On a conjecture of Davies and Levitin

Hasen Mekki Öztürk

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Abstract

Let H_c be a $(2n)\times(2n)$ symmetric tridiagonal matrix with diagonal elements $c\in\mathbb{R}$ and off-diagonal elements one, and S be a $(2n)\times(2n)$ diagonal matrix with the first n diagonal elements being plus ones and the last n being minus ones. Davies and Levitin studied the eigenvalues of a linear pencil $\mathcal{A}_c = H_c - \lambda S$ as $2n$ approaches to infinity. It was conjectured by DL that for any $n\in\mathbb{N}$ the non-real eigenvalues λ of $\mathcal{A}_c$ satisfy both $|\lambda+c|<2$ and $|\lambda-c|<2$. The conjecture has been verified numerically for a wide range of n and c, but so far the full proof is missing. The purpose of the paper is to support this conjecture with a partial proof and several numerical experiments which allow to get some insight in the behaviour of the non-real eigenvalues of $\mathcal{A}_c$. We provide a proof of the conjecture for $n\le 3$, and also in the case where $|\lambda+c|=|\lambda-c|$. In addition, numerics indicate that some phenomena may occur for more general linear pencils.

1 Introduction

In recent years, the spectral theory of operator pencils has been a very active research area from both a theoretical and a numerical point of view. Applications can be found for instance in control theory [2], vibrating structures [8], mathematical physics and quantum mechanics [1], and more recently electron waveguide in graphene [4]. For a historical survey on the field, the reader is referred to [9].

We are interested in a particular example of a finite dimensional, sign-indefinite, self-adjoint linear pencil, studied by Davies and Levitin [3] (hereafter abbreviated as DL). Namely, let $m,n\in\mathbb{N}$ and $N=m+n$, then consider the linear operator pencil

$$\mathcal{A}_c = \mathcal{A}_c(\lambda) := H_c^{(N)} - \lambda S_{m,n}, \tag{1.1}$$

where

$$H_c^{(N)} = \begin{pmatrix} c & 1 & & & & \\ 1 & c & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & c & 1 \\ & & & & 1 & c \end{pmatrix}, \qquad S_{m,n} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}, \tag{1.2}$$

with omitted entries equal to zero, and $c\in\mathbb{R}$ is a parameter. The size of both matrices is $N\times N$, and the diagonal matrix $S_{m,n}$ has m plus ones and n minus ones. The complex number λ_0 is said to be an eigenvalue of the pencil $\mathcal{A}_c$ if there exists a non-zero solution $\mathbf{x}\in\mathbb{C}^N$ of $\left(H_c^{(N)} - \lambda_0 S_{m,n}\right)\mathbf{x}=\mathbf{0}$. $\mathrm{Spec}(\mathcal{A}_c)$ denotes the spectrum of the pencil $\mathcal{A}_c$ which is the set of all its eigenvalues.

A matrix is called sign-definite if all of its eigenvalues have the same sign. It is well-known (cf. [7, Chapter 1.2]) that if either H_c or S is a sign-definite matrix, then the pencil problem can be reduced to a standard eigenvalue problem for a self-adjoint operator, therefore the spectrum of a self-adjoint linear pencil $\mathcal{A}_c$ is real. However, if both H_c and S are indefinite, then there may be some complex eigenvalues in the spectrum of $\mathcal{A}_c$.

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HMÖ: Department of Mathematics, Faculty of Arts and Sciences, Ordu University, Altınordu, Ordu, PK 52200, Turkey, Hasenozturk@gmail.com

It is straightforward to see that the matrix $S_{m,n}$ is indefinite. It is also known (cf. [3, Lemma 2.1(c)]) that if either $c \geq 2$ or $c \leq -2$, then the matrix $H_c^{(N)}$ is sign-definite, and therefore the spectrum $\mathrm{Spec}(\mathcal{A}_c)$ is real when $|c| \geq 2$. However, if $|c| < 2$, then $H_c^{(N)}$ is indefinite, thus the spectrum $\mathrm{Spec}(\mathcal{A}_c)$ may contain non-real eigenvalues.

Numerical experiments indicate that the eigenvalues of a sign-indefinite self-adjoint pencil often lie on or under a set of curves. DL proved that this is true in some cases. DL pursued an asymptotic approach for large-size matrices (i.e. $N \to \infty$) and obtained the asymptotics of the complex eigenvalues of $\mathcal{A}_0$ (i.e. when the parameter c is equal to zero). It was proved by DL that the eigenvalues of $\mathcal{A}_0$ are all non-real, and approximately lie on the same curve, which is independent of N, in coordinates $(\mathrm{Re}(\lambda), N\mathrm{Im}(\lambda))$. In addition, it was observed that when $c \neq 0$ there is a common bounding curve if one superimposes all the eigenvalues of the pencil $\mathcal{A}_c$ by taking some values of n with imaginary parts scaled by its size N. DL derived an explicit expression for the curve which bounds the whole spectrum of $\mathcal{A}_c$. Nevertheless, the crucial step when proving the result was the following conjecture.

Conjecture 1.1 ([3, Conjecture 5.3]). *Let $c > 0$ and $m = n$. If λ is a non-real eigenvalue of $\mathcal{A}_c$, then*

$$|\lambda \pm c| < 2, \tag{1.3}$$

and therefore

$$|\mathrm{Re}(\lambda)| < 2 - c. \tag{1.4}$$

We emphasise that extensive numerics confirm Conjecture 1.1, however the conjecture is still open. We conducted several numerical experiments taking n between 2 to 500, and taking very small step sizes c, and we failed to find a counter example to the conjecture. We will give numerical experiments to support this conjecture throughout this paper. One of these experiments, for instance, is illustrated in Figure 1, showing the complicated interplay between $\max\limits_{\lambda \in \mathrm{Spec}(\mathcal{A}_c)\setminus\mathbb{R}} \{|\lambda \pm c|\}$ and c. As can be seen from the figure that the maximum value of $|\lambda \pm c|$ (red line) never exceeds two (blue dotted line).

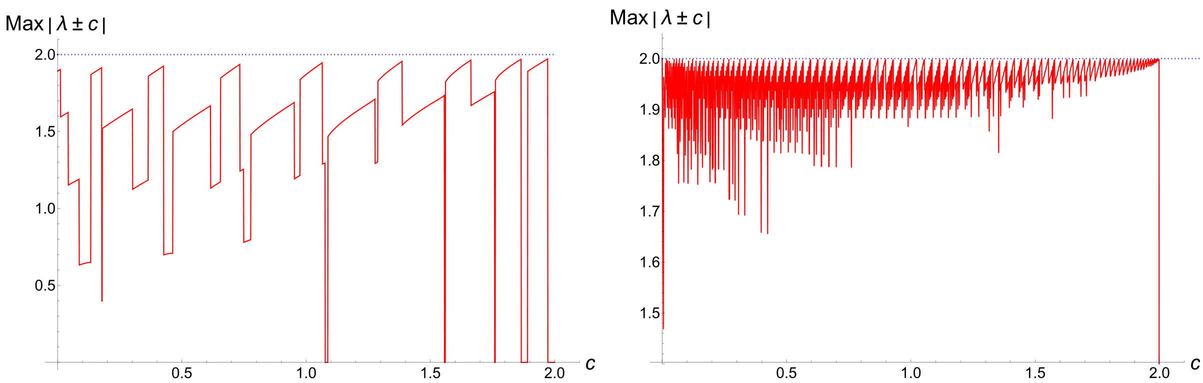

Figure 1: $\max\{|\lambda + c|, |\lambda - c|\}$ (red lines) among all non-real eigenvalues of $\mathcal{A}_c$ is drawn as c ranges from 0 to 2 with the step-size 0.001 and the constant line at 2 (blue dotted lines). Left: $n = 9$. Right: Zooming in near the line at 2 for $n = 100$.

Due to our interest, we take $m = n$ and $N = 2n$. There are several ways to reformulate Conjecture 1.1 by looking at the characteristic equation of the pencil $\mathcal{A}_c$ from different angles. One of which is to act by $\mathcal{A}_c$ on vectors which we will write as

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = (u_1, \ldots, u_n, v_n, \ldots, v_1)^T \tag{1.5}$$

With this convention, the spectral problem for the linear pencil $\mathcal{A}_c$ can be written in the block-matrix form as

$$\begin{pmatrix} H_0 - \sigma I & B \\ B & H_0 + \tau I \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \mathbf{0}, \tag{1.6}$$

where

$$\sigma = \lambda - c; \qquad \tau = \lambda + c. \tag{1.7}$$

Here all sub-blocks are $n \times n$, $H_0 = H_0^{(n)}$, and $B = B^*$ with $B_{nn} = 1$ and all other entries of B are zeros.

Another way to treat the pencil problem (1.1) is using iterative functions that turn out to be related to ratios of Chebyshev polynomials of the second kind U_n.

Remark 1.2. Conjecture 1.1 is satisfied if and only if the following statement holds: Let σ, $\tau \in \mathbb{C}$, $\mathrm{Im}(\sigma) = \mathrm{Im}(\tau) > 0$. If, for some $n \in \mathbb{N}$,

$$U_n(\sigma)\, U_n(\tau) + U_{n-1}(\sigma)\, U_{n-1}(\tau) = 0, \tag{1.8}$$

then $|\sigma| < 1$ and $|\tau| < 1$.

The statement (1.8) is apparently a new statement for Chebyshev polynomials. One approach is to study the system as the ratios of Chebyshev polynomials of the second kind. However, it is still an open question to estimate the ratio for orthogonal polynomials of finite index. We will clarify the reformulation given in Remark 1.2 in Section 2.1.

The paper is organised as follows. In Section 2.3 and Section 3, we provide a full proof of the statement of Conjecture 1.1 in two cases:

- when $|\sigma| = |\tau|$,
- when $n \in \{1, 2, 3\}$.

We shall see that even for small size, the result is non-trivial. The aim of the rest of the study is to treat the pencil problem for $\mathcal{A}_c$ differently in order to gain meaningful information. Section 4 looks at the non-real eigenvalues of $\mathcal{A}_c$ under a mapping. There is a complicated interplay between the eigenvalues of the pencil $\mathcal{A}_c$. In Section 5, we shall describe the dynamics of the eigenvalues of $\mathcal{A}_c$ as the parameter c changes. In Section 6, we give a brief discussion of the double eigenvalues of $\mathcal{A}_c$. Section 7 deals with the problem in a more generalised setting. Our numerics indicate that the second part of the conjecture, i.e. (1.4), seems true for a larger class of problems. Overall, we shall look at the non-real eigenvalues of $\mathcal{A}_c$ in different coordinate systems and we illustrate some interesting phenomena in their location and behaviour. We propose several conjectures about the location and the quantity of double eigenvalues based on numerical studies.

2 Preliminaries

2.1 Properties of the Chebyshev polynomials of the second kind

Chebyshev polynomials are special cases of the Jacobi polynomials, which are a class of classical orthogonal polynomials. We shall focus our attention only on Chebyshev polynomials of the second kind, denoted by $U_n(x)$. In this subsection, we mention some basic properties of $U_n(x)$ and build the connection with the matrix $H_c^{(n)}$. For more details, we refer reader to [6].

