SOME PARAMETERIZED QUANTUM SIMPSON'S AND QUANTUM NEWTON'S INTEGRAL INEQUALITIES VIA QUANTUM DIFFERENTIABLE CONVEX MAPPINGS

XUEXIAO YOU, MUHAMMAD AAMIR ALI, HÜSEYIN BUDAK, AND DAFANG ZHAO

Abstract. In this work, two generalized quantum integral identities are proved by using some parameters. By utilizing these equalities we present several parameterized quantum inequalities for convex mappings. These quantum inequalities generalize many of the important inequalities that exist in the literature, such as quantum trapezoid inequalities, quantum Simpson's inequalities and quantum Newton's inequalities. We also give some new midpoint type inequalities as special cases. The results in this work naturally generalize the results for the Riemann integral.

1. Introduction

Thomas Simpson's has developed crucial methods for the numerical integration and estimation of definite integrals considered as Simpson's rule during (1710-1761). Nevertheless, a similar approximation was used by J. Kepler almost one hundred years earlier, so it is also known as Kepler's rule. Simpson's rule includes the three-point Newton-Cotes quadrature rule, so estimation based on three steps quadratic kernel is sometimes called Newton-type results. 1) Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)dx \approx \frac{\kappa_2-\kappa_1}{6}\left[\mathcal{F}(\kappa_1)+4\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathcal{F}(\kappa_2)\right].$$

2) Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule)

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)dx \approx \frac{\kappa_2-\kappa_1}{8}\left[\mathcal{F}(\kappa_1)+3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathcal{F}(\kappa_2)\right].$$

There are a large number of estimations related to these quadrature rules in the literature, one of them is the following estimation known as Simpson's inequality:

Theorem 1. *Suppose that $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ is a four times continuously differentiable mapping on (κ_1,κ_2), and let $\left\|\mathcal{F}^{(4)}\right\|_\infty=\sup_{x\in(\kappa_1,\kappa_2)}\left|\mathcal{F}^{(4)}(x)\right|<\infty$. Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}+2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)dx\right|\leq\frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_\infty(\kappa_2-\kappa_1)^4.$$

In the recent era, Simpson's type of inequalities has been emphasized by many authors for numerous types of functions. Convexity is useful and potent for solving different problems that appear within various branches of applied and pure mathematics. For an instance, Dragomir demonstrated novel Simpson's type consequences and their applications to quadratic formulas in numerical integration in [13]. Furthermore, Alomari [4] has presented Simpson's type of inequalities for s-convex functions. The refinements of Simpson's type of inequalities depended on convexity have been visualized by Sarikaya et al. in [31]. For the further studies of this area, one can consult [15, 19, 30].

On the other side, in the domain of q-analysis, many works are being carried out initiating from Euler in order to attain adeptness in mathematics that constructs quantum computing q-calculus considered as a relationship between physics and mathematics. In different areas of mathematics, it has numerous

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applications such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other sciences, mechanics, the theory of relativity, and quantum theory [16–18, 20, 22]. Apparently, Euler invented this important mathematics branch. He used the q parameter in Newton's work on infinite series. Later, in a methodical manner, the q-calculus that knew without limits calculus was firstly given by Jackson [16, 20]. In 1966, Al-Salam [5] introduced a q-analogue of the q-fractional integral and q-Riemann-Liouville fractional. Since then, the related research has gradually increased. In particular, In 2013, Tariboon introduced ${}_{\kappa_1}D_q$-difference operator [6]. In 2020, Bermudo et al. introduced the notion of ${}^{\kappa_2}D_q$ derivative and integral [8].

Many well-known integral inequalities such as Hölder inequality, Hermite-Hadamard inequality, Simpson's inequality, Newton's inequality, Ostrowski inequality, Gruss inequality and other integral inequalities have been studied in the setup of q-calculus using the concept of classical convexity. For more results in this direction, we refer to [1, 2, 6, 7, 10–12, 18, 21, 23–29, 32, 35, 36].

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we first present some known definitions and related inequalities in q-calculus. Set the following notation(see, [22]):

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k, \quad q \in (0,1).$$

Jackson [20] defined the q-integral of a given function $\mathcal{F}$ from 0 to κ_2 as follows:

$$\int_0^{\kappa_2} \mathcal{F}(x) \ d_qx \ = (1-q)\kappa_2 \sum_{n=0}^{\infty} q^n \mathcal{F}(\kappa_2 q^n), \text{ where } 0 < q < 1 \tag{2.1}$$

provided that the sum converges absolutely. Moreover, he defined the q-integral of a given function over the interval $[\kappa_1, \kappa_2]$ as follows:

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x) \ d_qx \ = \int_0^{\kappa_2} \mathcal{F}(x) \ d_qx \ - \int_0^{\kappa_1} \mathcal{F}(x) \ d_qx \ .$$

Definition 1. *[33] We consider the mapping $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is continuous. Then, the q_{κ_1}-derivative of $\mathcal{F}$ at $x \in [\kappa_1, \kappa_2]$ is defined by the the following expression*

$$_{\kappa_1}D_q\mathcal{F}(x) \ = \frac{\mathcal{F}(x) - \mathcal{F}(qx + (1-q)\kappa_1)}{(1-q)(x-\kappa_1)}, \ x \neq \kappa_1. \tag{2.2}$$

Since $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a continuous function, we can define

$$_{\kappa_1}D_q\mathcal{F}(\kappa_1) \ = \lim_{x\to\kappa_1} \ _{\kappa_1}D_q\mathcal{F}(x) \ .$$

We can say that the function $\mathcal{F}$ is q_{κ_1}-differentiable on $[\kappa_1, \kappa_2]$ if ${}_{\kappa_1}D_q\mathcal{F}(x)$ exists for all $x \in [\kappa_1, \kappa_2]$. If we take $\kappa_1 = 0$ in (2.2), then we have ${}_0D_q\mathcal{F}(x) = D_q\mathcal{F}(x)$, where $D_q\mathcal{F}(x)$ is a known q-derivative of $\mathcal{F}$ at $x \in [0, \kappa_2]$ in (see, [22]) given by

$$D_q\mathcal{F}(x) \ = \frac{\mathcal{F}(x) - \mathcal{F}(qx)}{(1-q)x}, \ x \neq 0.$$

