Existence of infinitely many solutions for an anisotropic equation using genus theory

A. Razani*[a] and Giovany M. Figueiredo†[b]

[a]Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Postal code: 34149-16818, Qazvin, Iran.
[b]Departamento de Matemática, Universidade de Brasilia, 70.910-900–Brasilia (DF), Brazil.

Abstract

Using genus theory, the existence of infinitely many solutions for an anisotropic equation involves subcritical growth is proved. Also by using Krasnoselskii genus and Clark's theorem, the existence of k-pairs of distinct solutions is proved. Finally, the anisotropic equation involves critical growth is considered and the existence of infinitely many solutions is proved.

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1 Introduction

Let $F(\xi) = |\xi| = \left(\sum_{i=1}^{N} |\xi_i|^2\right)^{\frac{1}{2}}$ be a norm on $\mathbb{R}^N$, the anisotropic Laplacian is defined by

$$Lu := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(F(\nabla u)F_{\xi_i}(\nabla u)\Big) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial \xi_i}\left(\frac{1}{2}F^2\right)(\nabla u)\right).$$

*razani@sci.ikiu.ac.ir
†giovany@unb.br

In general, the anisotropic Laplacian is a quasilinear elliptic operator of divergent type. It appears in the Euler-Lagrange equations which involve functionals containing the expressions $\int_\Omega F(\nabla u(x))^2 dx$, which is in fact the Dirichlet energy of u in Minkowski space. Anisotropic Laplacian is closely related to a convex hypersurface in $\mathbb{R}^N$, which is called the Wulff shape (or equilibrium crystal shape) of F. Wulff [22] considered a variational problem of an anisotropic geometric functional in the physical model of crystal growth. He stated without proof that among closed convex hypersurfaces with constant enclosed volume, the so-called Wulff shape minimizes the anisotropic surface energy. His work initiated lots of works on the theory of phase transitions, in particular in the case of anisotropic and nonhomogenous media. Recently, there some articles about the existence of solutions for anisotropic problems (see [5, 8–13, 20, 23] and the references therein).

In this paper we are concerned with the multiplicity of nontrivial solutions for the following classes of nonlinear anisotropic problems

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\right) = \lambda |u|^{\gamma-2}u \quad \text{in} \quad \Omega, \\ u \in D_0^{1,\overrightarrow{p}}(\Omega) \cap D_0^{1,\overrightarrow{q}}(\Omega), \quad \gamma \in (1, q_N), \end{cases} \tag{1.1}$$

and

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\right) \\ \qquad = \lambda |u|^{\gamma-2}u + |u|^{q^*-2}u \quad \text{in} \quad \Omega, \\ u \in D_0^{1,\overrightarrow{p}}(\Omega) \cap D_0^{1,\overrightarrow{q}}(\Omega), \ \gamma \in (1, p_1), \end{cases} \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, λ is a positive parameter, $\overrightarrow{p} = (p_1, \cdots, p_N)$, $\overrightarrow{q} = (q_1, \cdots, q_N)$,

$$1 < p_1 \leq p_2 \leq \ldots \leq p_N, \qquad \sum_{i=1}^{N} \frac{1}{p_i} > 1,$$
$$1 < q_1 \leq q_2 \leq \ldots \leq q_N, \qquad \sum_{i=1}^{N} \frac{1}{q_i} > 1.$$

Also we define

$$D_0^{1,\overrightarrow{p}}(\Omega) := \{u \in L^{p^*}(\Omega) : \tfrac{\partial u}{\partial x_i} \in L^{p_i}(\Omega); i = 1, \cdots, N\},$$
$$D_0^{1,\overrightarrow{q}}(\Omega) := \{u \in L^{q^*}(\Omega) : \tfrac{\partial u}{\partial x_i} \in L^{q_i}(\Omega); i = 1, \cdots, N\}$$

and

$$p^* := \frac{N}{\left(\sum_{i=1}^{N} \frac{1}{p_i}\right) - 1} = \frac{N\overline{p}}{N - \overline{p}}, q^* := \frac{N}{\left(\sum_{i=1}^{N} \frac{1}{q_i}\right) - 1} = \frac{N\overline{q}}{N - \overline{q}},$$

where $\overline{p}$ and $\overline{q}$ denote the harmonic means $\overline{p} = N/\left(\sum_{i=1}^{N} \frac{1}{p_i}\right)$ and $\overline{q} = N/\left(\sum_{i=1}^{N} \frac{1}{q_i}\right)$, respectively.

We say

$$\overrightarrow{p} \leq \overrightarrow{q} \quad \text{iff} \quad p_i \leq q_i \quad \text{for all} \quad i = 1, \cdots, N. \tag{1.3}$$

Throughout all the paper, we assume that

$$\overrightarrow{p} \leq \overrightarrow{q}, p_N < p^* \text{ and } q_N < q^*$$

Also with respect to the (1.3), we have $p^* < q^*$.

In this paper, by intuition of [9] we study the existence of infinity many solutions for problems (1.1) and (1.2), respectively. With respect to this, the rest of the paper is as follows: In Section 2 the suitable function space is introduced and some facts about variational methods (such as Palais-Smale condition) and genus theory are recalled. In Section 3 the existence of infinity many solutions for problem (1.1) is studied by genus theory. This problem involves a subcritical growth and we can prove that the problem (1.1) has at least k pairs of distinct solutions, by applying Clark's Theorem 2.8 which implies the existence of at least k pairs of distinct critical points. In Section 4 the existence of infinity many solutions for problem (1.2) which involves critical growth, is proved. In this critical case, the energy functional is not bounded from below and we need to make a truncation in it such that critical points preserves.

2 Function space

It is well known that $D_0^{1,\overrightarrow{q}}(\Omega)$, which is the completion of the space $\mathcal{D}(\Omega) := \{\varphi \in C^\infty(\Omega) : supp\ \varphi \subset\subset \Omega\}$ with respect to the norm

$$\|u\|_q = \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{q_i},$$

is a reflexive Banach space and is continuously embedded in $L^{q^*}(\Omega)$. Here $|.|_{q_i}$ is the usual norm in $L^{q_i}(\Omega)$.