The sequence of polynomials $\{U_n(x)\}_{n=0}^{\infty}$ was first studied by the mathematician P. L. Chebyshev (1821-1894). Formally, they are defined by

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}, \qquad \text{when } x = \cos\theta. \tag{2.1}$$

It is well known that $U_n(x)$ satisfies the recurrence relation: let $j \geq 2$, then

$$U_{j+1}(x) = 2xU_j(x) - U_{j-1}(x), \quad U_0(x) = 1, \quad U_1(x) = 2x. \tag{2.2}$$

It can be observed from (2.1) that the θ-zeros in $[0, \pi]$ of $\sin[(n+1)\theta]$ must correspond to the x-zeros in $[-1, 1]$ of $U_n(x)$. Therefore, the zeros of $U_n(x)$ are

$$x_j = \cos\left(\frac{j\pi}{n+1}\right), \qquad j = 1, 2, \ldots, n. \tag{2.3}$$

Denote the monic orthogonal polynomial $F_n(x) = U_n(x/2)$. Then we have for $j \geq 2$ that

$$F_j(x) = xF_{j-1}(x) - F_{j-2}(x), \quad F_0 x = 1, \quad F_1(x) = x. \tag{2.4}$$

In addition, it can be deduced by (2.2) that the functions $F_j(x)$ obey the determinant identity

$$F_n(x) = U_n(x/2) = \det\left(H_x^{(n)}\right),$$

where $H_x^{(n)}$ is the $n \times n$ tridiagonal matrix given as in (1.2) with c replaced by x. Therefore, the eigenvalues of $H_0^{(n)}$ correspond to two times the zeros of $U_n(x)$, that is,

$$\mu_j^{(n)} = 2\cos\left(\frac{\pi j}{n+1}\right), \qquad j = 1, 2, \ldots, n. \tag{2.5}$$

Remark 2.1. Since $\operatorname{Spec}\left(H_c^{(N)}\right) \equiv \operatorname{Spec}\left(H_0^{(N)} + cI\right)$ and the eigenvalues of $H_0^{(N)}$ is given by (2.5), the matrix $H_c^{(N)}$ is indefinite when $c \in \left(\mu_N^{(N)}, \mu_1^{(N)}\right) \subset (-2, 2)$. Therefore, the spectrum $\operatorname{Spec}(\mathcal{A}_c)$ may contain non-real eigenvalues when $|c| < \mu_1^{(N)}$.

2.2 Explicit expressions for eigenfunctions of $\mathcal{A}_c$

The purpose of this subsection is to derive the exact expressions of the characteristic equation of the problem (1.6).

Lemma 2.2. *Let $\zeta \notin \operatorname{Spec}(H_0)$, and let $\mathbf{w}$ solve*

$$\left(H_0^{(n)} - \zeta I\right)\mathbf{w} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{2.6}$$

Then

$$\mathbf{w} = \mathbf{w}(\zeta) = -\frac{1}{F_n(\zeta)} \begin{pmatrix} F_0(\zeta) \\ F_1(\zeta) \\ \vdots \\ F_{n-2}(\zeta) \\ F_{n-1}(\zeta) \end{pmatrix} = -\frac{F_{n-1}(\zeta)}{F_n(\zeta)} \begin{pmatrix} F_0(\zeta)/F_{n-1}(\zeta) \\ F_1(\zeta)/F_{n-1}(\zeta) \\ \vdots \\ F_{n-2}(\zeta)/F_{n-1}(\zeta) \\ 1 \end{pmatrix}$$

with

$$\|\mathbf{w}(\zeta)\|^2 = \frac{\mathcal{F}_n(\zeta)}{|F_n(\zeta)|^2},$$

where

$$\mathcal{F}_n(\zeta) := \sum_{j=0}^{n-1} |F_j(\zeta)|^2.$$

Proof. The equation (2.6) is written in components as

$$\begin{aligned} w_2 - \zeta w_1 &= 0 \\ w_{j-1} - \zeta w_j + w_{j+1} &= 0, \qquad \text{for } j = 2, \ldots, (n-1), \\ w_{n-1} - \zeta w_n &= 1. \end{aligned}$$

Substituting $w_j = -\frac{F_{j-1}(\zeta)}{F_n(\zeta)}$ we see that all n equations hold by (2.4). ☐

Corollary 2.3. *Let $\mathbf{u}$, $\mathbf{v}$, σ, τ solve* (2.9)-(2.10), *and let $\sigma, \tau \notin \{\mu_n, \ldots, \mu_1\}$. Then*

$$\mathbf{u} = -v_n \mathbf{w}(\sigma) = \frac{v_n}{F_n(\sigma)} \begin{pmatrix} F_0(\sigma) \\ F_1(\sigma) \\ \vdots \\ F_{n-1}(\sigma) \end{pmatrix},$$

$$\mathbf{v} = -u_n \mathbf{w}(-\tau) = \frac{u_n}{F_n(\tau)} \begin{pmatrix} (-1)^n F_0(\tau) \\ (-1)^{n+1} F_1(\tau) \\ \vdots \\ -F_{n-1}(\tau) \end{pmatrix}.$$

Proof. Immediate from (2.9)-(2.10) and Lemma 2.2. □

The next result gives some information for the eigenvectors of a certain type of a block matrix.

Lemma 2.4. *Let $\gamma \in \mathbb{R}$. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of*

$$\mathbf{X}_\gamma = \begin{pmatrix} A_{11} - \gamma & A_{12} \\ -A_{12}^* & A_{22} + \gamma \end{pmatrix},$$

where $A_{jj} = A_{jj}^*$, $j = 1, 2$, then the corresponding eigenvectors $(\mathbf{u}, \mathbf{v})^T$ satisfy $\|\mathbf{u}\| = \|\mathbf{v}\|$. In addition,

$$|\mathrm{Im}(\lambda)| \leq \|A_{12}\|.$$

Proof. Consider the spectral problem

$$\begin{pmatrix} A_{11} - \gamma & A_{12} \\ -A_{12}^* & A_{22} + \gamma \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

Let $\lambda = x + iy$ where $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \{0\}$. Then re-writing the problem as

$$\begin{cases} (A_{11} - \gamma)\mathbf{u} + A_{12}\mathbf{v} & = (x + iy)\mathbf{u}, \\ -A_{12}^*\mathbf{u} + (A_{22} + \gamma)\mathbf{v} & = (x + iy)\mathbf{v}, \end{cases}$$

and multiplying the first equation by $\mathbf{u}$ and the second by $\mathbf{v}$ gives

$$\begin{cases} \langle A_{11}\mathbf{u}, \mathbf{u}\rangle - \gamma\|\mathbf{u}\|^2 + \langle A_{12}\mathbf{v}, \mathbf{u}\rangle = (x + iy)\|\mathbf{u}\|^2, \\ -\langle A_{12}^*\mathbf{u}, \mathbf{v}\rangle + \langle A_{22}\mathbf{v}, \mathbf{v}\rangle + \gamma\|\mathbf{v}\|^2 = (x + iy)\|\mathbf{v}\|^2. \end{cases} \tag{2.7}$$

Considering the imaginary parts of (2.7), we get

$$\begin{cases} -y\|\mathbf{u}\|^2 + \mathrm{Im}\langle A_{12}\mathbf{v}, \mathbf{u}\rangle & = 0 \\ \mathrm{Im}\langle A_{12}^*\mathbf{u}, \mathbf{v}\rangle + y\|\mathbf{v}\|^2 & = 0, \end{cases} \tag{2.8}$$

and since $\langle A_{12}\mathbf{v}, \mathbf{u}\rangle = \overline{\langle A_{12}^*\mathbf{u}, \mathbf{v}\rangle}$, summing the two equations in (2.8) gives the first result. In addition, we have from (2.8) that

$$|y| = \frac{|\mathrm{Im}\langle A_{12}\mathbf{v}, \mathbf{u}\rangle|}{\|\mathbf{u}\|^2} \leq \frac{\|A_{12}\mathbf{v}\|\|\mathbf{u}\|}{\|\mathbf{u}\|^2},$$

and since $\|\mathbf{u}\| = \|\mathbf{v}\|$, we obtain $|y| \leq \|A_{12}\|$. □

Remark 2.5. Note that we have the relation

$$\lambda \in \mathrm{Spec} \begin{pmatrix} A_{11} - \gamma & A_{12} \\ -A_{12}^* & A_{22} + \gamma \end{pmatrix} \quad \Leftrightarrow \quad \gamma \in \mathrm{Spec} \begin{pmatrix} A_{11} - \lambda & A_{12} \\ A_{12}^* & -A_{22} + \lambda \end{pmatrix}.$$

Therefore, it follows that for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, eigenvectors $(\mathbf{u}, \mathbf{v})^T$ corresponding to the real eigenvalues of

$$\begin{pmatrix} A_{11} - \lambda & A_{12} \\ A_{12}^* & -A_{22} + \lambda \end{pmatrix}$$

satisfy $\|\mathbf{u}\| = \|\mathbf{v}\|$.

The problem (1.6) can be re-written as

$$(H_0 - \sigma I)\mathbf{u} = -B\mathbf{v}, \tag{2.9}$$
$$(H_0 + \tau I)\mathbf{v} = -B\mathbf{u}. \tag{2.10}$$

Using Corollary 2.3, we can deduce the characteristic equation of the pencil $\mathcal{A}_c$.

Lemma 2.6. *Let* $\mathbf{u}$, $\mathbf{v}$, σ, τ *solve* (2.9)-(2.10), *and in addition let* $\sigma, \tau \notin \{\mu_n, \dots, \mu_1\}$ *and* $\|\mathbf{u}\| = 1$. *Then* $\lambda = (\sigma + \tau)/2$ *is a simple eigenvalue of* $\mathcal{A}_c$, *and*

$$F_{n-1}(\sigma)F_{n-1}(\tau) + F_n(\sigma)F_n(\tau) = 0, \tag{2.11}$$

where λ, c, σ, τ *be related by* (1.7). *Moreover*

$$|v_n|^2 = \frac{|F_n(\sigma)|^2}{\mathcal{F}_n(\sigma)} = \frac{|F_{n-1}(\tau)|^2}{\mathcal{F}_n(\tau)}, \tag{2.12}$$

$$|u_n|^2 = \frac{|F_n(\tau)|^2}{\mathcal{F}_n(\tau)} = \frac{|F_{n-1}(\sigma)|^2}{\mathcal{F}_n(\sigma)}. \tag{2.13}$$

Additionally,

$$\mathbf{u} = \frac{e^{is}}{\sqrt{\mathcal{F}_n(\sigma)}} \begin{pmatrix} F_0(\sigma) \\ F_1(\sigma) \\ \vdots \\ F_{n-1}(\sigma) \end{pmatrix}, \qquad \mathbf{v} = \frac{e^{it}}{\sqrt{\mathcal{F}_n(\tau)}} \begin{pmatrix} (-1)^n F_0(\tau) \\ (-1)^{n+1} F_1(\tau) \\ \vdots \\ -F_{n-1}(\tau) \end{pmatrix}.$$

for some s, $t \in \mathbb{R}$.