Definition 2. *[8] We consider the mapping $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is continuous. Then, the q^{κ_2}-derivative of $\mathcal{F}$ at $x \in [\kappa_1, \kappa_2]$ is defined by*

$$^{\kappa_2}D_q\mathcal{F}(x) \ = \frac{\mathcal{F}(qx + (1-q)\kappa_2) - \mathcal{F}(x)}{(1-q)(\kappa_2 - x)}, \ x \neq \kappa_2. \tag{2.3}$$

Definition 3. *[33] We consider the mapping $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is continuous. Then, the q_{κ_1}-definite integral on $[\kappa_1, \kappa_2]$ is defined by*

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x) \ _{\kappa_1}d_qx \ = (1-q)(\kappa_2-\kappa_1) \sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\kappa_2 + (1-q^n)\kappa_1) = (\kappa_2-\kappa_1) \int_0^1 \mathcal{F}((1-\tau)\kappa_1 + \tau\kappa_2) \ d_q\tau \ .$$

In [6, 26], the authors proved quantum Hermite-Hadamard type inequalities and their estimations by using the notions of q_{κ_1}-derivative and q_{κ_1}-integral.

On the other hand, in [8], Bermudo et al. gave the following definition and obtained the related Hermite-Hadamard type inequalities:

Definition 4. *[8] We consider the mapping $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ is continuous. Then, the q^{κ_2}-definite integral on $[\kappa_1,\kappa_2]$ is defined by*

$$\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx=(1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2)=(\kappa_2-\kappa_1)\int_0^1\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\ d_q\tau\ .$$

Theorem 2. *[8] Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a convex function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then, q^{κ_2}-Hermite-Hadamard inequalities are given as follows:*

$$\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\ \leq\frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}. \tag{2.4}$$

In [9], Budak proved the left and right bounds of the inequality (2.4).

The present paper aims to generalize the results proved in [9, 10, 14] and obtain some new inequalities of Simpson's type, Newton's type, midpoint type, and trapezoidal type for differentiable convex functions.

3. Crucial Identities

We deal with the three identities which are necessary to obtain our main results in this section.

Let's start with the following useful Lemma.

Lemma 1. *If $\mathcal{F}:[\kappa_1,\kappa_2]\subset\mathbb{R}\to\mathbb{R}$ is a q^{κ_2}-differentiable function on (κ_1,κ_2) such that ${}^{\kappa_2}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1,\kappa_2]$, then we have the following identity:*

$$\begin{aligned}&\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\\&-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\=\ &(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}}(q\tau-\beta)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau\right.\\&\left.+\int_{\frac{1}{2}}^1(q\tau-\alpha)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau\right]\end{aligned} \tag{3.1}$$

where $q\in(0,1)$.

Proof. By the Definition 2, we see that

$${}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)=\frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}. \tag{3.2}$$

After applying the fundamental properties of quantum integrals, we deduce that

$$\begin{aligned}
(3.3)\qquad & \int_0^{\frac{1}{2}} (q\tau-\beta)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau+\int_{\frac{1}{2}}^1 (q\tau-\alpha)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau \\
= & \int_0^{\frac{1}{2}} (\alpha-\beta)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau+\int_0^1 (q\tau-\alpha)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau \\
= & (\alpha-\beta)\int_0^{\frac{1}{2}} \frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau \\
& +q\int_0^1 \frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)}d_q\tau \\
& -\alpha\int_0^1 \frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau.
\end{aligned}$$

From Definition 4, we conclude that

$$\begin{aligned}
(3.4)\qquad & \int_0^{\frac{1}{2}} \frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau \\
= & \frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}\mathcal{F}\left(\frac{q^{n+1}}{2}\kappa_1+\left(1-\frac{q^{n+1}}{2}\right)\kappa_2\right)-\sum_{n=0}^{\infty}\mathcal{F}\left(\frac{q^{n}}{2}\kappa_1+\left(1-\frac{q^{n}}{2}\right)\kappa_2\right)\right] \\
= & \frac{1}{\kappa_2-\kappa_1}\left[\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right],
\end{aligned}$$

$$\begin{aligned}
(3.5)\qquad & \int_0^1 \frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau \\
= & \frac{1}{\kappa_2-\kappa_1}\left[\mathcal{F}(\kappa_2)-\mathcal{F}(\kappa_1)\right]
\end{aligned}$$

and

$$\begin{aligned}
(3.6)\qquad & \int_0^1 \frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau \\
= & \frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}q^n\mathcal{F}\left(q^{n+1}\kappa_1+\left(1-q^{n+1}\right)\kappa_2\right)-\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2)\right] \\
= & \frac{1}{\kappa_2-\kappa_1}\left[\frac{1}{q}\sum_{n=1}^{\infty}q^n\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2)-\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2)\right] \\
= & \frac{1}{\kappa_2-\kappa_1}\left[\frac{1}{q}\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2)-\frac{1}{q}\mathcal{F}(\kappa_1)-\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2)\right] \\
= & \frac{1}{\kappa_2-\kappa_1}\left[\frac{1}{q(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx-\frac{1}{q}\mathcal{F}(\kappa_1)\right].
\end{aligned}$$

We can obtain the required identity (3.1) by putting the computed integrals (3.4)-(3.6) in (3.3). □

Remark 1. *If we assume $\beta=\frac{1}{6}$ and $\alpha=\frac{5}{6}$ in Lemma 1, then we obtain [10, Lemma 2].*

Remark 2. *In Lemma 1, by taking the limit $q\to 1^-$, we have [14, Lemma 2.1 for $m=1$].*

Remark 3. *In Lemma 1, if we choose $\beta=\alpha=\frac{q}{[2]_q}$, then we obtain the following identity*

$$\begin{aligned}
(3.7)\qquad & \frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx \\
= & \frac{q(\kappa_2-\kappa_1)}{[2]_q}\int_0^1\left(1-[2]_q\tau\right)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau
\end{aligned}$$

which is proved by Budak in [9, Lemma 1].

