algebras (15), j is the number of the generators $X_{k+8}^{(i)}$, $(k = 1, 2, ..., j)$ in ith extension. Figure 4 presents the tree of the study for isentropic flows. The Lie algebras corresponding to the extensions i $(i = 1, 2, ..., 19)$ are finite dimensional, the Lie algebras corresponding to other extensions are infinite dimensional.

As mentioned above, finding an admitted Lie group is one of the first and necessary steps in application of the group analysis method for constructing invariant and partially invariant solutions. Because the equations (7) are variational, the symmetries found can also be used to derive conservation laws using Noether's theorem. The wide variety of these symmetries allows us to expect the derivation of new conservation laws. The search for invariant solutions, as well as the derivation of conservation laws, are the subject of further applications of the symmetries obtained in the present work.

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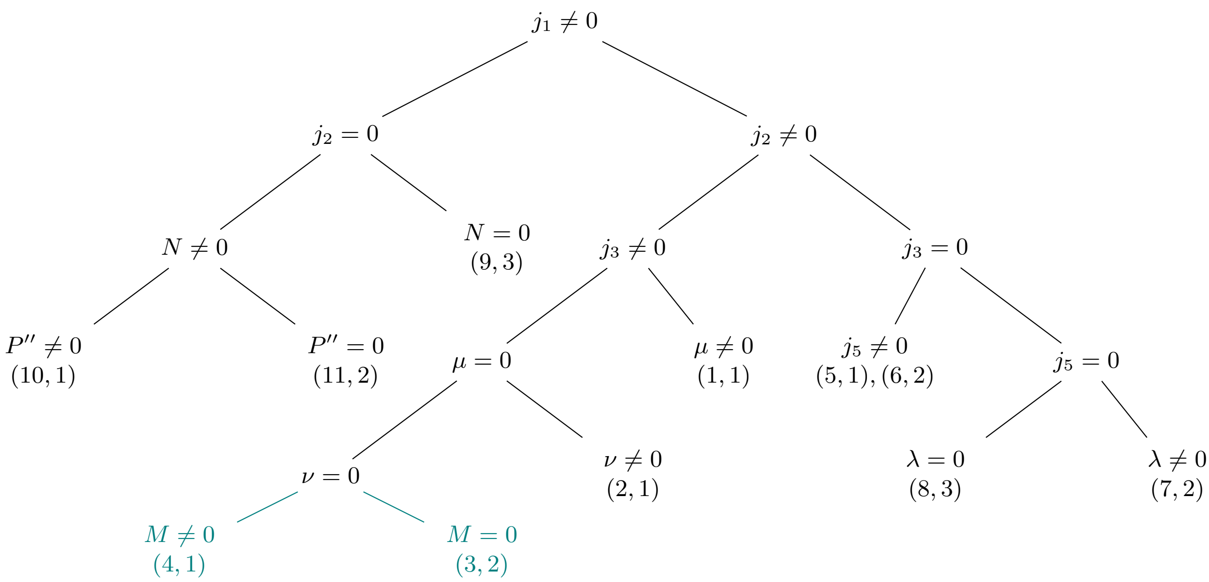

Figure 1. Tree of the study for $j_1 \neq 0$, $b_{01}^2 + b_{02}^2 \neq 0$ and $S \neq const$

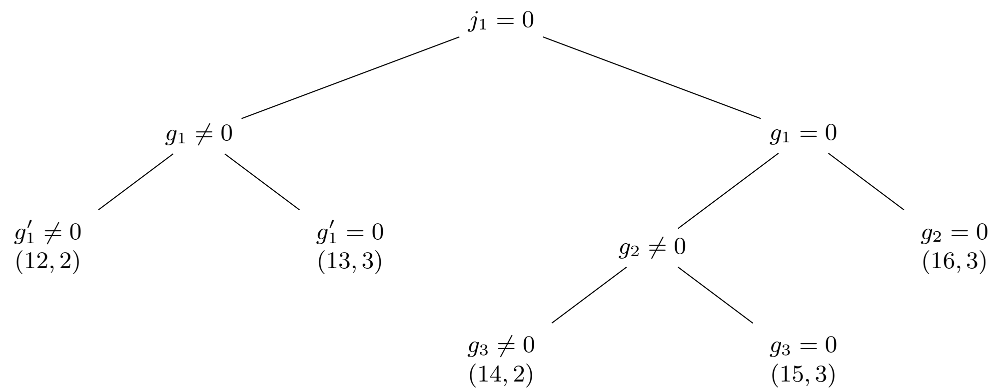

Figure 2. Tree of the study for $j_1 = 0$, $b_{01}^2 + b_{02}^2 \neq 0$ and $S \neq const$

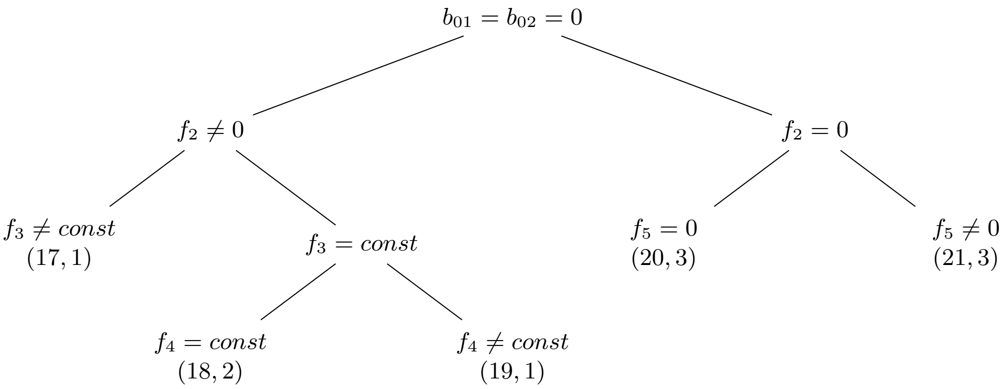

Figure 3. Tree of the study for $b_{01} = 0$, $b_{02} = 0$ and $S \neq const$

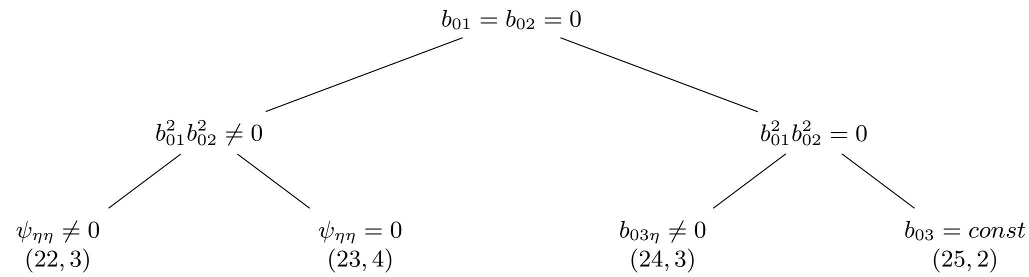

Figure 4. Tree of the study for isentropic flows $S = const$.

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