Proof. By Corollary 2.3, we have

$$u_n = v_n \frac{F_{n-1}(\sigma)}{F_n(\sigma)}, \qquad v_n = -u_n \frac{F_{n-1}(\tau)}{F_n(\tau)},$$

which implies (2.11). The eigenvalue is geometrically simple since the eigenspace is one-dimensional (with u_n or v_n being a simple parameter). By Lemma 2.4, we know that $\|\mathbf{u}\| = \|\mathbf{v}\|$. The first equalities in each of (2.12) and (2.13) are then obtained from the normalising conditions

$$\|\mathbf{v}\|^2 = |u_n|^2 \|\mathbf{w}(-\tau)\|^2 = |u_n|^2 \frac{\mathcal{F}_n(\tau)}{|F_n(\tau)|^2} = 1,$$

$$\|\mathbf{u}\|^2 = |v_n|^2 \|\mathbf{w}(\sigma)\|^2 = |v_n|^2 \frac{\mathcal{F}_n(\sigma)}{|F_n(\sigma)|^2} = 1.$$

The second ones are obtained from writing normalising conditions as

$$\|\mathbf{v}\|^2 = |v_n|^2 \sum_{j=1}^{n} \frac{|v_j|^2}{|v_n|^2} = |v_n|^2 \frac{\mathcal{F}_n(\tau)}{|F_{n-1}(\tau)|^2} = 1,$$

and similarly for $\mathbf{u}$. [Remark: the second inequalities in (2.12) and (2.13) can be also deduced from the first ones and (2.11).] The expressions for $\mathbf{u}$ and $\mathbf{v}$ result from a substitution. ☐

Remark 2.7. It would be nice to obtain additional restrictions on σ and τ from the fact that the quantities in (2.12) and (2.13) should be less than one. Unfortunately this is not the case - the numerics of the contour plot of the RH sides in (2.12) and (2.13) show that the curves on which $\frac{|F_n(\tau)|^2}{\mathcal{F}_n(\tau)} = 1$ protrude (by a small amount) into the domain $|\tau| > 2$ (or $\mathrm{Re}(\tau) > 2$) for $n \geq 4$. These curves are illustrated in Figure 2 as n increases. Note that this is because the construction above does not use the crucial condition $\mathrm{Im}(\sigma) = \mathrm{Im}(\tau)$.

Another way of treating the characteristic equation (2.11) is the following. Define the family of meromorphic function $\widetilde{F}_n : \mathbb{C} \to \mathbb{C}$ by

$$\widetilde{F}_n(\zeta) := \frac{F_n(\zeta)}{F_{n-1}(\zeta)}. \tag{2.14}$$

Then the characteristic equation (2.11) can be re-written as

$$\widetilde{F}_n(\sigma)\widetilde{F}_n(\tau) = -1. \tag{2.15}$$

Then another reformulation of Conjecture 1.1 can be written as the following.

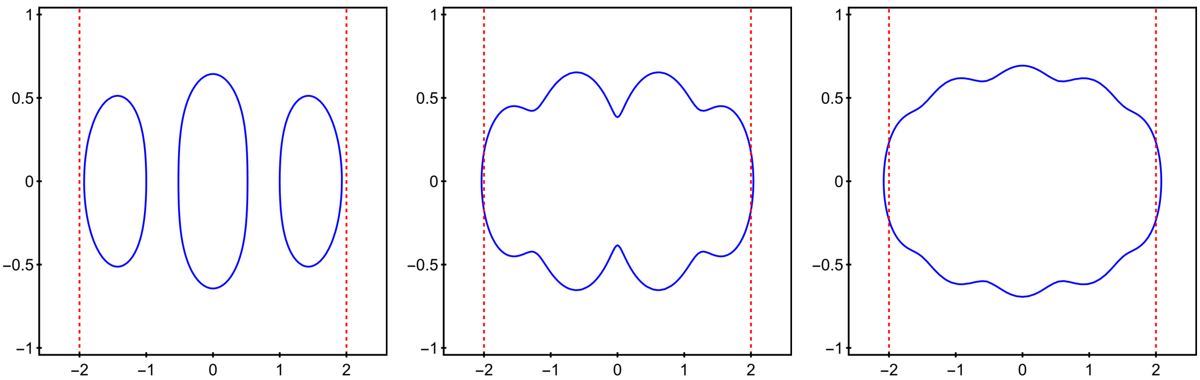

Figure 2: Contour plots of $\frac{|F_n(\tau)|^2}{\mathcal{F}_n(\tau)} = 1$ (blue curves) in the $(\mathrm{Re}(\tau), \mathrm{Im}(\tau))$-plane and $\mathrm{Re}(\tau) = \pm 2$ (red dashed lines) for $n = 3$, $n = 4$ and $n = 5$ from left to right.

Remark 2.8. Conjecture 1.1 is satisfied if and only if any solution $(\sigma, \tau) \in \mathbb{C}^2$ of

$$\widetilde{F}_n(\sigma)\widetilde{F}_n(\tau) = -1, \qquad \mathrm{Im}(\sigma) = \mathrm{Im}(\tau) > 0,$$

has both $|\sigma| < 2$ and $|\tau| < 2$.

Using the fact that the function $\widetilde{F}_n(\zeta)$ is equal to the ratios of Chebyshev polynomials of the second kind $U_n(\zeta/2) = F_n(\zeta)$, we obtain the another reformulation of Conjecture 1.1 as given in Remark 1.2.

The following recurrence relation is immediate.

Proposition 2.9. *Let $n \geq 1$. Then $\widetilde{F}_n(\zeta)$ satisfies the recurrence relation*

$$\widetilde{F}_{n+1}(\zeta) = \zeta - \frac{1}{\widetilde{F}_n(\zeta)}, \qquad \widetilde{F}_1(\zeta) = \zeta. \tag{2.16}$$

Proof. Consider the recurrence relation (2.4). Then for $n = 1$

$$\widetilde{F}_1(\zeta) = \frac{F_1(\zeta)}{F_0(\zeta)} = \zeta.$$

For $n \geq 2$, dividing the last equation of (2.4) by $F_{j-1}(\zeta)$ we arrive at (2.16). □

Conjecture 1.1 claims that any non-real eigenvalues λ of the pencil $\mathcal{A}_c$ satisfies both $|\sigma| < 2$ and $|\tau| < 2$. First, it is easy to see that two of these conditions cannot be broken simultaneously, that is if $|\sigma| \geq 2$ and $|\tau| \geq 2$, then there is no such $(\sigma, \tau) \in \mathbb{C}^2$ which satisfies (2.15). This is due to the following lemma. Nevertheless, it is not clear why one of these conditions cannot be broken for the non-real eigenvalues of $\mathcal{A}_c$.

Lemma 2.10. *If $|\zeta| \geq 2$, then $\left|\widetilde{F}_n(\zeta)\right| > 1$ for all $n \geq 1$.*

Proof. We proceed by induction. For $n = 1$, the statement is obvious. Now assume that $\left|\widetilde{F}_n(\zeta)\right| > 1$ holds for some $n \in \mathbb{N}$. Then

$$\left| |\zeta| - \frac{1}{\left|\widetilde{F}_n(\zeta)\right|} \right| \leq \left| \zeta - \frac{1}{\widetilde{F}_n(\zeta)} \right| = \left|\widetilde{F}_{n+1}(\zeta)\right|.$$

Since $|\zeta| \geq 2$ and $\frac{1}{\left|\widetilde{F}_n(\zeta)\right|} < 1$, the left-hand side of the inequality is greater than one, and therefore $\left|\widetilde{F}_{n+1}(\zeta)\right| > 1$. Then, by induction, the statement holds for all $n \in \mathbb{N}$. □

2.3 Gershgorin-type localisation

One of the most famous methods which can be used for eigenvalue localization regions is given by a Soviet mathematician Semyon Aranovich Gershgorin in 1931 [10]. He observed that the spectrum of a complex matrix is contained in a union of disks in the complex plane, which are nowadays called Gershgorin disks. These disks are centred at the diagonal entries of the matrix and their radii are the row (or column) sum of the absolute values of the non-diagonal entries, the formal definition of Gershgorin's theorem is stated below without proof.

Let $\mathcal{D}(o,r)$ be the disk of radius r around a point o, that is,

$$\mathcal{D}(o,r) := \{\zeta \in \mathbb{C} : |\zeta - o| < r\}.$$

We shall use $\partial\mathcal{D}(o,r)$ and $\overline{\mathcal{D}}(o,r)$ to denote the boundary of the disk $\mathcal{D}(o,r)$ and its closure, respectively.

Theorem 2.11 (Gershgorin Theorem, [10])**.** *Let $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$, $n \geq 2$. Then*

$$\operatorname{Spec}(A) \subseteq \bigcup_{i=1}^{n} \overline{\mathcal{D}}\left(a_{i,i}, r_i(A)\right),$$

where $r_i(A)$ denotes the i-th deleted absolute row sum of A, i.e.

$$r_i(A) = \sum_{j\in J_i} |a_{i,j}|, \qquad J_i = \{1,2,\dots,n\} \setminus \{i\}. \tag{2.17}$$

It can be said in the pencil problem (1.1) that since S is invertible, the spectrum of the pencil $\mathcal{A}_c$ equals that of the non-self-adjoint matrix $S^{-1}H_c^{(N)}$. Using (1.5), the spectral problem for the block matrix $S^{-1}H_c^{(N)}$ can be written as

$$\mathbf{A}_c \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} H_c & B \\ -B & -H_c \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}. \tag{2.18}$$

Applying Theorem 2.11 to the non-self-adjoint matrix $\mathbf{A}_c$, it can be seen that there are four Gershgorin disks, two of them are contained in the other two disks. Therefore we obtain

$$\operatorname{Spec}(\mathbf{A}_c) \subseteq \overline{\mathcal{D}}(c,2) \cup \overline{\mathcal{D}}(-c,2).$$

However, when we focus on the non-real eigenvalues of $\mathbf{A}_c$, we observe from numerics that they all seem to lie in the intersection of the two Gershgorin disks, which is

$$\mathcal{D}(c,2) \cap \mathcal{D}(-c,2) = \{\lambda \in \mathbb{C} : |\lambda - c| < 2 \text{ and } |\lambda + c| < 2\}.$$

Then this leads us to another reformulation in terms of the Gershgorin disks as the following

Remark 2.12. Conjecture 1.1 is satisfied if and only if $\lambda \in \mathcal{D}(c,2) \cap \mathcal{D}(-c,2)$ where $\lambda \in \operatorname{Spec}(\mathbf{A}_c) \setminus \mathbb{R}$.

Gershgorin's result is an easy tool to obtain such bounds without assuming any structure on the matrix. However, in the literature, there exists no Gershgorin-type localisation result which gives a bound only for the non-real eigenvalues of a matrix. All existing results give a bound for the whole spectrum of a matrix. We refer the reader to [7] for an overview of the Gershgorin-type results. In this case, we see that Conjecture 1.1 is somewhat stronger than Gershgorin Theorem. Nevertheless, we deduce the following result for $\mathbf{A}_c$.

