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An Euler-Maruyama Method and Its Fast Implementation for Multi-Term Fractional Stochastic Differential Equations

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Summary

In this paper, we derive an Euler-Maruyama (EM) method for a class of multi-term fractional stochastic nonlinear differential equations, and prove its strong convergence. The strong convergence order of this EM method is $\min\{\alpha_m - 0.5,\ \alpha_m - \alpha_{m-1}\}$, where $\{\alpha_i\}_{i=1}^m$ is the order of Caputo fractional derivative satisfying that $1 > \alpha_m > \alpha_{m-1} > \cdots > \alpha_2 > \alpha_1 > 0$, $\alpha_m > 0.5$, and $\alpha_m + \alpha_{m-1} > 1$. Then, a fast implementation of this proposed EM method is also presented based on the sum-of-exponentials approximation technique. Finally, some numerical experiments are given to verify the theoretical results and computational efficiency of our EM method.

KEYWORDS:
Fractional stochastic differential equations, Multi-term fractional derivatives, Euler-Maruyama method, Strong convergence, Fast implementation

1 | INTRODUCTION

In recent twenty years, fractional differential equations have attracted a large number of scholars' increasing interest due to their wide applications in disciplines such as mechanics, physics, electrical engineering and control theory, see [1,2] for examples. Among them, multi-term fractional models are known as the potential mathematical tools to describe the complex systems and phenomena caused by different anomalous relaxations. Therefore, many works focus on the theory and modelling of multi-term fractional differential equations, such as the fractional Bagley-Torvik equations [3,4] and Basset equations [5]. Meanwhile, lots of numerical methods for above multi-term fractional differential equations have been developed. For examples, Edwards in 2002 presented a numerical method for a class of linear fractional differential equations with initial value conditions, and this method was applied to numerically solve Basset equations [6]. Kukla in 2020 used the Mittag-Leffler function to obtain another numerical method for solving the initial value problem of Basset equations [7].

As is known that almost all of the mathematical models are influenced by noisy factors. Thus, researchers in various fields pay more attention to a novel model that is fractional stochastic differential equations (FSDEs) [8,9]. Obviously, it is very difficult or even impossible to analytically solve FSDEs, so people often focus on constructing numerical methods to solve this problem. For examples, Zhang in [10] proposed an Euler method for numerically solving stochastic Volterra equations with singular kernels. Doan et al. in [11] proposed an Euler-Maruyama (EM) method for a kind of single-term fractional stochastic differential equations driven by a multiplicative white noise with the fractional order $\alpha \in (0.5, 1)$. Anh et al. in [12] presented a variation of constant formula to solve a Caputo stochastic fractional differential equation. Additionally, there also exist many works of variable-order FSDEs, see [13,14] for examples. Zheng et al. in [13] analysed a nonlinear variable-order FSDE, and proved the well-posedness of this equation. Yang et al. in [14] studied the strong convergence of an EM scheme to a variable-order FSDE driven by a multiplicative white noise.

Herein, we in this paper will construct and analyze an EM method for the following multi-term FSDEs with the initial value condition $X(0) = \eta$,

$$\sum_{i=1}^{m} {}_0^C D_t^{\alpha_i} X(t) = f(t, X(t)) + g(t, X(t)) \frac{dW_t}{dt},\ 0 < t \leq T, \tag{1}$$

where ${}_0^C D_t^{\alpha_i} X(t)$, $(i = 1, 2, \cdots, m)$ are the Caputo fractional derivatives with $1 > \alpha_m > \alpha_{m-1} > \cdots > \alpha_2 > \alpha_1 > 0$, $\alpha_m > 0.5$, $\alpha_m + \alpha_{m-1} > 1$, and its definition is

$${}_0^C D_t^{\alpha_i} X(t) = \frac{1}{\Gamma(1-\alpha_i)} \int_0^t (t-s)^{-\alpha_i} X'(s) ds,$$

where $X : [0, T] \to \mathbb{R}^d$ with the positive integer d, $f, g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, here $\mathbb{R}^d$ is the d-dimensional Euclidean space. $(W_t)_{t\in[0,\infty)}$ denotes a one dimensional independent standard Winner process on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$. And then the strong convergence of the proposed EM method is proved. Furthermore, it is worth mentioning that the above result of strong convergence can be extended to the EM method for more general multi-term FSDEs with vector-valued noise. Additionally, due to the nonlocal property of Riemann-Liouville fractional integral operator, the computational cost of the proposed EM method is very expensive for the small step size. Thus, a fast implementation of the proposed EM method is also discussed by using the sum-of-exponentials (SOE) approximation for the weakly singular kernel in the Riemann-Liouville fractional integral.

The rest of this paper is organized as follows: in Section 2, some preliminaries are provided, and then the EM method is derived. In Section 3, the related numerical theoretical results and the strong convergence of the presented EM method are obtained and proved. Then, the main result of Section 3 is extended to more general case in Section 4. And also in Section 4, the fast implementation of the EM method is presented. In Section 5, some numerical experiments are given to illustrate the effectiveness of the EM method and verify the theoretical results. Finally, some concluding remarks are given.

2 | PRELIMINARIES AND THE DERIVATION OF EM METHOD

2.1 | Preliminaries

For each $t \in [0, \infty)$, let $\mathfrak{X}_t = \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ denote the space of all $\mathcal{F}_t$ -measurable, mean square integrable functions $f = (f_1, \cdots, f_d)^T : \Omega \to \mathbb{R}^d$ with the following standard norm

$$\|f\|_{ms} = \sqrt{\sum_{1\leq i\leq d} \mathbb{E}(|f_i|^2)}.$$

To equivalently transform Eq. (1) into its integral form, the following Riemann-Liouville fractional integral will be used,

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2}$$

where $\alpha > 0$. It is easy to check that $J^\alpha f(t) \to f(t)$ as $\alpha \to 0$, thus define $J^0 f(t) = f(t)$.

Now let us turn to Eq. (1). To guarantee Eq. (1) with the initial condition exists a unique solution, the following necessary assumptions are required:

(H1) Lipschitz continuity in $\mathbb{R}^d$ of the drift and diffusion: There exists a constant $L > 0$ such that for all $x, y \in \mathbb{R}^d,\ t \in [0, T]$,

$$\|f(t, x) - f(t, y)\| \vee \|g(t, x) - g(t, y)\| \leq\ L\|x - y\|,$$

where $\|\cdot\|$ is the standard d-dimensional Euclidean norm.

(H2) Lipschitz continuity in $[0, T]$ of the drift and diffusion: There exists a constant $L_1 > 0$ such that for all $x \in \mathbb{R}^d,\ t, s \in [0, T]$

$$\|f(t, x) - f(s, x)\| \vee \|g(t, x) - g(s, x)\| \leq\ L_1|t - s|.$$

(H3) Linear growth bound: There exists a constant $K > 0$ such that for all $t \in [0, T],\ x \in \mathbb{R}^d$

$$\|f(t, x)\| \vee \|g(t, x)\| \leq K(1 + \|x\|).$$

2.2 | Derivation of EM Method for multi-term FSDEs

Now let us left multiply the Riemann-Liouville fractional integral operator J^{α_m} defined by (2) on both sides of Eq. (1), that is,

$$J^{\alpha_m}\Big(\sum_{i=1}^{m} {}_{\mathbf{0}}^{\mathbf{C}}\mathbf{D}_{\mathbf{t}}^{\boldsymbol{\alpha}_\mathbf{i}} X(t)\Big) = J^{\alpha_m} f(t, X(t)) + J^{\alpha_m} g(t, X(t))\frac{dW_t}{dt}. \tag{3}$$

According to the property (see [15] for example) of Riemann-Liouville fractional integral and Caputo fractional derivative, it yields

$$J^{\alpha_m}({}_{\mathbf{0}}^{\mathbf{C}}\mathbf{D}_{\mathbf{t}}^{\boldsymbol{\alpha}_\mathbf{i}} X(t)) = J^{\alpha_m-\alpha_i}[X(t) - X(0)], \qquad i = 1, 2, \cdots, m,$$

which together with (3) implies that

$$\begin{aligned}\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_m-\alpha_i)}\int_0^t (t-s)^{\alpha_m-\alpha_i-1}X(s)ds \;=\; &\sum_{i=1}^{m}\frac{t^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}X(0)\\ &+\frac{1}{\Gamma(\alpha_m)}\int_0^t (t-s)^{\alpha_m-1}f(s,X(s))ds\\ &+\frac{1}{\Gamma(\alpha_m)}\int_0^t (t-s)^{\alpha_m-1}g(s,X(s))dW_s\end{aligned}$$

by using the following equality

$$J^{\alpha_m-\alpha_i}X(0) = \frac{t^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}X(0), \qquad i = 1, 2, \cdots, m.$$

