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RESEARCH ARTICLE

Inertial method for a solution of Split Equality of Monotone Inclusion and the f-Fixed Point Problems in Banach Spaces

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Summary

In this paper, we propose an inertial algorithm for solving split equality of monotone inclusion and f-fixed point of Bregman relatively f-nonexpansive mapping problems in reflexive real Banach spaces. Using the Bregman distance function, we prove a strong convergence theorem for the algorithm produced by the method in real reflexive Banach spaces. As an application, we provide several applications of our method. Furthermore, we give a numerical example to demonstrate the behavior of the convergence of the algorithm.

KEYWORDS:

Inertial method, Split Equality of Monotone Inclusion Problem, Bregman relatively f-nonexpansive mapping, reflexive Banach spaces

1 | INTRODUCTION

Let E be a reflexive real Banach space with its dual E^*. Let $f : E \to (-\infty, +\infty]$ be a function. We denote the domain of f by $domf$, that is, $domf = \{x \in E : f(x) < \infty\}$. A function f is said to be *proper* if $domf \neq \emptyset$. It is said to be *lower semi-continuous* if the set $\{x \in E : f(x) \leq r\}$ is closed for all $r \in \mathbb{R}$. The function f is called *convex* if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in E$ and $\alpha \in [0,1]$. The function f is called *uniformly convex* if there exists a continuous increasing function $\psi : [0, +\infty) \to \mathbb{R}$, $\psi(0) = 0$, such that $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t(1-t)\psi(\|x-y\|)$, for all $x, y \in domf$. The function ψ is called a *modulus of convexity* of f. It is called *strongly convex* if f is uniformly convex with the modulus of convexity $\psi(t) = ct^2$, $c > 0$. A function f is said to be *strongly coercive* if $\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty$.

For a proper, lower semi-continuous and convex function $f : E \to (-\infty, +\infty]$, the *subdifferential* of f at x is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle, \forall y \in E\}.$$

The *Fenchel conjugate* of f is a function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) : x \in E\}.$$

For any $x \in int(domf)$ and any $y \in E$, we denote by $f^0(x, y)$ the right-hand derivative of f at x in the direction of y, that is,

$$f^0(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{1}$$

The function f is called *Gâteaux differentiable* at x if the limit (1) exists for any $y \in E$. In this case, the gradient of f at x, $\nabla f(x)$, coincides with $f^0(x, y)$ for all $y \in E$. It is called *Gâteaux differentiable* if it is Gâteaux differentiable at every point $x \in int(domf)$. We note that if the subdifferential of f is single-valued, then $\partial f = \nabla f$. The function f is said to be *Fréchet differentiable* at x if the limit (1) is attained uniformly for every $y \in E$ with $\|y\| = 1$ and f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit (1) is attained uniformly for $x \in C$ and $\|y\| = 1$. If f is a uniformly convex and Gâteaux differentiable function in $(domf)$ with modulus of convexity ψ, then $\langle x - y, \nabla f(x) - \nabla f(y)\rangle \geq 2\psi(\|x - y\|), \forall x, y \in domf$, or equivalently, $f(y) \geq f(x) + \langle y - x, \nabla f(x)\rangle + \psi(\|x - y\|), \forall x, y \in domf$. If a function f is strongly convex with constant $\mu > 0$ and Gâteaux differentiable in $(domf)$, then $\langle x - y, \nabla f(x) - \nabla f(y)\rangle \geq \mu\|x - y\|^2, \forall x, y \in domf$, or equivalently, $f(y) \geq f(x) + \langle y - x, \nabla f(x)\rangle + \frac{\mu}{2}\|x - y\|^2, \forall x, y \in domf$. Note that, if f is strongly convex with constant μ, then f^* has a Lipschitz gradient with parameter $\frac{1}{\mu}$ and if f has a Lipschitz gradient with parameter L, then f^* is strongly convex with parameter $\frac{1}{L}$ (see,[34]). If E is a smooth and strictly convex Banach space, the function $f(x) = \|x\|^2, \forall x \in E$ is strongly convex with constant $\mu \in (0, 1]$ (see, Phelps[13]).

A function $f : E \to (-\infty, +\infty]$ is called Legendre if it satisfies the following two properties:

(L1) the interior of the domain of f, $int(domf)$, is nonempty, f is Gâteaux differentiable and $dom(\nabla f) = int(domf)$;

(L2) the interior of the domain of f^*, $int(domf^*)$, is nonempty, f^* is Gâteaux differentiable and $dom(\nabla f^*) = int(domf^*)$;

One of the important and interesting Legendre function in a smooth and strictly convex Banach space is $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < +\infty)$ with its conjugate function $f^*(x^*) = \frac{1}{q}\|x^*\|^q$ $(1 < q < +\infty)$ (see, for example, Bauschke[2] and Bauschke et al.[3]), where $\frac{1}{p} + \frac{1}{q} = 1$. In this case, the gradient of f, ∇f, coincides with the generalized duality mapping, J_p, of E, that is, $\nabla f = J_p$, where $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{y^* \in E^* : \langle x, y^*\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}, \forall x \in E.$$

If $p = 2$, we write $J_2 = J$, called the *normalized duality mapping* and if $E = H$, a real Hilbert space, then $J = I$, where I is the identity mapping on H. If the function f is a Legendre function and E is a reflexive Banach space, then $\nabla f^* = (\nabla f)^{-1}$(see, Bonnans and Shapiro[4]).

Let $f : E \to (-\infty, \infty]$ be a Gâteaux differentiable convex function. A mapping $T : E \to E^*$ is said to be *f-nonexpansive* if

$$\|Tx - Ty\| \leq \|\nabla f(x) - \nabla f(y)\|,$$

for all $x, y \in E$. The set of f-fixed points of a mapping T denoted by $F_f(T)$ is defined by $F_f(T) = \{p : Tp = \nabla f(p)\}$. A point $x \in E$ is called an *f-asymptotic fixed point*[31] of T if E contains a sequence $\{x_n\}$ which converges weakly to x and $\lim_{n\to\infty} \|T(x_n) - \nabla f(x_n)\| = 0$. We denote the set of asymptotic fixed points of T by $\widetilde{F_f}(T)$.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable convex function. The function $D_f : domf \times int(domf) \to [0, +\infty)$, defined by

$$D_f(y, x) = f(y) - f(x) - \langle y - x, \nabla f(x)\rangle, \forall x, y \in E. \tag{2}$$

is called the *Bregman distance* with respect to f (see, Bregman[5]).
The Bregman distance has the following two important properties (see, Reich and Sabach[15]), called the *three-point identity*: for any $x \in domf$ and $y, z \in int(domf)$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle x - y, \nabla f(z) - \nabla f(y)\rangle, \tag{3}$$

and the *four-point identity*: for any $y, w \in domf$ and $x, z \in int(domf)$,

$$D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle y - w, \nabla f(z) - \nabla f(x)\rangle. \tag{4}$$

Note that if E is a smooth and strictly convex Banach space and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then we have that $\nabla f = J$, where J is the normalized duality mapping from E into 2^{E^*} and the Bregman distance with respect to f, D_f, reduces to the Lyapunov functional $\phi : E \times E \to [0, +\infty)$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \forall x, y \in E. \tag{5}$$

Definition 1. (Wega and Zegeye[31]) A mapping T is called a Bregman relatively f-nonexpansive if

$$D_f(p, \nabla f^*(Tx)) \leq D_f(p, x), \forall x \in E, p \in \widetilde{F_f}(T) \text{ and } F_f(T) = \widetilde{F_f}(T) \neq \emptyset.$$

We remark that if $f(x) = \frac{1}{2}\|x\|^2$, for all $x \in E$, then $\nabla f = J$ and $D_f(y, x) = \phi(y, x)$ for all $x, y \in E$ and hence the f-nonexpansive and Bregman relatively f-nonexpansive mappings reduce to the semi-nonexpansive and ϕ-relatively J-nonexpansive mappings, respectively. Moreover, f-fixed point and f-asymptotic fixed point of T reduce to semi-fixed point and semi-asymptotic fixed point of T, respectively. If, in addition, $E = H$, a real Hilbert space, then f-nonexpansive and Bregman relatively f-nonexpansive mappings become nonexpansive and relatively nonexpansive mappings, respectively.

A mapping $A : E \to E^*$ is called *monotone* if

$$\langle x - y, A(x) - A(y)\rangle \geq 0, \forall x, y \in E.$$

It is called *α-inverse strongly monotone* if

$$\langle x - y, A(x) - A(y)\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in E.$$

A multi-valued mapping $B : E \to 2^{E^*}$ with domain $domB := \{x \in E : B(x) \neq \emptyset\}$ is said to be (i) *monotone* if, for every $x, y \in E$, we have $\langle x-y, u-v\rangle \geq 0, \forall u \in B(x), \forall v \in B(y)$; (ii) maximal monotone if it is monotone and if, for $(x, u) \in E\times E^*$, $\langle x-y, u-v\rangle \geq 0$ for all $(y, v) \in grh(B)$ implies $u \in B(x)$, where the set $grh(B) := \{(x, y) \in E\times E^* : y \in B(x)\}$ is graph of B.

Let $f : E \to \mathbb{R}$ be a convex and smooth function and $g : E \to \mathbb{R}$ be a convex and lower semicontinuous function. Consider the following minimization problem:

$$\min_{x\in E}\{f(x) + g(x)\}. \tag{6}$$

By Fermat's rule, problem (6) is equivalent to the problem of finding a point $p \in E$ such that

$$0 \in (\nabla f + \partial g)(p), \tag{7}$$

where ∇f is the gradient of f and ∂g is the subdifferential of g.

The general form of problem (7) is called *monotone inclusion problem* (MIP) which is to find an element $x \in E$ such that

$$0 \in (A + B)x, \tag{8}$$

where $A : E \to E^*$ is a monotone mapping and $B : E \to 2^{E^*}$ is a maximal monotone mapping. We denote the solution set of (8) by $(A + B)^{-1}0$, that is, $(A + B)^{-1}0 = \{x \in E : 0 \in A(x) + B(x)\}$.

We define the *resolvent* of a maximal monotone mapping B for $\lambda > 0$ by

$$J^B_\lambda(x) = (\nabla f + \lambda B)^{-1}\nabla f(x), \forall x \in E, \tag{9}$$

where $f : E \to (-\infty, +\infty]$ is a Gâteaux differentiable convex function. We note that, $F(J^B_\lambda) = B^{-1}(0)$.

Moudafi[12] introduced and studied a new generalization of the monotone inclusion problem in Hilbert spaces. It is called *Split Monotone Inclusion Problem (SMIP)* which is defined as finding a point $p \in H_1$ such that

$$(p, S(p)) \in (A + B)^{-1}0 \times (C + D)^{-1}0, \tag{10}$$

where $A : H_1 \to H_1$ and $C : H_2 \to H_2$ are inverse strongly monotone mappings, $B : H_1 \to 2^{H_1}$ and $D : H_2 \to 2^{H_2}$ are maximal monotone mappings and $S : H_1 \to H_2$ is bounded linear mapping, where $H_i, i = 1, 2$ are Hilbert spaces. If $S = I$, then the SMIP reduces to the *Common Solution Monotone Inclusion Problem (CSMIP)*. He proposed the following iterative algorithm for approximating the solution of SMIP and proved its weak convergence. For $x_1 \in H_1$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = U(x_n + \gamma S^*(T - I)Sx_n), \tag{11}$$

where S^* is the adjoint mapping of $S, T = J^B_\lambda(I - \lambda A)$, and $U = J^D_\lambda(I - \lambda C)$, where $\lambda > 0$.