Theorem 2.13. *Conjecture 1.1 is satisfied for those $\lambda \in \operatorname{Spec}(\mathcal{A}_c) \setminus \mathbb{R}$ for which $|\sigma| = |\tau|$ with λ, σ, τ related by* (1.7)*.*

Proof. The condition $|\sigma| = |\tau|$ implies that

$$|\lambda - c| = |\lambda + c| \;\Leftrightarrow\; c = 0 \;\text{ or }\; \operatorname{Re}(\lambda) = 0.$$

If $c = 0$, then the intersection and the union of the two Gershgorin disks yield the same set, therefore the statement of the conjecture holds trivially. The statement $\operatorname{Re}(\lambda) = 0$ implies that λ is equidistant from both c and $-c$. In other words, these eigenvalues lie on the imaginary axis in the complex plane. Since the Gershgorin disks $\overline{\mathcal{D}}(c,2)$ and $\overline{\mathcal{D}}(-c,2)$ are symmetric with respect to the imaginary axis, the intersection as well as the union of two Gershgorin disks always cover the imaginary axis by Gershgorin Theorem. Therefore these purely imaginary eigenvalues always lie in the intersection of two Gershgorin disks which implies that both $|\sigma| = |\lambda - c|$ and $|\tau| = |\lambda + c|$ are smaller than two. Note that the boundaries $|\sigma| = |\tau| = 2$ does not play a role since the characteristic equation (2.15) is not satisfied by Lemma 2.10. $\square$

3 Partial cases

The goal of this section is to prove that the statement of Conjecture 1.1 holds for $n \leq 3$. We will prove the equivalent statement of the conjecture given in Remark 2.8. The main objective is to present that even for small size, the result is non-trivial, computations get messy and there is no pattern.

When $n = 1$, it is straightforward to see that Conjecture 1.1 is satisfied:

$$\sigma\tau = -1 \quad \Rightarrow \quad \tau = -\frac{1}{\sigma} = -\frac{\mathrm{Re}(\sigma) - i\mathrm{Im}(\sigma)}{|\sigma|^2}$$

and since $\mathrm{Im}(\sigma) = \mathrm{Im}(\tau)$, we deduce

$$\mathrm{Im}(\tau) = \frac{\mathrm{Im}(\sigma)}{|\sigma|^2} \quad \Rightarrow \quad |\sigma| = 1 \quad \Rightarrow \quad |\tau| = \frac{1}{|\sigma|} = 1,$$

as required.

Now, set

$$\sigma = x + iy, \qquad \tau = t + iy, \tag{3.1}$$

where $x, t \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \{0\}$, and look for solutions of the characteristic equation (2.15) when $n = 2$ and $n = 3$. We will see in each situation that there are two cases: either $|\sigma| = |\tau|$ or $|\sigma| \neq |\tau|$. We know by Theorem 2.13 that Conjecture 1.1 is satisfied in the case $|\sigma| = |\tau|$ for all $n \in \mathbb{N}$. Nevertheless, we emphasise that we will include the proof of this case for the sake of completeness. In addition, we know by Lemma 2.4 that all the eigenvalues of $\lambda \in \mathrm{Spec}\,(\mathbf{A}_c)$ satisfy $|\mathrm{Im}\lambda| \leq 1$, i.e. $|y| \leq 1$ for all $n \in \mathbb{N}$.

3.1 The case $n = 2$

For $n = 2$, the characteristic equation (2.15) becomes

$$\left(\sigma - \frac{1}{\sigma}\right)\left(\tau - \frac{1}{\tau}\right) = -1 \quad \Leftrightarrow \quad -\sigma\tau = (\sigma^2 - 1)(\tau^2 - 1),$$

substituting (3.1) into above gives

$$-(x + iy)(t + iy) = (x^2 - y^2 - 1 + i2xy)(t^2 - y^2 - 1 + i2ty).$$

Separating the real and imaginary parts of the equation gives the system of two equations

$$y^2 - xt = (x^2 - y^2 - 1)(t^2 - y^2 - 1) - 4xty^2, \tag{3.2}$$

$$-(xy + yt) = 2ty(x^2 - y^2 - 1) + 2xy(t^2 - y^2 - 1). \tag{3.3}$$

Re-arranging equation (3.3) yields

$$(x + t)(1 + 2y^2 - 2xt) = 0. \tag{3.4}$$

Since (3.4) must be satisfied, we have either $t = -x$ or $1 + 2y^2 - 2xt = 0$. If $t = -x$, then $|\sigma| = |\tau|$. This means that we have two cases, either $|\sigma| = |\tau|$ or $|\sigma| \neq |\tau|$.

The first case: $|\sigma| = |\tau|$

Let $t = -x$. Then using this in (3.2), we have, after simplification, that

$$4x^2 - 1 = (x^2 + y^2)^2 + (x^2 + y^2),$$

and since $4x^2 - 1 < 4|\sigma|^2 - 1$, the above equation reduces to

$$|\sigma|^4 + |\sigma|^2 < 4|\sigma|^2 - 1$$

which implies

$$\left(|\sigma|^2 - \frac{3}{2}\right)^2 - \frac{5}{4} < 0 \quad \Leftrightarrow \quad \frac{3}{2} - \frac{\sqrt{5}}{2} < |\sigma|^2 < \frac{3}{2} + \frac{\sqrt{5}}{2},$$

and since $|\sigma| = |\tau|$, we obtain $|\sigma|, |\tau| < 2$.

The second case: $|\sigma| \neq |\tau|$

Take $t \neq -x$. Re-arranging (3.2), we have

$$\begin{aligned} 0 &= (x^2-y^2-1)(t^2-y^2-1) - 4xty^2 - y^2 + xt \\ &= (xt)^2 - (y^2+1)(x^2+t^2) + y^4 + y^2 + 1 - 4y^2(xt) + xt \end{aligned}$$

Then using $xt = y^2 + 1/2$ into the above equation, we obtain

$$\begin{aligned} 0 &= \left(y^4 + y^2 + \frac{1}{4}\right) - (y^2+1)(x^2+t^2) + y^4 + y^2 + 1 - 4y^4 - 2y^2 + y^2 + \frac{1}{2} \\ &= -(y^2+1)(x^2+t^2+2y^2) + 3y^2 + \frac{7}{4} \\ &= -(y^2+1)(|\sigma|^2 + |\tau|^2) + 3y^2 + \frac{7}{4}, \end{aligned}$$

which implies

$$|\sigma|^2 + |\tau|^2 = \frac{3y^2 + \frac{7}{4}}{y^2+1} = 3 - \frac{5}{4(y^2+1)} < 3$$

for all y. Thus, $|\sigma|, |\tau| < 2$. Hence the conjecture is true for $n = 2$.

3.2 The case $n = 3$

When $n = 3$, simplifying (2.15) with account of (2.16) we arrive at

$$(\sigma^3 - 2\sigma)(\tau^3 - 2\tau) = -(\sigma^2 - 1)(\tau^2 - 1). \tag{3.5}$$

First, we look at the imaginary parts of (3.5). Re-arranging the LHS of (3.5), we get

$$\begin{aligned} &\mathrm{Im}[(\sigma^3 - 2\sigma)(\tau^3 - 2\tau)] \\ &= \mathrm{Im}[(x^3 - 3xy^2 - 2x + i(3x^2y - 2y - y^3))(t^3 - 3ty^2 - 2t + i(3t^2y - 2y - y^3))] \\ &= (x^3 - 3xy^2 - 2x)(3t^2y - 2y - y^3) + (t^3 - 3ty^2 - 2t)(3x^2y - 2y - y^3) \\ &= (x+t)y[(-y^2-2)(x^2+t^2-xt) + xt(3xt - 9y^2 - 6) + 3y^4 + 8y^2 + 4], \end{aligned} \tag{3.6}$$

and re-arranging the imaginary part of RHS of (3.5), we have

$$\begin{aligned} \mathrm{Im}[-(\sigma^2-1)(\tau^2-1)] &= \mathrm{Im}[-(x^2-y^2-1+i2xy)(t^2-y^2-1+i2ty)] \\ &= -2ty(x^2-y^2-1) - 2xy(t^2-y^2-1) \\ &= -2y(x+t)(xt-y^2-1). \end{aligned} \tag{3.7}$$

Equating (3.6) and (3.7) we obtain

$$(x+t)y\left[(-y^2-2)(x^2+t^2-xt) + xt(3xt - 9y^2 - 4) + 3y^4 + 6y^2 + 2\right] = 0. \tag{3.8}$$

Since $y \neq 0$, we again have two cases; either $|\sigma| = |\tau|$ or $|\sigma| \neq |\tau|$.

The first case: $|\sigma| = |\tau|$

Let $t = -x$. We then have from the real part of LHS of (3.5) that

$$\begin{aligned} \mathrm{Re}[(\sigma^3 - 2\sigma)(\tau^3 - 2\tau)] &= -(x^3 - 3xy^2 - 2x)^2 - (3x^2y - 2y - y^3)^2 \\ &= -[(x^2+y^2)^3 + 4(x^2+y^2) - 4(x^2+y^2)^2 + 8y^2(x^2+y^2)] \\ &= -|\sigma|^6 - 4|\sigma|^2 + 4|\sigma|^4 - 8y^2|\sigma|^2, \end{aligned} \tag{3.9}$$

and we get from the real part of RHS of (3.5) that

$$\begin{aligned} \mathrm{Re}[-(\sigma^2-1)(\tau^2-1)] &= -(x^2-y^2-1)^2 - 4x^2y^2 \\ &= -[(x^2+y^2)^2 - 2x^2 + 2y^2 + 1] \\ &= -|\sigma|^4 + 2|\sigma|^2 - 4y^2 - 1. \end{aligned} \tag{3.10}$$

Equating (3.9) and (3.10) yields

$$|\sigma|^6 - 5|\sigma|^4 + |\sigma|^2(6 + 8y^2) - 4y^2 - 1 = 0. \tag{3.11}$$

Let $s := |\sigma|^2$ and denote

$$f_y(s) := s^3 - 5s^2 + (6 + 8y^2)s - 4y^2 - 1,$$

then the equation (3.11) becomes

$$f_y(s) = 0.$$

We now need to show that if s_0 is a root of $f_y(s)$ for some $y > 0$, then $s_0 < 4$. First, note that when $s = 4$, the function is positive for all y, i.e.

$$f_y(4) = 7 + 28y^2 > 0.$$

Now, we want to show that the function is increasing for all $y > 0$ and $s > 4$. Taking the derivative with respect to s gives

$$f'_y(s) = 3s^2 - 10s + 6 + 8y^2.$$

It can be seen that $f'_y(s)$ is increasing in s for $s \geq 5/3$. Therefore, for $s \geq 4$,

$$f'_y(s) \geq f'_y(4) = 14 + 8y^2 > 0.$$

Thus, $f'_y(s) > 0$ for $s \geq 4$, which means that if there exists s_0 such that $f_y(s_0) = 0$, then $s_0 < 4$. Hence we prove that if $|\sigma| = |\tau|$, then $|\sigma|, |\tau| < 2$.