Here we consider the $D_0^{1,\overrightarrow{p}}(\Omega) \cap D_0^{1,\overrightarrow{q}}(\Omega)$ with respect to the norm

$$\|u\| = \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|_{p_i} + \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|_{q_i}.$$

Remark 2.1. *Notice that if $\overrightarrow{p} \leq \overrightarrow{q}$ and Ω is a bounded domain in $\mathbb{R}^N$ then $D_0^{1,\overrightarrow{p}}(\Omega) \subset D_0^{1,\overrightarrow{q}}(\Omega)$. To see this, we have*

$$|\frac{\partial u}{\partial x_i}|_{p_i}^{p_i} = \int_\Omega 1 \cdot |\frac{\partial u}{\partial x_i}|^{p_i} dx \leq |\Omega|^{\frac{q_i - p_i}{q_i}} \left(\int_\Omega \cdot |\frac{\partial u}{\partial x_i}|^{q_i} dx \right)^{\frac{p_i}{q_i}}.$$

Hence

$$\begin{aligned} \|u\|_p = \sum_{i=1}^{N} |\tfrac{\partial u}{\partial x_i}|_{p_i} \leq & \quad N \max_{i=1,\cdots,N} |\Omega|^{\frac{q_i - p_i}{q_i}} \sum_{i=1}^{N} |\tfrac{\partial u}{\partial x_i}|_{q_i} \\ = & \quad N \max_{i=1,\cdots,N} |\Omega|^{\frac{q_i - p_i}{q_i}} \|u\|_q. \end{aligned}$$

Then $D_0^{1,\overrightarrow{p}}(\Omega) \subset D_0^{1,\overrightarrow{q}}(\Omega)$.

Since Ω is a bounded domain of $\mathbb{R}^N$, from [14, Theorem 1], the continuity of the embedding $D_0^{1,\overrightarrow{q}}(\Omega) \hookrightarrow L^s(\Omega)$, for all $s \in [1, q_N]$ relies on a well-known Poincaré type inequality. More precisely, denoting by $e_1, \cdots, e_n$ the canonical basis of $\mathbb{R}^N$, assume that Ω has width $a > 0$ in the direction of e_i, namely $\sup_{x,y\in\Omega}(x - y, e_i) = a$. Thus, for every $\gamma \geq 1$, we have

$$|u|_\gamma \leq \frac{a\gamma}{2} \left|\frac{\partial u}{\partial x_i}\right|_\gamma, \quad \text{for all } u \in \mathcal{D}(\Omega). \tag{2.1}$$

On the other hand, from [21, Theorem 1.2], is also continuous the embedding $D_0^{1,\overrightarrow{q}}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ and, since $q_N < q^*$, we obtain $D_0^{1,\overrightarrow{q}}(\Omega) \hookrightarrow L^s(\Omega)$, for all $s \in [1, q^*]$.

The function space that we study the problem on it is $X := D_0^{1,\overrightarrow{p}}(\Omega) \cap D_0^{1,\overrightarrow{q}}(\Omega)$ which respect to the Remark 2.1, we have $X = D_0^{1,\overrightarrow{q}}(\Omega)$, with the norm

$$\|u\| = \|u\|_p + \|u\|_q$$

which is equivalent to $\|u\|_q$.

Definition 2.2. *We say that $u \in X$ is a weak solution of the problem* (1.1) *(and* (1.2)*) if it verifies*

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}\,dx+\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}\,dx\\ -\lambda\int_{\Omega}|u|^{\gamma-2}u\phi\,dx-\int_{\Omega}h(u)\phi\,dx=0,\end{aligned} \tag{2.2}$$

for all $\phi \in X$, where $h(t)=0$ in problem (1.1) *and $h(t)=|t|^{q^*-2}t$ in problem* (1.2)*.*

If a function $u \in X \bigcap L^{\infty}(\Omega)$ satisfies (2.2), then u is a strong solution of the problems (1.1) (and (1.2)). From [2, Proposition 4.1] and [14, Theorem 4], weak solutions of problems (1.1) (and (1.2)) are strong solutions.

We will look for solutions of (1.1) (and (1.2)) by finding critical points of the C^1-functional $I : X \to \mathbb{R}$ given by

$$I(u)=\sum_{i=1}^{N}\int_{\Omega}\frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+\sum_{i=1}^{N}\int_{\Omega}\frac{1}{q_i}\left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx-\lambda\frac{1}{\gamma}\int_{\Omega}|u|^{\gamma}\,dx-\int_{\Omega}H(u)\,dx,$$

where $H(t)=\displaystyle\int_0^t h(\tau)\,d\tau$.

Note that

$$\begin{aligned}I'(u)\phi=\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}\,dx+\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}\,dx\\ -\lambda\int_{\Omega}|u|^{\gamma-2}u\phi\,dx-\int_{\Omega}h(u)\phi\,dx,\end{aligned} \tag{2.3}$$

for all $\phi \in X$. Hence critical points of I are weak solutions for (1.1) (and (1.2)).

In order to use variational methods, we first derive some results related to the Palais-Smale compactness condition.

Definition 2.3. *A sequence $(u_n) \subset X$ is a Palais-Smale sequence for the functional I if*

$$I(u_n) \to d \text{ and } \|I'(u_n)\| \to 0 \text{ in } X', \tag{2.4}$$

for some $d \in \mathbb{R}$.

If (2.4) implies the existence of a subsequence $(u_{n_j}) \subset (u_n)$ which converges in X for all $d \in \mathbb{R}$, we say that I satisfies the Palais-Smale condition. If this strongly convergent subsequence exists only for some d values, we say that I verifies a local Palais-Smale condition.

We recall some basic notions on the Krasnoselskii genus which we will use in the proofs of our main results.

Let E be a real Banach space. Let us denote by $\mathfrak{A}$ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 2.4. *Let $A \in \mathfrak{A}$. The Krasnoselskii genus $\gamma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for all $x \in A$. If k does not exist we set $\gamma(A) = \infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.*

In the sequel we will establish only the properties of the genus that will be used through this work. More information on this subject may be found in the references [1], [3], [7] and [18].

Proposition 2.5. *Let A and B be sets in $\mathfrak{A}$.*

(i) If there exists an odd application $\varphi \in C(A, B)$ then $\gamma(A) \leq \gamma(B)$.

(ii) If there exists an odd homeomorphism $\varphi : A \to B$ then $\gamma(A) = \gamma(B)$.

(iii) If A is a compact set, then there exists a neighborhood $K \in \mathfrak{A}$ of A such that $\gamma(A) = \gamma(K)$.

(iv) If $\gamma(B) < \infty$, then$\gamma(\overline{A \backslash B}) \geq \gamma(A) - \gamma(B)$.

(v) If $\gamma(A) \geq 2$, then A has infinitely many points.

Proposition 2.6. *Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Corollary 2.7. *$\gamma(\mathcal{S}^{N-1}) = N$ where $\mathcal{S}^{N-1}$ is a unit sphere of $\mathbb{R}^N$.*

The proofs of these results can be found, for example, [19, Proposition 7.5, Remark 7.6 and Proposition 7.7]. We now establish a result due to Clark [6].