The Split Monotone Inclusion and Fixed Point Problem (SMIFPP)[24] is the generalization of SMIP which is defined as finding a point $(p, q) \in H_1 \times H_2$ such that

$$p \in F(T) \cap (A + B)^{-1}0, q \in F(G) \cap (C + D)^{-1}0 \text{ and } S(p) = K(q), \tag{12}$$

where $A : H_1 \to H_1$ and $C : H_2 \to H_2$ are inverse strongly monotone mappings, $B : H_1 \to 2^{H_1}$ and $D : H_2 \to 2^{H_2}$ are maximal monotone mappings, $T : H_1 \to H_1$ and $G : H_2 \to H_2$ are demi-contractive mappings, $S : H_1 \to H_3$ and

$K : H_2 \to H_3$ are bounded linear mappings, where $H_i, i = 1, 2, 3$ are Hilbert spaces. If $C = 0 = D$, $T = G = I$, and $K = I$, where I is identity mapping, the SMIFPP reduces to the Split Monotone Inclusion Problem (SMIP).

In 2021, Taiwo et al.[24] studied the split equality problem for systems of monotone inclusions and fixed point problems of set-valued demi-contractive mappings in real Hilbert spaces. They proposed a viscosity type algorithm and proved its strong convergence under some mild assumptions.

The need to speed up the convergence of iterative algorithms has always been of great importance. In 1964, Polyak[14] proposed an *inertial algorithm* which can be seen as a discrete version of a second order time dynamical system to speed up convergence rate of smooth convex minimization problem. The main idea of this method is to make use of two previous iterates in order to update the next iterate, which results in speeding up the algorithm's convergence. Very recently, some authors have proposed viscosity-type algorithm with different inertial parameters for solving equilibrium and fixed point problems; see for example[9,32].

In 2021, Yao et al.[32] proposed the following iterative algorithm with inertial extrapolation step for approximating a solution of *SMIP* in real Hilbert spaces and proved weak convergence of the sequence generated by the proposed algorithm under some mild assumptions. Let $A : H_1 \to H_1$ and $C : H_2 \to H_2$ be inverse strongly monotone mappings, $B : H_1 \to 2^{H_1}$ and $D : H_2 \to 2^{H_2}$ be maximal monotone mappings. For arbitrary $x_0, x_1 \in H_1$, define the sequences $\{w_n\}$ and $\{x_n\}$ by

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}) \\ x_{n+1} = U(w_n + \gamma S^*(T - I)Sw_n), \end{cases} \tag{13}$$

where S^* is the adjoint mapping of S, $T = J_\lambda^B(I - \lambda A)$, and $U = J_\lambda^D(I - \lambda C)$, where $\lambda > 0$, $0 \le \alpha_n \le \overline{\alpha}_n$,where $\overline{\alpha}_n = \theta$ if $x_n = x_{n-1}$, otherwise $\overline{\alpha}_n = \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}$, where $\theta \in [0, 1)$ and $\{\epsilon_n\} \subset \ell_1$ and $\gamma_n = \gamma > 0$ if $(T - I)Sw_n = 0$, otherwise $\gamma_n = \frac{\sigma_n \|(T-I)Sw_n\|^2}{\|S^*(T-I)Sw_n\|^2}$, where $0 < \sigma_n < 1$.

In 2020, Izuchukwu et al.[9] proposed and studied a new inertial extrapolation method for solving the split feasibility problems over the solution set of monotone inclusion problems in real Hilbert spaces. Let $A : H_1 \to H_1$ be Lipschitz monotone mapping, $T : H_1 \to H_1$ be nonexpansive mappings, $B : H_1 \to 2^{H_1}$ be maximal monotone mapping and $S : H_1 \to H_3$ be bounded linear mapping such that $\|S\| \neq 0$. For arbitrary $x_0, x_1 \in H_1$, define the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{x_n\}$ by

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}) \\ w_n = u_n - \gamma_n S^*(I - T)Su_n \\ y_n = (I + \lambda_n B)^{-1}(1 - \lambda_n A)w_n \\ z_n = y_n - \lambda_n(Ay_n - Aw_n) \\ x_{n+1} = (1 - \theta_n - \beta_n)w_n + \theta_n z_n, \end{cases} \tag{14}$$

where $0 \le \alpha_n \le \overline{\alpha}_n$,where $\overline{\alpha}_n = \frac{n-1}{n+\alpha-1}$ if $x_n = x_{n-1}$, otherwise $\overline{\alpha}_n = \min\{\frac{n-1}{n+\alpha-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}$, $0 \le b \le \gamma_n \le c < \frac{1}{\|S\|^2}$, and $\lambda_{n+1} = \min\{\frac{\mu\|w_n - v_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\}$ if $Aw_n \neq Ay_n$, otherwise $\lambda_n, \forall n > 0$ and $\{\beta_n\}$, $\{\theta_n\}$, and $\{\epsilon_n\}$ are sequence of positive real numbers. They proved that the proposed method converges strongly to x^*, where $x^* \in (A + B)^{-1}(0)$ and $S(x^*) \in F(T)$, under some condition on the control parameters β_n, θ_n, and ϵ_n, provided that $\{p \in (A + B)^{-1}(0) : S(p) \in F(T)\} \neq \emptyset$.

All the results addressed above deal with either of the following: split monotone inclusion and fixed point problem or split feasibility problems over the solution set of monotone inclusion problems in real Hilbert spaces. Based on these results, the following important question arises:

Question 1. Can we obtain an inertial method for approximating a solution of split equality of monotone inclusion and f-fixed point problems in real Banach spaces?

The Split Equality of Monotone Inclusion and f-Fixed Point Problems (SEMIfFPP) is defined as finding a point $(p, q) \in E_1 \times E_2$ such that

$$(p, q) \in (F_f(T) \cap (A + B)^{-1}0) \times (F_g(G) \cap (C + D)^{-1}0) \text{ and } S(p) = K(q), \tag{15}$$

where $T : E_1 \to E_1^*$ and $G : E_2 \to E_2^*$ are Bregman relatively f-nonexpansive and Bregman relatively g-nonexpansive mappings, respectively, $A : E_1 \to E_1^*$ and $C : E_2 \to E_2^*$ are monotone mappings, and $B : E_1 \to 2^{E_1^*}$ and $D : E_2 \to 2^{E_2^*}$ are maximal monotone mappings, $S : E_1 \to E_3$ and $K : E_2 \to E_3$ are bounded linear mappings with adjoints $S^* : E_3^* \to E_1^*$ and $K^* : E_3^* \to E_2^*$, respectively. If $E_i = H_i$, $i = 1, 2, 3$ are real Hilbert spaces, then the *SEMIfFPP* reduces to *the split equality of monotone inclusion and fixed point problems (SEMIFPP).*

Many mathematical models in the fields of machine learning, statistical regression, image processing and signal recovery are reformulated as problem (15) (see,[10],[26],[27],[28],[29]). In addition, the problem includes the core of many mathematical problems, as special cases, such as: split monotone inclusion and semi-fixed (J-fixed) point problem, split monotone inclusion and fixed point problem, common solutions of monotone inclusion and fixed point problems, split equality monotone inclusion problem, split equality fixed point problem and many important optimization problems such as, split feasibility problems, split minimization problems, split equilibrium problems, split saddle-point problems (see for example,[8],[10],[20],[23],[24],[25],[26]).

Motivated and inspired by the works of Moudaf[12], Taiwo et al.[23,24], Izuchukwu et al.[9] and Sunthrayuth et al.[22], we introduce and study an inertial algorithm which converges strongly to a solution of split equality of monotone inclusion and f, g-fixed point of Bregman relatively f, g-nonexpansive mapping problems (15) in reflexive real Banach spaces. In addition, we provide several applications of our method and provide a numerical example to demonstrate the behavior of the convergence of the algorithm to a solution of the indicated problems.

2 | PRELIMINARIES

Let E be a reflexive real Banach space with its dual E^*. Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable convex function. The function $v_f : int(domf) \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$v_f(x,t) = \inf_{y \in int(domf)} \{D_f(y,x) : \|x-y\| = t\}$$

is called the *Modulus of total convexity* of f at $x \in int(domf)$ and f is called *totally convex* if

$$v_f(x,t) > 0, \text{ for all } (x,t) \in int(domf) \times \mathbb{R}^+.$$

We remark that f is totally convex on bounded subsets of E if and only if f is uniformly convex on bounded subsets of E (see, Butnariu and Resmerita[7], Theorem 2.10, Page 9).
The Bregman projection of $x \in int(domf)$ onto the nonempty, closed and convex set $C \subset domf$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y,x) : y \in C\}.$$

The well known Bregman projection properties are:

Lemma 1. (Bunariu and Resmerita[7]) Let f be a totally convex and Gâteaux differentiable function on the $int(domf)$ and $x \in int(domf)$. Let C be a nonempty, closed and convex subset of $int(domf)$. Then,

(i) $z = P_C^f(x)$ if and only if $\langle y - z, \nabla f(x) - \nabla f(z)\rangle \leq 0, \forall y \in C$;

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y,x), \forall y \in C$.

Let $f : E \to \mathbb{R}$ be a Legendre function. We make use of the function $V_f : E \times E^* \to \mathbb{R}$ defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \text{ for all } x \in E \text{ and } x^* \in E^*.$$

We note that V_f is a nonnegative function which satisfies (see, Senakka and Cholamjiak[19])

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \text{ for all } x \in E \text{ and } x^* \in E^*, \tag{16}$$

and

$$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^*\rangle \leq V_f(x, x^* + y^*), \text{ for all } x \in E \text{ and } x^*, y^* \in E^*. \tag{17}$$

Lemma 2. (Wega and Zegeye[30]) Let f be a strongly convex function with constant $\mu > 0$. Then, for all $y \in domf$ and $x \in int(domf)$,

$$D_f(y,x) \geq \frac{\mu}{2}\|x-y\|^2,$$

where $D_f(y,x)$ is a Bregman distance with respect to f .

Lemma 3. (Phelps[13]) If $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak lower semi-continuous and convex function and for any $x \in E$, $\{y_k\}_{k=1}^N \subseteq E$ and $\{c_k\}_{k=1}^N \subseteq (0,1)$

with $\sum_{k=1}^{N} c_k = 1$ the following holds:

$$D_f(x, \nabla f^* \left(\sum_{k=1}^{N} c_k \nabla f(y_k) \right)) \leq \sum_{k=1}^{N} c_k D_f(x, y_k). \tag{18}$$

Lemma 4. (Reich and Sabach[17]) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subset of E. If $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E, then $\lim_{n\to\infty} D_f(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 5. (Reich and Sabach[17]) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x \in E$ and the sequence $D_f(x_n, x)$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 6. (Butnariu and Iusem[6]) Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function. Then the following properties are satisfied:

(i) $\nabla f : E \to E^*$ is one-to-one, onto and norm-to-weak continuous;

(ii) the set $\{x \in E : D_f(x, y) \leq r\}$ is bounded for all $y \in E$ and $r > 0$;

(iii) $dom f^* = E^*$, f^* is Gâteaux differentiable and $\nabla f^* = (\nabla f)^{-1}$.

Lemma 7. (Reich and Sabach[16]) If $f : E \to \mathbb{R}$ is a uniformly Fréchet differentiable and bounded function on bounded subsets of E, then ∇f is norm-to-norm uniformly continuous on bounded subsets of E and hence both f and ∇f are bounded on bounded subset of E.

Lemma 8. (Wega and Zegeye[31]) If $T : E \to E^*$ is a Bregman relatively f-nonexpansive mapping, then $F_f(T)$ is closed and convex.

Lemma 9. (Saejung and Yotkaew[18]) Let $\{b_n\} \subset \mathbb{R}$ and let $\{a_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, n \geq 1.$$

If for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\liminf_{k\to\infty} \left(a_{n_k+1} - a_{n_k}\right) \geq 0$ we have $\limsup_{k\to\infty} b_{n_k} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 10. (Barbu[1]) Let $A : E \to E^*$ be a monotone, hemicontinuous and bounded mapping, and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Then $A + B$ is maximal monotone.

Let E_1 and E_2 be reflexive real Banach spaces with duals E_1^* and E_2^*, respectively. Let $E = E_1 \times E_2$ with dual $E^* = E_1^* \times E_2^*$ and duality pairing

$$\langle x, y^* \rangle = \langle x_1, y_1^* \rangle + \langle x_2, y_2^* \rangle,$$

where $x = (x_1, x_2) \in E$, $y^* = (y_1^*, y_2^*) \in E^*$.