The second case: $|\sigma| \neq |\tau|$

Proving this case is not as easy as the first case. We will see that all calculations get messy and the problem involves finding zeros of high degree polynomials.

Dividing (3.8) by $y(x + t)$ and re-arranging it gives

$$3x^2t^2 - 2xt(4y^2 + 1) - (x^2 + t^2)(y^2 + 2) + 3y^4 + 6y^2 + 2 = 0,$$

which implies

$$3x^2t^2 = 2xt(4y^2 + 1) + (|\sigma|^2 + |\tau|^2)(y^2 + 2) - 2y^4 - 4y^2 - 3y^4 - 6y^2 - 2.$$

Let $p := xt$ and $A := |\sigma|^2 + |\tau|^2$, then

$$3p^2 = p\left(8y^2 + 2\right) + A\left(y^2 + 2\right) - \left(5y^4 + 10y^2 + 2\right). \tag{3.12}$$

Now, the real part of (3.5), after simplification, is

$$\begin{aligned} x^3t^3 + x^2t^2(-9y^2 + 1) + xt(-(3y^2 + 2)(x^2 + t^2) + 9y^4 + 8y^2 + 4) \\ +(x^2 + t^2)(3y^4 + 5y^2 - 1) - y^6 - 3y^4 - 2y^2 + 1 = 0. \end{aligned}$$

Using $p = xt$ and $A = |\sigma|^2 + |\tau|^2$ yields

$$\begin{aligned} p^3 + p^2(-9y^2 + 1) + p\left(-A(3y^2 + 2) + 15y^4 + 12y^2 + 4\right) \\ + A(3y^4 + 5y^2 - 1) - 7y^6 - 13y^4 + 1 = 0. \end{aligned} \tag{3.13}$$

Inserting equation (3.12) into equation (3.13) yields

$$\begin{aligned} \left(-19y^2 + 5\right)\left[p\left(8y^2 + 2\right) + A\left(y^2 + 2\right) - \left(5y^4 + 10y^2 + 2\right)\right] \\ +3p\left(A(-8y^2 - 4) + 40y^4 + 26y^2 + 10\right) \\ +9A(3y^4 + 5y^2 - 1) - 9(7y^6 + 13y^4 - 1) = 0, \end{aligned}$$

which implies

$$p\left(3A(8y^2+4)+32y^4-80y^2-40\right)=A(8y^4+12y^2+1)+32y^6+48y^4-12y^2-1.$$

Resolving the last equation with respect to p and substituting back into (3.12) gives

$$\begin{aligned} g_y(A):=&\frac{-144A^3(y^2+2)(2y^2+1)^2+9A^2(640y^6+1776y^4+1208y^2+243)}{16(A(6y^2+3)+8y^4-20y^2-10)^2}\\ &-9A\frac{(512y^8+3584y^6+6256y^4+3328y^2+558)}{16(A(6y^2+3)+8y^4-20y^2-10)^2}\\ &+\frac{20736y^8+75456y^6+77328y^4+27432y^2+3123}{16(A(6y^2+3)+8y^4-20y^2-10)^2}=0. \end{aligned}$$

We want to show that if A_0 is a root of $g_y(A)$ for some $0<y\le 1$, then $A_0<4$. First, we show that $g_y(4)$ is negative for all $y\in(0,1]$:

$$g_y(4)=\frac{9(256y^8+192y^6-304y^4-152y^2-45)}{64(4y^4+2y^2+1)^2}.$$

The denominator of $g_y(4)$ is positive. By using $y^8<y^6<y^4$ and $-y^2<-y^4$ we obtain

$$\begin{aligned} 64(4y^4+2y^2+1)^2g_y(4)&=2304y^8+1728y^6-2736y^4-1368y^2-405\\ &<2304y^4+1728y^4-2736y^4-1368y^4-405\\ &=-72y^4-405<0 \end{aligned}$$

which implies

$$g_y(4)<0,\qquad \forall y\in(0,1]. \tag{3.14}$$

Second, taking the derivative with respect to A, we will show that $g_y(A)$ is decreasing for all $0<y\le 1$ and $A>4$. A direct calculation gives

$$g_y'(A)=\frac{k_3A^3+k_2A^2+k_1A+k_0}{k_{-1}},$$

where

$$\begin{aligned} k_3(y)&=-1728y^8-6048y^6-6480y^4-2808y^2-432,\\ k_2(y)&=-6912y^{10}-3456y^8+44928y^6+61344y^4+28080y^2+4320,\\ k_1(y)&=59904y^{10}+116352y^8-73008y^6-185472y^4-92466y^2-14337,\\ k_0(y)&=-18432y^{12}-207360y^{10}-394560y^8-85824y^6+164376y^4+98946y^2+15741\\ k_{-1}(A,y)&=8(A(6y^2+3)+8y^4-20y^2-10)^3. \end{aligned}$$

It is easy to see that the denominator $k_{-1}(A,y)$ is always positive when $A>4$, and therefore we will ignore it since it does not play a role to the sign of $g_y'(A)$. To understand the sign of the numerator of $g_y'(A)$, we need the following result.

Lemma 3.1. *For any* $0<y\le 1$,

(i) $4^3k_3+4^2k_2+4k_1+k_0<0$.

(ii) $k_1+k_2+k_3<0$.

(iii) $k_1+k_3<0$.

Proof.

(i) We have

$$\begin{aligned} 4^3k_3+4^2k_2+4k_1+k_0=&-18432y^{12}-78336y^{10}-95040y^8-46080y^6-10728y^4\\ &-1350y^2-135 \end{aligned}$$

which is always negative.

(ii) We have

$$k_1 + k_2 + k_3 = 52992y^{10} + 111168y^8 - 34128y^6 - 130608y^4 - 67194y^2 - 10449.$$

By using $y^{10} < y^8 < y^4$ and $-y^2 < -y^4$, we obtain

$$\begin{aligned} k_1 + k_2 + k_3 &< 52992y^4 + 111168y^4 - 34128y^6 - 130608y^4 - 67194y^4 - 10449 \\ &= -34128y^6 - 33642y^4 - 10449 \end{aligned}$$

which is also negative.

(iii) We have

$$k_1 + k_3 = 59904y^{10} + 114624y^8 - 79056y^6 - 191952y^4 - 95274y^2 - 14769.$$

By using $y^{10} < y^8 < y^6$ and $-y^4 < -y^6$, we get

$$\begin{aligned} k_1 + k_3 &< 59904y^6 + 114624y^6 - 79056y^6 - 191952y^6 - 95274y^2 - 14769 \\ &= -96480y^6 - 95274y^2 - 14769 < 0 \end{aligned}$$

as required. □

Now, consider the numerator of $g_y'(A)$. Using, consecutively, Lemma 3.1(i), (ii) and then (iii), we have for all $0 < y \leq 1$ and all $A \geq 4$ that

$$\begin{aligned} k_3A^3 + k_2A^2 + k_1A + k_0 &< k_3(A^3 - 4^3) + k_2(A^2 - 4^2) + k_1(A - 4) \\ &< k_3(A^3 - 4^3 - A^2 + 4^2) + k_1(-A^2 + 4^2 + A - 4) \\ &< k_3(A^3 - 4^3 - A^2 + 4^2 + A^2 - 4^2 - A + 4) \\ &= k_3(A - 4)(A^2 + 4A + 15) < 0, \end{aligned}$$

since $k_3 < 0$. Thus,

$$g_y'(A) < 0 \qquad \forall A \geq 4, \forall y \in (0, 1]. \tag{3.15}$$

Combining (3.14) and (3.15), we conclude that if there exists A_0 such that $g_y(A_0) = 0$, then $A_0 < 4$.

4 Mapping $z + 1/z$

Let $\lambda \in \operatorname{Spec}(\mathcal{A}_c)$, and define

$$\lambda - c := z + \frac{1}{z}, \qquad \lambda + c := w + \frac{1}{w}, \tag{4.1}$$

where z and w are some complex numbers. Note that each λ corresponds to two values of z (which are inverses of each other) and two values of w (which are also inverses of each other). These values are the solutions of the quadratic equations

$$z^2 - (\lambda - c)z + 1 = 0, \tag{4.2}$$
$$w^2 - (\lambda + c)w + 1 = 0, \tag{4.3}$$

respectively. If λ is non-real, then z and w are defined to be the unique solutions of (4.2) and (4.3), respectively, which satisfy

$$|z| > 1, \qquad |w| > 1. \tag{4.4}$$

We will see in our numerical experiments that there are some symmetries in the spectrum $\operatorname{Spec}(\mathcal{A}_c)$.

Lemma 4.1 ([3, Lemma 2.1(a) and Lemma 5.1])**.** *The spectrum* $\operatorname{Spec}(\mathcal{A}_c)$ *is invariant under the symmetry* $\lambda \to \overline{\lambda}$ *and the symmetry* $\lambda \to -\lambda$*. Moreover* $\operatorname{Spec}(\mathcal{A}_c)$ *is symmetric with respect to* $c \to -c$*.*

Define the meromorphic function $f_n : \mathbb{C}^2 \to \mathbb{C}$ and $G_n : \mathbb{C} \to \mathbb{C}$ by

$$f_n(z,w) := (z^{n+1} - z^{-n-1})(w^{n+1} - w^{-n-1}) + (z^n - z^{-n})(w^n - w^{-n}),$$
$$G_n(\xi) := \frac{\xi^{n+1} - \xi^{-n-1}}{\xi^n - \xi^{-n}}.$$

Now, we quote two results from [3]; the first one concerns the localisation of eigenvalues of the pencil $\mathcal{A}_c$, and the second one relates the eigenvalues of the pencil $\mathcal{A}_c$ to the function f_n and G_n.

Theorem 4.2 ([3, Theorem 2.3]). *Let $\lambda \in \mathbb{C} \setminus \{-2-c, -2+c, 2-c, 2+c\}$ and let λ, z, w be related by* (4.1)-(4.4). *Then*

(a) λ is an eigenvalue of the pencil $\mathcal{A}_c$ if and only if

$$f_n(z,w) = 0. \tag{4.5}$$

(b) If λ is an eigenvalue of $\mathcal{A}_c$, then it is real if and only if both z and w lie in the set

$$\partial\mathcal{D}(0,1) \cup (\mathbb{R} \setminus \{0\}).$$

(c) If $\lambda \notin \mathbb{R}$ then λ is an eigenvalue of $\mathcal{A}_c$ if and only if

$$G_n(z)G_n(w) = -1. \tag{4.6}$$

Remark 4.3.

(i) Looking at the imaginary parts of (4.1), we see that

$$\mathrm{Im}\lambda = \mathrm{Im}\left(z + \frac{1}{z}\right) = \mathrm{Im}\left(w + \frac{1}{w}\right),$$

and re-arranging this we obtain

$$\mathrm{Im}\lambda = \mathrm{Im}(z)\left(1 - \frac{1}{|z|^2}\right) = \mathrm{Im}(w)\left(1 - \frac{1}{|w|^2}\right). \tag{4.7}$$

If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then using $|z| > 1$ and $|w| > 1$ implies that corresponding $\mathrm{Im}(z)$ and $\mathrm{Im}(w)$ are of the same sign.