Theorem 2.8. *Suppose $J \in C^1(X, \mathbb{R})$ is a functional satisfying the Palais-Smale condition and*

A_1) J is bounded from below and even.

A_2) *there is a compact set* $K \in \mathfrak{A}$ *such that* $\gamma(K) = k$ *and* $\sup_{x \in K} J(x) < J(0)$.

Then J *possesses at least* k *pairs of distinct critical points and their corresponding critical values* c_j *are less than* $J(0)$.

3 Subcritical problem

As we mentioned in the last section, we consider the space $X = D_0^{1,\overrightarrow{q}}(\Omega) = D_0^{1,\overrightarrow{p}}(\Omega) \cap D_0^{1,\overrightarrow{q}}(\Omega)$ endowed with the norm

$$\|u\| = \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i} + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{q_i}.$$

which is equivalent to the norm on $D_0^{1,\overrightarrow{q}}(\Omega)$.
First, we study some properties related to the C^1-functional $I : X \to \mathbb{R}$, given by

$$I(u) = \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx + \sum_{i=1}^{N} \int_\Omega \frac{1}{q_i} \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx - \lambda \frac{1}{\gamma} \int_\Omega |u|^\gamma \, dx. \quad (3.1)$$

The next two lemmas are true for $\gamma \in (1, q_N)$. In [4] the authors showed that I is coercive when $\gamma \in (p_1, q_N)$, by using the boundedness of levels sets $I^b = \{u \in X : I(u) \leq b\}$.

Lemma 3.1. *I is bounded from below.*

Proof. I is coercive. In fact, suppose by contradiction that $\|u\| \to \infty$. Unfortunately, we can not assure that $\left| \frac{\partial u}{\partial x_i} \right|_{p_i}, \left| \frac{\partial u}{\partial x_i} \right|_{q_i} \to \infty$ for all $i \in \{1, \cdots, N\}$. Hence, we will consider two cases.

If $|u|_\gamma$ is bounded, then we have already $I(u) \to \infty$. On the other hand, if $|u|_\gamma \to \infty$ then, by using Holder's inequality and (2.1), we conclude that

$$\left| \frac{\partial u}{\partial x_i} \right|_{p_i}, \left| \frac{\partial u}{\partial x_i} \right|_{q_i} \to \infty, \ \gamma \leq p_i \leq q_i. \quad (3.2)$$

Moreover, since $\gamma < q_N$, we have

$$I(u) \geq \frac{1}{q_N} \left| \frac{\partial u}{\partial x_i} \right|_{q_N}^{q_N} - \frac{C}{\gamma} \lambda \left| \frac{\partial u}{\partial x_i} \right|_{q_N}^{\gamma} dx.$$

It follows from (3.2) that $I(u) \to \infty$. In any case, I is coercive and, therefore, I is bounded from below. $\square$

Lemma 3.2. *I satisfies the (PS) condition.*

Proof. Let (u_n) be a sequence in X such that

$$I(u_n) \to C \quad \text{and} \quad I'(u_n) \to 0.$$

Since I is coercive, we conclude that (u_n) is bounded in X. Thus, passing to a subsequence, if necessary, we have

$$u_n \rightharpoonup u \quad \text{in} \quad X,$$

$$u_n \to u \quad \text{in} \quad L^\sigma(\Omega) \quad \text{with} \quad \sigma \in [1, q^*),$$

and

$$u_n(x) \to u(x) \quad \text{a.e in} \quad \Omega.$$

Thus, from convergence in $L^\sigma(\Omega)$ we get

$$\int_\Omega |u_n|^\gamma \, dx - \int_\Omega |u_n|^{\gamma-2} u_n u \, dx = o_n(1), \tag{3.3}$$

and from weak convergence

$$\begin{aligned} \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial u_n}{\partial x_i} \, dx + \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q_i-2} \frac{\partial u}{\partial x_i}\frac{\partial u_n}{\partial x_i} \, dx \\ - \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx - \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx = o_n(1). \end{aligned} \tag{3.4}$$

Hence, from (3.4) we obtain

$$\begin{aligned} 0 \quad &\leq \quad C_p \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right|_{p_i}^{p_i} + C_q \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right|_{q_i}^{q_i} \\ &\leq \quad \sum_{i=1}^N \int_\Omega \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dx \\ &\quad + \sum_{i=1}^N \int_\Omega \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{q_i-2}\frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dx \\ &= \quad \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} dx - \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i}\frac{\partial u}{\partial x_i} \, dx \\ &\quad + \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i} dx - \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i-2}\frac{\partial u_n}{\partial x_i}\frac{\partial u}{\partial x_i} \, dx + o_n(1). \end{aligned}$$

From (3.3), we derive

$$0 \le C_p \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right|_{p_i}^{p_i} + C_q \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right|_{q_i}^{q_i} \qquad (3.5)$$
$$\le I'(u_n)u_n - I'(u_n)u + o_n(1),$$

where C_p and C_q are constants..

Thus, we conclude that $u_n \to u$ in X and the proof is complete. □

Now we can state the existence of infinitely many solutions for problem (1.1).

Theorem 3.3. *Assume that $\gamma \in (1, p_1)$. Then, problem (1.1) has infinitely many solutions, for all $\lambda \in (0, +\infty)$.*

Proof. Let $X_k = span\{e_1, e_2, \cdots, e_k\}$ be a subspace of X with $\dim X_k = k$. Note that X_k is continuously embedded in $L^\gamma(\Omega)$. Thus, the norms of X and $L^\gamma(\Omega)$ are equivalent on X_k and there exists a positive constant $C(k)$ which depends on k, such that

$$-C(k)\|u\|^\gamma \ge -\int_\Omega |u|^\gamma \, dx, \quad \text{for all} \quad u \in X_k.$$

Thus we conclude that

$$I(u) \le \sum_{i=1}^{N} \frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|_{p_i}^{p_i} + \sum_{i=1}^{N} \frac{1}{q_i} \left| \frac{\partial u}{\partial x_i} \right|_{q_i}^{q_i} - \lambda C(k) \frac{1}{\gamma} \|u\|^\gamma.$$