Let $h : E = E_1 \times E_2 \to (-\infty, +\infty]$ be defined by $h(x_1, x_2) = f(x_1) + g(x_2), \forall (x_1, x_2) \in E_1 \times E_2$, where $f : E_1 \to (-\infty, +\infty]$ and $g : E_2 \to (-\infty, +\infty]$ are proper, lower semi-continuous and convex functions. Then h is a proper, lower semi-continuous and convex function and the subdifferential of h at $x = (x_1, x_2)$ is the convex set given by

$$\begin{aligned} \partial h(x) &= \{x^* \in E^* : h(y) - h(x) \geq \langle y - x, x^* \rangle, \forall y \in E\} \\ &= \{(x_1^*, x_2^*) \in E_1^* \times E_2^* : x_1^* \in \partial f(x_1) \text{ and } x_2^* \in \partial g(x_2)\}. \end{aligned}$$

If $f : E_1 \to (-\infty, +\infty]$ and $g : E_2 \to (-\infty, +\infty]$ are Gâteaux differentiable convex functions, then h is Gâteaux differentiable convex function and $\nabla h(x_1, x_2) = (\nabla f(x_1), \nabla g(x_2)), \forall (x_1, x_2) \in E_1 \times E_2$.

3 MAIN RESULTS

In this section, we propose an inertial algorithm to solve the split equality of monotone inclusion and f, g-fixed point of Bregman relatively f, g-nonexpansive mapping problems in reflexive real Banach spaces. The following assumptions will be used in the sequel.

Assumptions

(A1) Let $E_i, i = 1, 2, 3$ be reflexive real Banach spaces with their respective duals E_i^*, $i = 1, 2, 3$;

(A2) Let $f : E_1 \to \mathbb{R}$ and $g : E_2 \to \mathbb{R}$ be strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre functions on bounded subsets with strongly convex conjugate f^* and g^*, respectively. Let the strong convexity constants of f and g be μ_1 and μ_2, respectively, and let $\mu = \min\left\{\mu_1, \mu_2\right\}$;

(A3) Let $T : E_1 \to E_1^*$ and $G : E_2 \to E_2^*$ be Bregman relatively f-nonexpansive and Bregman relatively g-nonexpansive mappings, respectively;

(A4) Let $A : E_1 \to E_1^*$ and $C : E_2 \to E_2^*$ be uniformly continuous monotone mappings;

(A5) Let $B : E_1 \to 2^{E_1^*}$ and $D : E_2 \to 2^{E_2^*}$ be maximal monotone mappings;

(A6) Let $S : E_1 \to E_3$ and $K : E_2 \to E_3$ be bounded linear mappings with adjoints $S^* : E_3^* \to E_1^*$ and $K^* : E_3^* \to E_2^*$, respectively;

(A7) Let $\Omega = \{(a, b) \in (F_f(T) \cap (A + B)^{-1}(0)) \times (F_g(G) \cap (C + D)^{-1}(0)) : S(a) = K(b)\} \neq \emptyset$.

(A8) Let $\{\alpha_n\} \subset (0, 1)$ be such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(A9) Let $\{r_n\}$ be sequence in $(0, \frac{\mu}{2})$ such that $\lim_{n\to\infty} \frac{r_n}{\alpha_n} = 0$;

(A10) Let J_{E_3} be a normalized duality mapping on E_3.

Algorithm 3.1

Initialization: Choose $(x, w), (x_0, w_0), (x_1, w_1) \in E_1 \times E_2$, $\beta \in (0, 1)$, $\theta \in (0, \mu)$, $0 < \sigma, \rho, \eta, \lambda_1, \delta_1$. Define the algorithm as follows:

Step 0: Choose σ_n such that $0 \leq \sigma_n \leq \overline{\sigma}_n$ where

$$\overline{\sigma}_n = \begin{cases} \min\left\{\frac{r_n}{\|\nabla f(x_n)-\nabla f(x_{n-1})\|+\|\nabla g(w_n)-\nabla g(w_{n-1})\|}, \sigma\right\} & \text{if } x_n \neq x_{n-1} \ \ \& \ \ w_n \neq w_{n-1} \\ \sigma & \text{otherwise} \end{cases} \tag{19}$$

Step 1: Compute

$$\begin{aligned} a_n &= \nabla f^*(\nabla f(x_n) + \sigma_n(\nabla f(x_n) - \nabla f(x_{n-1}))), \\ b_n &= \nabla g^*(\nabla g(w_n) + \sigma_n(\nabla g(w_n) - \nabla g(w_{n-1}))). \end{aligned} \tag{20}$$

Step 2: Choose γ_n such that $\rho \leq \gamma_n \leq \rho_n$ for $S(a_n) \neq K(b_n)$ otherwise $\gamma_n = \rho$, for some $\rho > 0$, where

$$\rho_n = \min\left\{\rho + 1, \frac{\mu\|S(a_n) - K(b_n)\|^2}{2\left[\|S^* J_{E_3}(S(a_n)) - K(b_n)\|^2 + \|K^* J_{E_3}(K(b_n)) - S(a_n)\|^2\right]}\right\}. \tag{21}$$

Step 3: Compute

$$\begin{aligned} d_n &= \nabla f^*(\nabla f(a_n) - \gamma_n S^* J_{E_3}(S(a_n) - K(b_n))), \\ e_n &= \nabla g^*(\nabla g(b_n) - \gamma_n K^* J_{E_3}(K(b_n) - S(a_n))), \end{aligned} \tag{22}$$

Step 4: Compute

$$\begin{aligned} y_n &= J_{\lambda_n}^{B} \nabla f^*(\nabla f(d_n) - \lambda_n A(d_n)), \\ z_n &= J_{\lambda_n}^{D} \nabla g^*(\nabla g(e_n) - \delta_n C(e_n)). \end{aligned}$$

Step 5: Compute

$$\begin{aligned} u_n &= \nabla f^*(\nabla f(y_n) - \lambda_n(A(y_n) - A(d_n))), \\ v_n &= \nabla g^*(\nabla g(z_n) - \delta_n(C(z_n) - C(e_n))), \\ x_{n+1} &= \nabla f^*(\alpha_n \nabla f(x) + (1 - \alpha_n)\left[\beta \nabla f(u_n) + (1 - \beta)T(u_n)\right]), && (23) \\ w_{n+1} &= \nabla g^*(\alpha_n \nabla g(w) + (1 - \alpha_n)\left[\beta \nabla g(v_n) + (1 - \beta)G(v_n)\right]). && (24) \end{aligned}$$

Step 6: Choose λ_{n+1} and δ_{n+1} such that $\eta \leq \lambda_{n+1} \leq \overline{\lambda_{n+1}}$ and $\eta \leq \delta_{n+1} \leq \overline{\delta_{n+1}}$, for some $\eta > 0$, where

$$\overline{\lambda_{n+1}} = \begin{cases} \min\left\{\lambda_n, \frac{\theta\|y_n - d_n\|}{\|A(y_n)-A(d_n)\|}\right\}, & \text{if } A(y_n) \neq A(d_n), \\ \lambda_n, & \text{otherwise,} \end{cases} \tag{25}$$

and

$$\overline{\delta_{n+1}} = \begin{cases} \min\left\{\delta_n, \frac{\theta\|z_n - e_n\|}{\|C(z_n)-C(e_n)\|}\right\} & \text{if } C(z_n) \neq C(e_n), \\ \delta_n, & \text{otherwise.} \end{cases} \tag{26}$$

Set $n := n+1$ and go to **Step 0**.

Remark 1. We note that if A and C are Lipschitz monotone mappings with Lipschitz constants L_1 and L_2, respectively, then following the method in[21], we obtain $\eta = \min\left\{\frac{\theta}{L_1}, \lambda_1, \frac{\theta}{L_2}, \delta_1\right\}$ and hence $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lim_{n\to\infty} \delta_n = \delta$, where $\lambda, \delta \geq \eta$.

Lemma 11. Suppose that the assumptions (A1)- (A10) hold. Then the sequences $\{x_n\}$ and $\{w_n\}$ generated by Algorithm 3.1 are bounded.

Proof. Let $(p, q) \in \Omega$. By the definition of the Bregman distance, (25) and Lemma 2, we have

$$\begin{aligned} D_f(p, u_n) &= D_f(p, \nabla f^*(\nabla f(y_n) - \lambda_n(A(y_n) - A(d_n)))) \\ &= f(p) - f(u_n) - \langle p - u_n, \nabla f(y_n)\rangle + \langle p - u_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &= f(p) - f(y_n) - \langle p - y_n, \nabla f(y_n)\rangle - [f(u_n) - f(y_n) - \langle u_n - y_n, \nabla f(y_n)\rangle] + \langle p - u_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &= D_f(p, y_n) - D_f(u_n, y_n) + \langle p - u_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &= D_f(p, y_n) - D_f(u_n, y_n) + \langle p - y_n, \lambda_n(A(y_n) - A(d_n))\rangle + \langle y_n - u_n, \lambda_n(A(y_n) - A(d_n))\rangle \\ &\leq D_f(p, y_n) - D_f(u_n, y_n) + \langle p - y_n, \lambda_n(A(y_n) - A(d_n))\rangle + \lambda_n\|A(y_n) - A(d_n)\|\|y_n - u_n\| \\ &\leq D_f(p, y_n) - D_f(u_n, y_n) + \langle p - y_n, \lambda_n(A(y_n) - A(d_n))\rangle + \left(\frac{\lambda_n}{\lambda_{n+1}}\right)\theta\|y_n - d_n\|\|y_n - u_n\| \\ &\leq D_f(p, y_n) - D_f(u_n, y_n) + \langle p - y_n, \lambda_n(A(y_n) - A(d_n))\rangle + \frac{\theta\lambda_n}{2\lambda_{n+1}}\|y_n - d_n\|^2 + \frac{\theta\lambda_n}{2\lambda_{n+1}}\|y_n - u_n\|^2 \\ &\leq D_f(p, y_n) - D_f(u_n, y_n) + \langle p - y_n, \lambda_n(A(y_n) - A(d_n))\rangle + \frac{\theta\lambda_n}{2\lambda_{n+1}}\frac{2}{\mu}D_f(y_n, d_n) + \frac{\theta\lambda_n}{2\lambda_{n+1}}\frac{2}{\mu}D_f(u_n, y_n) \\ &\leq D_f(p, y_n) - \left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n, y_n) + \frac{\theta\lambda_n}{\mu\lambda_{n+1}}D_f(y_n, d_n) + \langle p - y_n, \lambda_n(A(y_n) - A(d_n))\rangle. \end{aligned} \tag{27}$$

From (3), we have

$$D_f(p, y_n) = D_f(p, d_n) - D_f(y_n, d_n) + \langle p - y_n, \nabla f(d_n) - \nabla f(y_n)\rangle. \tag{28}$$

Furthermore, from (16) and (17), we obtain

$$\begin{aligned} D_f(p, d_n) &= D_f(p, \nabla f^*(\nabla f(a_n) - \gamma_n S^* J_{E_3}(S(a_n) - K(b_n)))) \\ &= V_f(p, \nabla f(a_n) - \gamma_n S^* J_{E_3}(S(a_n) - K(b_n))) \\ &\leq V_f(p, \nabla f(a_n)) - \gamma_n \left\langle d_n - p, S^* J_{E_3}(S(a_n) - K(b_n))\right\rangle \\ &= D_f(p, a_n) - \gamma_n \left\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\right\rangle. \end{aligned} \tag{29}$$

Substituting (29) into (28) gives

$$\begin{aligned} D_f(p, y_n) &\leq D_f(p, a_n) - \gamma_n \left\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\right\rangle - D_f(y_n, d_n) + \langle p - y_n, \nabla f(d_n) - \nabla f(y_n)\rangle \\ &= D_f(p, a_n) - D_f(y_n, d_n) + \langle p - y_n, \nabla f(d_n) - \nabla f(y_n)\rangle - \gamma_n \left\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\right\rangle. \end{aligned} \tag{30}$$