(ii) Let $\lambda \in \mathbb{R}$. Then Theorem 4.2(b) immediately follows from (4.7). In addition, we have from (4.2) and (4.3) that

$$z = \frac{\lambda - c \pm \sqrt{(\lambda-c)^2 - 4}}{2}, \qquad w = \frac{\lambda + c \pm \sqrt{(\lambda+c)^2 - 4}}{2}. \tag{4.8}$$

It is known (cf. [3, Lemma 2.1]) that all the eigenvalues $\lambda \in \mathrm{Spec}(\mathcal{A}_c)$ satisfy $|\lambda| < 2 + |c|$. Then we deduce from (4.8) with account of Theorem 4.2(b) that

$$\lambda \in (-2-c, -2+c] \quad \Leftrightarrow \quad z \in \mathbb{R}, \qquad w \in \partial\mathcal{D}(0,1), \tag{4.9}$$
$$\lambda \in (-2+c, 2-c) \quad \Leftrightarrow \quad z \in \partial\mathcal{D}(0,1), \quad w \in \partial\mathcal{D}(0,1), \tag{4.10}$$
$$\lambda \in [2-c, 2+c) \quad \Leftrightarrow \quad z \in \partial\mathcal{D}(0,1), \quad w \in \mathbb{R}. \tag{4.11}$$

(iii) The characteristic equation of the pencil problem for non-real eigenvalues was given in Theorem 4.2(c). In fact, the equation (4.6) holds also for the real eigenvalues of $\mathcal{A}_c$ as long as $z^n \neq z^{-n}$ and $w^n \neq w^{-n}$. This is due to Theorem 4.2(a), and that

$$\frac{f_n(z,w)}{(z^n - z^{-n})(w^n - w^{-n})} = G_n(z)G_n(w) - 1.$$

It is also easy to see that if λ is non-real, then by Theorem 4.2(b), we have $z^n \neq z^{-n}$ and $w^n \neq w^{-n}$.

A direct calculation leads to the following auxiliary result.

Lemma 4.4. *The roots of* $|z + 1/z| = 2$ *are exactly* $z = \pm i + \sqrt{2}e^{i\varphi}$ *where* $\varphi \in [0, 2\pi]$*, and therefore*

$$\left|z + \frac{1}{z}\right| < 2 \;\Leftrightarrow\; |z \pm i| < \sqrt{2}. \tag{4.12}$$

The following is the reformulation of Conjecture 1.1 in terms of corresponding z and w.

Remark 4.5. Conjecture 1.1 is satisfied if and only if the following statement hold: Let z, $w \in \mathbb{C} \setminus \mathbb{R}$ such that $|z|, |w| > 1$. Then any solutions z, w of

$$G_n(z)G_n(w) = -1,$$

satisfy

$$z, w \in \mathcal{Z}_1 \cup \mathcal{Z}_2 \tag{4.13}$$

where

$$\mathcal{Z}_1 := \left\{\zeta \in \mathbb{C} : |\zeta| > 1 \text{ and } |\zeta - i| < \sqrt{2}\right\}, \qquad \mathcal{Z}_2 := \left\{\zeta \in \mathbb{C} : |\zeta| > 1 \text{ and } |\zeta + i| < \sqrt{2}\right\}.$$

To clarify the reformulation, recall that if λ is a non-real eigenvalue of the pencil $\mathcal{A}_c$, we then define z, w to be the unique solutions of (4.2) and (4.3), respectively, which satisfy $|z| > 1$ and $|w| > 1$. Then by Theorem 4.2(b), λ is non-real if and only if $z, w \in \mathbb{C} \setminus \mathbb{R}$ with $|z| > 1$ and $|w| > 1$. By Lemma 4.4, we can re-write the statement of the conjecture as any solutions z, w of (4.6) should lie in the set $\mathcal{Z}_1 \cup \mathcal{Z}_2$ where

$$\mathcal{Z}_1 = \mathcal{D}(i, \sqrt{2}) \setminus \overline{\mathcal{D}}(0, 1), \qquad \mathcal{Z}_2 = \mathcal{D}(-i, \sqrt{2}) \setminus \overline{\mathcal{D}}(0, 1).$$

We present some numerical experiments in Figure 3. For simplicity, we omit the values of z and w which lie on the unit circle $\partial\mathcal{D}(0, 1)$ and on the real line. It is clear from (4.9)-(4.11) that these values correspond to the real eigenvalues of $\mathcal{A}_c$. We take the set of non-real eigenvalues in the λ-plane and plot them in a different complex plane using (4.1). We superimpose corresponding values of z and w in the same complex plane by solving the quadratic equations (4.2) and (4.3) on a finely spaced grid of c values. The black dashed circle represents the unit circle $\partial\mathcal{D}(0, 1)$, the orange circle represents $\partial\mathcal{D}(i, \sqrt{2})$ and the light-orange shaded area represents the set $\mathcal{Z}_1$. The green circle represents $\partial\mathcal{D}(-i, \sqrt{2})$ and the light-green shaded area represents the set $\mathcal{Z}_2$. The red and blue curves, which correspond to values of z and w respectively, are exactly the non-real eigenvalue curves of the pencil $\mathcal{A}_c$ under the change of variables (4.1). For presentation purposes, in our illustrations we zoom in on an area near the unit circle to see the z and w curves closer.

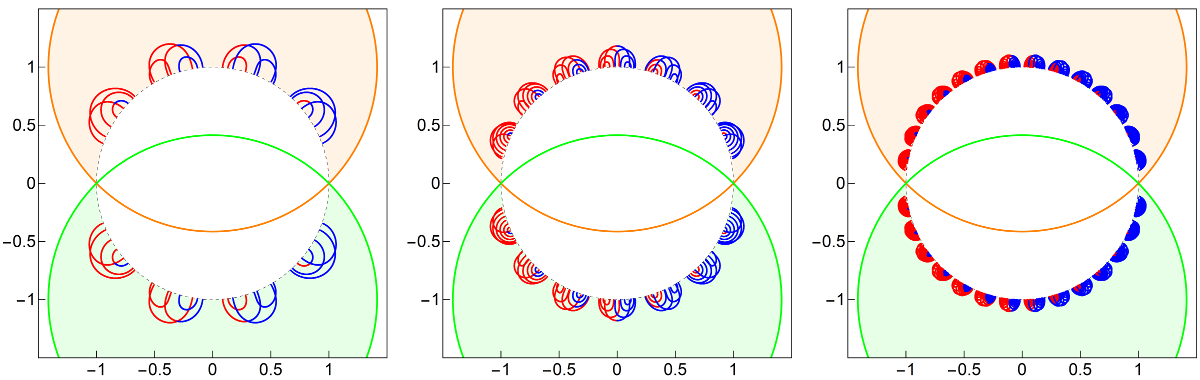

Figure 3: $\partial\mathcal{D}(i, \sqrt{2})$ (orange circle), $\mathcal{Z}_1$ (orange shaded region), $\partial\mathcal{D}(-i, \sqrt{2})$ (green circle), $\mathcal{Z}_2$ (green shaded region), $\partial\mathcal{D}(0, 1)$ (black dashed circle), drawn in the complex plane, together with the superimposition of z (red curves) and w (blue curves) values, which correspond to the non-real eigenvalues only, for c values between 0 and 2 with step-size of 10^{-3}. Left: $n = 4$. Middle: $n = 7$. Right: $n = 14$.

In Figure 3 we illustrate all non-real eigenvalue curves, however we would like to know only whether (4.13) is satisfied. Without seeing the dynamics, it is difficult to understand which two values (z and w) correspond to the same c. The only thing we know from Remark i is that $\mathrm{Im}\lambda > 0$ implies $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(w) > 0$. According to the figure, the eigenvalues (in terms of z and w) always appear from the unit circle and disappear to the unit circle. Conjecture 1.1 will be broken if one eigenvalue escapes from the unit circle and then travels to the real axis. In this case, it would have to cross the forbidden region. The important thing in this figure is that they do not cross the boundary of $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Conjecture 1.1 appears to be true since whatever escapes from the unit circle, returns to the unit circle without going through the forbidden area. Hence, both z and w, which correspond to the non-real eigenvalues, seem to lie in $\mathcal{Z}_1 \cup \mathcal{Z}_2$.

5 Dynamics of the eigenvalues of the pencil $\mathcal{A}_c$

As we mentioned earlier, when $c = 0$, the eigenvalues of $\mathcal{A}_c$ are all non-real and DL were able to determine the asymptotics of complex eigenvalues of $\mathcal{A}_c$ as $n \to \infty$. However, as c changes, tracing the behaviour of the eigenvalues of $\mathcal{A}_c$ remains difficult. In this section, we describe the dynamics of the eigenvalues of $\mathcal{A}_c$ as c increases from 0 to 2. Since the spectrum $\mathrm{Spec}(\mathcal{A}_c)$ is invariant under reflections about the real and imaginary axes, we are able to describe the behaviour that occurs in the right half plane, i.e. $\mathrm{Re}(\lambda) > 0$. The real and non-real eigenvalue curves $\lambda(c)$ may collide, with two possible types of collisions: those when two real eigenvalues collide and produce a complex conjugate pair, called Type-A, and those when a pair of complex conjugate eigenvalues collide and become real, called Type-B.

Numerical experiments demonstrate that as c goes from 0 to 2, for arbitrary n, each complex conjugate pair approach to each other along a curve, as opposed to a straight line, until they collide for the first time on the real axis. We refer to this as Level-1 which consists only of Type-B collisions that occur for the first time. All collisions in Level-1 occur in order, i.e. the pair with the greatest real part collide first and the pair who has the least real part collide last.

After the first collisions occur, each pair travels along the real axis, where one heads to the right and the other to the left. If two real eigenvalues meet, they become a double eigenvalue at the point at which they meet. Subsequently they split and escape from the real line as a conjugate pair. In the complex plane, they make a small jump (in the sense that the eigenvalues travels a half-loop) in directions opposite to the origin. Then this jump is followed by another collision on the real line. This is what we refer to as Level-2, where eigenvalues have two collisions in total; first Type-A collisions occur and each complex conjugate pair makes a small jump in the complex plane and then Type-B collisions occur. The process in Level-2 is repeated in the following levels.

These collisions and jumps in each level are illustrated in Table 1 for $n = 7$. For simplicity, we plot jumps that occur only in the first quadrant. There are some collisions which takes place at the origin, i.e. $\lambda = 0$. When Type-A collisions occur at the origin, they then escape to the complex plane as a purely imaginary pair and they never leave the imaginary axis. They come back and collide on the real line at the origin again. Note that in Table 1 we do not show the jumps that occur at the origin in any level as the eigenvalues move along the imaginary axis only. Nevertheless, the final collision always occurs at the origin.

We observe the following generic behaviour from numerics:

(i) Each pair always collide, separately and independently of other pairs.

(ii) All jumps occur in the direction opposite to the origin.

(iii) At each Level-j, $j = 2, \ldots, n-1$, the eigenvalue whose real part is closest to the origin reaches higher point while jumping in the complex plane.