Let $0 < R < 1$ and $u \in X$ be such that $\|u\| \le R$. Thus

$$I(u) \le \frac{D}{p_1}\|u\|^{p_1} - \lambda C(k)\frac{1}{\gamma}\|u\|^\gamma = \|u\|^\gamma \left[\frac{D}{p_1}\|u\|^{p_1-\gamma} - \lambda C(k)\frac{1}{\gamma} \right],$$

where D is a constant. Since $\gamma < p_1$, choosing $0 < R < \min\left\{1, \left(\frac{\lambda C(k) p_1}{D\gamma}\right)^{\frac{1}{p_1-\gamma}}\right\}$ we have

$$I(u) \le R^\gamma \left[\frac{D}{p_1} R^{p_1-\gamma} - \lambda C(k)\frac{1}{\gamma} \right] < 0 = I(0),$$

for all $u \in K = \{u \in X_k : \|u\| = R\}$. This inequality implies

$$\sup_{u \in K} I(u) < 0 = I(0).$$

Since X_k and $\mathbb{R}^k$ are isomorphic and K and $\mathcal{S}^{k-1}$ are homeomorphic, we conclude that $\gamma(K) = k$. Moreover, I is even. By Clarke's Theorem (Theorem 2.8), I has at least k pairs of different critical points. Since k is arbitrary, we found infinitely many critical points of I. $\square$

We point out that in order to apply the Clarke's Theorem in the previous proof, we use Lemmas 3.1 and 3.2, which hold for any $\gamma \in (1, q_N)$.

Here we stat a lemma which we need it for proving the existence of at least k pairs of solutions for problem (1.1).

Lemma 3.4. *Let A be the set defined by*

$$A = \left\{ u \in X \backslash \{0\} : \left| \frac{\partial u}{\partial x_i} \right|_{\ell_i}^{\ell_i} \leq \left| \frac{\partial u}{\partial x_N} \right|_{q_N}^{q_N}, \text{ for all } 1 \leq i \leq N \right\},$$

where $\ell_i = p_i$ or $\ell_i = q_i$ for $1 \leq i \leq N-1$ and $\ell_N = p_N$. For each compact set $K \subset X\backslash\{0\}$, there exists $t_K > 0$ such that $tK \subset A$ for all $t \geq t_K$, where $tK = \{tu : u \in K\}$.

Proof. We define functions $h_{\ell_i} : X\backslash\{0\} \to \mathbb{R}$ by

$$h_{\ell_i}(u) = \frac{\left| \frac{\partial u}{\partial x_i} \right|_{\ell_i}^{\ell_i}}{\left| \frac{\partial u}{\partial x_N} \right|_{q_N}^{q_N}} \text{ for all } 1 \leq i \leq N.$$

By using (2.1), we conclude that h_{ℓ_i} is well defined in the domain $X\backslash\{0\}$, namely, $\left| \frac{\partial u}{\partial x_N} \right|_{q_N}^{q_N}$ is not null. Moreover, the inequality

$$\begin{aligned} \left| \left| \frac{\partial u}{\partial x_i} \right|_{\ell_i} - \left| \frac{\partial v}{\partial x_i} \right|_{\ell_i} \right| \leq & \left| \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right|_{\ell_i} \\ \leq & \|u - v\|, \end{aligned}$$

for all $u, v \in X,\ 1 \leq i \leq N$, says us that h_{ℓ_i} is a continuous function for $1 \leq i \leq N$.

Since K is compact, there exists $u_{\ell_i} \in K$ such that $h_{\ell_i}(u) \leq h_{\ell_i}(u_{\ell_i})$ for all $u \in K$. Define still $t_{\ell_i} := [h_{\ell_i}(u_{\ell_i})]^{\frac{1}{q_N - \ell_i}}$, $t_j = \max\limits_{1 \leq i \leq N} t_{\ell_i}$ and choose $t_K = t_j$. Thus, if $t \geq t_K$ we have $t \geq t_{\ell_i}$ and $t^{q_N - \ell_i} \geq t_{\ell_i}^{q_N - \ell_i} = h_{\ell_i}(u_{\ell_i})$. Consequently,

$$\frac{\left| \frac{\partial u}{\partial x_i} \right|_{\ell_i}^{\ell_i}}{\left| \frac{\partial u}{\partial x_N} \right|_{q_N}^{q_N}} \leq t^{q_N - \ell_i},$$

and

$$\left|\frac{\partial(tu)}{\partial x_i}\right|_{\ell_i}^{\ell_i} \leq \left|\frac{\partial(tu)}{\partial x_N}\right|_{q_N}^{q_N},$$

for all $u \in K$ and for all $i \in \{1, \cdots, N\}$. □

The next theorem is the second main result of this paper.

Theorem 3.5. *Assume that $\gamma \in [p_1, q_N)$. Then, for each $k \in \mathbb{N}$, there exists $\lambda_k > 0$ such that problem (1.1) has at least k pairs of solutions, for all $\lambda \in (\lambda_k, +\infty)$.*

Proof. In a similar way to the previous theorem, for each $k \in \mathbb{N}$, we consider a k-dimensional subspace $X_k = span\{e_1, e_2, \cdots, e_k\}$ of X, continuously embedded in $L^{q_N}(\Omega)$. This is, there exists a positive constant $C(k)$ which depends on k, such that

$$C(k)\left|\frac{\partial u}{\partial x_N}\right|_{q_N} \leq C(k)\|u\| \leq |u|_\gamma.$$

So,

$$-C(k)\left|\frac{\partial u}{\partial x_N}\right|_{q_N}^{\gamma} \geq -C(k)\|u\|^\gamma \geq -|u|_\gamma^\gamma, \quad \text{for all} \quad u \in X_k, \tag{3.6}$$

where we are using the same notation $C(k)$ to denote different constants.

Denoting by S_k the unit sphere of X_k and noting that $S_k \subset X\backslash\{0\}$ is a compact set, it follows from previous lemma that there exists $t_k > 0$ such that $tS_k \subset A$, for all $t \geq t_k$. Thus, for each $u \in t_k S_k$, we have

$$\begin{aligned} I(u) &= \sum_{i=1}^N \int_\Omega \frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx + \sum_{i=1}^N \int_\Omega \frac{1}{q_i}\left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx - \lambda\frac{1}{\gamma}\int_\Omega |u|^\gamma \, dx \\ &\leq \left(\sum_{i=1}^N \frac{1}{p_i}\right)\int_\Omega \left|\frac{\partial u}{\partial x_N}\right|^{q_N} dx + \left(\sum_{i=1}^N \frac{1}{q_i}\right)\int_\Omega \left|\frac{\partial u}{\partial x_N}\right|^{q_N} dx - \lambda\frac{1}{\gamma}|u|_\gamma^\gamma \\ &= \left(\frac{N}{\overline{p}} + \frac{N}{\overline{q}}\right)\left|\frac{\partial u}{\partial x_N}\right|_{q_N}^{q_N} - \lambda\frac{1}{\gamma}|u|_\gamma^\gamma. \end{aligned}$$