Again, from (3), we have

$$D_f(p, a_n) = D_f(p, x_n) - D_f(a_n, x_n) + \langle p - a_n, \nabla f(x_n) - \nabla f(a_n)\rangle. \tag{31}$$

Now, from (19), (20) and Lemma 2 we obtain that

$$\begin{aligned}\langle p-a_n, \nabla f(x_n)-\nabla f(a_n)\rangle &\le \|\nabla f(x_n)-\nabla f(a_n)\|\|p-a_n\| \\ &= \sigma_n\|\nabla f(x_n)-\nabla f(x_{n-1})\|\|p-a_n\| \\ &\le \frac{\sigma_n}{2}\|\nabla f(x_n)-\nabla f(x_{n-1})\|\left[\|p-a_n\|^2+1\right] \\ &\le \sigma_n\|\nabla f(x_n)-\nabla f(x_{n-1})\|\left[\|p-x_n\|^2+\|x_n-a_n\|^2\right]+\frac{\sigma_n}{2}\|\nabla f(x_n)-\nabla f(x_{n-1})\| \\ &\le \frac{2r_n}{\mu}D_f(p,x_n)+\frac{2r_n}{\mu}D_f(a_n,x_n)+\frac{r_n}{2}. \end{aligned} \tag{32}$$

Combining (27), (30), (31) and (32), we find

$$\begin{aligned} D_f(p,u_n) &\le D_f(p,y_n)-\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n)+\frac{\theta\lambda_n}{\mu\lambda_{n+1}}D_f(y_n,d_n)+\langle p-y_n,\lambda_n(A(y_n)-A(d_n))\rangle \\ &\le D_f(p,a_n)-D_f(y_n,d_n)+\langle p-y_n,\nabla f(d_n)-\nabla f(y_n)\rangle-\gamma_n\left\langle S(d_n)-S(p),J_{E_3}(S(a_n)-K(b_n))\right\rangle \\ &\quad -\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n)+\frac{\theta\lambda_n}{\mu\lambda_{n+1}}D_f(y_n,d_n)+\langle p-y_n,\lambda_n(A(y_n)-A(d_n))\rangle \\ &= D_f(p,a_n)-D_f(y_n,d_n)-\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n)+\frac{\theta\lambda_n}{\mu\lambda_{n+1}}D_f(y_n,d_n) \\ &\quad -\gamma_n\left\langle S(d_n)-S(p),J_{E_3}(S(a_n)-K(b_n))\right\rangle+\langle p-y_n,\nabla f(d_n)-\nabla f(y_n)+\lambda_n(A(y_n)-A(d_n))\rangle \\ &= D_f(p,x_n)-D_f(a_n,x_n)+\langle p-a_n,\nabla f(x_n)-\nabla f(a_n)\rangle-D_f(y_n,d_n)-\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n) \\ &\quad +\frac{\theta\lambda_n}{\mu\lambda_{n+1}}D_f(y_n,d_n)-\gamma_n\left\langle S(d_n)-S(p),J_{E_3}(S(a_n)-K(b_n))\right\rangle+\langle p-y_n,\nabla f(d_n)-\nabla f(y_n)+\lambda_n(A(y_n)-A(d_n))\rangle \\ &\le D_f(p,x_n)-D_f(a_n,x_n)+\frac{2r_n}{\mu}D_f(p,x_n)+\frac{2r_n}{\mu}D_f(a_n,x_n)+\frac{r_n}{2}-\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n,d_n) \\ &\quad -\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n)-\gamma_n\left\langle S(d_n)-S(p),J_{E_3}(S(a_n)-K(b_n))\right\rangle \\ &\quad +\langle p-y_n,\nabla f(d_n)-\nabla f(y_n)+\lambda_n(A(y_n)-A(d_n))\rangle \\ &= \left(1+\frac{2r_n}{\mu}\right)D_f(p,x_n)-\left(1-\frac{2r_n}{\mu}\right)D_f(a_n,x_n)-\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n,d_n)-\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n) \\ &\quad -\gamma_n\left\langle S(d_n)-S(p),J_{E_3}(S(a_n)-K(b_n))\right\rangle+\frac{r_n}{2}+\langle p-y_n,\nabla f(d_n)-\nabla f(y_n)+\lambda_n(A(y_n)-A(d_n))\rangle. \end{aligned} \tag{33}$$

By the definition of y_n, we have $\nabla f(d_n)-\lambda_n A(d_n)\in\nabla f(y_n)+\lambda_n B(y_n)$. Since B is maximal monotone, there exists $h_n\in B(y_n)$ such that

$$\nabla f(d_n)-\lambda_n A(d_n)=\nabla f(y_n)+\lambda_n h_n.$$

This implies that

$$h_n=\frac{1}{\lambda_n}(\nabla f(d_n)-\nabla f(y_n)-\lambda_n A(d_n))\in B(y_n). \tag{34}$$

Since $0\in(A+B)(p)$, $A(y_n)+h_n\in(A+B)(y_n)$ and $A+B$ is monotone, we get

$$\langle p-y_n, A(y_n)+h_n\rangle\le 0. \tag{35}$$

From (34) and (35), we have

$$\langle p-y_n,\nabla f(d_n)-\nabla f(y_n)+\lambda_n(A(y_n)-A(d_n))\rangle\le 0. \tag{36}$$

Thus, from (33) and (36) we get

$$\begin{aligned} D_f(p,u_n) &\le \left(1+\frac{2r_n}{\mu}\right)D_f(p,x_n)-\left(1-\frac{2r_n}{\mu}\right)D_f(a_n,x_n)-\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n,d_n) \\ &\quad -\left(1-\frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n,y_n)-\gamma_n\left\langle S(d_n)-S(p),J_{E_3}(S(a_n)-K(b_n))\right\rangle+\frac{r_n}{2}. \end{aligned} \tag{37}$$

Furthermore, from (23), (37) and the Bregman relatively f-nonexpansiveness of T, we have

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_n \nabla f(x) + (1-\alpha_n)\left[\beta \nabla f(u_n) + (1-\beta)T(u_n)\right])) \\ &\le \alpha_n D_f(p, x) + (1-\alpha_n) D_f(p, \nabla f^*(\beta \nabla f(u_n) + (1-\beta)T(u_n))) \\ &\le \alpha_n D_f(p, x) + (1-\alpha_n)\left[\beta D_f(p, u_n) + (1-\beta) D_f(p, \nabla f^*(T(u_n)))\right] \\ &\le \alpha_n D_f(p, x) + (1-\alpha_n) D_f(p, u_n) \\ &\le \alpha_n D_f(p, x) + (1-\alpha_n)\left(1 + \frac{2r_n}{\mu}\right) D_f(p, x_n) - (1-\alpha_n)\left(1 - \frac{2r_n}{\mu}\right) D_f(a_n, x_n) \\ &\quad -(1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(y_n, d_n) - (1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n, y_n) \\ &\quad -(1-\alpha_n)\gamma_n \left\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\right\rangle + \frac{r_n}{2}(1-\alpha_n). \end{aligned} \tag{38}$$

Similarly, we have

$$\begin{aligned} D_g(q, w_{n+1}) &\le \alpha_n D_g(q, w) + (1-\alpha_n)\left(1 + \frac{2r_n}{\mu}\right) D_g(q, w_n) - (1-\alpha_n)\left(1 - \frac{2r_n}{\mu}\right) D_g(b_n, w_n) \\ &\quad -(1-\alpha_n)\left(1 - \frac{\theta\delta_n}{\mu\delta_{n+1}}\right) D_g(z_n, e_n) - (1-\alpha_n)\left(1 - \frac{\theta\delta_n}{\mu\delta_{n+1}}\right) D_g(v_n, z_n) \\ &\quad -(1-\alpha_n)\gamma_n \left\langle K(e_n) - K(q), J_{E_3}(K(b_n) - S(a_n))\right\rangle + \frac{r_n}{2}(1-\alpha_n). \end{aligned} \tag{39}$$

Since $\lim_{n\to\infty} \lambda_n$ and $\lim_{n\to\infty} \delta_n$ exist and $\theta \in (0, \mu)$, we obtain that

$$\lim_{n\to\infty}\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) = \lim_{n\to\infty}\left(1 - \frac{\theta\delta_n}{\mu\delta_{n+1}}\right) = 1 - \frac{\theta}{\mu} > 0. \tag{40}$$

Take $\epsilon \in (0, \frac{\mu}{2})$. Then, from (A9) and (40), there exists $N \in \mathbb{N}$ such that

$$\frac{2r_n}{\mu} < \alpha_n\epsilon, \quad \left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) > 0 \text{ and } \left(1 - \frac{\theta\delta_n}{\mu\delta_{n+1}}\right) > 0, \forall n \ge N. \tag{41}$$

Now, from (38), (39) and (41), we have

$$\begin{aligned} D_f(p, x_{n+1}) + D_g(q, w_{n+1}) &\le \alpha_n(D_f(p, x) + D_g(q, w)) + (1-\alpha_n)(D_f(p, x_n) + D_g(q, w_n)) \\ &\quad +\alpha_n\epsilon(D_f(p, x_n) + D_g(q, w_n)) + \alpha_n\epsilon\frac{\mu}{2} - (1-\alpha_n)\gamma_n \left\langle K(e_n) - S(d_n), J_{E_3}(K(b_n) - S(a_n))\right\rangle, \end{aligned} \tag{42}$$

for all $n \ge N$. But,

$$\begin{aligned} &-\left\langle K(e_n) - S(d_n), J_{E_3}(K(b_n) - S(a_n))\right\rangle \\ &= -\left\langle K(b_n) - S(a_n), J_{E_3}(K(b_n) - S(a_n))\right\rangle - \left\langle K(e_n) - K(b_n), J_{E_3}(K(b_n) - S(a_n))\right\rangle \\ &\quad -\left\langle S(a_n) - S(d_n), J_{E_3}(K(b_n) - S(a_n))\right\rangle \\ &= -\|K(b_n) - S(a_n)\|^2 - \left\langle e_n - b_n, K^* J_{E_3}(K(b_n) - S(a_n))\right\rangle - \left\langle a_n - d_n, S^* J_{E_3}(K(b_n) - S(a_n))\right\rangle \\ &\le -\|K(b_n) - S(a_n)\|^2 + \|e_n - b_n\|\|K^* J_{E_3}(K(b_n) - S(a_n))\| + \|a_n - d_n\|\|S^* J_{E_3}(K(b_n) - S(a_n))\|. \end{aligned}$$

From the fact that ∇g^* is a Lipschitz mapping with constant $\frac{1}{\mu}$ and the definition of e_n, we obtain that

$$\|e_n - b_n\| = \|\nabla g^*(\nabla g(b_n) - \gamma_n K^* J_{E_3}(K(b_n) - S(a_n))) - b_n\| \le \frac{\gamma_n}{\mu}\|K^* J_{E_3}(K(b_n) - S(a_n))\|. \tag{43}$$

Similarly, by the Lipschitz property of ∇f^* and the definition of d_n gives

$$\|d_n - a_n\| \le \frac{\gamma_n}{\mu}\|S^* J_{E_3}(S(a_n) - K(b_n))\|. \tag{44}$$

Then, from (21), (43), (43) and (44), we get

$$\begin{aligned}-\gamma_n \left\langle K(e_n) - S(d_n), J_{E_3}(K(b_n) - S(a_n))\right\rangle &\leq -\gamma_n\|K(b_n) - S(a_n)\|^2 + \frac{\gamma_n^2}{\mu}\|K^* J_{E_3}(K(b_n) - S(a_n))\|^2 \\ &\quad + \frac{\gamma_n^2}{\mu}\|S^* J_{E_3}(S(a_n) - K(b_n))\|^2 \\ &\leq -\gamma_n\|K(b_n) - S(a_n)\|^2 + \frac{\gamma_n}{2}\|K(e_n) - S(d_n)\|^2 \\ &= -\frac{\gamma_n}{2}\|K(b_n) - S(a_n)\|^2 \\ &\leq -\frac{\rho}{2}\|K(b_n) - S(a_n)\|^2. \end{aligned} \tag{45}$$