Corollary 1. *In Lemma 1, if we choose $\beta = 0$ and $\alpha = 1$, then we obtain the following new identity*

$$\begin{aligned}
&\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx-\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\\
=\ &(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}} q\tau\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau+\int_{\frac{1}{2}}^{1}(q\tau-1)^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau\right].
\end{aligned}$$

Lemma 2. *If $\mathcal{F} : [\kappa_1, \kappa_2] \subset \mathbb{R} \to \mathbb{R}$ is a q^{κ_2}-differentiable function on (κ_1, κ_2) such that ${}^{\kappa_2}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1, \kappa_2]$, then we have the following identity:*

$$\begin{aligned}
&\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\\
&-\left[\beta\mathcal{F}(\kappa_2)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(\gamma-\alpha)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(1-\gamma)\mathcal{F}(\kappa_1)\right]\\
=\ &(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{3}}(q\tau-\beta)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau\right.\\
&+\int_{\frac{1}{3}}^{\frac{2}{3}}(q\tau-\alpha)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau\\
&\left.+\int_{\frac{2}{3}}^{1}(q\tau-\gamma)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau\right]
\end{aligned}\tag{3.8}$$

where $q \in (0, 1)$.

Proof. After applying the fundamental properties of quantum integrals, we deduce that

$$\begin{aligned}
&\int_0^{\frac{1}{3}}(q\tau-\beta)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau+\int_{\frac{1}{3}}^{\frac{2}{3}}(q\tau-\alpha)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau\\
&+\int_{\frac{2}{3}}^{1}(q\tau-\gamma)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau\\
=\ &\int_0^{\frac{1}{3}}(\alpha-\beta)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau+\int_0^{\frac{2}{3}}(\gamma-\alpha)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau\\
&+\int_0^{1}(q\tau-\gamma)\ {}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\,d_q\tau.
\end{aligned}$$

If the same steps in the proof of Lemma 1 are applied for the rest of this proof, we can obtain the desired identity (3.8). □

Remark 4. *By assuming $\beta = \frac{1}{8}$, $\alpha = \frac{1}{2}$, and $\gamma = \frac{7}{8}$ in Lemma 2, we obtain [10, Lemma 3].*

Remark 5. *If we take $\beta = \alpha = \gamma = \frac{q}{[2]_q}$, in Lemma 2, then we recapture the identity (3.7).*

Corollary 2. *If we take the limit $q \to 1^-$ in Lemma 2, then we obtain the following new identity*

$$\begin{aligned}
&\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx\\
&-\left[\beta\mathcal{F}(\kappa_2)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(\gamma-\alpha)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(1-\gamma)\mathcal{F}(\kappa_1)\right]\\
=\ &(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{3}}(\tau-\beta)\mathcal{F}'(\tau\kappa_1+(1-\tau)\kappa_2)\,d\tau+\int_{\frac{1}{3}}^{\frac{2}{3}}(\tau-\alpha)\mathcal{F}'(\tau\kappa_1+(1-\tau)\kappa_2)\,d\tau\right.\\
&\left.+\int_{\frac{2}{3}}^{1}(\tau-\gamma)\mathcal{F}'(\tau\kappa_1+(1-\tau)\kappa_2)\,d\tau\right].
\end{aligned}$$

Now, we calculate the integrals in the following lemma which will be used in our next section.

Lemma 3. *The subsequent quantum integrals are true:*

$$\Xi_{11}=\int_0^{\frac{1}{2}}|q\tau-\beta|\,d_q\tau=\begin{cases}\frac{8\beta^2+q}{4[2]_q}-\frac{\beta}{2}, & q>2\beta,\\ \frac{\beta}{2}-\frac{q}{4[2]_q}, & q\le 2\beta,\end{cases}\tag{3.9}$$

$$\Xi_{12}=\int_{\frac{1}{2}}^{1}|q\tau-\alpha|\,d_q\tau=\begin{cases}\frac{\alpha}{2}-\frac{3q}{4[2]_q}, & q<\alpha,\\ \frac{8\alpha^2+5q}{4[2]_q}-\frac{3\alpha}{2}, & \alpha\le q\le 2\alpha,\\ \frac{3q}{4[2]_q}-\frac{\alpha}{2}, & q>2\alpha,\end{cases}\tag{3.10}$$

$$\Xi_{13}=\int_0^{\frac{1}{3}}|q\tau-\beta|\,d_q\tau=\begin{cases}\frac{2\beta^2}{[2]_q}+\frac{q}{9[2]_q}-\frac{\beta}{3}, & q>3\beta,\\ \frac{\beta}{3}-\frac{q}{9[2]_q}, & q\le 3\beta,\end{cases}\tag{3.11}$$

$$\Xi_{14}=\int_{\frac{1}{3}}^{\frac{2}{3}}|q\tau-\alpha|\,d_q\tau=\begin{cases}\frac{\alpha}{3}-\frac{q}{3[2]_q}, & q<\frac{3\alpha}{2},\\ \frac{18\alpha^2+5q}{9[2]_q}-\alpha, & \frac{3\alpha}{2}\le q\le 3\alpha,\\ \frac{q}{3[2]_q}-\frac{\alpha}{3}, & q>3\alpha,\end{cases}\tag{3.12}$$

$$\Xi_{15}=\int_{\frac{2}{3}}^{1}|q\tau-\gamma|\,d_q\tau=\begin{cases}\frac{\gamma}{3}-\frac{5q}{9[2]_q}, & q<\gamma,\\ \frac{18\gamma^2+13q}{9[2]_q}-\frac{5\gamma}{3}, & \gamma\le q\le \frac{3\gamma}{2},\\ \frac{5q}{9[2]_q}-\frac{\gamma}{3}, & q>\frac{3\gamma}{2},\end{cases}\tag{3.13}$$