For each $\eta \in \mathfrak{X}_0$, a $\mathbb{F}$-adapted process $X(t)$ is called a solution of (1) on the interval $[0, T]$ if the following equality holds for $t \in [0, T]$

$$\begin{aligned}X(t) \;=\; &\eta\sum_{i=1}^{m}\frac{t^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)} - \sum_{i=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_i)}\int_0^t (t-s)^{\alpha_m-\alpha_i-1}X(s)ds\\ &+\int_0^t \frac{(t-s)^{\alpha_m-1}}{\Gamma(\alpha_m)}f(s,X(s))ds + \int_0^t \frac{(t-s)^{\alpha_m-1}}{\Gamma(\alpha_m)}g(s,X(s))dW_s.\end{aligned} \tag{4}$$

We are now in a position to derive the EM method for Eq. (1) based on its equivalent integral form Eq. (4). That is, for each $n \in \mathbb{N}^*$, where $\mathbb{N}^*$ denotes the set of positive integer numbers, the EM approximate solution $X^{(n)}(t)$ is defined by $X^{(n)}(0) = \eta$ and for $t \in (0, T]$,

$$\begin{aligned}X^{(n)}(t) \;=\; &\eta\sum_{i=1}^{m}\frac{t^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}\\ &-\sum_{i=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_i)}\int_0^t (t-s)^{\alpha_m-\alpha_i-1}X^{(n)}(\tau_n(s))ds\\ &+\frac{1}{\Gamma(\alpha_m)}\int_0^t (t-s)^{\alpha_m-1}f(\tau_n(s),X^{(n)}(\tau_n(s)))ds\\ &+\frac{1}{\Gamma(\alpha_m)}\int_0^t (\rho_n(t)-\tau_n(s))^{\alpha_m-1}g(\tau_n(s),X^{(n)}(\tau_n(s))dW_s,\end{aligned} \tag{5}$$

where $\tau_n(s) = \frac{kT}{n}$ and $\rho_n(s) = \frac{(k+1)T}{n}$ for $s \in (\frac{kT}{n}, \frac{(k+1)T}{n}]$. It is clear that the above EM method can be performed step by step on each interval $(\frac{kT}{n}, \frac{(k+1)T}{n}]$, $k = 0, 1, \cdots, n-1$.

3 | STRONG CONVERGENCE OF THE PROPOSED EM METHOD

Before going to present the strong convergence of the proposed EM method, some preparatory lemmas are needed. Firstly, we will prove that $\sup\limits_{0\le t\le T}\|X^{(n)}(t)\|_{ms}$ is bounded.

Lemma 1. For all $x_i\in\mathbb{R}^d,\ \ i=1,2,\cdots,n$, we have that

$$\|\sum_{i=1}^{n}x_i\|^2\le n\sum_{i=1}^{n}\|x_i\|^2.$$

The proof of Lemma 1 can easily be obtained by the property of norm.

Lemma 2. Let

$$C_0=E_{\alpha_m-\alpha_{m-1}}((\frac{8Q_1K^2(T+1)}{\Gamma^2(\alpha_m)}+\frac{4Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})})t^{\alpha_m-\alpha_{m-1}}\Gamma(\alpha_m-\alpha_{m-1})),$$

$$C_1=4\|\eta\|_{ms}^2\left(\sum_{i=1}^{m}\frac{T^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}\right)^2+\frac{8Q_1K^2(T^{\alpha_m-\alpha_{m-1}+1}+T^{\alpha_m-\alpha_{m-1}})}{(\alpha_m-\alpha_{m-1})\Gamma^2(\alpha_m)}\cdot C_0, \tag{6}$$

where $E_\gamma(\cdot)$ in C_0 denotes the Mittag-Leffler function defined as

$$E_\gamma(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\gamma k+1)},\quad \gamma>0,$$

$Q=\max\{1,T^{\alpha_m+\alpha_{m-1}-2\alpha_1}\}$, $Q_1=\max\{1,T^{\alpha_m+\alpha_{m-1}-1}\}$, and $m_1=\min\{\Gamma(\alpha_m-\alpha_{m-1}),\cdots,\Gamma(\alpha_m-\alpha_1)\}$. Then, for all $n\in\mathbb{N}^*$, we have

$$\sup_{0\le t\le T}\|X^{(n)}(t)\|_{ms}^2\le C_1.$$

Proof. From (5) and Lemma 1, it is deduced that

$$\begin{aligned}\mathbb{E}(\|X^{(n)}(t)\|^2)\le\ & 4\mathbb{E}\|\eta\sum_{i=1}^{m}\frac{t^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}\|^2\\ &+4\mathbb{E}\|\sum_{i=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_i)}\int_0^t(t-s)^{\alpha_m-\alpha_i-1}X^{(n)}(\tau_n(s))ds\|^2\\ &+\frac{4}{\Gamma^2(\alpha_m)}\mathbb{E}(\|\int_0^t\frac{f(\tau_n(s),X^{(n)}(\tau_n(s)))}{(t-s)^{1-\alpha_m}}ds\|^2)\\ &+\frac{4}{\Gamma^2(\alpha_m)}\mathbb{E}(\|\int_0^t\frac{g(\tau_n(s),X^{(n)}(\tau_n(s)))}{(\rho_n(t)-\tau_n(s))^{1-\alpha_m}}dW_s\|^2).\end{aligned}$$

Using Lemma 1 and Hölder inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\|\sum_{i=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_i)}\int_0^t (t-s)^{\alpha_m-\alpha_i-1}X^{(n)}(\tau_n(s))ds\|^2\\
&\le (m-1)\sum_{i=1}^{m-1}\mathbb{E}\|\frac{1}{\Gamma(\alpha_m-\alpha_i)}\int_0^t (t-s)^{\frac{\alpha_m-\alpha_i-1}{2}}(t-s)^{\frac{\alpha_m-\alpha_i-1}{2}}X^{(n)}(\tau_n(s))ds\|^2\\
&\le \frac{(m-1)}{m_1^2(\alpha_m-\alpha_{m-1})}\sum_{i=1}^{m-1}t^{\alpha_m-\alpha_i}\int_0^t (t-s)^{\alpha_m-\alpha_i-1}\mathbb{E}\|X^{(n)}(\tau_n(s))\|^2ds\\
&\le \frac{(m-1)}{m_1^2(\alpha_m-\alpha_{m-1})}\sum_{i=1}^{m-1}t^{\alpha_m-\alpha_i}\int_0^t (t-s)^{\alpha_m-\alpha_{m-1}-1}(t-s)^{\alpha_{m-1}-\alpha_i}\mathbb{E}\|X^{(n)}(\tau_n(s))\|^2ds\\
&\le \frac{(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})}\max\{1,T^{\alpha_m+\alpha_{m-1}-2\alpha_1}\}\int_0^t (t-s)^{\alpha_m-\alpha_{m-1}-1}\mathbb{E}\|X^{(n)}(\tau_n(s))\|^2ds
\end{aligned}
$$

where $m_1=\min\{\Gamma(\alpha_m-\alpha_{m-1}),\cdots,\Gamma(\alpha_m-\alpha_1)\}$. For convenience, let $Q=\max\{1,T^{\alpha_m+\alpha_{m-1}-2\alpha_1}\}$, then using Hölder inequality and Itô's isometry, we can get

$$
\begin{aligned}
\|X^{(n)}(t)\|_{ms}^2 \le\ & 4\|\eta\sum_{i=1}^{m}\frac{t^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}\|_{ms}^2\\
&+\frac{4Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})}\int_0^t\frac{\mathbb{E}\|X^{(n)}(\tau_n(s))\|^2}{(t-s)^{1+\alpha_{m-1}-\alpha_m}}ds\\
&+\frac{4t}{\Gamma^2(\alpha_m)}\int_0^t\frac{\mathbb{E}\|f(\tau_n(s),X^{(n)}(\tau_n(s)))\|^2}{(t-s)^{2-2\alpha_m}}ds\\
&+\frac{4}{\Gamma^2(\alpha_m)}\int_0^t\frac{\mathbb{E}\|g(\tau_n(s),X^{(n)}(\tau_n(s)))\|^2}{(\rho_n(t)-\tau_n(s))^{2-2\alpha_m}}ds.
\end{aligned}
$$