Thus, from (42) and (45), we obtain for all $n \geq N$

$$\begin{aligned}D_f(p, x_{n+1}) + D_g(q, w_{n+1}) &\leq \alpha_n(D_f(p, x) + D_g(q, w)) + (1 - \alpha_n)(D_f(p, x_n) + D_g(q, w_n)) \\ &\quad + \alpha_n\epsilon(D_f(p, x_n) + D_g(q, w_n)) + \alpha_n\epsilon\frac{\mu}{2} - \frac{\rho}{2}(1 - \alpha_n)\|K(b_n) - S(a_n)\|^2 \\ &\leq \alpha_n(D_f(p, x) + D_g(q, w)) + (1 - \alpha_n(1 - \epsilon))(D_f(p, x_n) + D_g(q, w_n)) + \alpha_n\epsilon\frac{\mu}{2} \\ &\leq (1 - \alpha_n(1 - \epsilon))(D_f(p, x_n) + D_g(q, w_n)) + \alpha_n(1 - \epsilon)\left[\frac{1}{1 - \epsilon}(D_f(p, x) + D_g(q, w)) + \frac{\mu\epsilon}{2(1 - \epsilon)}\right] \\ &\leq \max\left\{D_f(p, x_n) + D_g(q, w_n), \frac{1}{1 - \epsilon}(D_f(p, x) + D_g(q, w)) + \frac{\mu\epsilon}{2(1 - \epsilon)}\right\}. \end{aligned} \tag{46}$$

Therefore, by induction, for all $n \geq N$, we have that

$$D_f(p, x_n) + D_g(q, w_n) \leq \max\left\{D_f(p, x_N) + D_g(q, w_N), \frac{1}{1 - \epsilon}(D_f(p, x) + D_g(q, w)) + \frac{\mu\epsilon}{2(1 - \epsilon)}\right\},$$

and hence $\{D_f(p, x_n) + D_g(q, w_n)\}$ is bounded which implies that the sequences $\{D_f(p, x_n)\}$ and $\{D_g(q, w_n)\}$ are bounded. Furthermore, by Lemma 5, we have $\{x_n\}$ and $\{w_n\}$ are bounded. □

Theorem 1. Suppose that assumption (A1)- A(10) are satisfied. Then, the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 converges strongly to (p, q) in Ω, where $(p, q) = P^h_\Omega(x, w)$, where $h : E_1 \times E_2 \to \mathbb{R}$ is given by $h(x, y) = f(x) + g(y)$.

Proof. Let $(p, q) = P^h_\Omega(x, w)$. Then, from Algorithm 3.1, Lemma 3, (16), and (17), we obtain

$$\begin{aligned}D_f(p, x_{n+1}) &= D_f(p, \nabla f^*\left(\alpha_n\nabla f(x) + (1 - \alpha_n)[\beta\nabla f(u_n) + (1 - \beta)T(u_n)]\right)) \\ &= V_f(p, \alpha_n\nabla f(x) + (1 - \alpha_n)[\beta\nabla f(u_n) + (1 - \beta)T(u_n)]) \\ &\leq V_f(p, \alpha_n\nabla f(p) + (1 - \alpha_n)[\beta\nabla f(u_n) + (1 - \beta)T(u_n)]) - \alpha_n\left\langle x_{n+1} - p, \nabla f(p) - \nabla f(x)\right\rangle \\ &= D_f(p, \nabla f^*\left(\alpha_n\nabla f(p) + (1 - \alpha_n)[\beta\nabla f(u_n) + (1 - \beta)T(u_n)]\right)) - \alpha_n\left\langle x_{n+1} - p, \nabla f(p) - \nabla f(x)\right\rangle \\ &\leq \alpha_n D_f(p, p) + (1 - \alpha_n)D_f(p, \nabla f^*\left(\beta\nabla f(u_n) + (1 - \beta)T(u_n)\right)) - \alpha_n\left\langle x_{n+1} - p, \nabla f(p) - \nabla f(x)\right\rangle \\ &= (1 - \alpha_n)D_f(p, \nabla f^*\left(\beta\nabla f(u_n) + (1 - \beta)T(u_n)\right)) - \alpha_n\left\langle x_{n+1} - p, \nabla f(p) - \nabla f(x)\right\rangle \\ &= (1 - \alpha_n)V_f(p, \left(\beta\nabla f(u_n) + (1 - \beta)T(u_n)\right)) + \alpha_n\left\langle x_{n+1} - x_n, \nabla f(x) - \nabla f(p)\right\rangle + \alpha_n\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle \\ &\leq (1 - \alpha_n)V_f(p, \beta\nabla f(u_n) + (1 - \beta)T(u_n)) + \alpha_n\|x_{n+1} - x_n\|\|\nabla f(x) - \nabla f(p)\| \\ &\quad + \alpha_n\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle. \end{aligned} \tag{47}$$

Thus, from the definition of V_f, uniformly convexity of f^* and the Bregman relatively f-nonexpansiveness of T, we obtain that

$$
\begin{aligned}
V_f(p, \beta\nabla f(u_n) + (1-\beta)T(u_n)) &= f(p) - \langle p, \beta\nabla f(u_n) + (1-\beta)T(u_n)\rangle + f^*(\beta\nabla f(u_n) + (1-\beta)T(u_n)) \\
&\leq f(p) - \beta\langle p, \nabla f(u_n)\rangle - (1-\beta)\langle p, T(u_n)\rangle + \beta f^*(\nabla f(u_n)) + (1-\beta)f^*(T(u_n)) \\
&\quad -\beta(1-\beta)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&= \beta V_f(p, \nabla f(u_n)) + (1-\beta)V_f(p, T(u_n)) - \beta(1-\beta)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&= \beta D_f(p, u_n) + (1-\beta)D_f(p, \nabla f^*(T(u_n))) - \beta(1-\beta)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&\leq \beta D_f(p, u_n) + (1-\beta)D_f(p, u_n) - \beta(1-\beta)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&= D_f(p, u_n) - \beta(1-\beta)\psi_1(\|\nabla f(u_n) - T(u_n)\|), \qquad (48)
\end{aligned}
$$

where ψ_1 is the modulus of convexity of f.
From (37), (47) and (48), we obtain that

$$
\begin{aligned}
D_f(p, x_{n+1}) &\leq (1-\alpha_n)D_f(p, u_n) - \beta(1-\beta)(1-\alpha_n)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&\quad +\alpha_n\|x_{n+1} - x_n\|\|\nabla f(x) - \nabla f(p)\| + \alpha_n\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle \\
&\leq (1-\alpha_n)\left(1 + \frac{2r_n}{\mu}\right)D_f(p, x_n) - (1-\alpha_n)\left(1 - \frac{2r_n}{\mu}\right)D_f(a_n, x_n) \qquad (49) \\
&\quad -(1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n, d_n) - (1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n, y_n) \\
&\quad -\gamma_n(1-\alpha_n)\left\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\right\rangle + (1-\alpha_n)\frac{r_n}{2} - \beta(1-\beta)(1-\alpha_n)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&\quad +\alpha_n\|x_{n+1} - x_n\|\|\nabla f(x) - \nabla f(p)\| + \alpha_n\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle,
\end{aligned}
$$

and hence from (41) and (49), we get

$$
\begin{aligned}
D_f(p, x_{n+1}) &\leq (1-\alpha_n)D_f(p, x_n) + \frac{2r_n}{\mu}D_f(p, x_n) + \frac{r_n}{2} + \alpha_n\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle + \alpha_n\|x_{n+1} - x_n\|\|\nabla f(x) - \nabla f(p)\| \\
&\quad -(1-\alpha_n)\left(1 - \frac{2r_n}{\mu}\right)D_f(a_n, x_n) - (1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n, d_n) - (1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n, y_n) \\
&\quad -\gamma_n(1-\alpha_n)\left\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\right\rangle - \beta(1-\beta)(1-\alpha_n)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&\leq (1-\alpha_n)D_f(p, x_n) + \alpha_n\epsilon D_f(p, x_n) + \frac{r_n}{2} + \alpha_n\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle + \alpha_n\|x_{n+1} - x_n\|\|\nabla f(x) - \nabla f(p)\| \\
&\quad -(1-\alpha_n)\left(1 - \alpha_n\epsilon\right)D_f(a_n, x_n) - (1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n, d_n) - (1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n, y_n) \\
&\quad -\gamma_n(1-\alpha_n)\left\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\right\rangle - \beta(1-\beta)(1-\alpha_n)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&= (1-\alpha_n(1-\epsilon))D_f(p, x_n) + \alpha_n(1-\epsilon)\left[\frac{1}{1-\epsilon}\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle\right] + \frac{r_n}{2} \qquad (50) \\
&\quad +\alpha_n\|x_{n+1} - x_n\|\|\nabla f(x) - \nabla f(p)\| - (1-\alpha_n)\left(1 - \alpha_n\epsilon\right)D_f(a_n, x_n) - (1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(y_n, d_n) \\
&\quad -(1-\alpha_n)\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right)D_f(u_n, y_n) - \beta(1-\beta)(1-\alpha_n)\psi_1(\|\nabla f(u_n) - T(u_n)\|) \\
&\quad -\gamma_n(1-\alpha_n)\left\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\right\rangle.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
D_g(q, w_{n+1}) &\leq (1-\alpha_n(1-\epsilon))D_g(q, w_n) + \alpha_n(1-\epsilon)\left[\frac{1}{1-\epsilon}\langle w_n - q, \nabla g(w) - \nabla g(q)\rangle\right] + \frac{r_n}{2} \\
&\quad +\alpha_n\|w_{n+1} - w_n\|\|\nabla g(w) - \nabla g(q)\| - (1-\alpha_n)\left(1 - \alpha_n\epsilon\right)D_g(b_n, w_n) - (1-\alpha_n)\left(1 - \frac{\theta\delta_n}{\mu\delta_{n+1}}\right)D_g(v_n, z_n) \\
&\quad -(1-\alpha_n)\left(1 - \frac{\theta\delta_n}{\mu\delta_{n+1}}\right)D_g(z_n, e_n) - \beta(1-\beta)(1-\alpha_n)\psi_2(\|\nabla g(v_n) - G(v_n)\|) \\
&\quad -(1-\alpha_n)\gamma_n\left\langle K(e_n) - K(q), J_{E_3}(K(b_n) - S(a_n))\right\rangle, \qquad (51)
\end{aligned}
$$

where ψ_2 is the modulus of convexity of g.
From the fact that $\{r_n\} \subset (0, \frac{\mu}{2})$, $\theta \in (0, \mu)$, (A7), (50) and (51) we obtain that

$$\begin{aligned} D_f(p, x_{n+1}) + D_g(q, w_{n+1}) \leq\ & (1-\alpha_n(1-\epsilon))(D_f(p, x_n) + D_g(q, w_n)) \\ & +\alpha_n(1-\epsilon)\frac{1}{1-\epsilon}\left[\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle + \langle w_n - q, \nabla g(w) - \nabla g(q)\rangle\right] \\ & +\alpha_n D(\|x_{n+1} - x_n\| + \|w_{n+1} - w_n\|) + r_n - (1-\alpha_n)(1-\epsilon\alpha_n)(D_f(a_n, x_n) + D_g(b_n, w_n)) \\ & -(1-\alpha_n)\left[\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(y_n, d_n) + \left(1 - \frac{\theta\delta_n}{\mu\delta_{n+1}}\right) D_g(z_n, e_n)\right] \\ & -(1-\alpha_n)\left[\left(1 - \frac{\theta\lambda_n}{\mu\lambda_{n+1}}\right) D_f(u_n, y_n) + \left(1 - \frac{\theta\delta_n}{\mu\delta_{n+1}}\right) D_g(v_n, z_n)\right] \\ & -M(1-\alpha_n)(\psi_1(\|\nabla f(u_n) - T(u_n)\|) + \psi_2(\|\nabla g(v_n) - G(v_n)\|)) \\ & -(1-\alpha_n)\frac{\rho}{2}\|S(a_n) - K(b_n)\|^2 \end{aligned} \tag{52}$$

$$\begin{aligned} \leq\ & (1-\alpha_n(1-\epsilon))(D_f(p, x_n) + D_g(q, w_n)) \\ & +\alpha_n(1-\epsilon)\frac{1}{1-\epsilon}\left[\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle + \langle w_n - q, \nabla g(w) - \nabla g(q)\rangle\right] \\ & +r_n + \alpha_n D(\|x_{n+1} - x_n\| + \|w_{n+1} - w_n\|), \end{aligned} \tag{53}$$

where $D = \max\{\|\nabla f(x) - \nabla f(p)\|, \|\nabla g(w) - \nabla g(q)\|\}$ and $M = \beta(1-\beta)$.
Suppose that $\{D_f(p, x_{n_k}) + D_g(q, w_{n_k})\}$ is a subsequence of $\{D_f(p, x_n) + D_g(q, w_n)\}$ such that

$$\liminf_{k\to\infty} \left[\left(D_f(p, x_{n_k+1}) + D_g(q, w_{n_k+1})\right) - \left(D_f(p, x_{n_k}) + D_g(q, w_{n_k})\right)\right] \geq 0. \tag{54}$$