$$\Xi_{1}=\int_0^{\frac{1}{2}}\tau\,|q\tau-\beta|\,d_q\tau=\begin{cases}\frac{2\beta^3}{[2]_q[3]_q}+\frac{q}{8[3]_q}-\frac{\beta}{4[2]_q}, & q>2\beta,\\ \frac{\beta}{4[2]_q}-\frac{q}{8[3]_q}, & q\le 2\beta,\end{cases}\tag{3.14}$$

$$\begin{aligned}\Xi_2 &= \int_0^{\frac{1}{2}}(1-\tau)\,|q\tau-\beta|\,d_q\tau\\ &= \Xi_{11}-\Xi_1\\ &= \begin{cases}\frac{8\beta^2+\beta+q}{4[2]_q}-\frac{\beta}{2}-\frac{q}{8[3]_q}-\frac{2\beta^3}{[2]_q[3]_q}, & q>2\beta,\\ \frac{\beta}{2}-\frac{\beta+q}{4[2]_q}+\frac{q}{8[3]_q}, & q\le 2\beta,\end{cases}\end{aligned}\tag{3.15}$$

$$\begin{aligned}\Xi_3 &= \int_{\frac{1}{2}}^{1}\tau\,|q\tau-\alpha|\,d_q\tau=\\ &= \begin{cases}\frac{3\alpha}{4[2]_q}-\frac{7q}{8[3]_q}, & q<\alpha,\\ \frac{2\alpha^3}{[2]_q[3]_q}-\frac{5\alpha}{4[2]_q}+\frac{9q}{8[3]_q}, & \alpha\le q\le 2\alpha,\\ \frac{7q}{8[3]_q}-\frac{3\alpha}{4[2]_q}, & q>2\alpha,\end{cases}\end{aligned}\tag{3.16}$$

$$\begin{aligned}\Xi_4 &= \int_{\frac{1}{2}}^{1}(1-\tau)\left|q\tau-\alpha\right|d_q\tau = \\ &= \Xi_{12}-\Xi_3 \\ &= \begin{cases} \frac{\alpha}{2}-\frac{3(\alpha+q)}{4[2]_q}+\frac{7q}{8[3]_q}, & q<\alpha, \\ \frac{8\alpha^2+5q-5\alpha}{4[2]_q}-\frac{3\alpha}{2}-\frac{9q}{8[3]_q}-\frac{2\alpha^3}{[2]_q[3]_q}, & \alpha\le q\le 2\alpha, \\ \frac{3(\alpha+q)}{4[2]_q}-\frac{\alpha}{2}-\frac{7q}{8[3]_q}, & q>2\alpha, \end{cases}\end{aligned} \tag{3.17}$$

$$\Xi_5 = \int_0^{\frac{1}{3}}\tau\left|q\tau-\beta\right|d_q\tau = \begin{cases} \frac{2\beta^3}{[2]_q[3]_q}+\frac{q}{27[3]_q}-\frac{\beta}{9[2]_q}, & q>3\beta, \\ \frac{\beta}{9[2]_q}-\frac{q}{27[3]_q}, & q\le 3\beta, \end{cases} \tag{3.18}$$

$$\begin{aligned}\Xi_6 &= \int_0^{\frac{1}{3}}(1-\tau)\left|q\tau-\beta\right|d_q\tau = \\ &= \Xi_{13}-\Xi_5 \\ &= \begin{cases} \frac{18\beta^2+\beta+q}{9[2]_q}-\frac{\beta}{3}-\frac{q}{27[3]_q}-\frac{2\beta^3}{[2]_q[3]_q}, & q>2\beta, \\ \frac{\beta}{3}-\frac{\beta+q}{9[2]_q}+\frac{q}{27[3]_q}, & q\le 2\beta, \end{cases}\end{aligned} \tag{3.19}$$

$$\Xi_7 = \int_{\frac{1}{3}}^{\frac{2}{3}}\tau\left|q\tau-\alpha\right|d_q\tau = \begin{cases} \frac{\alpha}{3[2]_q}-\frac{7q}{27[3]_q}, & q<\frac{3\alpha}{2}, \\ \frac{2\alpha^3}{[2]_q[3]_q}-\frac{5\alpha}{9[2]_q}+\frac{q}{3[3]_q}, & \frac{3\alpha}{2}\le q\le 3\alpha, \\ \frac{7q}{27[3]_q}-\frac{\alpha}{3[2]_q}, & q>3\alpha, \end{cases} \tag{3.20}$$

$$\begin{aligned}\Xi_8 &= \int_{\frac{1}{3}}^{\frac{2}{3}}(1-\tau)\left|q\tau-\alpha\right|d_q\tau \\ &= \Xi_{14}-\Xi_7 \\ &= \begin{cases} \frac{\alpha}{3}-\frac{q+\alpha}{3[2]_q}+\frac{7q}{27[3]_q}, & q<\frac{3\alpha}{2}, \\ \frac{18\alpha^2+5q+5\alpha}{9[2]_q}-\alpha-\frac{q}{3[3]_q}-\frac{2\alpha^3}{[2]_q[3]_q}, & \frac{3\alpha}{2}\le q\le 3\alpha, \\ \frac{q+\alpha}{3[2]_q}-\frac{\alpha}{3}-\frac{7q}{27[3]_q}, & q>3\alpha, \end{cases}\end{aligned} \tag{3.21}$$

$$\Xi_9 = \int_{\frac{2}{3}}^{1}\tau\left|q\tau-\gamma\right|d_q\tau = \begin{cases} \frac{5\gamma}{9[2]_q}-\frac{19q}{27[3]_q}, & q<\gamma, \\ \frac{2\gamma^3}{[2]_q[3]_q}-\frac{13\gamma}{9[2]_q}+\frac{35q}{27[3]_q}, & \gamma\le q\le\frac{3\gamma}{2}, \\ \frac{19q}{27[3]_q}-\frac{5\gamma}{9[2]_q}, & q>\frac{3\gamma}{2}, \end{cases} \tag{3.22}$$

$$\begin{aligned}
\Xi_{10} &= \int_{\frac{2}{3}}^{1} (1-\tau)\left|q\tau-\gamma\right| d_q\tau \\
&= \Xi_{15}-\Xi_{9} \\
&= \begin{cases} \frac{\gamma}{3}-\frac{5(q+\gamma)}{9[2]_q}+\frac{19q}{27[3]_q}, & q<\gamma, \\ \frac{18\gamma^2+13q+13\gamma}{9[2]_q}-\frac{5\gamma}{3}-\frac{35q}{27[3]_q}-\frac{2\gamma^3}{[2]_q[3]_q}, & \gamma\leq q\leq \frac{3\gamma}{2}, \\ \frac{5(q+\gamma)}{9[2]_q}-\frac{\gamma}{3}-\frac{19q}{27[3]_q}, & q>\frac{3\gamma}{2}. \end{cases}
\end{aligned} \tag{3.23}$$

Proof. Case I: Let $q>2\beta$.