Let $Q_1 = \max\{1, T^{\alpha_m+\alpha_{m-1}-1}\}$, this together with the fact that $|\rho_n(t) - \tau_n(s)| \geq |t-s|$ and the linear growth condition **(H3)** implies that

$$\begin{aligned}
\|X^{(n)}(t)\|_{ms}^2 \leq 4\|\eta \sum_{i=1}^{m} \frac{t^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}\|_{ms}^2 \\
&+\frac{4Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})}\int_0^t \frac{\mathbb{E}\|X^{(n)}(\tau_n(s))\|^2}{(t-s)^{1+\alpha_{m-1}-\alpha_m}}ds \\
&+\frac{8Q_1 t}{\Gamma^2(\alpha_m)}\int_0^t \frac{K^2}{(t-s)^{1+\alpha_{m-1}-\alpha_m}}ds + \frac{8Q_1 tK^2}{\Gamma^2(\alpha_m)}\int_0^t \frac{\mathbb{E}\|X^{(n)}(\tau_n(s))\|^2}{(t-s)^{1+\alpha_{m-1}-\alpha_m}}ds \\
&+\frac{8Q_1}{\Gamma^2(\alpha_m)}\int_0^t \frac{K^2}{(t-s)^{1+\alpha_{m-1}-\alpha_m}}ds + \frac{8Q_1 K^2}{\Gamma^2(\alpha_m)}\int_0^t \frac{\mathbb{E}\|X^{(n)}(\tau_n(s))\|^2}{(t-s)^{1+\alpha_{m-1}-\alpha_m}}ds \\
&\leq 4\|\eta \sum_{i=1}^{m} \frac{t^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}\|_{ms}^2 + \frac{8Q_1K^2(t^{\alpha_m-\alpha_{m-1}+1}+t^{\alpha_m-\alpha_{m-1}})}{(\alpha_m-\alpha_{m-1})\Gamma^2(\alpha_m)} \\
&+(\frac{8Q_1K^2(t+1)}{\Gamma^2(\alpha_m)} + \frac{4Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})})\int_0^t \frac{\mathbb{E}\|X^{(n)}(\tau_n(s))\|^2}{(t-s)^{1+\alpha_{m-1}-\alpha_m}}ds.
\end{aligned}$$

Let $m_t = \sup\limits_{0\leq s\leq t} \|X^{(n)}(s)\|_{ms}^2$, using the generalized Gronwall's inequality [16], we arrive at

$$\begin{aligned}
m_t \leq\ & (4\|\eta \sum_{i=1}^{m} \frac{T^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}\|_{ms}^2 + \frac{8Q_1K^2(T^{\alpha_m-\alpha_{m-1}+1}+T^{\alpha_m-\alpha_{m-1}})}{(\alpha_m-\alpha_{m-1})\Gamma^2(\alpha_m)}) \\
&\cdot E_{\alpha_m-\alpha_{m-1}}((\frac{8Q_1K^2(T+1)}{\Gamma^2(\alpha_m)} + \frac{4Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})})t^{\alpha_m-\alpha_{m-1}}\Gamma(\alpha_m-\alpha_{m-1})).
\end{aligned}$$

This completes the proof. □

The following lemma concerns the generalized continuity of EM solution with respect to the variable $t \in [0, T]$.

Lemma 3. Let

$$C_2 = 6(\frac{(m-1)^2(\|\eta\|_{ms}^2 + 4Q_2C_1)}{m_1^2(\alpha_m-\alpha_{m-1})^2} + \frac{2K^2(1+C_1)(T+2)}{(2\alpha_m-1)\Gamma^2(\alpha_m)}),\ C_3 = \frac{12K^2(1+C_1)}{(2\alpha_m-1)\Gamma^2(\alpha_m)}T^{2\alpha_m-1}, \tag{7}$$

where C_1 is defined in (6) and $Q_2 = \max\{1, T^{\alpha_{m-1}-\alpha_1}\}$. Then, for all $n \in \mathbb{N}^*$ and $t, \tilde{t} \in [0, T]$, we have

$$\|X^{(n)}(t) - X^{(n)}(\tilde{t})\|_{ms}^2 \leq C_2|t-\tilde{t}|^{2\alpha_m-2\alpha_{m-1}} + \frac{C_3}{n^{2\alpha_m-1}}.$$

Proof. For any $t, \tilde{t} \in [0, T]$ with $t > \tilde{t}$. By (5), it is clear that

$$
\begin{aligned}
& X^{(n)}(t) - X^{(n)}(\tilde{t}) \\
= & \eta \sum_{i=1}^{m-1} \frac{t^{\alpha_m-\alpha_i} - \tilde{t}^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m - \alpha_i + 1)} - \sum_{i=1}^{m-1} \frac{1}{\Gamma(\alpha_m - \alpha_i)} \int_{\tilde{t}}^{t} (t-s)^{\alpha_m-\alpha_i-1} X^{(n)}(\tau_n(s)) ds \\
& + \sum_{i=1}^{m-1} \frac{1}{\Gamma(\alpha_m - \alpha_i)} \int_0^{\tilde{t}} ((\tilde{t}-s)^{\alpha_m-\alpha_i-1} - (t-s)^{\alpha_m-\alpha_i-1}) X^{(n)}(\tau_n(s)) ds \\
& + \frac{1}{\Gamma(\alpha_m)} \int_0^{\tilde{t}} ((t-s)^{\alpha_m-1} - (\tilde{t}-s)^{\alpha_m-1}) f(\tau_n(s), X^{(n)}(\tau_n(s))) ds \\
& + \frac{1}{\Gamma(\alpha_m)} \int_{\tilde{t}}^{t} (t-s)^{\alpha_m-1} f(\tau_n(s), X^{(n)}(\tau_n(s))) ds \\
& + \frac{1}{\Gamma(\alpha_m)} \int_0^{\tilde{t}} ((\rho_n(t) - \tau_n(s))^{\alpha_m-1} - (\rho_n(\tilde{t}) - \tau_n(s))^{\alpha_m-1}) g(\tau_n(s), X^{(n)}(\tau_n(s))) dW_s \\
& + \frac{1}{\Gamma(\alpha_m)} \int_{\tilde{t}}^{t} (\rho_n(t) - \tau_n(s))^{\alpha_m-1} g(\tau_n(s), X^{(n)}(\tau_n(s))) dW_s.
\end{aligned}
$$

By using Lemma 1, we have

$$
\begin{aligned}
& \frac{1}{6} \|X^{(n)}(t) - X^{(n)}(\tilde{t})\|_{ms}^2 \\
\leq & \mathbb{E} \|\eta \sum_{i=1}^{m-1} \frac{t^{\alpha_m-\alpha_i} - \tilde{t}^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m - \alpha_i + 1)}\|^2 \\
& + \mathbb{E} \| \sum_{i=1}^{m-1} \frac{1}{\Gamma(\alpha_m - \alpha_i)} \int_0^{\tilde{t}} ((\tilde{t}-s)^{\alpha_m-\alpha_i-1} - (t-s)^{\alpha_m-\alpha_i-1}) X^{(n)}(\tau_n(s)) ds \\
& \quad - \sum_{i=1}^{m-1} \frac{1}{\Gamma(\alpha_m - \alpha_i)} \int_{\tilde{t}}^{t} (t-s)^{\alpha_m-\alpha_i-1} X^{(n)}(\tau_n(s)) ds \|^2 \\
& + \mathbb{E} \| \frac{1}{\Gamma(\alpha_m)} \int_0^{\tilde{t}} ((t-s)^{\alpha_m-1} - (\tilde{t}-s)^{\alpha_m-1}) f(\tau_n(s), X^{(n)}(\tau_n(s))) ds \|^2 \\
& + \mathbb{E} \| \frac{1}{\Gamma(\alpha_m)} \int_{\tilde{t}}^{t} (t-s)^{\alpha_m-1} f(\tau_n(s), X^{(n)}(\tau_n(s))) ds \|^2 \\
& + \mathbb{E} \| \frac{1}{\Gamma(\alpha_m)} \int_0^{\tilde{t}} ((\rho_n(t) - \tau_n(s))^{\alpha_m-1} - (\rho_n(\tilde{t}) - \tau_n(s))^{\alpha_m-1}) g(\tau_n(s), X^{(n)}(\tau_n(s))) dW_s \|^2 \\
& + \mathbb{E} \| \frac{1}{\Gamma(\alpha_m)} \int_{\tilde{t}}^{t} (\rho_n(t) - \tau_n(s))^{\alpha_m-1} g(\tau_n(s), X^{(n)}(\tau_n(s))) dW_s \|^2.
\end{aligned}
$$