(iv) At Level-1, the collisions are monotonic in c. This, however, is not true for any other levels. The collisions occur without order, that is for sufficiently large n, some eigenvalues at Level-j may be produced earlier than those which are still in Level-$(j-1)$ or Level-$(j-2)$. Nevertheless, collisions and jumps occur with sufficient speed such that the eigenvalues are unable to overtake one another on the real line.

6 Double eigenvalues of $\mathcal{A}_c$

In this section we provide illustrative numerical examples for the location and the number of double eigenvalues of the pencil $\mathcal{A}_c$. Recall that the pencil $\mathcal{A}_c$ has dimension $2n$. Most conjectures in this section are supported by numerical calculations for $n = 1, \ldots, 30$ and for a wide range of c. As n gets bigger, it takes a relatively large amount of time to find the location of the double eigenvalues of $\mathcal{A}_c$. First, we note the following:

Conjecture 6.1. *For each given value of $n \in \mathbb{N}$ and $c \in (0, 2)$ there are no non-real double eigenvalues of $\mathcal{A}_c$.*

We provide two examples as illustrations. First, we plot all the double eigenvalues in the (c, λ)-plane for a fixed size, $n = 20$, in the left of Figure 4. Although some double eigenvalues appear to lie on the same line, it is difficult to see a pattern. However, if one superimposes the double eigenvalues by taking the values of n from 2 to 30, then one would see that they form an interesting pattern in the (c, λ)-plane, as shown in the right of Figure 4.

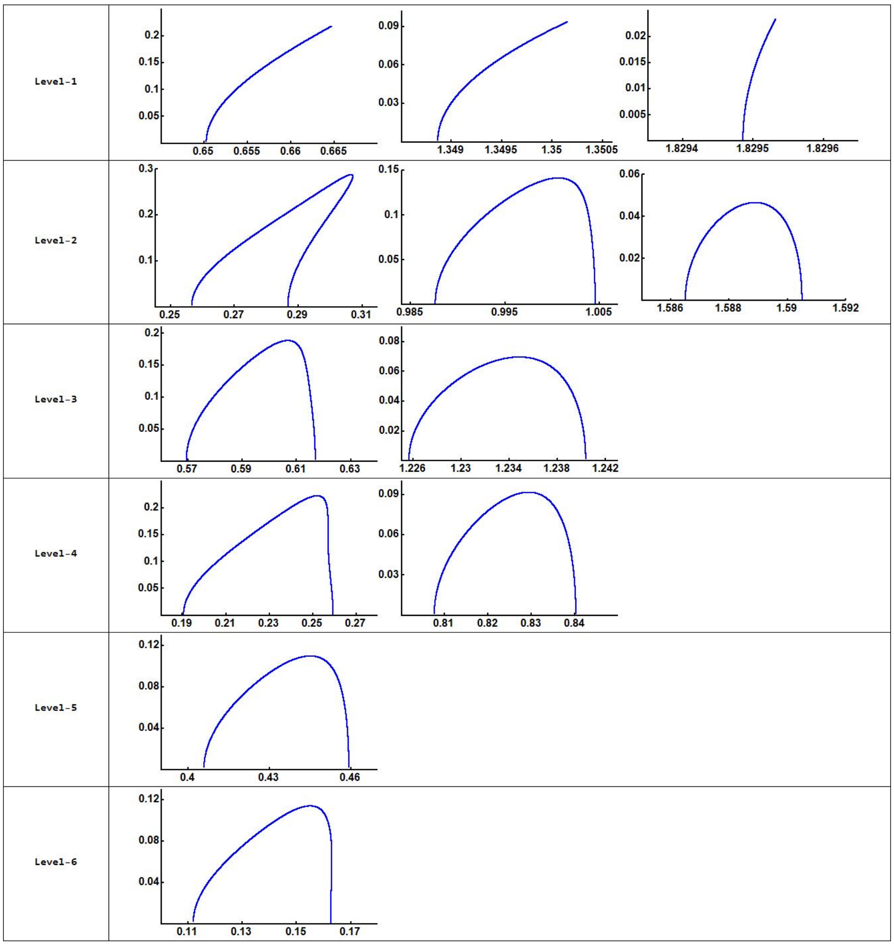

Table 1: For $n = 7$, blue dots represents the dynamics of the non-real eigenvalues of $\mathcal{A}_c$ in the first quadrant of the complex plane $(\mathrm{Re}(\lambda), \mathrm{Im}(\lambda))$, as c increases from 0 to 2.

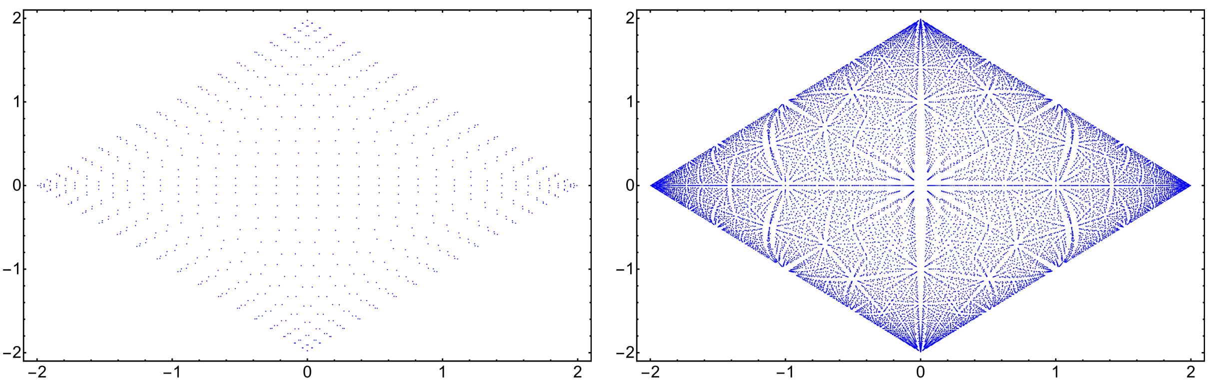

Figure 4: Left: The double eigenvalues of $\mathcal{A}_c$ in the (c, λ)-plane when $n = 20$. Right:The superimposition of the double eigenvalues of $\mathcal{A}_c$ in the (c, λ)-plane when n is between 2 and 30.

Let us denote

$$\alpha = \lambda - c, \qquad \beta = -\lambda - c. \tag{6.1}$$

If we treat α and β as two spectral parameters, then one can transform the linear pencil problem for $\mathcal{A}_c$ (or the problem (1.6)) into a two-parameter eigenvalue problem by setting (6.1). The real spectrum of the two-parameter eigenvalue problem in a more generalised setting was discussed in [5] and the localisation result for the real spectrum was given.

It is known in [5, Teorem 2.3(c)] that the real eigenvalues of the pencil $\mathcal{A}_c$ consists of curves $(\alpha, \beta(\alpha))$ which are continuous except at $\alpha \in \mathrm{Spec}(H_0)$. We denote by Λ_i, $i = 1, \dots, n$, both branches of the curve $(\alpha, \beta_i(\alpha))$, such that $\beta_i(\alpha_i \pm 0) \to \pm\infty$. A typical enumeration of the curves Λ_i can be seen in Figure 5 for $n = 6$.

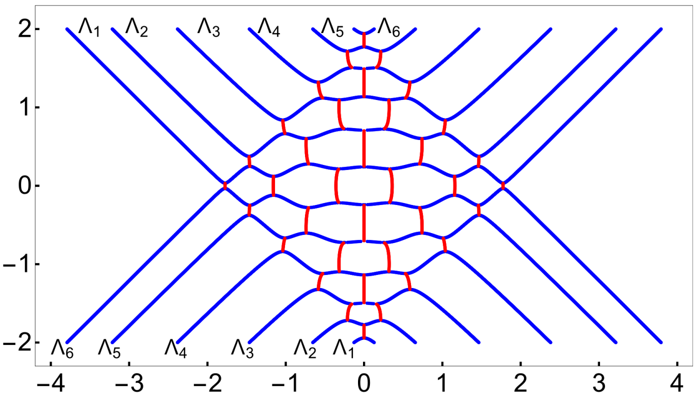

Figure 5: For $n = 6$, $\mathrm{Spec}(\mathcal{A}_c)$ when $c \in (-2, 2)$ in the $(\mathrm{Re}(\lambda), c)$-plane. Blue curves represent the real eigenvalues and the red curves represent the real part of the non-real eigenvalues of $\mathcal{A}_c$.

As we described in Section 5, the real and non-real eigenvalue curves $\lambda(c)$ may collide, with two possible types of collisions: Type-A and Type-B collisions. These collision locations, where the blue and the red curves meet in Figure 5, are also the double eigenvalue locations. It was shown in [5, Lemma 6.1] that the collisions of the eigenvalues (or, equivalently, double eigenvalues) of the pencil $\mathcal{A}_c$ occur when $dc/d\lambda = 0$. These are the critical points along eigenvalue curves. From numerical experiments we observe the following.

Conjecture 6.2. *Let $c \in (-2, 2)$. Then each curve Λ_i, $i = 1, \dots, n$ has exactly $2n$ critical points such that*

$$\frac{dc(\lambda)}{d\lambda} = 0.$$

It can be seen from Figure 5, when $n = 6$, each curve has exactly 6 Type-A collisions and 6 Type-B collisions. We may be able to determine the location of some double eigenvalues of the pencil $\mathcal{A}_c$. Indeed, we can count the number of double eigenvalues of $\mathcal{A}_c$ which occur at $\lambda = 0$. Recall the pencil problem

$$\mathcal{A}_c \mathbf{f} = \left(H_c^{(2n)} - \lambda S\right) \mathbf{f} = \left(H_0^{(2n)} + cI - \lambda S\right) \mathbf{f} = 0.$$

We observe that if $\lambda = 0$, then

$$H_0^{(2n)} \mathbf{f} = -c\mathbf{f},$$

which implies $c = \mathrm{Spec}\left(-H_0^{(2n)}\right)$, and since the spectrum of $H_0^{(2n)}$ is symmetric by (2.5), the double eigenvalues λ^* of $\mathcal{A}_c$ that occur at the origin (i.e. at $\lambda = 0$) occur when

$$c = \mu_j^{(2n)} = 2\cos\frac{\pi j}{2n+1}, \qquad j = 1, \dots, 2n,$$

and therefore

$$\# \bigcup_{c \in (-2,2)} \left\{\lambda^* \in \mathrm{Spec}\left(\mathcal{A}_c\right) : \lambda^* = 0\right\} = 2n. \tag{6.2}$$

Note that for some particular c's, there may be or may not be any double eigenvalues of $\mathcal{A}_c$ since it is rare. So the number of all critical points as a union of all c's and all λ's is the number of double eigenvalues. Therefore λ^* do not depend continuously on c and they form a discrete set. If we look at the union of all c's, then we look at the whole picture, c.f. the left of Figure 4 for $n = 20$. On the other hand, in the left of Figure 4, if one wants to look at the picture or count the double eigenvalues of $\mathcal{A}_c$ for a given c, then one needs to take a straight vertical line in the (c, λ)-plane, and count how many of them are on the line, which will give the number of double eigenvalues.