By the definition q-integral, we have

$$\begin{aligned}
\Xi_1 &= \int_0^{\frac{1}{2}} \tau\left|q\tau-\beta\right| d_q\tau \\
&= \int_0^{\frac{\beta}{q}} \tau(\beta-q\tau)\,d_q\tau+\int_{\frac{\beta}{q}}^{\frac{1}{2}} \tau(q\tau-\beta)\,d_q\tau \\
&= 2\int_0^{\frac{\beta}{q}} \tau(\beta-q\tau)\,d_q\tau+\int_0^{\frac{1}{2}} \tau(q\tau-\beta)\,d_q\tau \\
&= \frac{2\beta^3}{[2]_q[3]_q}+\frac{q}{8[3]_q}-\frac{\beta}{4[2]_q}.
\end{aligned}$$

Case II: Let $q\leq 2\beta$.

From definition q-integral, we get

$$\Xi_1=\int_0^{\frac{1}{2}} \tau\left|q\tau-\beta\right| d_q\tau=\int_0^{\frac{1}{2}} \tau(\beta-q\tau)\,d_q\tau=\frac{\beta}{4[2]_q}-\frac{q}{8[3]_q}.$$

This gives the proof of the equality (3.14). In similar way, we can prove the others. □

4. Simpson's type inequalities for quantum integrals

An extension of quantum Simpson's inequalities for quantum differentiable convex functions using the quantum integrals are given in this section.

Theorem 3. *We assume that the given conditions of Lemma 1 hold. If the mapping $|{}^{\kappa_2}D_q\mathcal{F}|$ is convex on $[\kappa_1,\kappa_2]$, then the following Simpson's type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}^{\kappa_2}d_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\leq (\kappa_2-\kappa_1)\left[(\Xi_1+\Xi_3)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|+(\Xi_2+\Xi_4)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\right]
\end{aligned} \tag{4.1}$$

where Ξ_1-Ξ_4 are given in (3.14)-(3.17), respectively.

Proof. By the Lemma 1 and the convexity of $|^{\kappa_2}D_q\mathcal{F}|$, we conclude that

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
\leq\ &(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}}|q\tau-\beta|\left|{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\right|d_q\tau\right.\\
&\left.+\int_{\frac{1}{2}}^{1}|q\tau-\alpha|\left|{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\right|d_q\tau\right]\\
\leq\ &(\kappa_2-\kappa_1)\left[\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|\left\{\int_0^{\frac{1}{2}}\tau|q\tau-\beta|d_q\tau+\int_{\frac{1}{2}}^{1}\tau|q\tau-\alpha|d_q\tau\right\}\right.\\
&\left.+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\left\{\int_0^{\frac{1}{2}}(1-\tau)|q\tau-\beta|d_q\tau+\int_{\frac{1}{2}}^{1}(1-\tau)|q\tau-\alpha|d_q\tau\right\}\right]\\
=\ &(\kappa_2-\kappa_1)\left[(\Xi_1+\Xi_3)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|+(\Xi_2+\Xi_4)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\right]
\end{aligned}$$

which is the desired conclusion. □

Remark 6. *By taking the limit $q\to 1^-$ in Theorem 3, we have [14, Theorem 2.1 for $s=m=1$].*

Remark 7. *If we assume $\beta=\alpha=\frac{q}{[2]_q}$ in Theorem 3, then we obtain the following trapezoidal type inequality*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\right|\\
\leq\ &(\kappa_2-\kappa_1)\left[\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|\frac{q^2\left([3]_q+3q\right)}{[3]_q[2]_q^4}+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\frac{q^2\left(1+3q^2+2q^3\right)}{[3]_q[2]_q^4}\right]
\end{aligned}\tag{4.2}$$

which is established by Budak in [9, Theorem 3].

Corollary 3. *In Theorem 3, if we choose $\beta=0$ and $\alpha=1$, then we obtain the following midpoint type inequality*

$$\begin{aligned}
&\left|\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\right|\\
\leq\ &(\kappa_2-\kappa_1)\left[\frac{3}{4[2]_q[3]_q}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|+\frac{2q^2+2q-1}{4[2]_q[3]_q}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\right].
\end{aligned}$$

Remark 8. *In Theorem 3, if we assume $\beta=\frac{1}{6}$ and $\alpha=\frac{5}{6}$, then Theorem 3 reduces to [10, Theorem 4].*

Theorem 4. *We assume that the given conditions of Lemma 1 hold. If the mapping $|^{\kappa_2}D_q\mathcal{F}|^{p_1}$, $p_1\geq 1$ is convex on $[\kappa_1,\kappa_2]$, then the following Simpson's type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
\leq\ &(\kappa_2-\kappa_1)\left[\Xi_{11}^{1-\frac{1}{p_1}}\left(\Xi_1\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_2\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\Xi_{12}^{1-\frac{1}{p_1}}\left(\Xi_3\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_4\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}\tag{4.3}$$

where Ξ_{11}, Ξ_{12} and Ξ_1-Ξ_4 are given in (3.9), (3.10), and (3.14)-(3.17), respectively.