Let $Q_2 = \max\{1, T^{\alpha_{m-1}-\alpha_1}\}$, the second term of the right-hand side can be deduced as

$$
\begin{aligned}
&\mathbb{E}\|\sum_{i=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_i)}[\int_0^{\tilde{t}}((\tilde{t}-s)^{\alpha_m-\alpha_i-1}-(t-s)^{\alpha_m-\alpha_i-1})X^{(n)}(\tau_n(s))ds\\
&\qquad -\int_{\tilde{t}}^{t}(t-s)^{\alpha_m-\alpha_i-1}X^{(n)}(\tau_n(s))ds]\|^2\\
\leq&\sum_{i=1}^{m-1}\frac{(m-1)}{\Gamma^2(\alpha_m-\alpha_i)}\mathbb{E}\|\int_0^{\tilde{t}}((\tilde{t}-s)^{\alpha_m-\alpha_i-1}-(t-s)^{\alpha_m-\alpha_i-1})X^{(n)}(\tau_n(s))ds\\
&\qquad -\int_{\tilde{t}}^{t}(t-s)^{\alpha_m-\alpha_i-1}X^{(n)}(\tau_n(s))ds\|^2\\
\leq&\sum_{i=1}^{m-1}\frac{2(m-1)}{\Gamma^2(\alpha_m-\alpha_i)}\mathbb{E}(\|\int_0^{\tilde{t}}((\tilde{t}-s)^{\alpha_m-\alpha_i-1}-(t-s)^{\alpha_m-\alpha_i-1})^{\frac{1}{2}\cdot 2}X^{(n)}(\tau_n(s))ds\|^2\\
&\qquad +\|\int_{\tilde{t}}^{t}(t-s)^{\frac{\alpha_m-\alpha_i-1}{2}}(t-s)^{\frac{\alpha_m-\alpha_i-1}{2}}X^{(n)}(\tau_n(s))ds\|^2)\\
\leq&\sum_{i=1}^{m-1}\frac{2(m-1)(t-\tilde{t})^{\alpha_m-\alpha_i}}{(\alpha_m-\alpha_i)\Gamma^2(\alpha_m-\alpha_i)}(\int_0^{\tilde{t}}((\tilde{t}-s)^{\alpha_m-\alpha_i-1}-(t-s)^{\alpha_m-\alpha_i-1})\|X^{(n)}(\tau_n(s))\|_{ms}^2ds\\
&\qquad +\int_{\tilde{t}}^{t}(t-s)^{\alpha_m-\alpha_i-1}\|X^{(n)}(\tau_n(s))\|_{ms}^2ds)\\
\leq&\frac{2Q_2(m-1)^2(t-\tilde{t})^{\alpha_m-\alpha_{m-1}}}{m_1^2(\alpha_m-\alpha_{m-1})}(\int_0^{\tilde{t}}((\tilde{t}-s)^{\alpha_m-\alpha_{m-1}-1}-(t-s)^{\alpha_m-\alpha_{m-1}-1})\|X^{(n)}(\tau_n(s))\|_{ms}^2ds\\
&\qquad +\int_{\tilde{t}}^{t}(t-s)^{\alpha_m-\alpha_{m-1}-1}\|X^{(n)}(\tau_n(s))\|_{ms}^2ds),
\end{aligned}
$$

By using Hölder inequality and Itô's isometry, we have that

$$
\begin{aligned}
&\frac{1}{6}\|X^{(n)}(t)-X^{(n)}(\tilde{t})\|_{ms}^2\\
\leq\ &\mathbb{E}\|\eta\sum_{i=1}^{m-1}\frac{t^{\alpha_m-\alpha_i}-\tilde{t}^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)}\|^2\\
&+\frac{2Q_2(m-1)^2(t-\tilde{t})^{\alpha_m-\alpha_{m-1}}}{m_1^2(\alpha_m-\alpha_{m-1})}\int_0^{\tilde{t}}((\tilde{t}-s)^{\alpha_m-\alpha_{m-1}-1}-(t-s)^{\alpha_m-\alpha_{m-1}-1})\|X^{(n)}(\tau_n(s))\|_{ms}^2ds\\
&+\frac{2Q_2(m-1)^2(t-\tilde{t})^{\alpha_m-\alpha_{m-1}}}{m_1^2(\alpha_m-\alpha_{m-1})}\int_{\tilde{t}}^{t}(t-s)^{\alpha_m-\alpha_{m-1}-1}\|X^{(n)}(\tau_n(s))\|_{ms}^2ds\\
&+\frac{\tilde{t}}{\Gamma^2(\alpha_m)}\int_0^{\tilde{t}}((t-s)^{\alpha_m-1}-(\tilde{t}-s)^{\alpha_m-1})^2\|f(\tau_n(s),X^{(n)}(\tau_n(s)))\|_{ms}^2ds\\
&+\frac{t-\tilde{t}}{\Gamma^2(\alpha_m)}\int_{\tilde{t}}^{t}(t-s)^{2\alpha_m-2}\|f(\tau_n(s),X^{(n)}(\tau_n(s)))\|_{ms}^2ds\\
&+\frac{1}{\Gamma^2(\alpha_m)}\int_0^{\tilde{t}}((\rho_n(t)-\tau_n(s))^{\alpha_m-1}-(\rho_n(\tilde{t})-\tau_n(s))^{\alpha_m-1})^2\|g(\tau_n(s),X^{(n)}(\tau_n(s)))\|_{ms}^2ds\\
&+\frac{1}{\Gamma^2(\alpha_m)}\int_{\tilde{t}}^{t}((\rho_n(t)-\tau_n(s))^{2\alpha_m-2}\|g(\tau_n(s),X^{(n)}(\tau_n(s)))\|_{ms}^2ds.
\end{aligned}
$$

This together with Lemma 2 and the following two inequalities

$$
(\frac{1}{(t-s)^{1-\alpha_m}}-\frac{1}{(\tilde{t}-s)^{1-\alpha_m}})^2<\frac{1}{(\tilde{t}-s)^{2-2\alpha_m}}-\frac{1}{(t-s)^{2-2\alpha_m}},
$$

and

$$
\frac{1}{(\rho_n(\tilde{t})-\tau_n(s))^{2-2\alpha_m}}-\frac{1}{(\rho_n(t)-\tau_n(s))^{2-2\alpha_m}}\leq\frac{1}{(\rho_n(\tilde{t})-s)^{2-2\alpha_m}}-\frac{1}{(\rho_n(t)-s)^{2-2\alpha_m}},
$$

implies that

$$
\begin{aligned}
&\frac{1}{6}\|X^{(n)}(t)-X^{(n)}(\tilde{t})\|_{ms}^2\\
\leq\ &\frac{(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})^2}\|\eta\|_{ms}^2(t-\tilde{t})^{2\alpha_m-2\alpha_{m-1}}\\
&+\frac{2Q_2(m-1)^2C_1}{m_1^2(\alpha_m-\alpha_{m-1})^2}(t-\tilde{t})^{\alpha_m-\alpha_{m-1}}(2(t-\tilde{t})^{\alpha_m-\alpha_{m-1}}+\tilde{t}^{\alpha_m-\alpha_{m-1}}-t^{\alpha_m-\alpha_{m-1}})\\
&+\frac{2K^2\tilde{t}(1+C_1)}{(2\alpha_m-1)\Gamma^2(\alpha_m)}((t-\tilde{t})^{2\alpha_m-1}+\tilde{t}^{2\alpha_m-1}-t^{2\alpha_m-1})\\
&+\frac{2K^2(t-\tilde{t})(1+C_1)}{(2\alpha_m-1)\Gamma^2(\alpha_m)}(t-\tilde{t})^{2\alpha_m-1}\\
&+\frac{2K^2(1+C_1)}{(2\alpha_m-1)\Gamma^2(\alpha_m)}(t-\tilde{t})^{2\alpha_m-1}\\
&+\frac{2K^2(1+C_1)}{(2\alpha_m-1)\Gamma^2(\alpha_m)}((\rho_n(t)-\tilde{t})^{2\alpha_m-1}-(\rho_n(\tilde{t})-\tilde{t})^{2\alpha_m-1}).
\end{aligned}\tag{8}
$$

Noting $0<2\alpha_m-1<1$, and using the inequality $|x+y|^{2\alpha_m-1}\leq|x|^{2\alpha_m-1}+|y|^{2\alpha_m-1}$, we obtain that

$$
(\rho_n(t)-\tilde{t})^{2\alpha_m-1}-(\rho_n(\tilde{t})-\tilde{t})^{2\alpha_m-1}\leq(\rho_n(t)-\rho_n(\tilde{t}))^{2\alpha_m-1}.
$$