From (3.6), we get

$$I(u) \leq \left(\frac{N}{\overline{p}} + \frac{N}{\overline{q}}\right)\left|\frac{\partial u}{\partial x_N}\right|_{q_N}^{q_N} - \frac{C(k)}{\gamma}\lambda\left|\frac{\partial u}{\partial x_N}\right|_{q_N}^{\gamma},$$

and so

$$I(u) \leq \left|\frac{\partial u}{\partial x_N}\right|_{q_N}^{\gamma} \left(\left(\frac{N}{\overline{p}} + \frac{N}{\overline{q}}\right)\left|\frac{\partial u}{\partial x_N}\right|_{q_N}^{q_N-\gamma} - \frac{C(k)}{\gamma}\lambda\right). \tag{3.7}$$

From (2.1) we conclude that $\alpha := \min_{u \in t_k S_k} \left|\frac{\partial u}{\partial x_N}\right|_{q_N} > 0$. Hence,

$$I(u) \leq \alpha^{\gamma}\left(\left(\frac{N}{\overline{p}} + \frac{N}{\overline{q}}\right)t_k^{q_N-\gamma} - \frac{C(k)}{\gamma}\lambda\right) < 0,$$

when $\lambda > \lambda_k = \frac{\gamma N\left(\frac{1}{\overline{p}}+\frac{1}{\overline{q}}\right)}{C(k)} t_k^{q_N-\gamma}$. Therefore,

$$\sup_{t_k S_k} I_\lambda < 0,$$

for all $\lambda > \lambda_k$, with $\gamma(t_k S_k) = k$. Arguing as in the proof of Theorem 3.3, the result follows from Clarke's Theorem 2.8. $\square$

4 Critical case

Since I is not bounded from below, in the critical case, to apply genus theory, we will need to make a truncation in the functional I. In fact, the idea is to get a truncated functional J such that critical points u of J with $J(u) < 0$ are also critical points of I.

However, the anisotropy of (1.2) becomes our job somewhat more complicated. To overcome the difficulties , we need to consider separately the cases $\|u\| \leq 1$ and $\|u\| > 1$ in the building of J.

Case 1: $\|u\| \leq 1$.

In this case, we have $\left|\frac{\partial u}{\partial x_i}\right|_{p_i} \leq 1$ and $\left|\frac{\partial u}{\partial x_i}\right|_{q_i} \leq 1$ for all $i \in \{1, \ldots, N\}$, and consequently

$$\left|\frac{\partial u}{\partial x_i}\right|_{p_i}^{q_N} \leq \left|\frac{\partial u}{\partial x_i}\right|_{p_i}^{p_i} \text{ and } \left|\frac{\partial u}{\partial x_i}\right|_{q_i}^{q_N} \leq \left|\frac{\partial u}{\partial x_i}\right|_{q_i}^{q_i}.$$

Hence

$$I(u) \geq \frac{1}{p_N}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|_{p_i}^{q_N} + \frac{1}{q_N}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|_{q_i}^{q_N} - \lambda\frac{1}{\gamma}\int_\Omega |u|^\gamma\, dx - \frac{1}{q^*}\int_\Omega |u|^{q^*}\, dx.$$

From continuous embedding,

$$\int_\Omega |u|^s \, dx \leq C\|u\|^s, \quad s \in [1, q^*].$$

From previous inequality we obtain

$$I(u) \geq C_1\left(\frac{1}{p_N} + \frac{1}{q_N}\right)\|u\|^{q_N} - \lambda C\|u\|^\gamma - C_2\|u\|^{q^*} = g(\|u\|), \tag{4.1}$$

where $g(t) := C_1\left(\frac{1}{p_N} + \frac{1}{q_N}\right)t^{q_N} - \lambda C t^\gamma - C_2 t^{q^*}$. So, there exists $\lambda^* > 0$ such that, if $\lambda \in (0, \lambda^*)$, then g attains its positive maximum.

We denote by $0 < R_0(\lambda) < R_1(\lambda)$ the unique two roots of g. The next lemma is essential to construct the truncated functional.

Lemma 4.1. *$R_0(\lambda) \to 0$ as $\lambda \to 0$.*

Proof. Indeed, from $g(R_0(\lambda)) = 0$ and $g'(R_0(\lambda)) > 0$, we have

$$C_1\left(\frac{1}{p_N} + \frac{1}{q_N}\right)R_0(\lambda)^{q_N} = \lambda C R_0(\lambda)^\gamma + C_2 R_0(\lambda)^{q^*} \tag{4.2}$$

and

$$C_1\left(\frac{q_N}{p_N} + 1\right)R_0(\lambda)^{q_N - 1} > \lambda C\gamma R_0(\lambda)^{\gamma-1} + C_2 q^* R_0(\lambda)^{q^*-1}, \tag{4.3}$$

for all $\lambda \in (0, \lambda^*)$. From (4.2), we conclude that $R_0(\lambda)$ is bounded. Suppose that $R_0(\lambda) \to R_0 > 0$ as $\lambda \to 0$. Then,

$$C_1\left(\frac{1}{p_N} + \frac{1}{q_N}\right)R_0^{q_N} = C_2 R_0^{q^*} \text{ and } C_1\left(\frac{q_N}{p_N} + 1\right)R_0^{q_N-1} \geq C_2 q^* R_0^{q^*-1},$$

a contradiction, because $q^* > q_N$. Therefore $R_0 = 0$. $\square$

Now we consider the following truncation in the functional I:
From Lemma 4.1, we have $R_0(\lambda) < 1$ for small λ. So $R_0(\lambda) < \min\{R_1(\lambda), 1\}$ and we can take $\phi \in C_0^\infty([0, +\infty))$, $0 \leq \phi(t) \leq 1$, for all $t \in [0, +\infty)$, such that

$$\phi(t) = \begin{cases} 1\,, & t \in [0, R_0(\lambda)], \\ 0\,, & t \in [\min\{R_1(\lambda), 1\}, +\infty). \end{cases}$$

We define the functional

$$J(u) = \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx + \sum_{i=1}^{N} \int_\Omega \frac{1}{q_i} \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx$$
$$- \lambda \frac{1}{\gamma} \int_\Omega |u|^\gamma \, dx - \phi(\|u\|) \frac{1}{q^*} \int_\Omega |u|^{q^*} dx.$$

Notice that $J \in C^1(X, \mathbb{R})$ and, as in (4.1), $J(u) \geq \overline{g}(\|u\|)$, for all $u \in X$ with $\|u\| < 1$, where

$$\overline{g}(t) = C_1 \left(\frac{1}{p_N} + \frac{1}{q_N} \right) t^{q_N} - \lambda C t^\gamma - C_2 \phi(t) t^{q^*} \geq 0, \tag{4.4}$$

for all $t \in (R_0(\lambda), \min\{R_1(\lambda), 1\})$.