Proof. From Lemma 1 and the power mean inequality, we obtain

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
\leq\ &(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\beta|\,d_q\tau\right)^{1-\frac{1}{p_1}}\left(\int_0^{\frac{1}{2}}|q\tau-\beta|\left|{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}|q\tau-\alpha|\,d_q\tau\right)^{1-\frac{1}{p_1}}\left(\int_{\frac{1}{2}}^{1}|q\tau-\alpha|\left|{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

By applying the convexity of $|{}^{\kappa_2}D_q\mathcal{F}|^{p_1}$, we obtain

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
\leq\ &(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\beta|\,d_q\tau\right)^{1-\frac{1}{p_1}}\right.\\
&\times\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_0^{\frac{1}{2}}\tau|q\tau-\beta|\,d_q\tau+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_0^{\frac{1}{2}}(1-\tau)|q\tau-\beta|\,d_q\tau\right)^{\frac{1}{p_1}}\\
&+\left(\int_{\frac{1}{2}}^{1}|q\tau-\alpha|\,d_q\tau\right)^{1-\frac{1}{p_1}}\\
&\left.\times\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_{\frac{1}{2}}^{1}\tau|q\tau-\alpha|\,d_q\tau+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_{\frac{1}{2}}^{1}(1-\tau)|q\tau-\alpha|\,d_q\tau\right)^{\frac{1}{p_1}}\right]\\
=\ &(\kappa_2-\kappa_1)\left[\Xi_{11}^{1-\frac{1}{p_1}}\left(\Xi_1\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_2\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\Xi_{12}^{1-\frac{1}{p_1}}\left(\Xi_3\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_4\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

and the proof is completed. □

Remark 9. *If we take the limit $q\to1^-$ in Theorem 4, then we have [14, Theorem 2.3 for $s=m=1$].*

Remark 10. *If we assume $\beta=\alpha=\frac{q}{[2]_q}$ in Theorem 4, then we obtain the following trapezoidal type inequality*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\right|\\
\leq\ &\frac{q(\kappa_2-\kappa_1)}{[2]_q}\left(\frac{q\left(2+q+q^3\right)}{[2]_q^3}\right)^{1-\frac{1}{p_1}}\\
&\times\left[\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\frac{q\left([3]_q+3q\right)}{[3]_q[2]_q^3}+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\frac{q\left(1+3q^2+2q^3\right)}{[3]_q[2]_q^3}\right]
\end{aligned}\tag{4.4}$$

which is established by Budak in [9, Theorem 4].

Remark 11. *If we assume $\beta=\frac{1}{6}$ and $\alpha=\frac{5}{6}$ in Theorem 4, then Theorem 4 reduces to [10, Theorem 6].*

Corollary 4. *In Theorem 4, if we choose $\beta = 0$ and $\alpha = 1$, then we obtain the following midpoint type inequality*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx-\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right| \\
\leq\ & (\kappa_2-\kappa_1)\left(\frac{q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\frac{q}{8[3]_q}+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\frac{q\left([3]_q+q^2\right)}{8[2]_q[3]_q}\right)^{\frac{1}{p_1}} \\
&+\left(\frac{2-q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\frac{6[3]_q-7q[2]_q}{8[2]_q[3]_q}+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\left(\frac{1}{2}-\frac{3q}{4[2]_q}-\frac{6[3]_q-7q[2]_q}{8[2]_q[3]_q}\right)\right)^{\frac{1}{p_1}}.
\end{aligned}$$

Theorem 5. *Assume that the given conditions of Lemma 1 hold. If the mapping $\left|{}^{\kappa_2}D_q\mathcal{F}\right|^{p_1}$, $p_1>1$ is convex on $[\kappa_1,\kappa_2]$, then the following Simpson's type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\Xi_{16}^{\frac{1}{r_1}}\left(\frac{\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}+\frac{(2q+1)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&\left.+\Xi_{17}^{\frac{1}{r_1}}\left(\frac{3\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}+\frac{(2q-1)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned}\tag{4.5}$$

where $p_1^{-1}+r_1^{-1}=1$ and

$$\Xi_{16}=\int_0^{\frac{1}{2}}\left|q\tau-\beta\right|^{r_1}d_q\tau,\ \ \Xi_{17}=\int_{\frac{1}{2}}^1\left|q\tau-\alpha\right|^{r_1}d_q\tau.$$

Proof. From Lemma 1 and the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}\left|q\tau-\beta\right|^{r_1}d_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^{\frac{1}{2}}\left|{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|q\tau-\alpha\right|^{r_1}d_q\tau\right)^{\frac{1}{r_1}}\left(\int_{\frac{1}{2}}^1\left|{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

By using the convexity of $|^{\kappa_2}D_q\mathcal{F}|^{p_1}$, we obtain

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
\leq\ &(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\beta|^{r_1}d_q\tau\right)^{\frac{1}{r_1}}\left(|^{\kappa_2}D_q\mathcal{F}(\kappa_1)|^{p_1}\int_0^{\frac{1}{2}}\tau d_q\tau+|^{\kappa_2}D_q\mathcal{F}(\kappa_2)|^{p_1}\int_0^{\frac{1}{2}}(1-\tau)d_q\tau\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}|q\tau-\alpha|^{r_1}d_q\tau\right)^{\frac{1}{r_1}}\left(|^{\kappa_2}D_q\mathcal{F}(\kappa_1)|^{p_1}\int_{\frac{1}{2}}^{1}\tau d_q\tau+|^{\kappa_2}D_q\mathcal{F}(\kappa_2)|^{p_1}\int_{\frac{1}{2}}^{1}(1-\tau)d_q\tau\right)^{\frac{1}{p_1}}\right]\\
=\ &(\kappa_2-\kappa_1)\left[\Xi_{16}^{\frac{1}{r_1}}\left(\frac{|^{\kappa_2}D_q\mathcal{F}(\kappa_1)|^{p_1}}{4[2]_q}+\frac{(2q+1)|^{\kappa_2}D_q\mathcal{F}(\kappa_2)|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\Xi_{17}^{\frac{1}{r_1}}\left(\frac{3|^{\kappa_2}D_q\mathcal{F}(\kappa_1)|^{p_1}}{4[2]_q}+\frac{(2q-1)|^{\kappa_2}D_q\mathcal{F}(\kappa_2)|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

and the proof is completed. □

Remark 12. *If we take the limit $q\to1^-$ in Theorem 5, then Theorem 5 becomes [14, Theorem 2.2 for $s=m=1$].*

Remark 13. *If we assume $\beta=\frac{1}{6}$ and $\alpha=\frac{5}{6}$ in Theorem 5, then Theorem 5 becomes [10, Theorem 5].*

5. Newton's type inequalities for quantum integrals

In this section, a new extension of quantum Newton's inequalities for quantum differentiable convex functions is given.