Lemma 6.3. *Subject to Conjecture 6.2, for a fixed n, the double real eigenvalues λ^* of $\mathcal{A}_c$ satisfy*

(i) $\# \bigcup_{c\in(-2,2)} \{\lambda^* \in \mathrm{Spec}(\mathcal{A}_c) : -4 \le \lambda^* \le 4\} = 2n^2.$

(ii) $\# \bigcup_{c\in(-2,2)} \{\lambda^* \in \mathrm{Spec}(\mathcal{A}_c) : 0 < \lambda^*\} = n(n-1).$

Proof. The statement (i) follows by multiplying the number of curves, which is n, and the number of critical points in each curve, which is $2n$ by Conjecture 6.2. To show (ii), we see by using (6.2) that there are $2n^2 - 2n$ critical points which do not lie at $\lambda = 0$. Then the statement follows from dividing by 2, as we consider the upper-half plane $\lambda > 0$. □

In the next result, we conjecture that all double eigenvalues of the pencil $\mathcal{A}_c$ are localized to a specific area. Numerical experiments suggest that double eigenvalues of $\mathcal{A}_c$ occur only when $\lambda < 2 - |c|$. In fact, if the only time when the eigenvalues of $\mathcal{A}_c$ escape from the real line and rejoin the real line are after the collisions, then Conjecture 1.1 is a consequence of the following conjecture.

Conjecture 6.4. *The double eigenvalues λ^* of the pencil $\mathcal{A}_c$ satisfy*

(i) $\# \bigcup_{c\in(-2,2)} \{\lambda^* \in \mathrm{Spec}(\mathcal{A}_c) : 0 < \lambda^* + |c| < 2\} = n(n-1),$

(ii) $\# \bigcup_{c\in(-2,2)} \{\lambda^* \in \mathrm{Spec}(\mathcal{A}_c) : 2 \le \lambda^* + |c| \le 4\} = 0.$

Remark 6.5. Note that as c increases from 0 to 2, the double eigenvalues do not overtake each other when they collide and while they jump. Thus, Conjecture 6.4 implies that the rate at which eigenvalues become double eigenvalues should be lower than the rate at which c decreases.

7 Heuristics for a class of block operator matrices

In this section we discuss some heuristic results for the behaviour of the non-real eigenvalues of the linear pencil problem in the aspect of a class of block operator matrices. We shall consider the eigenvalue problem (2.18) in a more general setting:

$$\mathbf{A}_{c,\kappa}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix} H_c & \kappa B \\ -\kappa B & -H_c\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} = \lambda\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix},$$

where $\kappa \in \mathbb{R}$, and H_c and B are given as in (2.18). Note that when $\kappa = 1$, the problem is reduced to the pencil problem considered in [3], i.e. $\mathbf{A}_c = \mathbf{A}_{c,1}$. In the following, we present some illustrations about the non-real-eigenvalues of $\mathbf{A}_{c,\kappa}$. It appears from numerics that the patterns we show in here occur only for our specific block matrix $\mathbf{A}_{c,\kappa}$.

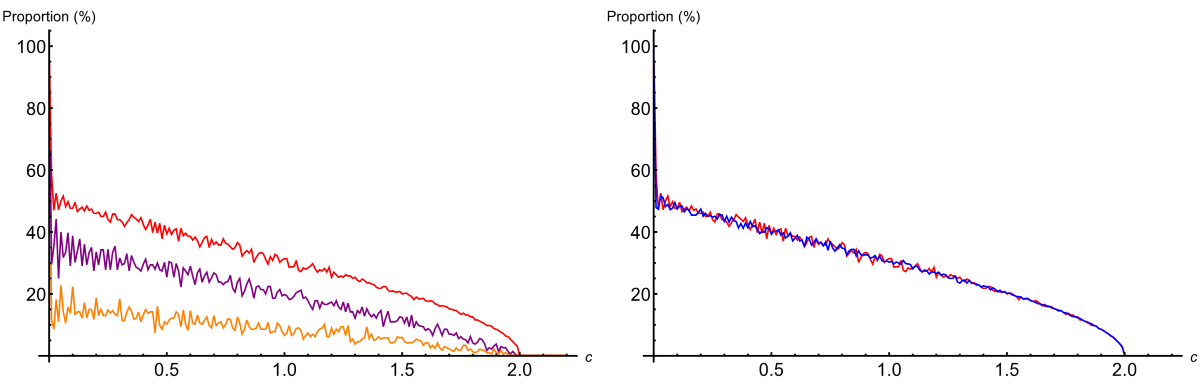

Figure 6: Proportion of the non-real eigenvalues of $\mathbf{A}_{c,\kappa}$ as c goes from 0 to 2.2. Left: for $n = 500$, with $\kappa = 1$ (red line), $\kappa = 2$ (purple line), $\kappa = 5$ (orange line). Right: for $\kappa = 1$, with $n = 500$ (red line), $n = 1000$ (blue line).

When $\kappa \geq 1$, the proportion of non-real eigenvalues of $\mathbf{A}_{c,\kappa}$ tends to usually decrease with κ, whereas $\kappa \leq 1$ it tends to increase with κ. We note that we show a typical picture for some $\kappa \geq 1$ in Figure 6(left). On the other hand, if we fix κ and change the half-size of the matrix n, then proportion of non-real eigenvalues becomes more stable, see Figure 6(right). Although the proportion of non-real eigenvalues become more stable for a specific value of c and κ as n increases, it remains difficult to estimate the range of possible values.

Another example is given in Figure 7 where we illustrate the non-real eigenvalues of $\mathbf{A}_{c,\kappa}$ in the $(c, \mathrm{Re}(\lambda))$-plane by taking $n = 10$ and superimposing the real parts of the non-real eigenvalues of $\mathbf{A}_{c,\kappa}$, with respect to c, for different values of κ between 0.001 and 5. We also plot the lines $|\mathrm{Re}(\lambda)| = 2 - c$ and $|\mathrm{Re}(\lambda)| = 2 + c$ shown in blue. We note that the condition (1.4) is given for $c > 0$ whereas here we take $c \in (-2, 2)$. One can increase the maximum value of κ, nevertheless, the figure does not change much.

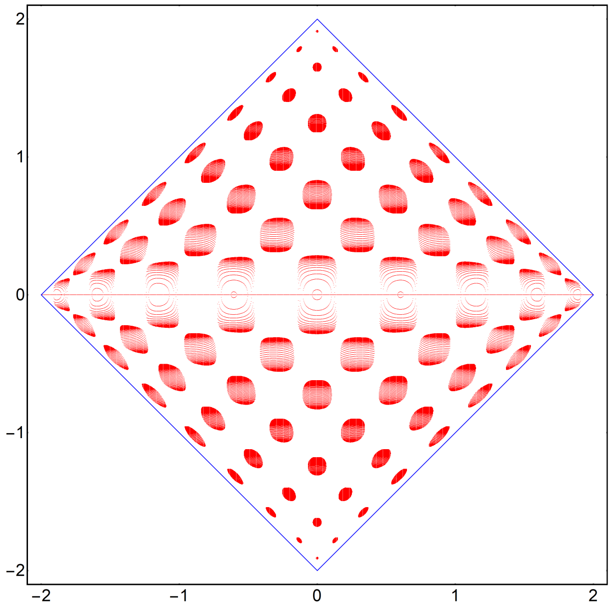

Figure 7: For $n = 10$, this figure shows $|\mathrm{Re}(\lambda)| = 2 \pm c$ (blue lines) and the superimposition of the non-real eigenvalues (red dots) of $\mathbf{A}_{c,\kappa}$ in the $(c, \mathrm{Re}(\lambda))$-plane, when the values of κ ranges from 0.001 to 5 with the step-size of 0.04.

When the similar experiment is repeated for the real eigenvalues of $\mathbf{A}_{c,\kappa}$, one can see a pattern called Chess Board structure by dividing the complex plane in rectangular regions (some of them are semi-infinite). It was shown in [5] that the real spectra is contained in half of the rectangles, and the other half does not contain any eigenvalue. One can observe a similar pattern to the Chess Board structure in Figure 7, however, not in the real eigenvalues but in the real parts of the non-real eigenvalues of $\mathbf{A}_{c,\kappa}$.

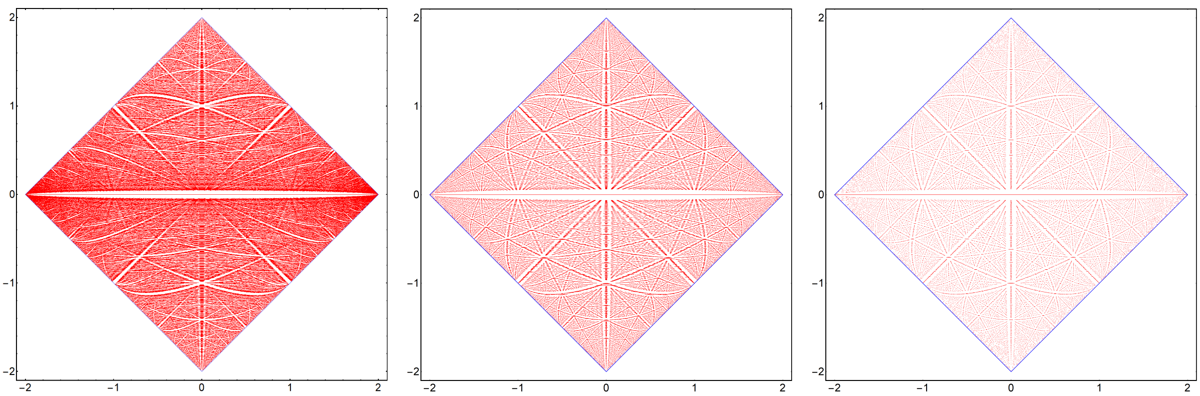

Figure 8: In the $(c, \mathrm{Re}(\lambda))$-plane, $|\mathrm{Re}(\lambda)| = 2 \pm c$ (blue lines) and the superimposition of the non-real eigenvalues (red dots) of $\mathbf{A}_{c,\kappa}$ by taking n from 2 to 50. Left: for $\kappa = 1$. Middle: for $\kappa = 4$. Right: for $\kappa = 13$.

In Figure 8, the non-real eigenvalues of $\mathbf{A}_{c,\kappa}$ is superimposed in the coordinates $(c, \mathrm{Re}(\lambda))$, by taking n from 2 to 50. If one takes κ to be either closer to zero or large, then this pattern somehow gets clearer, for instance see Figure 8 when $\kappa = 1$, $\kappa = 4$ and $\kappa = 13$ from left to right respectively. Both Figure 7 and Figure 8 indicate that the second part of Conjecture 1.1, i.e. the equation (1.4), seems true for a larger class of problems. However, the first part of Conjecture 1.1 seems true only for the pencil $\mathcal{A}_c$. We therefore claim the following.

Conjecture 7.1. *Let $c \in [-2, 2], \kappa \in \mathbb{R}$ and $n \in \mathbb{N}$. If λ is a non-real eigenvalue of $\mathbf{A}_{c,\kappa}$, then*

$$|\mathrm{Re}(\lambda) \pm c| < 2.$$

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