According to the definition of $\rho_n(\cdot)$ in (5), it holds $\rho_n(t) - \rho_n(\tilde{t}) \leq t - \tilde{t} + \frac{T}{n}$. Thus,

$$\frac{(\rho_n(t) - \rho_n(\tilde{t}))^{2\alpha_m - 1}}{2\alpha_m - 1} \leq \frac{(t - \tilde{t})^{2\alpha_m - 1} + \left(\frac{T}{n}\right)^{2\alpha_m - 1}}{2\alpha_m - 1}.$$

This together with (8) implies that

$$\begin{aligned}
&\frac{1}{6}\|X^{(n)}(t) - X^{(n)}(\tilde{t})\|_{ms}^2 \\
\leq\ &\frac{(m-1)^2(4Q_2C_1 + \|\eta\|_{ms}^2)}{m_1^2(\alpha_m - \alpha_{m-1})^2}(t - \tilde{t})^{2\alpha_m - 2\alpha_{m-1}} + \frac{2K^2(1 + C_1)}{(2\alpha_m - 1)\Gamma^2(\alpha_m)}(t + 2)(t - \tilde{t})^{2\alpha_m - 1} \\
&+ \frac{2K^2(1 + C_1)}{(2\alpha_m - 1)\Gamma^2(\alpha_m)} \cdot \frac{T^{2\alpha_m - 1}}{n^{2\alpha_m - 1}} \\
\leq\ &\left(\frac{(m-1)^2(\|\eta\|_{ms}^2 + 4Q_2C_1)}{m_1^2(\alpha_m - \alpha_{m-1})^2} + \frac{2K^2(1 + C_1)}{(2\alpha_m - 1)\Gamma^2(\alpha_m)}(T + 2)\right)(t - \tilde{t})^{2\alpha_m - 2\alpha_{m-1}} \\
&+ \frac{2K^2(1 + C_1)}{(2\alpha_m - 1)\Gamma^2(\alpha_m)} \cdot \frac{T^{2\alpha_m - 1}}{n^{2\alpha_m - 1}}.
\end{aligned}$$

This proof is completed. □

Theorem 1. (Strong convergence of the EM method for multi-term Caputo SFDEs)
Let $\kappa = \min\{\alpha_m - 0.5,\ \alpha_m - \alpha_{m-1}\}$. Then, there exists a constant C depending only on T, L, L_1, α_i, K such that

$$\sup_{0 \leq t \leq T} \|X^{(n)}(t) - X(t)\|_{ms}^2 \leq \frac{C}{n^{2\kappa}}. \tag{9}$$

Proof. For a fixed $\eta \in \mathfrak{X}_0$. From (4) and (5), it deduces that

$$\begin{aligned}
&X^{(n)}(t) - X(t) \\
=\ &\sum_{i=1}^{m-1} \frac{1}{\Gamma(\alpha_m - \alpha_i)} \int_0^t (t - s)^{\alpha_m - \alpha_i - 1}(X(s) - X^{(n)}(\tau_n(s)))ds \\
&+ \frac{1}{\Gamma(\alpha_m)} \int_0^t (t - s)^{\alpha_m - 1}(f(\tau_n(s), X^{(n)}(\tau_n(s))) - f(s, X(s)))ds \\
&+ \frac{1}{\Gamma(\alpha_m)} \int_0^t \frac{g(\tau_n(s), X^{(n)}(\tau_n(s))) - g(s, X(s))}{(t - s)^{1 - \alpha_m}} dW_s \\
&+ \frac{1}{\Gamma(\alpha_m)} \int_0^t \left(\frac{g(\tau_n(s), X^{(n)}(\tau_n(s)))}{(\rho_n(t) - \tau_n(s))^{1 - \alpha_m}} - \frac{g(\tau_n(s), X^{(n)}(\tau_n(s)))}{(t - s)^{1 - \alpha_m}}\right) dW_s.
\end{aligned}$$

By Lemma 1, Hölder inequality and Itô's isometry, we have

$$\begin{aligned}
&\frac{1}{4}\|X^{(n)}(t)-X(t)\|_{ms}^2 \\
\le\ &\frac{Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})}\int_0^t (t-s)^{\alpha_m-\alpha_{m-1}-1}\|X(s)-X^{(n)}(\tau_n(s))\|_{ms}^2 ds \\
&+\frac{t}{\Gamma^2(\alpha_m)}\int_0^t \frac{\|f(\tau_n(s),X^{(n)}(\tau_n(s)))-f(s,X(s))\|_{ms}^2}{(t-s)^{2-2\alpha_m}}ds \\
&+\frac{1}{\Gamma^2(\alpha_m)}\int_0^t \Big(\frac{\|g(\tau_n(s),X^{(n)}(\tau_n(s)))\|_{ms}}{(\rho_n(t)-\tau_n(s))^{1-\alpha_m}}-\frac{\|g(\tau_n(s),X^{(n)}(\tau_n(s)))\|_{ms}}{(t-s)^{1-\alpha_m}}\Big)^2 ds \\
&+\frac{1}{\Gamma^2(\alpha_m)}\int_0^t \frac{\|g(\tau_n(s),X^{(n)}(\tau_n(s)))-g(s,X(s))\|_{ms}^2}{(t-s)^{2-2\alpha_m}}ds.
\end{aligned}\tag{10}$$

Moreover, based on **(H1)** and **(H2)**, it is easily obtained that

$$\begin{aligned}
&\|f(\tau_n(s),X^{(n)}(\tau_n(s)))-f(s,X(s))\|_{ms}^2 \\
\le\ &2L^2\|X^{(n)}(\tau_n(s))-X(s)\|_{ms}^2+2L_1^2|\tau_n(s)-s|^2,
\end{aligned}\tag{11}$$

and

$$\begin{aligned}
&\|g(\tau_n(s),X^{(n)}(\tau_n(s)))-g(s,X(s))\|_{ms}^2 \\
\le\ &2L^2\|X^{(n)}(\tau_n(s))-X(s)\|_{ms}^2+2L_1^2|\tau_n(s)-s|^2.
\end{aligned}\tag{12}$$

By **(H3)**, Lemma 1, and the following inequality

$$\begin{aligned}
\Big(\frac{1}{(\rho_n(t)-\tau_n(s))^{1-\alpha_m}}-\frac{1}{(t-s)^{1-\alpha_m}}\Big)^2 &\le \frac{1}{(t-s)^{2-2\alpha_m}}-\frac{1}{(\rho_n(t)-\tau_n(s))^{2-2\alpha_m}} \\
&\le \frac{1}{(t-s)^{2-2\alpha_m}}-\frac{1}{(\frac{2T}{n}+t-s)^{2-2\alpha_m}},
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^t \Big(\frac{\|g(\tau_n(s),X^{(n)}(\tau_n(s)))\|_{ms}}{(\rho_n(t)-\tau_n(s))^{1-\alpha_m}}-\frac{\|g(\tau_n(s),X^{(n)}(\tau_n(s)))\|_{ms}}{(t-s)^{1-\alpha_m}}\Big)^2 ds \\
\le\ &2K^2(1+C_1)\int_0^t \Big(\frac{1}{(t-s)^{2-2\alpha_m}}-\frac{1}{(\frac{2T}{n}+t-s)^{2-2\alpha_m}}\Big)ds \\
\le\ &\frac{2K^2(1+C_1)(2T)^{2\alpha_m-1}}{2\alpha_m-1}\frac{1}{n^{2\alpha_m-1}}.
\end{aligned}$$

This together with (10)-(12) implies that

$$\begin{aligned}
&\|X^{(n)}(t)-X(t)\|_{ms}^2 \\
\le\ &\frac{4Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})}\int_0^t (t-s)^{\alpha_m-\alpha_{m-1}-1}\|X(s)-X^{(n)}(\tau_n(s))\|_{ms}^2 ds \\
&+\frac{8L^2(t+1)}{\Gamma^2(\alpha_m)}\int_0^t \frac{\|X^{(n)}(\tau_n(s))-X(s)\|_{ms}^2}{(t-s)^{2-2\alpha_m}}ds+\frac{8L_1^2 t}{\Gamma^2(\alpha_m)}\int_0^t \frac{|\tau_n(s)-s|^2}{(t-s)^{2-2\alpha_m}}ds \\
&+\frac{8L_1^2}{\Gamma^2(\alpha_m)}\int_0^t \frac{|\tau_n(s)-s|^2}{(t-s)^{2-2\alpha_m}}ds+\frac{8K^2(1+C_1)(2T)^{2\alpha_m-1}}{(2\alpha_m-1)\Gamma^2(\alpha_m)}\frac{1}{n^{2\alpha_m-1}}.
\end{aligned}\tag{13}$$