Theorem 6. *We assume that the given conditions of Lemma 2 hold. If the mapping $|^{\kappa_2}D_q\mathcal{F}|$ is convex on $[\kappa_1,\kappa_2]$, then the following Newton's type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\right.\\
&\left.-\left[\beta\mathcal{F}(\kappa_2)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(\gamma-\alpha)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(1-\gamma)\mathcal{F}(\kappa_1)\right]\right|\\
\leq\ &(\kappa_2-\kappa_1)\left[(\Xi_5+\Xi_7+\Xi_9)|^{\kappa_2}D_q\mathcal{F}(\kappa_1)|+(\Xi_6+\Xi_8+\Xi_{10})|^{\kappa_2}D_q\mathcal{F}(\kappa_2)|\right]
\end{aligned}\tag{5.1}$$

where Ξ_5-Ξ_{10} are given in (3.18)-(3.23), respectively.

Proof. If we consider Lemma 2 and apply the same method that used in the proof of Theorem 3, then we can obtain the desired inequality (5.1). □

Corollary 5. *If we take the limit $q\to1^-$ in Theorem 6, then we obtain the following Newton's type inequality*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)dx\right.\\
&\left.-\left[\beta\mathcal{F}(\kappa_2)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(\gamma-\alpha)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(1-\gamma)\mathcal{F}(\kappa_1)\right]\right|\\
\leq\ &(\kappa_2-\kappa_1)\left[(\Xi_5^*+\Xi_7^*+\Xi_9^*)|^{\kappa_2}D_q\mathcal{F}(\kappa_1)|+(\Xi_6^*+\Xi_8^*+\Xi_{10}^*)|^{\kappa_2}D_q\mathcal{F}(\kappa_2)|\right]
\end{aligned}$$

where

$$\Xi_5^*=\int_0^{\frac{1}{3}}\tau|\tau-\beta|d\tau=\frac{\beta^3}{3}+\frac{1}{81}-\frac{\beta}{18},$$

$$\Xi_6^*=\int_0^{\frac{1}{3}}(1-\tau)|\tau-\beta|d\tau=\frac{18\beta^2+\beta+1}{18}-\frac{28}{81}-\frac{\beta^3}{3},$$

$$\Xi_7^* = \int_{\frac{1}{3}}^{\frac{2}{3}} \tau \left| q\tau - \alpha \right| d\tau = \frac{\alpha^3}{3} - \frac{5\alpha}{18} + \frac{1}{9}$$

$$\Xi_8^* = \int_{\frac{1}{3}}^{\frac{2}{3}} (1-\tau) \left| \tau - \alpha \right| d\tau = \frac{18\alpha^2 + 5 + 5\alpha}{18} - \alpha - \frac{1}{9} - \frac{\alpha^3}{3},$$

$$\Xi_9^* = \int_{\frac{2}{3}}^{1} \tau \left| \tau - \gamma \right| d\tau = \frac{\gamma^3}{3} - \frac{13\gamma}{18} + \frac{35}{81},$$

$$\Xi_{10}^* = \int_{\frac{2}{3}}^{1} (1-\tau) \left| \tau - \gamma \right| d\tau = \frac{18\gamma^2 + 13 + 13\gamma}{18} - \frac{5\gamma}{3} - \frac{35}{81} - \frac{\gamma^3}{3}.$$

Remark 14. *If we take $\beta = \frac{1}{8}$, $\alpha = \frac{1}{2}$, and $\gamma = \frac{7}{8}$ in Theorem 6, then Theorem 6 reduces to [10, Theorem 7].*

Remark 15. *If we assume $\beta = \alpha = \gamma = \frac{q}{[2]_q}$ in Theorem 6, then we recapture the inequality (4.2).*

Theorem 7. *We assume that the given conditions of Lemma 2 hold. If the mapping $\left| {}^{\kappa_2}D_q \mathcal{F} \right|^{p_1}$, $p_1 \geq 1$ is convex on $[\kappa_1, \kappa_2]$, then the following Newton's type inequality holds:*

$$\begin{aligned}
(5.2) \qquad & \left| \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(x) \ {}^{\kappa_2}d_q x \right. \\
& \left. - \left[\beta \mathcal{F}(\kappa_2) + (\alpha - \beta) \mathcal{F}\left(\frac{\kappa_1 + 2\kappa_2}{3} \right) + (\gamma - \alpha) \mathcal{F}\left(\frac{2\kappa_1 + \kappa_2}{3} \right) + (1-\gamma) \mathcal{F}(\kappa_1) \right] \right| \\
\leq \ & (\kappa_2 - \kappa_1) \left[\Xi_{13}^{1-\frac{1}{p_1}} \left(\Xi_5 \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_1) \right|^{p_1} + \Xi_6 \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_2) \right|^{p_1} \right)^{\frac{1}{p_1}} \right. \\
& + \Xi_{14}^{1-\frac{1}{p_1}} \left(\left(\Xi_7 \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_1) \right|^{p_1} + \Xi_8 \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_2) \right|^{p_1} \right)^{\frac{1}{p_1}} \right) \\
& \left. + \Xi_{15}^{1-\frac{1}{p_1}} \left(\Xi_9 \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_1) \right|^{p_1} + \Xi_{10} \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_2) \right|^{p_1} \right)^{\frac{1}{p_1}} \right]
\end{aligned}$$

where Ξ_5-Ξ_{10} and Ξ_{13}-Ξ_{15} are given in (3.18)-(3.23) and (3.11)-(3.13), respectively.