By definition, if $\|u\| \leq R_0(\lambda) < \min\{R_1(\lambda), 1\}$ then $J(u) = I(u)$. Once we will obtain critical points u of J with $J(u) < 0$, to show that these critical points verify $\|u\| < R_0(\lambda)$ is important to ensure that $J(u) \geq 0$ when $\|u\| > 1$.
In fact, suppose just for a moment that $J(u) \geq 0$ when $\|u\| > 1$. Let $\overline{u}$ be a critical point of J such that

$$J(\overline{u}) < 0. \tag{4.5}$$

So $\|\overline{u}\| \leq 1$. If $\min\{R_1(\lambda), 1\} = 1$, follows from (4.4) and (4.5) that $\|\overline{u}\| < R_0(\lambda)$. On the other hand, if $\min\{R_1(\lambda), 1\} = R_1(\lambda)$, we conclude again from (4.4), (4.5) and definition of J that $\|\overline{u}\| < R_0(\lambda)$. It remains to prove that $J(u) \geq 0$ when $\|u\| > 1$.

Case 2: $\|u\| > 1$.
Note that in this case we have $\phi(\|u\|) = 0$, and there exists $i = i(u) \in \{1, 2, \ldots, N\}$ such that $\left|\frac{\partial u}{\partial x_i}\right|_{\ell_i} \geq \frac{1}{N}$ where $\ell_i = p_i$ or $\ell_i = q_i$. So,

$$\begin{aligned} J(u) &= \sum_{i=1}^{N} \frac{1}{p_i} \left|\frac{\partial u}{\partial x_i}\right|_{p_i}^{p_i} + \sum_{i=1}^{N} \frac{1}{q_i} \left|\frac{\partial u}{\partial x_i}\right|_{q_i}^{q_i} - \lambda \frac{1}{\gamma} |u|_\gamma^\gamma \\ &\geq \frac{1}{\ell_i} \left|\frac{\partial u}{\partial x_i}\right|_{\ell_i}^{\ell_i} - \frac{C}{\gamma} \lambda \left|\frac{\partial u}{\partial x_i}\right|_{\ell_i}^{\gamma} \\ &= g_i \left(\left|\frac{\partial u}{\partial x_i}\right|_{\ell_i} \right), \end{aligned}$$

where $g_i : [1/N, \infty) \to \mathbb{R}$ is defined by

$$g_i(t) = \frac{1}{\ell_i} t^{\ell_i} - \frac{C}{\gamma} \lambda t^{\gamma}, \quad \text{with } i = i(u),$$

which has a global minimum point at $t_i = (C\lambda)^{\frac{1}{\ell_i - \gamma}}$ and

$$g_i(t_i) = (C\lambda)^{\frac{\ell_i}{\ell_i - \gamma}} \left(\frac{1}{\ell_i} - \frac{1}{\gamma} \right) < 0.$$

Observe that $g_i(t) \geq 0$ if, and only if, $t \geq (\frac{C\ell_i}{\gamma}\lambda)^{\frac{1}{\ell_i - \gamma}}$. Hence, to ensure that $\min_{t \geq \frac{1}{N}} g_i(t) \geq 0$, we take $\lambda^* \leq \dfrac{\gamma}{C\ell_i N^{\ell_i - \gamma}}$. Therefore, for each $\lambda \in (0, \lambda^*)$ we have $J(u) \geq 0$ for all $\|u\| \geq 1$. Moreover, we conclude that the functional J is coercive and bounded from below.

Now, we will show that J satisfies the local Palais-Smale condition. For this, we need the following technical result.

Lemma 4.2. *Let $(u_n) \subset X$ be a bounded sequence such that*

$$I(u_n) \to c \quad \textit{and} \quad I'(u_n) \to 0.$$

If

$$\begin{aligned} c \quad < \quad & \left(\frac{1}{q_N} - \frac{1}{q^*} \right) S^{q^*/(q^* - q_N)} - \left[\frac{\left(\frac{1}{\gamma} - \frac{1}{q_N} \right) |\Omega|^{\frac{q^* - \gamma}{q^*}}}{\left(\frac{1}{q_N} - \frac{1}{q^*} \right)} \right]^{q^*/(q^* - \gamma)} \\ \times \quad & \left[\left(\frac{\gamma}{q^*} \right)^{\gamma/(q^* - \gamma)} - \left(\frac{\gamma}{q^*} \right)^{q^*/(q^* - \gamma)} \right] \left(\frac{1}{q_N} - \frac{1}{q^*} \right) \lambda^{q^*/(q^* - \gamma)} \end{aligned}$$

hold, then there exists $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*)$, we have that, up to a subsequence, (u_n) is strongly convergent in X.

Proof. Using a version of Lions's concentration compactness-principle (see [16, Corollary 1 of Lemma 5]), we obtain at most a countable index set Λ, sequences $(x_j) \subset \overline{\Omega}$, $(b_j), (a_j) \subset (0, \infty)$, such that

$$\sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i} + \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} \right|^{q_i} \rightharpoonup \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{q_i} + \mu \tag{4.6}$$

and

$$|u_n|^{q^*} \rightharpoonup |u|^{q^*} + \nu \tag{4.7}$$

in the weak*-sense of measures, where

$$\mu \geq \sum_{j\in\Lambda} b_j \delta_{x_j}, \quad \nu = \sum_{j\in\Lambda} a_j \delta_{x_j}, \quad S a_j^{q_N/q^*} \leq b_j, \tag{4.8}$$

for all $j \in \Lambda$ and δ_{x_j} is the Dirac mass at $x_j \in \overline{\Omega}$.