Then, from (52) and the fact that $\{\alpha_n\} \subset (0, 1)$, $r_n \leq \frac{\mu}{2}$ and $\theta \leq \mu$ for all $n \geq 0$, $\alpha_n \to 0$ and $r_n \to 0$ as $n \to \infty$, we obtain

$$D_f(a_{n_k}, x_{n_k}) \to 0, D_g(b_{n_k}, w_{n_k}) \to 0 \text{ as } k \to \infty, \tag{55}$$

$$D_f(y_{n_k}, d_{n_k}) \to 0, D_g(z_{n_k}, e_{n_k}) \to 0 \text{ as } k \to \infty, \tag{56}$$

$$D_f(u_{n_k}, y_{n_k}) \to 0, D_g(v_{n_k}, z_{n_k}) \to 0, \text{ as } k \to \infty, \tag{57}$$

$$\psi_1(\|\nabla f(u_{n_k}) - T(u_{n_k})\|) \to 0, \psi_2(\|\nabla g(v_{n_k}) - G(v_{n_k})\|) \to 0 \text{ as } k \to \infty, \tag{58}$$

and

$$\|S(a_{n_k}) - K(b_{n_k})\| \to 0 \text{ as } k \to \infty. \tag{59}$$

Moreover, from (55), (56) (57),(58), Lemma 4 and the property of ψ_1 and ψ_2, we have

$$\|a_{n_k} - x_{n_k}\| \to 0, \|y_{n_k} - d_{n_k}\| \to 0, \|u_{n_k} - y_{n_k}\| \to 0 \text{ as } k \to \infty, \tag{60}$$

$$\|b_{n_k} - w_{n_k}\| \to 0, \|z_{n_k} - e_{n_k}\| \to 0, \|v_{n_k} - z_{n_k}\| \to 0 \text{ as } k \to \infty, \tag{61}$$

and

$$\|\nabla f(u_{n_k}) - T(u_{n_k})\| \to 0, \|\nabla g(v_{n_k}) - G(v_{n_k})\| \to 0 \text{ as } k \to \infty. \tag{62}$$

From (23), (24), (62) and the fact that $\alpha_{n_k} \to 0$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \|\nabla f(x_{n_k+1}) - \nabla f(u_{n_k})\| \leq \lim_{k\to\infty} \left[\alpha_{n_k}\|\nabla f(x) - \nabla f(u_{n_k})\| + (1-\beta)\|\nabla f(u_{n_k}) - T(u_{n_k})\|\right] = 0 \tag{63}$$

and

$$\lim_{k\to\infty} \|\nabla g(w_{n_k+1}) - \nabla g(v_{n_k})\| = 0. \tag{64}$$

Now, from (22) and (59) we obtain that

$$\begin{aligned} \|\nabla f(a_{n_k}) - \nabla f(d_{n_k})\| &= \gamma_n\|S^* J_3(S(a_{n_k}) - K(b_{n_k}))\| \\ &\leq (\rho+1)\|S^*\|\|J_3(S(a_{n_k}) - K(b_{n_k}))\| \to 0 \text{ as } k \to \infty. \end{aligned} \tag{65}$$

Similarly, we get

$$\|\nabla g(b_{n_k}) - \nabla g(e_{n_k})\| \to 0 \text{ as } k \to \infty. \tag{66}$$

From (63), (64), (65), (66) and the uniformly continuity of ∇f^* and ∇g^*, we have

$$\begin{aligned} \|x_{n_k+1} - u_{n_k}\| \to 0, \text{ and } \|w_{n_k+1} - v_{n_k}\| \to 0 \text{ as } k \to \infty, \\ \|a_{n_k} - d_{n_k}\| \to 0 \text{ and } \|b_{n_k} - e_{n_k}\| \to 0 \text{ as } k \to \infty. \end{aligned} \tag{67}$$

Then, from (60), (61) and (67), we obtain that

$$\begin{aligned} \lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| \le & \lim_{k\to\infty} \|x_{n_k+1} - u_{n_k}\| + \lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| + \lim_{k\to\infty} \|y_{n_k} - d_{n_k}\| \\ & + \lim_{k\to\infty} \|d_{n_k} - a_{n_k}\| + \lim_{k\to\infty} \|a_{n_k} - x_{n_k}\| = 0. \end{aligned} \tag{68}$$

Similarly, we have

$$\lim_{k\to\infty} \|w_{n_k+1} - w_{n_k}\| = 0. \tag{69}$$

Now, since $\{(x_{n_k}, w_{n_k})\}$ is bounded in $E_1 \times E_2$, there exists $(x^*, w^*) \in E_1 \times E_2$ and a subsequence $\{(x_{n_{k_j}}, w_{n_{k_j}})\}$ of $\{(x_{n_k}, w_{n_k})\}$ such that $(x_{n_{k_j}}, w_{n_{k_j}}) \rightharpoonup (x^*, w^*)$ and

$$\begin{aligned} & \limsup_{k\to\infty} \langle (x_{n_k}, w_{n_k}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q))\rangle \\ & = \lim_{j\to\infty} \langle (x_{n_{k_j}}, w_{n_{k_j}}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q))\rangle. \end{aligned} \tag{70}$$

But $(x_{n_{k_j}}, w_{n_{k_j}}) \rightharpoonup (x^*, w^*)$ implies that $x_{n_{k_j}} \rightharpoonup x^*$ and $w_{n_{k_j}} \rightharpoonup w^*$. Moreover, from (60), (61) and (67), we have

$$a_{n_{k_j}} \rightharpoonup x^*, d_{n_{k_j}} \rightharpoonup x^*, y_{n_{k_j}} \rightharpoonup x^*, u_{n_{k_j}} \rightharpoonup x^* \text{ as } j \to \infty, \tag{71}$$

and

$$b_{n_{k_j}} \rightharpoonup w^*, e_{n_{k_j}} \rightharpoonup w^*, z_{n_{k_j}} \rightharpoonup w^*, v_{n_{k_j}} \rightharpoonup w^* \text{ as } j \to \infty. \tag{72}$$

In addition, from (62), (71), (72), the fact that T is Bregman relatively f-nonexpansive and G is Bregman relatively g-nonexpansive mapping, we conclude that $x^* \in \widetilde{F}_f(T)$ and $w^* \in \widetilde{F}_g(G)$, and hence $x^* \in F_f(T)$ and $w^* \in F_g(G)$. Next, we need to show that $(x^*, w^*) \in \Omega$. Let $s \in (A+B)(e)$. Then, there exists $h \in B(e)$ such that $s = A(e) + h$.
Thus, from (34) and monotonicity of A and B, we have

$$\begin{aligned} \langle e - y_{n_{k_j}}, s\rangle &= \langle e - y_{n_{k_j}}, A(e) + h\rangle \\ &= \langle e - y_{n_{k_j}}, A(e) - A(y_{n_{k_j}})\rangle + \langle e - y_{n_{k_j}}, A(y_{n_{k_j}}) - A(d_{n_{k_j}})\rangle + \langle e - y_{n_{k_j}}, A(d_{n_{k_j}}) + h_{n_{k_j}}\rangle \\ &\quad + \langle e - y_{n_{k_j}}, h - h_{n_{k_j}}\rangle \\ &\ge \langle e - y_{n_{k_j}}, A(y_{n_{k_j}}) - A(d_{n_{k_j}})\rangle + \langle e - y_{n_{k_j}}, A(d_{n_{k_j}}) + \frac{1}{\lambda_{n_{k_j}}}(\nabla f(d_{n_{k_j}}) - \nabla f(y_{n_{k_j}}) - \lambda_{n_{k_j}} A(d_{n_{k_j}}))\rangle \\ &= \langle e - y_{n_{k_j}}, A(y_{n_{k_j}}) - A(d_{n_{k_j}})\rangle + \frac{1}{\lambda_{n_{k_j}}}\langle e - y_{n_{k_j}}, \nabla f(d_{n_{k_j}}) - \nabla f(y_{n_{k_j}}))\rangle \end{aligned} \tag{73}$$

Taking limits on both sides of the inequality (73) as $j \to \infty$ and using the fact that ∇f and A are uniformly continuous, (60) and (71), we have

$$\langle e - x^*, s\rangle \ge 0. \tag{74}$$

Then, by the maximal monotonicity of $A + B$, we get $0 \in (A+B)x^*$, that is, $x^* \in (A+B)^{-1}(0)$. Similarly we obtain that, $w^* \in (C+D)^{-1}(0)$. Now, from (59), (71), (72) and the fact that S and K are bounded linear mappings we have $Sx^* = Kw^*$ and hence $(x^*, w^*) \in \Omega$. Therefore, from (70) and Lemma 1, we obtain that

$$\begin{aligned} & \limsup_{k\to\infty} \langle (x_{n_k}, w_{n_k}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q))\rangle \\ & = \lim_{j\to\infty} \langle (x_{n_{k_j}}, w_{n_{k_j}}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q))\rangle \\ & = \langle (x^*, w^*) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q))\rangle \\ & \le 0. \end{aligned} \tag{75}$$

Let $s_n = \alpha_n(1-\epsilon)$ and

$$b_n = \frac{1}{1-\epsilon}\left[\langle x_n - p, \nabla f(x) - \nabla f(p)\rangle + \langle w_n - q, \nabla g(w) - \nabla g(q)\rangle\right] + \frac{r_n}{\alpha_n(1-\epsilon)} + \frac{1}{1-\epsilon} D\left[\|x_{n+1} - x_n\| + \|w_{n+1} - w_n\|\|\|\right].$$

Then, inequality (53) implies that

$$\left\{D_f(p, x_{n+1}) + D_g(q, w_{n+1})\right\} \le (1 - s_n)\left\{D_f(p, x_n) + D_g(q, w_n)\right\} + s_n b_n \tag{76}$$

From (68), (69), (75) and using the fact that $\alpha_n \to 0$ and $\frac{r_n}{\alpha_n} \to 0$ as $n \to \infty$, we obtain $\lim_{n\to\infty} s_n = 0$, and $\limsup_{k\to\infty} b_{n_k} \le 0$. From Lemma 9, we conclude that $\lim_{n\to\infty}\left(D_f(p, x_n) + D_g(q, w_n)\right) = 0$ and so $\lim_{n\to\infty} D_f(p, x_n) = 0$ and $\lim_{n\to\infty} D_g(q, w_n) = 0$. Hence, by Lemma 4 we obtain $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} w_n = q$.

Therefore, the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 converges strongly to $(p, q) = P^h_\Omega(x, w)$ and this completes the proof. □

If, in Theorem 1, we assume that $E_1 = E_2 = E_3 = E$, $C = 0 = D$, $S = 0 = K$ and $G = \nabla g$, then we obtain the following theorem.