By the definition of $\tau_n(\cdot)$ in (5), we have $|\tau_n(s)-s| \le \frac{T}{n}$ for any $s \in [0,T]$. Hence, a direct computation yields that

$$\begin{aligned}
&\frac{8L_1^2 t}{\Gamma^2(\alpha_m)}\int_0^t \frac{|\tau_n(s)-s|^2}{(t-s)^{2-2\alpha_m}}ds + \frac{8L_1^2}{\Gamma^2(\alpha_m)}\int_0^t \frac{|\tau_n(s)-s|^2}{(t-s)^{2-2\alpha_m}}ds \\
\le\ &\frac{8L_1^2 T^{2\alpha_m+1}(T+1)}{(2\alpha_m-1)\Gamma^2(\alpha_m)}\cdot\frac{1}{n^2}.
\end{aligned} \tag{14}$$

On the other hand, by virtue of Lemma 2, we have

$$\begin{aligned}
&\|X^{(n)}(\tau_n(s))-X(s)\|_{ms}^2 \\
\le\ &2\|X^{(n)}(\tau_n(s))-X^{(n)}(s)\|_{ms}^2 + 2\|X^{(n)}(s)-X(s)\|_{ms}^2 \\
\le\ &2C_2|\tau_n(s)-s|^{2\alpha_m-2\alpha_{m-1}} + \frac{2C_3}{n^{2\alpha_m-1}} + 2\|X^{(n)}(s)-X(s)\|_{ms}^2 \\
\le\ &\frac{2T^{2\alpha_m-2\alpha_{m-1}}C_2}{n^{2\alpha_m-2\alpha_{m-1}}} + \frac{2C_3}{n^{2\alpha_m-1}} + 2\|X^{(n)}(s)-X(s)\|_{ms}^2,
\end{aligned}$$

where C_2 and C_3 are defined in (7). This together with (13) and (14) gives that

$$\begin{aligned}
&\sup_{0\le s\le t}\|X^{(n)}(s)-X(s)\|_{ms}^2 \\
\le\ &\left(\frac{8Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})} + \frac{16L^2(T+1)}{\Gamma^2(\alpha_m)}\right)\int_0^t \frac{sup_{0\le r\le s}\|X(s)-X^{(n)}(s)\|_{ms}^2}{(t-s)^{1+\alpha_{m-1}-\alpha_m}}ds \\
&+D_1\frac{1}{n^2} + D_2\frac{1}{n^{2\alpha_m-1}} + D_3\frac{1}{n^{2\alpha_m-2\alpha_{m-1}}},
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= \frac{8L_1^2T^{2\alpha_m+1}(T+1)}{(2\alpha_m-1)\Gamma^2(\alpha_m)}, \\
D_2 &= \left(\frac{8Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})} + \frac{16L^2(T+1)}{\Gamma^2(\alpha_m)}\right)\frac{C_3T^{\alpha_m-\alpha_{m-1}}}{\alpha_m-\alpha_{m-1}} + \frac{2^{2\alpha_m+2}K^2(1+C_1)T^{2\alpha_m-1}}{(2\alpha_m-1)\Gamma^2(\alpha_m)}, \\
D_3 &= \left(\frac{8Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})} + \frac{16L^2(T+1)}{\Gamma^2(\alpha_m)}\right)\frac{C_2T^{3\alpha_m-3\alpha_{m-1}}}{\alpha_m-\alpha_{m-1}}.
\end{aligned}$$

Applying the generalized Gronwall's inequality, we can arrive at

$$\begin{aligned}
&\sup_{0\le s\le t}\|X^{(n)}(s)-X(s)\|_{ms}^2 \\
\le\ &\left(\frac{D_1}{n^2} + \frac{D_2}{n^{2\alpha_m-1}} + \frac{D_3}{n^{2\alpha_m-2\alpha_{m-1}}}\right) \\
&\cdot E_{\alpha_m-\alpha_{m-1}}\left(\left(\frac{8Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})} + \frac{16L^2(T+1)}{\Gamma^2(\alpha_m)}\right)T^{\alpha_m-\alpha_{m-1}}\Gamma(\alpha_m-\alpha_{m-1})\right).
\end{aligned}$$

Hence, Inequality (9) can be obtained by denoting

$$C = \left(\sum_{i=1}^3 D_i\right)E_{\alpha_m-\alpha_{m-1}}\left(\left(\frac{8Q(m-1)^2}{m_1^2(\alpha_m-\alpha_{m-1})} + \frac{16L^2(T+1)}{\Gamma^2(\alpha_m)}\right)T^{\alpha_m-\alpha_{m-1}}\Gamma(\alpha_m-\alpha_{m-1})\right).$$

The proof is completed. □

4 | EXTENSION AND FAST IMPLEMENTATION OF THE EM METHOD

The proposed EM method can be easily extended to the following multi-term FSDEs with vector-valued noise and initial value condition $X(0)=\eta$,

$$\sum_{i=1}^m {}_0^C D_t^{\alpha_i} X(t) = f(t,X(t)) + \sum_{i=1}^r g_i(t,X(t))\frac{dW_t^i}{dt},\ 0 < t \le T,$$

where $1 > \alpha_m > \alpha_{m-1} > \cdots > \alpha_2 > \alpha_1 > 0$, $\alpha_m > 0.5$, $\alpha_m + \alpha_{m-1} > 1$, the functions $f(t, X(t))$ and $\{g_i(t, X(t))\}_{i=0}^r$ satisfy the assumptions as in **(H1)**, **(H2)** and **(H3)**. Then, the EM method for solving the above problem can be described as: for each $n \in \mathbb{N}^*$ and $t \in (0, T]$,

$$\begin{aligned}
X^{(n)}(t) = {} & \eta \sum_{i=1}^{m} \frac{t^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)} \\
& - \sum_{i=1}^{m-1} \frac{1}{\Gamma(\alpha_m-\alpha_i)} \int_0^t (t-s)^{\alpha_m-\alpha_i-1} X^{(n)}(\tau_n(s))ds \\
& + \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m-1} f(\tau_n(s), X^{(n)}(\tau_n(s)))ds \\
& + \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{r} \int_0^t (\rho_n(t)-\tau_n(s))^{\alpha_m-1} g_i(\tau_n(s), X^{(n)}(\tau_n(s))dW_s^i,
\end{aligned} \tag{15}$$

where $X^{(n)}(t)$ is the numerical solution with $X^{(n)}(0) = \eta$, $\tau_n(s) = \frac{kT}{n}$ and $\rho_n(s) = \frac{(k+1)T}{n}$ for $s \in (\frac{kT}{n}, \frac{(k+1)T}{n}]$. By using the similar analytical procedures to prove Theorem 1, it can be deduced that the strong convergence order of the above EM method is also to be $\min\{\alpha_m - 0.5,\ \alpha_m - \alpha_{m-1}\}$.

Now let us discuss the fast implementation of the EM method (5) based on SOE technique[17,18,19]. The fast implementation of (15) is very similar, thus we neglect it. For convenience $n \in \mathbb{N}^*$, denote $\tau = \frac{T}{n}$ and $t_k = k\tau$ with $k = 0, 1, \cdots, n$. Consider Equation (4) at $t = t_k$, where $k \geq 2$ (if $k = 1$, the EM method (5) is directly applied), that is

$$\begin{aligned}
X(t_k) = {} & \eta \sum_{i=1}^{m} \frac{t_k^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)} - \sum_{i=1}^{m-1} \int_0^{t_k} \frac{(t_k-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m-\alpha_i)} X(s)ds \\
& + \int_0^{t_k} \frac{(t_k-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} f(s, X(s))ds + \int_0^{t_k} \frac{(t_k-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} g(s, X(s))dW_s \\
= {} & \eta \sum_{i=1}^{m} \frac{t_k^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)} - \sum_{i=1}^{m-1} \int_0^{t_{k-1}} \frac{(t_k-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m-\alpha_i)} X(s)ds \\
& + \int_0^{t_{k-1}} \frac{(t_k-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} f(s, X(s))ds + \int_0^{t_{k-1}} \frac{(t_k-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} g(s, X(s))dW_s \\
& - \sum_{i=1}^{m-1} \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m-\alpha_i)} X(s)ds + \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} f(s, X(s))ds \\
& + \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} g(s, X(s))dW_s.
\end{aligned} \tag{16}$$

For the integrals from t_{k-1} to t_k in (16), we use the same approximations in the EM method (5) to discretize them, namely

$$\sum_{i=1}^{m-1} \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m-\alpha_i)} X(s)ds \approx \sum_{i=1}^{m-1} \frac{\tau^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)} X(t_{k-1}),$$

$$\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} f(s, X(s))ds \approx \frac{\tau^{\alpha_m}}{\Gamma(\alpha_m+1)} f(t_{k-1}, X(t_{k-1})),$$

and

$$\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} g(s, X(s)) dW_s \approx \frac{\tau^{\alpha_m - 1}}{\Gamma(\alpha_m)} g(t_{k-1}, X(t_{k-1})) \Delta W_k,$$

where $\Delta W_k = W(t_k) - W(t_{k-1})$.