Now, for every $\varrho > 0$, we set $\psi_\varrho(x) := \psi((x - x_k)/\varrho)$ where $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla\psi|_\infty \leq 2$. Since $(\psi_\varrho u_n)$ is bounded, $I'(u_n)(\psi_\varrho u_n) \to 0$, that is,

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \psi_\varrho \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} + \sum_{i=1}^N \int_\Omega \psi_\varrho \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i} \, dx \\
= \ & -\sum_{i=1}^N \int_\Omega u_n \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i}\frac{\partial \psi_\varrho}{\partial x_i} \, dx - \sum_{i=1}^N \int_\Omega u_n \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i-2} \frac{\partial u_n}{\partial x_i}\frac{\partial \psi_\varrho}{\partial x_i} \, dx \\
& + \lambda \int_\Omega |u_n|^\gamma \psi_\varrho \, dx + \int_\Omega \psi_\varrho |u_n|^{q^*} \, dx + o_n(1).
\end{aligned}$$

Arguing as [15], we can prove that

$$\lim_{\varrho\to 0}\left[\limsup_{n\to\infty}\left|\sum_{i=1}^N \int_\Omega u_n \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i}\frac{\partial \psi_\varrho}{\partial x_i} \, dx\right| + \left|\sum_{i=1}^N \int_\Omega u_n \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i-2} \frac{\partial u_n}{\partial x_i}\frac{\partial \psi_\varrho}{\partial x_i} \, dx\right|\right] = 0.$$

Moreover, since $u_n \to u$ in $L^\gamma(\Omega)$, ψ_ϱ has compact support and (4.6), (4.7) and (4.8) hold, if $n \to \infty$ in the above expression then

$$\begin{aligned}
& \textstyle\sum_{i=1}^N \int_\Omega \psi_\rho \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx + \sum_{i=1}^N \int_\Omega \psi_\rho \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx + \sum_{j\in\Lambda} b_j \psi_\rho(x_j) \\
\leq \ & \limsup_{n\to\infty}\left|\sum_{i=1}^N \int_\Omega u_n \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i}\frac{\partial \psi_\varrho}{\partial x_i} \, dx\right| \\
& + \limsup_{n\to\infty}\left|\sum_{i=1}^N \int_\Omega u_n \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i-2} \frac{\partial u_n}{\partial x_i}\frac{\partial \psi_\varrho}{\partial x_i} \, dx\right| \\
& \textstyle + \lambda \int_\Omega |u|^\gamma \psi_\rho dx + \int_\Omega \psi_\rho |u|^{q^*} dx + \sum_{j\in\Lambda} a_j \psi_\rho(x_j).
\end{aligned}$$

Letting $\varrho \to 0$ and using the definition of ψ_ρ we conclude that $b_k \leq a_k$. Since $S a_k^{q_N/q^*} \leq b_k$ we have that

$$S^{q^*/(q*-q_N)} \leq a_k. \tag{4.9}$$

Now we shall prove that the above expression cannot occur, and therefore the set Λ is empty. Indeed, arguing by contradiction, let us suppose that the inequality (4.9) holds for some $k \in \Lambda$. Thus, from

$$c \ = \ I(u_n) - \frac{1}{q_N} I'(u_n) u_n + o_n(1),$$

we obtain

$$\left(\frac{1}{q_N}-\frac{1}{q^*}\right)\int_\Omega \psi_\varrho |u_n|^{q^*}\ dx-\lambda\left(\frac{1}{\gamma}-\frac{1}{q_N}\right)\int_\Omega |u_n|^{\gamma}\ dx\le c+o_n(1),$$

because $0\le\psi_\rho\le 1$. Letting $n\to\infty$ and using (4.6) and (4.8), we get

$$\left(\frac{1}{q_N}-\frac{1}{q^*}\right)\int_\Omega \psi_\rho |u|^{q^*}\ dx+\left(\frac{1}{q_N}-\frac{1}{q^*}\right)\sum_{j\in\Lambda} a_j\psi_\rho(x_j)-\lambda\left(\frac{1}{\gamma}-\frac{1}{q_N}\right)\int_\Omega |u|^{\gamma}\ dx\le c.$$

Since Ω is a bounded domain, we can choose ρ such that $\overline{\Omega}\subset\subset supp\psi_\rho$, in this case, $\psi_\rho(x)=1$ for all $x\in\Omega$. Thus,

$$\left(\frac{1}{q_N}-\frac{1}{q^*}\right)\int_\Omega |u|^{q^*}\ dx+\left(\frac{1}{q_N}-\frac{1}{q^*}\right)\sum_{j\in\Lambda} a_j-\lambda\left(\frac{1}{\gamma}-\frac{1}{q_N}\right)\int_\Omega |u|^{\gamma}\ dx\le c.$$

By using (4.9), we obtain

$$\left(\frac{1}{q_N}-\frac{1}{q^*}\right)\int_\Omega |u|^{q^*}\ dx+\left(\frac{1}{q_N}-\frac{1}{q^*}\right)S^{q^*/(q^*-q_N)}-\lambda\left(\frac{1}{\gamma}-\frac{1}{q_N}\right)\int_\Omega |u|^{\gamma}\ dx\le c.$$

By Holder's inequality

$$\begin{aligned}\left(\frac{1}{q_N}-\frac{1}{q^*}\right)\int_\Omega |u|^{q^*}\ dx &+ \left(\frac{1}{q_N}-\frac{1}{q^*}\right)S^{q^*/(q^*-q_N)}\\ &- \lambda\left(\frac{1}{\gamma}-\frac{1}{q_N}\right)|\Omega|^{(q^*-\gamma)/q^*}\left(\int_\Omega |u|^{q^*}\ dx\right)^{\gamma/q^*}\le c.\end{aligned}$$

Let

$$f(t)=\left(\frac{1}{q_N}-\frac{1}{q^*}\right)t^{q^*}-\lambda\left(\frac{1}{\gamma}-\frac{1}{q_N}\right)|\Omega|^{\frac{q^*-\gamma}{q^*}}t^{\gamma}.$$

This function attains its absolute minimum, for $t>0$, at the point

$$t_0=\left[\frac{\gamma\lambda\left(\frac{1}{\gamma}-\frac{1}{q_N}\right)|\Omega|^{\frac{q^*-\gamma}{q^*}}}{q^*\left(\frac{1}{q_N}-\frac{1}{q^*}\right)}\right]^{1/(q^*-\gamma)}.$$

Thus, we conclude that

$$\begin{aligned}&\left(\frac{1}{q_N}-\frac{1}{q^*}\right)S^{q^*/(q^*-q_N)}-\left[\frac{\left(\frac{1}{\gamma}-\frac{1}{q_N}\right)|\Omega|^{\frac{q^*-\gamma}{q^*}}}{\left(\frac{1}{q_N}-\frac{1}{q^*}\right)}\right]^{q^*/(q^*-\gamma)}\\ \times\ &\left[\left(\frac{\gamma}{q^*}\right)^{\gamma/(q^*-\gamma)}-\left(\frac{\gamma}{q^*}\right)^{q^*/(q^*-\gamma)}\right]\left(\frac{1}{q_N}-\frac{1}{q^*}\right)\lambda^{q^*/(q^*-\gamma)}\\ \le\ & c.\end{aligned}$$