Theorem 2. Let E be a reflexive real Banach space with its dual space E^*. Let $f : E \to \mathbb{R}$ be a strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre function on bounded subsets with strongly convex conjugate f^*. Let $g : E \to \mathbb{R}$ be a strongly convex and Gâteaux differentiable function with the strong convexity constant of g be μ_2. Let $T : E \to E^*$ be Bregman relatively f-nonexpansive mapping. Let $A : E \to E^*$ and $B : E \to 2^{E^*}$ be uniformly continuous monotone and maximal monotone mappings, respectively. If the assumptions (A2),(A8), (A9) and (A10) are satisfied and $\Omega = \{a \in E : a \in (F_f(T) \cap (A + B)^{-1}(0))\} \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 3.1 with $E_1 = E_2 = E_3 = E$, $C = 0 = D$, $S = 0 = K$, $G = \nabla g$ and $w = w_0 = w_1$ converges strongly to $p = P^f_\Omega(x)$.

If E_i, $i = 1, 2, 3$ are strictly convex and smooth Banach spaces with their respective, duals E_i^*, $i = 1, 2, 3$ and $g(x) = f(x) = \frac{1}{2}\|x\|^2$, then $\nabla f = \nabla g = J$, $\nabla f^* = \nabla g^* = J^{-1}$, $P^f_\Omega = \Pi_\Omega = P^g_\Omega$ and the Bregman relatively f-nonexpansive mapping T and the Bregman relatively g-nonexpansive mapping G reduce to relatively semi-nonexpansive mappings. Thus, we obtain the following theorem.

Theorem 3. Let E_i, $i = 1, 2, 3$ be strictly convex and smooth reflexive real Banach spaces with their respective, duals E_i^*, $i = 1, 2, 3$ and let $T : E_1 \to E_1^*$ and $G : E_2 \to E_2^*$ be relatively semi-nonexpansive mappings. Suppose that the assumptions (A4)- (A6) and (A8)-(A10) are satisfied. If $\Omega = \{(a, b) \in (F_s(T) \cap (A + B)^{-1}(0)) \times (F_s(G) \cap (C + D)^{-1}(0)) : S(a) = K(b)\} \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 with $\nabla f = J = \nabla g$, $\nabla f^* = J^{-1} = \nabla g^*$ and $P^f_\Omega = \Pi_\Omega = P^g_\Omega$ converges strongly to (p, q) in Ω, where $(p, q) = \Pi_\Omega(x, w)$.

If E_i, $i = 1, 2, 3$ are real Hilbert spaces and $g(x) = f(x) = \frac{1}{2}\|x\|^2$ then $\nabla f = \nabla g = I$, $\nabla f^* = \nabla g^* = I$, $P^f_\Omega = P_\Omega = P^g_\Omega$, $J_{E_3} = I$ and the Bregman relatively f-nonexpansive mapping T and the Bregman relatively g-nonexpansive mapping G reduce to relatively nonexpansive mappings. Thus, we obtain the following corollary.

Corollary 1. Let H_i, $i = 1, 2, 3$ be real Hilbert spaces, $T : H_1 \to H_1$ and $G : H_2 \to H_2$ be relatively nonexpansive mappings. Suppose that the assumptions (A4)- (A6) and (A8)-(A9) are satisfied. If $\Omega = \{(a, b) \in (F(T) \cap (A + B)^{-1}(0)) \times (F(G) \cap (C + D)^{-1}(0)) : S(a) = K(b)\} \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 with $\nabla f = I = \nabla g$, $\nabla f^* = I = \nabla g^*$, $J_{E_3} = I$ and $P^f_\Omega = P_\Omega = P^g_\Omega$ converges strongly to $(p, q) = P_\Omega(x, w)$.

4 | APPLICATION

4.1 | Split Monotone Inclusion and f-fixed Point Problems

If $E_i = E$, $i = 1, 2, 3$, $K = I$, where I is identity mapping, then *SEMIfFPP* reduces to the split monotone inclusion and f, g-fixed point problems, which is defined as finding $p \in (F_f(T) \cap (A + B)^{-1}(0))$ such that $Sp \in (F_g(G) \cap (C + D)^{-1}(0))$.

Denote $\Psi = \{p \in (F_f(T) \cap (A + B)^{-1}(0)) : S(p) \in (F_g(G) \cap (C + D)^{-1}(0))\}$.

Corollary 2. Assume that conditions (A1)- (A6) and (A8)- (A10) are satisfied with $E_i = E$, $i = 1, 2, 3$ and $K = I$. If $\Psi \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 converges strongly to $(p, S(p)) = P^h_\Psi(x, w)$.

4.2 | Common Solutions of Monotone Inclusion and f-fixed Point Problems

Let $E_i = E$, $i = 1, 2, 3$, $S = K = I$. Then, *SEMIfFPP* reduces to a common solution of two monotone inclusion and f, g-fixed point problems, which is defined as finding a point $p \in E$ such that $p \in (F_f(T) \cap (A + B)^{-1}(0)) \cap (F_g(G) \cap (C + D)^{-1}(0))$.

$$\text{Denote } \Gamma = \{p \in E : p \in (F_f(T) \cap (A + B)^{-1}(0)) \cap (F_g(G) \cap (C + D)^{-1}(0))\}.$$

Corollary 3. Assume that conditions (A1)-(A5) and (A8)-(10) are satisfied with $E_i = E$, $i = 1, 2, 3$. If $\Gamma \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 with $S = K = I$ converges strongly to $(p, p) = P^h_\Gamma(x, w)$.

4.3 | Split Equality Monotone Inclusion Problem

If $T = \nabla f$ and $G = \nabla g$, then *SEMIfFPP* reduces to the split equality of monotone inclusion problems, which is finding a point $(p, q) \in E_1 \times E_2$ such that $p \in (A + B)^{-1}(0)$, $q \in (C + D)^{-1}(0)$ and $Sp = Kq$.

$$\text{Denote } \Lambda = \{(p, q) \in (A + B)^{-1}(0) \times (C + D)^{-1}(0) : Sp = Kq\}.$$

Corollary 4. If conditions (A1),(A2), (A4)-(A6) and (A8)-(A10) are satisfied and $\Lambda \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 with $T = \nabla f$ and $G = \nabla g$ converges strongly to (p, q) in Λ, where $(p, q) = P^h_\Lambda(x, w)$.

4.4 | Split Equality f-Fixed Point Problem

If $A = B = 0 = C = D$, then *SEMIfFPP* reduces to the split equality of f, g-fixed point problems, which is finding a point $(p, q) \in E_1 \times E_2$ such that $p \in F_f(T)$, $q \in F_g(G)$ and $Sp = Kq$.

$$\text{Denote } \Sigma = \{(p, q) \in E_1 \times E_2 : p \in F_f(T), q \in F_g(G) \text{ and } Sp = Kq\}.$$

Corollary 5. If conditions (A1)-(A3),(A6) and (A8)-(A10) are satisfied and $\Sigma \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 with $A = B = 0 = C = D$ converges strongly to $(p, q) = P^h_\Sigma(x, w)$.

4.5 | Optimization Problem

Let $E_i = E$, $i = 1, 2, 3$, be reflexive real Banach spaces. Let $f_i : E_i \to \mathbb{R}$ be convex smooth functions and $g_i : E_i \to \mathbb{R}$ be convex, lower semicontinuous functions, $i = 1, 2$. We consider the following minimization problem: Find $(p, q) \in E_1 \times E_2$ such that

$$p \text{ solves } \min_{x \in E_1} \{f_1(x) + g_1(x) : (\nabla f_1 - T)(x) = 0\}, \tag{77}$$

$$q \text{ solves } \min_{y \in E_2} \{f_2(y) + g_2(y) : (\nabla f_2 - G)(y) = 0\} \tag{78}$$

and

$$Sp = Kq, \tag{79}$$

where $T : E_1 \to E_1^*$ and $G : E_2 \to E_2^*$ are Bregman relatively f-nonexpansive mappings and $S : E_1 \to E_3$ and $K : E_2 \to E_3$ are bounded linear mappings.

$$\text{Denote } \Delta = \{(z, v) \in E_1 \times E_2 : z \text{ solves (77)}, v \text{ solves (78) and } Sz = Kv\}.$$

This problem is equivalent, by Fermat's rule, to the problem of finding $(p, q) \in E_1 \times E_2$ such that

$$(p, q) \in \left[F_{f_1}(T) \cap (\nabla f_1 + \partial g_1)^{-1}(0)\right] \times \left[F_{f_2}(G) \cap (\nabla f_2 + \partial g_2)^{-1}(0)\right] \text{ and } S(p) = K(q), \tag{80}$$

where ∇f_i are gradient of f_i and ∂g_i are subdifferential of g_i, $i = 1, 2$. Note that ∇f_i and ∂g_i are monotone and maximal monotone mappings, respectively.

Corollary 6. Let $f_i : E_i \to \mathbb{R}$ be convex smooth functions and $g_i : E_i \to \mathbb{R}$ be convex, lower semicontinuous functions, $i = 1, 2$. Assume that the conditions (A1),(A3), (A6) and (A8) - (A10) are satisfied. If $\Delta \neq \emptyset$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 with $A = \nabla f_1$, $C = \nabla f_2$, $B = \partial g_1$ and $D = \partial g_2$, converges strongly to (p, q) in Δ, where $(p, q) = P^h_\Delta(x, w)$, where $h = f_1 + f_2$.

5 | NUMERICAL EXPERIMENT

In this section, we provide some numerical examples to clearly exhibit the behavior of the convergence of the proposed method.
Example 1. Consider the minimization problem: find $(p, q) \in \mathbb{R}^4 \times \mathbb{R}^3$ such that

$$p \text{ solve } \quad \min_{x\in\mathbb{R}^4}\{\|x\|_1 + \frac{1}{2}\|x\|_2^2 + (3, 4, -2, 5)^t x + 3 : T(x) = (-2, -3, 1, -4)\}, \tag{81}$$

$$q \text{ solve } \quad \min_{y\in\mathbb{R}^3}\{\|y\|_1 + \|y\|_2^2 + (0, -5, 3)^t y + 2 : G(y) = (0, 2, -1)\} \tag{82}$$

and

$$Sp = Kq, \tag{83}$$

where $S(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3)$, $T(x_1, x_2, x_3, x_4) = (\frac{1}{2}x_1 - 1, \frac{2}{3}x_2 - 1, x_3, \frac{3}{4}x_4 - 1)$, $\forall (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $K(y_1, y_2, y_3) = (-2y_2 + y_3, -y_3)$, $G(y_1, y_2, y_3) = (y_1, \frac{1}{4}y_2 + \frac{3}{2}, -\frac{1}{2}y_3 - \frac{3}{2})$, $\forall (y_1, y_2, y_3) \in \mathbb{R}^3$.
By Fermat's rule, this problem is equivalent to the problem of finding a point $(p, q) \in \mathbb{R}^4 \times \mathbb{R}^3$ such that

$$(p, q) \in \Omega = \{(x, y) \in \left[F_f(T) \cap (A + B)^{-1}(0)\right] \times \left[F_g(G) \cap (C + D)^{-1}(0)\right] : S(x) = K(y)\}, \tag{84}$$

where $A(x) = \nabla(\frac{1}{2}\|x\|_2^2 + (3, 4, -2, 5)^t x + 3) = x + (3, 4, -2, 5)$, $B(x) = \partial(\|x\|_1)$ and $f(x) = \frac{1}{2}\|x\|_2^2$, $\forall x \in \mathbb{R}^4$ and $C(y) = \nabla(\|y\|_2^2 + (0, -5, 3)^t y + 2) = 2y + (0, -5, 3)$, $D(y) = \partial(\|y\|_1)$ and $g(y) = \frac{1}{2}\|y\|_2^2$, $y \in \mathbb{R}^3$. We note that the mappings $A : E_1 \to E_1$ and $C : E_2 \to E_2$ are monotone mappings, $B : E_1 \to E_1$ and $D : E_2 \to E_2$ are maximal monotone mappings, $S : E_1 \to E_3$ and $K : E_2 \to E_3$ are bounded linear mappings with adjoints $S^*(x_1, x_2) = (x_1, x_1, x_2, 0)$ and $K^*(x_1, x_2) = (0, -2x_1, x_1, -x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, respectively, where $E_1 = \mathbb{R}^4$, $E_2 = \mathbb{R}^3$ and $E_3 = \mathbb{R}^2$. Moreover, we observe that $\nabla f(x) = x$, $\nabla g(y) = y$, $J_{\mathbb{R}^2} = I$, where I is identity mapping on $\mathbb{R}^2$, and the mapping $T : E_1 \to E_1$ is a Bregman relatively f-nonexpansive and $G : E_2 \to E_2$ is a Bregman relatively g-nonexpansive mapping and $\Omega = \{((-2, -3, 1, -4), (0, 2, -1))\} \neq \emptyset$.