For the integrals from 0 to t_{k-1} in (16), the SOE approximation technique is used to discretize them. This is the key ingredient to construct the fast implementation of the EM method (5), that is

$$\sum_{i=1}^{m-1} \int_0^{t_{k-1}} \frac{(t_k - s)^{\alpha_m - \alpha_i - 1}}{\Gamma(\alpha_m - \alpha_i)} X(s) ds \approx \sum_{i=1}^{m-1} \frac{1}{\Gamma(\alpha_m - \alpha_i)} \sum_{j=1}^{N_{exp}} \omega_j^{(\alpha_m - \alpha_i)} U_{X,j}^{(\alpha_m - \alpha_i)}(t_k),$$

$$\int_0^{t_{k-1}} \frac{(t_k - s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f(s, X(s)) ds \approx \frac{1}{\Gamma(\alpha_m)} \sum_{j=1}^{N_{exp}} \omega_j^{(\alpha_m)} U_{f,j}^{(\alpha_m)}(t_k),$$

and

$$\int_0^{t_{k-1}} \frac{(t_k - s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} g(s, X(s)) dW_s \approx \frac{1}{\Gamma(\alpha_m)} \sum_{j=1}^{N_{exp}} \omega_j^{(\alpha_m)} U_{g,j}^{(\alpha_m)}(t_k),$$

where the summations are evaluated from the following SOE approximations on $[\tau, T]$

$$|t^{\alpha_m - 1} - \sum_{j=1}^{N_{exp}} \omega_j^{(\alpha_m)} e^{-s_j^{(\alpha_m)} t}| \vee |t^{\alpha_m - \alpha_i - 1} - \sum_{j=1}^{N_{exp}} \omega_j^{(\alpha_m - \alpha_i)} e^{-s_j^{(\alpha_m - \alpha_i)} t}| \leq \varepsilon, \ i = 1, \cdots, m-1,$$

and

$$U_{X,j}^{(\alpha_m - \alpha_i)}(t_k) = \int_0^{t_{k-1}} e^{-(t_k - s) s_j^{(\alpha_m - \alpha_i)}} X(s) ds, \ i = 1, \cdots, m-1$$

$$U_{f,j}^{(\alpha_m)}(t_k) = \int_0^{t_{k-1}} e^{-(t_k - s) s_j^{(\alpha_m)}} f(s, X(s)) ds,$$

$$U_{g,j}^{(\alpha_m)}(t_k) = \int_0^{t_{k-1}} e^{-(t_k - s) s_j^{(\alpha_m)}} g(s, X(s)) dW_s.$$

Where $\omega_j^{(\alpha_m - \alpha_i)}$, $\omega_j^{(\alpha_m)}$ and $s_j^{(\alpha_m - \alpha_i)}$, $s_j^{(\alpha_m)}$ are the Gaussian weights and nodes, ε is the uniform absolute error, and N_{exp} is the number of exponentials.

Now we can compute $U_{X,j}^{(\alpha_m - \alpha_i)}(t_k)$, $U_{f,j}^{(\alpha_m)}(t_k)$, and $U_{g,j}^{(\alpha_m)}(t_k)$ for $k = 2, \cdots, n$ by the following recurrence formulas

$$U_{X,j}^{(\alpha_m - \alpha_i)}(t_k) \approx e^{-\tau s_j^{(\alpha_m - \alpha_i)}} U_{X,j}^{(\alpha_m - \alpha_i)}(t_{k-1}) + \int_{t_{k-2}}^{t_{k-1}} e^{-(t_k - s) s_j^{(\alpha_m - \alpha_i)}} X(t_{k-2}) ds,$$

$$U_{f,j}^{(\alpha_m)}(t_k) \approx e^{-\tau s_j^{(\alpha_m)}} U_{f,j}^{(\alpha_m)}(t_{k-1}) + \int_{t_{k-2}}^{t_{k-1}} e^{-(t_k - s) s_j^{(\alpha_m)}} f(t_{k-2}, X(t_{k-2})) ds,$$

$$U_{g,j}^{(\alpha_m)}(t_k) \approx e^{-\tau s_j^{(\alpha_m)}} U_{g,j}^{(\alpha_m)}(t_{k-1}) + e^{-2\tau s_j^{(\alpha_m)}} g(t_{k-2}, X(t_{k-2})) \Delta W_{k-1},$$

with $U_{X,j}^{(\alpha_m-\alpha_i)}(t_1) = 0$, $U_{f,j}^{(\alpha_m)}(t_1) = 0$, and $U_{g,j}^{(\alpha_m)}(t_1) = 0$. Thus, the fast implementation of proposed EM method (5) can be described as

$$\begin{aligned}X^{(n)}(t_k) &= \eta\sum_{i=1}^{m}\frac{t_k^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)} - \sum_{i=1}^{m-1}\frac{1}{\Gamma(\alpha_m-\alpha_i)}\sum_{j=1}^{N_{exp}}\omega_j^{(\alpha_m-\alpha_i)}U_{X,j}^{(\alpha_m-\alpha_i)}(t_k)\\ &+\frac{1}{\Gamma(\alpha_m)}\sum_{j=1}^{N_{exp}}\omega_j^{(\alpha_m)}\left(U_{f,j}^{(\alpha_m)}(t_k)+U_{g,j}^{(\alpha_m)}(t_k)\right) - \sum_{i=1}^{m-1}\frac{\tau^{\alpha_m-\alpha_i}X^{(n)}(t_{k-1})}{\Gamma(\alpha_m-\alpha_i+1)}\\ &+\frac{\tau^{\alpha_m}}{\Gamma(\alpha_m+1)}f(t_{k-1},X^{(n)}(t_{k-1}))+\frac{\tau^{\alpha_m-1}}{\Gamma(\alpha_m)}g(t_{k-1},X^{(n)}(t_{k-1}))\Delta W_k.\end{aligned}$$

The above fast method requires $O(nN_{exp})$ computational work whereas the computational cost of direct EM method (5) is $O(n^2)$. Obviously, N_{exp} is greatly less than n when the step size τ becomes small. Thus, the fast method has the more powerful computational performance than the direct EM method (5).

5 | NUMERICAL EXAMPLES

In this section, three numerical examples are given to demonstrate the effectiveness of our EM method and its fast implementation for solving multi-term FSDEs. All of these computations are performed by using a MATLAB (R2017b) subroutine on a laptop (Lenovo G50) with the Intel(R) Core(TM) i7-5500U CPU 2.40 GHz and 4G RAM. The computational error is defined as

$$e_n = \max_{1\le k\le n}\left(\frac{1}{1000}\sum_{i=1}^{1000}\|X^{(n,i)}(t_k)-X^{(2n,i)}(t_k)\|^2\right)^{\frac{1}{2}},$$

where i denotes the ith sample path.

Example 1. Consider the following two-term FSDEs

$${}_{\mathbf{0}}^{\mathbf{C}}\mathbf{D}_{\mathbf{t}}^{\boldsymbol{\alpha_2}}X(t)+{}_{\mathbf{0}}^{\mathbf{C}}\mathbf{D}_{\mathbf{t}}^{\boldsymbol{\alpha_1}}X(t) = cos(X(t))+sin(X(t))\frac{dW_t}{dt},\ 0<t\le 1,$$

with initial value $X(0) = \eta = 0.1$. For $n = 128, 256, 512, 1024$, we use the proposed EM method and its fast implementation to compute the errors and convergence orders, see Table 1. Table 1 shows that for two-term FSDEs with different combinations of α_1 and α_2, the convergence orders of the EM method and its fast implementation both approximate to $\min\{\alpha_2-0.5,\ \alpha_2-\alpha_1\}$, which verifies the convergence result in our Theorem 1. To present the computational costs of EM method and its fast implementation, the mean CPU times (second) of the two methods with different step sizes are listed in Table 2. We can know from Table 2 that the computational performance of fast method is greatly more powerful than the EM method, especially for the small step sizes.

TABLE 1 Convergence orders of EM method and its fast implementation for Example 1.