But this is a contradiction. Thus Λ is empty and it follows that $u_n \to u$ in $L^{q^*}(\Omega)$. Arguing as in the proof of Lemma 3.2, we find

$$\|u_n - u\| = o_n(1).$$

□

By the Lemma 4.2 we conclude, for $\lambda > 0$ sufficiently small, that

$$\left(\frac{1}{q_N} - \frac{1}{q^*}\right) S^{q^*/(q^*-q_N)} - \left[\frac{\left(\frac{1}{\gamma} - \frac{1}{q_N}\right)|\Omega|^{\frac{q^*-\gamma}{q^*}}}{\left(\frac{1}{q_N} - \frac{1}{q^*}\right)}\right]^{q^*/(q^*-\gamma)}$$
$$\times \left[\left(\frac{\gamma}{q^*}\right)^{\gamma/(q^*-\gamma)} - \left(\frac{\gamma}{q^*}\right)^{q^*/(q^*-\gamma)}\right]\left(\frac{1}{q_N} - \frac{1}{q^*}\right)\lambda^{q^*/(q^*-\gamma)} > 0$$

and, hence, if (u_n) is a sequence bounded such that $I(u_n) \to c$, $I'(u_n) \to 0$ with $c < 0$, then (u_n) has a subsequence convergent.

Lemma 4.3. *If $J(u) < 0$, then $\|u\| < R_0(\lambda)$, and $J(v) = I(v)$, for all v in a small enough neighborhood of u. Moreover, J verifies a local Palais-Smale condition for $c < 0$.*

Proof. Since $\lambda \in (0, \lambda^*)$ then $J(u) \geq 0$ whenever $\|u\| \geq 1$. Hence, if $J(u) < 0$ we have $\|u\| < 1$ and consequently $\overline{g}(\|u\|) \leq J(u) < 0$, with $\overline{g}$ defined in (4.4). Therefore, $\|u\| < R_0(\lambda)$ and $J(u) = I(u)$. Moreover, we conclude that $J(v) = I(v)$, for all $\|v - u\| < R_0(\lambda) - \|u\|$. Moreover, if (u_n) is a sequence such that $J(u_n) \to c < 0$ and $J'(u_n) \to 0$, for n sufficiently large, $I(u_n) = J(u_n) \to c < 0$ and $I'(u_n) = J'(u_n) \to 0$. Since J is coercive, we get that (u_n) is bounded in X. From Lemma 4.2, for λ sufficiently small,

$$\begin{aligned} c \; < \; & \left(\frac{1}{q_N} - \frac{1}{q^*}\right) S^{q^*/(q^*-q_N)} - \left[\frac{\left(\frac{1}{\gamma} - \frac{1}{q_N}\right)|\Omega|^{\frac{q^*-\gamma}{q^*}}}{\left(\frac{1}{q_N} - \frac{1}{q^*}\right)}\right]^{q^*/(q^*-\gamma)} \\ & \times \left[\left(\frac{\gamma}{q^*}\right)^{\gamma/(q^*-\gamma)} - \left(\frac{\gamma}{q^*}\right)^{q^*/(q^*-\gamma)}\right]\left(\frac{1}{q_N} - \frac{1}{q^*}\right)\lambda^{q^*/(q^*-\gamma)} \end{aligned}$$

and, hence, up to a subsequence, (u_n) is strongly convergent in X. □

Now, we will construct an appropriate mini-max sequence of negative critical values for the functional J. Thus, for each real number ϵ, we consider the set

$$J^{-\epsilon} = \{u \in X : J(u) \leq -\epsilon\} \in \mathfrak{A}.$$

Lemma 4.4. *Given $k \in \mathbb{N}$, there exists $\epsilon = \epsilon(k) > 0$ such that*

$$\gamma(J^{-\epsilon}) \geq k.$$

Proof. Given $k \in \mathbb{N}$, consider a k-dimensional subspace

$$X_k = span\{e_1, \cdots, e_k\}$$

of X, such that

$$-C(k)\|u\|^\gamma \geq -\int_\Omega |u|^\gamma dx,$$

for all $u \in X_k$. Thus,

$$J(u) \leq \sum_{i=1}^{N} \frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|_{p_i}^{p_i} + \sum_{i=1}^{N} \frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|_{q_i}^{q_i} - \lambda\frac{1}{\gamma}C(k)\|u\|^\gamma.$$

By a similar argument of Theorem 3.3, there exists $R \in (0, 1)$ small enough, such that defining $K = \{u \in X_k : \|u\| = R\}$, we get

$$J(u) \leq \sup_{u \in K} J(u) = -\epsilon < J(0) = 0, \ \forall \ u \in K,$$

for some $\epsilon > 0$. Since $\gamma(K) = k$ and $K \subset J^{-\epsilon}$, it follows from (i) in the Proposition 2.5, that $\gamma(J^{-\epsilon}) \geq k$. □

For each $k \in \mathbb{N}$, one can define the sets

$$\Gamma_k = \{C \subset X : C \in \mathfrak{A} \ \text{ and } \ \gamma(C) \geq k\},$$

$$K_c = \{u \in X : J'(u) = 0 \ \text{ and } \ J(u) = c\}$$

and the number

$$c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J(u).$$

Lemma 4.5. *Given $k \in \mathbb{N}$, the number c_k is negative.*

Proof. It is sufficient to use Lemma 4.4 and to argument as in [15, Lemma 4.4]. □

The next Lemma is necessary for the existence of critical points of J. The proof is very similar to that in [15], we omit it here.

Lemma 4.6. *If $c = c_k = c_{k+1} = \cdots = c_{k+r}$ for some $r \in \mathbb{N}$, then there exists $\lambda^* > 0$ such that*

$$\gamma(K_c) \geq r + 1,$$

for $\lambda \in (0, \lambda^*)$.

Finally, we can prove the existence of infinitely many solutions of problem (1.2).

Theorem 4.7. *Assume that $q \in (1, p_1)$. Then, there exists $\lambda^* > 0$ such that problem (1.2) has infinitely many solutions, for all $\lambda \in (0, \lambda^*)$.*

Proof. If $-\infty < c_1 < c_2 < \cdots < c_k < \cdots < 0$ with $c_i \neq c_j$, since each c_k is critical value of J, the we obtain infinitely many critical points of J and, hence problem (1.2) has infinitely many solutions.
On the other hand, if there are two constants $c_k = c_{k+r}$, then $c = c_k = c_{k+1} = \cdots = c_{k+r}$ and from Lemma 4.6, there exists $\lambda^* > 0$ such that

$$\gamma(K_c) \geq r + 1 \geq 2$$

for all $\lambda \in (0, \lambda^*)$. From Proposition 2.5, K_c has infinitely many points, that is, problem (1.2) has infinitely many solutions. □

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