Proof. If we apply the steps used in the proof of Theorem 4 and taking into account Lemma 2, we can obtain the required inequality (5.2). □

Corollary 6. *If we take the limit $q \to 1^-$ in Theorem 7, then we obtain the following Newton's type inequality*

$$\begin{aligned}
& \left| \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(x) \, dx \right. \\
& \left. - \left[\beta \mathcal{F}(\kappa_2) + (\alpha - \beta) \mathcal{F}\left(\frac{\kappa_1 + 2\kappa_2}{3} \right) + (\gamma - \alpha) \mathcal{F}\left(\frac{2\kappa_1 + \kappa_2}{3} \right) + (1-\gamma) \mathcal{F}(\kappa_1) \right] \right| \\
\leq \ & (\kappa_2 - \kappa_1) \left[\Pi_{11}^{1-\frac{1}{p_1}} \left(\Xi_5^* \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_1) \right|^{p_1} + \Xi_6^* \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_2) \right|^{p_1} \right)^{\frac{1}{p_1}} \right. \\
& + \Pi_{12}^{1-\frac{1}{p_1}} \left(\left(\Xi_7^* \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_1) \right|^{p_1} + \Xi_8^* \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_2) \right|^{p_1} \right)^{\frac{1}{p_1}} \right) \\
& \left. + \Pi_{13}^{1-\frac{1}{p_1}} \left(\Xi_9^* \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_1) \right|^{p_1} + \Xi_{10}^* \left| {}^{\kappa_2}D_q \mathcal{F}(\kappa_2) \right|^{p_1} \right)^{\frac{1}{p_1}} \right]
\end{aligned}$$

where Ξ_5^*-Ξ_{10}^* are defined in Corollary 5 and

$$\Pi_{11} = \int_0^{\frac{1}{3}} \left| \tau - \beta \right| d\tau = \beta^2 + \frac{1}{9\,[2]_q} - \frac{\beta}{3},$$

$$\Pi_{12} = \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \tau - \alpha \right| d\tau = \frac{18\alpha^2 + 5}{18} - \alpha,$$

$$\Pi_{13}=\int_{\frac{2}{3}}^{1}|\tau-\gamma|\,d\tau=\frac{18\gamma^{2}+13}{18}-\frac{5\gamma}{3}.$$

Remark 16. *If we take $\beta=\frac{1}{8}$, $\alpha=\frac{1}{2}$, and $\gamma=\frac{7}{8}$ in Theorem 7, then Theorem 7 reduces to [10, Theorem 9].*

Remark 17. *If we assume $\beta=\alpha=\gamma=\frac{q}{[2]_q}$ in Theorem 7, then we recapture the inequality (4.4).*

Theorem 8. *We assume that the given conditions of Lemma 2 hold. If the mapping $|^{\kappa_2}D_q\mathcal{F}|^{p_1}$, $p_1>1$ is convex on $[\kappa_1,\kappa_2]$, then the following Newton's type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\right.\\
&\left.-\left[\beta\mathcal{F}(\kappa_2)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(\gamma-\alpha)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(1-\gamma)\mathcal{F}(\kappa_1)\right]\right|\\
\leq\ &(\kappa_2-\kappa_1)\left[\Xi_{18}^{\frac{1}{r_1}}\left(\frac{|^{\kappa_2}D_q\mathcal{F}(\kappa_1)|^{p_1}}{9[2]_q}+\frac{(3q+2)|^{\kappa_2}D_q\mathcal{F}(\kappa_2)|^{p_1}}{9[2]_q}\right)^{\frac{1}{p_1}}\right.\\
&+\Xi_{19}^{\frac{1}{r_1}}\left(\frac{|^{\kappa_2}D_q\mathcal{F}(\kappa_1)|^{p_1}}{3[2]_q}+\frac{q|^{\kappa_2}D_q\mathcal{F}(\kappa_2)|^{p_1}}{3[2]_q}\right)^{\frac{1}{p_1}}\\
&\left.+\Xi_{20}^{\frac{1}{r_1}}\left(\frac{5|^{\kappa_2}D_q\mathcal{F}(\kappa_1)|^{p_1}}{9[2]_q}+\frac{(3q-2)|^{\kappa_2}D_q\mathcal{F}(\kappa_2)|^{p_1}}{9[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned}\tag{5.3}$$

where $p_1^{-1}+r_1^{-1}=1$ and

$$\Xi_{18}=\int_0^{\frac{1}{3}}|q\tau-\beta|^{r_1}\,d_q\tau,\ \Xi_{19}=\int_{\frac{1}{3}}^{\frac{2}{3}}|q\tau-\alpha|^{r_1}\,d_q\tau,\ \Xi_{20}=\int_{\frac{2}{3}}^{1}|q\tau-\gamma|^{r_1}\,d_q\tau.$$

Proof. If we apply the steps used in the proof of Theorem 5 and taking into account Lemma 2, we can obtain the required inequality (5.3). □

Remark 18. *If we take $\beta=\frac{1}{8}$, $\alpha=\frac{1}{2}$, and $\gamma=\frac{7}{8}$ in Theorem 8, then Theorem 8 becomes [10, Theorem 8].*

6. Conclusions

In this investigation, we gave a new extension of quantum trapezoid, quantum Simpson's and quantum Newton's type estimations for quantum differentiable convex functions using the quantum integrals. It is also shown that the results given here are the generalizations of the results proved in [9, 10, 14]. We also obtained several new inequalities of Simpson's type, Newton's type, midpoint type, and trapezoidal type in the special cases of newly established results. It is an interesting and new problem that the forthcoming researchers can obtain similar inequalities for convex and co-ordinated convex functions in their research.

Author contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Availability of Data Materials

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

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CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

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School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China.
E-mail address: `youxuexiao@126.com`

Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, China.
E-mail address: `mahr.muhammad.aamir@gmail.com`

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY.
E-mail address: `hsyn.budak@gmail.com`

School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China.
E-mail address: `dafangzhao@163.com`