Now, we present the numerical experiment results for testing and comparing the performance of the control parameter by taking different values in the Algorithm 3.1. All experiments are performed by using MATLAB R2021b.

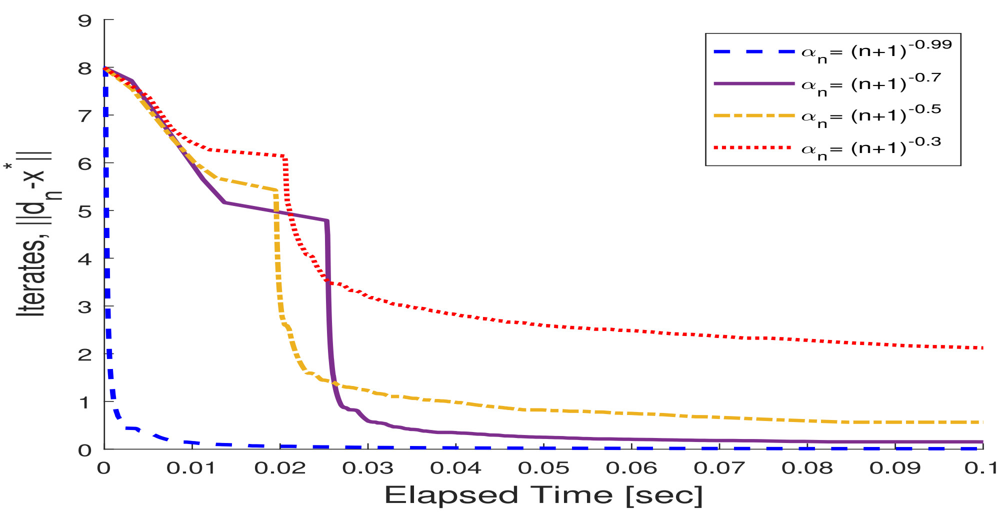

FIGURE 1 Algorithms 3.1. with $\beta = 0.5$, $\lambda_1 = 1 = \delta_1$ $\mu = 0.99$, $\theta = 0.1$, $\gamma = 10^{-4}$ and different α_n.

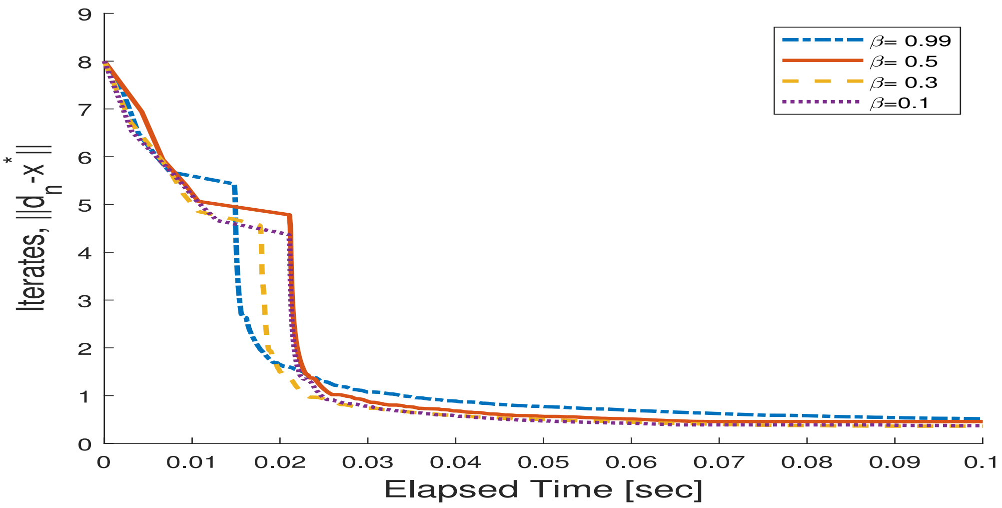

FIGURE 2 Algorithms 3.1. with $\alpha_n = (n+1)^{-0.5}$, $\lambda_1 = 1 = \delta_1$, $\mu = 0.99$, $\theta = 0.1$, $\gamma = 10^{-5}$ and different β.

From FIGURE 1 and 2, we observe that the sequence $d_n = (x_n, w_n)$ converges faster to $x^* = ((-2,-3,1,4),(0,2,-1))$, when the power a of the control parameters $\alpha_n = (n+1)^{-a}$ gets closer to one while the initial point and all other parameters are kept fixed, and the control parameter β gets closer to zero while the initial point and all other parameters are kept fixed, respectively.

6 | CONCLUSION

In this paper, we proposed an inertial type algorithm to approximate the solution of the split equality of monotone inclusion and f,g-fixed point of Bregman relatively f,g-nonexpansive mapping problems. We proved a strong convergence theorem for the developed algorithm in reflexive real Banach spaces. The main result of our method improves the result obtained by Izuchukwu et al.[9] from the split feasibility over the solution set of monotone variational inclusion problems in real Hilbert spaces to the split equality of monotone inclusion and f,g-fixed point of Bregman relatively f,g-nonexpansive mapping problems in reflexive real Banach spaces. As an application, we provided several applications of our method and provided a numerical result to demonstrate the behavior of the convergence of the algorithm. A numerical example is also provided to illustrate the behavior of the proposed algorithm.

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Conflict of Interest

There are no conflicts of interest to this work.

References

1. Barbu V, *Nonlinear differential equations of monotone types nonlinear differential in Banach spaces.* New York: Springer; 2010.

2. Bauschke HH, Borwein JM, *Legendre functions and the method of random Bregman projections.* J. Convex Anal. 1997; 4(1): 27-67.

3. Bauschke HH, Borwein JM, Combettes PL, *Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces.* Commun. Contemp. Math. 2001; 3(4): 615-647.

4. Bonnans FJ, Shapiro A, *Perturbation Analysis of Optimization Problem.* New York: Springer; 2000.

5. Bregman LM,*The relaxation method for finding common points of convex sets and its application to the solution of problems in convex programming.* USSR Comput. Math. Math. Phys. 1967; 7(3): 200-217.

6. Butnariu D, Iusem AN, *Totally Convex Functions for Fixed Points Computation and Infnite Dimentional Optimization.* vol. 40. Klumer Academic, Dodrecht, 2000.

7. Butnariu D, Resmerita E, *Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces.* Abstr. Appl. Anal. 2006: 1-39.

8. Izuchukwu C, Okeke CC, Isiogugu FO, *A viscosity iterative technique for split variational inclusion and fixed point problems between a Hilbert space and a Banach space.* J. Fixed Point Theory Appl. 2018: 20(157).

9. Izuchukwu C, Ogwo GN, Mewomo OT, *An inertial method for solving generalized split feasibility problems over the solution set of monotone variational inclusions.* Optimization 2020; 1-29, https://doi.org/10.1080/02331934.2020.1808648.

10. Jolaoso LO, Ogbuisi FU, and Mewomo OT, *On split equality variation inclusion problems in Banach spaces without operator norms.* Int. J. Nonlinear Anal. Appl. 2021; 12: 425-446.

11. Maingé PE, *Strong convergence of projected subgradiant method for nonsmooth and nonstrictily convex minimization.* Set-Valued Anal. 2008; 16: 899-912.

12. Moudafi A, *Split monotone variational inclusions.* J. Optim. Theory Appl. 2011; 150: 275-283.

13. Phelps RP, *Convex functions, monotone operators, and differentiability.* Lecture Notes in Mathematics, vol. 1364, 2nd edn. Berlin: Springer; 1993.

14. Polyak BT, *Some methods of speeding up the convergence of iteration methods.* USSR Comput. Math. Math. Phy. 1964; 4(5): 1-17.

15. Reich S, *Product formulas, nonlinear semigroups, and accretive operators*. J. Function. Anal. 1980; 36: 147-168.

16. Reich S and Sabach S, *strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces.* J. Nonlinear Convex Anal. 2009; 10(3): 471-485.

17. Reich S and Sabach S, *Two strong convergence theorems for a proximal method in reflexive Banach spaces.* Numer. Funct. Anal. Optim. 2010; 31(1): 22-44.

18. Saejung S, Yotkaew P, *Approximation of zeros of inverse strongly monotone operators in Banach spaces.* Nonlinear Anal. 2012; 75(2): 742-750.

19. Senakka P, Cholamjiak P, *Approximation method for solving fixed point problem of Bregman strongly nonexpansive mappings in reflexive Banach spaces.* Ric. Mat. 2016; 65(1): 209-220.

20. Shehu Y, Cholamjiak P, *Another look at the split common fixed point problem for demicontractive operators*. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM 2016; 110(1): 201-218.

21. Shehu Y, Iyiola OS, Yao JC, *New projection methods with inertial steps for variational inequalities.* Optimization 2021; 1-32, https://doi.org/10.1080/02331934.2021.1964079.

22. Sunthrayuth P, Pholasa N, Cholamjiak P, *Mann-type algorithms for solving the monotone inclusion problem and the fixed point problem in reflexive Banach spaces.* Ricerche di Matematica 2021; 1-28. https://doi.org/10.1007/s11587-021-00596-y

23. Taiwo A, Jolaoso LO, Mewomo OT, *Inertial-type algorithm for solving split common fixed point problems in Banach spaces.* J Sci Comput 2021; 86(12): 1-30.

24. Taiwo A, Owolabi AO-E, Jolaoso LO, Mewomo OT, A. Gibali, *A new approximation scheme for solving various split inverse problems.* Afr. Mat. 2021; (32): 369-401.

25. Taiwo A, Alakoya TO, Mewomo OT, *Halpern-type iterative process for solving split common fixed point and monotone variational inclusion problem between Banach spaces,* Numer. Algor. 2021; 86: 1359-1389.

26. Tang Y, Gibali A, *New self-adaptive step size algorithms for solving split variational inclusion problems and its applications.* Numer Algorithms. 2020; 83: 305-331.

27. Thong DV, Gibali A, Vuong PT, *An explicit algorithm for solving monotone variational inequalities.* Appl. Numer. Math. 2022; 171: 408-425.

28. Tibshirani R, *Regression shrinkage and selection via the Lasso.* J. R. Stat. Soc. B Methodol. 1996; 58(1): 267-288.

29. Tufa AR, Zegeye H, Thuto M, *Convergence theorems for non-self mappings in CAT(0) spaces*. Numer. Funct. Anal. Optim. 2017; 38(6): 705-722.

30. Wega GB, Zegeye H, *Convergence results of Forward-Backward method for a zero of the sum of maximally monotone mappings in Banach spaces.* Comput. Appl. Math. 2020; (39): 223. https://doi.org/10.1007/s40314-020-01246-z

31. Wega GB, Zegeye H, *Convergence theorems of common solutions of variational inequality and f-fixed point problems in Banach spaces.* Appl. Set-Valued Anal. Optim. 2021; 3(1): 55-73.

32. Yao Y, Shehu Y, Li XH, Dong QL, *A method with inertial extrapolation step for split monotone inclusion problems.* Optimization, 2021; 70(4): 741-761.

33. Zalinescu C, *Convex Analysis in General Vector Spaces.* World Scientifc, River Edge, 2002.

34. Zhou X, *On the fenchel duality between Strong convexity and Lipschitz continuous gradient.* arXiv:1803.06573 [math.OC], 2018.