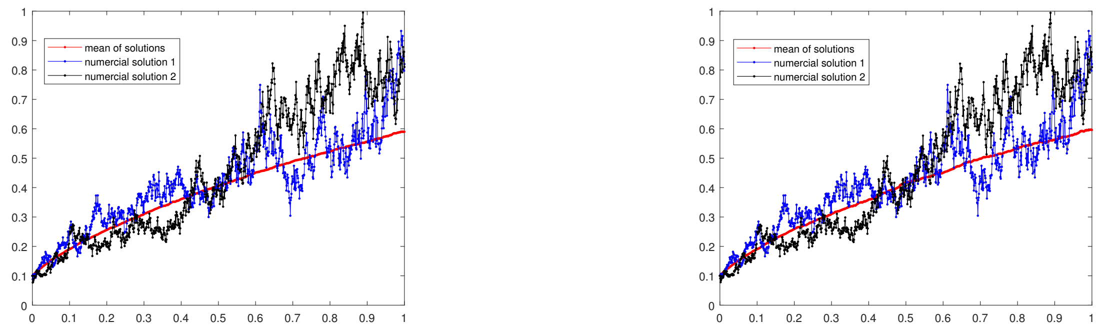
n	$\alpha_1 = 0.6, \alpha_2 = 0.9$				$\alpha_1 = 0.3, \alpha_2 = 0.9$			
	EM method		Fast method		EM method		Fast method	
	error	order	error	order	error	order	error	order
128	2.320e-2		2.400e-2		2.199e-2		2.198e-2	
256	1.922e-2	0.27	1.659e-2	0.53	1.668e-2	0.40	1.667e-2	0.40
512	1.581e-2	0.28	1.201e-2	0.46	1.235e-2	0.43	1.235e-2	0.43
1024	1.289e-2	0.29	9.281e-3	0.37	9.780e-2	0.34	9.780e-3	0.34

In Figures 1 and 2, we firstly use the EM method and its fast implementation to solve Example 1 with two different combinations of α_1 and α_2, then plot the means (red lines) of numerical solutions of 1000 sample paths, and randomly select the numerical solutions (blue lines and black lines) of two different sample paths. From these two figures, it can be found that the

TABLE 2 Comparisons of mean CPU time (second) of EM method and its fast implementation with different step sizes for Example 1.

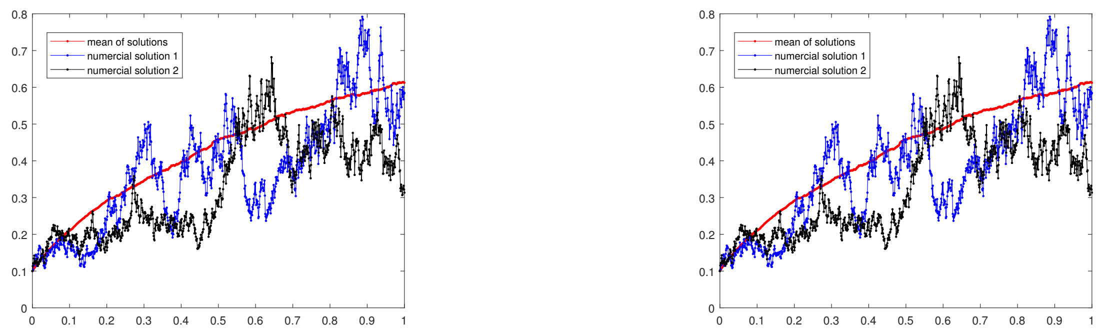
method \ n	128	256	512	1024
EM method	34.99	128.76	502.59	2000.42
Fast method	14.93	29.31	57.15	110.11

numerical solutions of EM method are consistent with the numerical solutions of fast EM method, and the values of α_1 and α_2 have remarkable influence on the numerical solutions.



(a) Solutions of EM method (b) Solutions of Fast method

FIGURE 1 For $n = 256$, numerical solutions of EM method (a) and fast method (b) when $\alpha_1 = 0.6$ and $\alpha_2 = 0.9$ in Example 1.



(a) Solutions of EM method (b) Solutions of Fast method

FIGURE 2 For $n = 256$, numerical solutions of EM method (a) and fast method (b) when $\alpha_1 = 0.3$ and $\alpha_2 = 0.9$ in Example 1.

Example 2. Consider the following three-term FSDEs

$$\sum_{i=1}^{3} {}_0^{\mathrm{C}}\mathbf{D}_t^{\alpha_i} X(t) = cos(X(t)) + sin(X(t))\frac{dW_t}{dt},\ 0 < t \leq 1,$$

with initial value $X(0) = \eta = 0.1$. Table 3 presents the errors and convergence orders of EM method and its fast implementation for solving Example 2 with two different combinations of α_1, α_2, and α_3. In Case 1, the values of α_1, α_2, and α_3 are prescribed as 0.1, 0.4, and 0.9, respectively. The left part of Table 3 tells us that the convergence order of EM method and fast method approach to $0.4 = \min\{\alpha_3 - 0.5,\ \alpha_3 - \alpha_2\}$. In Case 2, the values of α_1, α_2, and α_3 are given as 0.2, 0.55, and 0.95, respectively. The right part of Table 3 tells us that the convergence order of two methods approach to $0.4 = \min\{\alpha_3 - 0.5,\ \alpha_3 - \alpha_2\}$. Thus, Theorem 1 can be verified. Additionally, to compare the computational cost, for $n = 1024$, the mean CPU times of EM method and fast method are reported as 1964.48 seconds and 112.56 seconds, respectively.

TABLE 3 Convergence orders of EM method and its fast implementation for Example 2.

n	$\alpha_1 = 0.1, \alpha_2 = 0.4, \alpha_3 = 0.9$				$\alpha_1 = 0.2, \alpha_2 = 0.55, \alpha_3 = 0.95$			
	EM method		Fast method		EM method		Fast method	
	error	order	error	order	error	order	error	order
128	1.676e-2		1.711e-2		1.648e-2		1.374e-2	
256	1.263e-2	0.41	1.275e-2	0.42	1.229e-2	0.42	9.305e-3	0.56
512	9.250e-3	0.44	9.370e-3	0.44	9.161e-3	0.42	6.795e-3	0.45
1024	7.289e-3	0.34	7.426e-3	0.34	6.811e-3	0.43	5.337e-3	0.35

Example 3. Consider the following two-term FSDEs with vector-valued noise

$$\sum_{i=1}^{2} {}_{\mathbf{0}}^{\mathbf{C}}\mathbf{D}_{\mathbf{t}}^{\alpha_i} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} sin(X_1(t)) \\ cos(2X_2(t)) \end{pmatrix} + \begin{pmatrix} cos(X_1(t)) & cos(X_2(t)) \\ sin(X_2(t)) & sin(X_1(t)) \end{pmatrix} \begin{pmatrix} \frac{dW_1(t)}{dt} \\ \frac{dW_2(t)}{dt} \end{pmatrix}$$

for $t \in (0, 1]$, and the initial value $(X_1(0), X_2(0))^T = (0, 0)^T$. We in Table 4 list the errors and convergence orders of EM method and its fast implementation for solving Example 3, and can find that the convergence orders are close to $\min\{\alpha_2 - 0.5,\ \alpha_2 - \alpha_1\}$. And when $n = 256$, the mean CPU times of EM method and fast method are 331.09 seconds and 74.37 seconds, respectively. Obviously, the CPU time of the fast method is extremely less than that's of the direct EM method.

TABLE 4 Convergence orders of EM method and its fast implementation for Example 3.

n	$\alpha_1 = 0.3, \alpha_2 = 0.9$				$\alpha_1 = 0.6, \alpha_2 = 0.9$			
	EM method		Fast method		EM method		Fast method	
	error	order	error	order	error	order	error	order
32	2.491e-1		5.017e-1		2.232e-1		4.687e-1	
64	1.992e-1	0.32	4.044e-1	0.31	1.586e-1	0.49	3.471e-1	0.43
128	1.522e-1	0.39	3.109e-1	0.38	1.211e-1	0.38	2.669e-1	0.37
256	1.129e-1	0.43	2.400e-1	0.37	1.004e-1	0.27	2.234e-1	0.26

6 | CONCLUSION

In this paper, we have constructed the EM method for multi-term FSDEs and strictly established its strong convergence, i.e., the strong convergence order is $\min\{\alpha_m - 0.5,\ \alpha_m - \alpha_{m-1}\}$. The EM method and its theoretical results can be extended to multi-term FSDEs with vector-valued noise. Based on the SOE approximation technique, a fast implementation of this EM method is also presented. Three numerical examples are given to support our theoretical results, and demonstrate that the computational performance of the fast method has the overwhelming advantages over the direct EM method.

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Financial disclosure

None reported.

Conflict of interest

The authors declare no potential conflict of interests.

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