Blow-up Criterion for Compressible Navier-Stokes Equations with Degenerate Viscosities

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Abstract

The compressible Navier-Stokes system with degenerate viscosities was little known in many physical and mathematical considerations. Such a degeneracy is a highly challenging obstacle for development of well-posedness theory. In this paper, We establish the blow-up criterion of classical solutions to the Cauchy problem for the compressible Navier-Stokes equations, as well as Shallow water equations, with degenerate viscosities. Moreover, we prove that the maximum norm of the gradients of velocity controls the possible breakdown of regular solutions for compressible Navier-Stokes equations with degenerate viscosities.

Keyword: Compressible Navier-Stokes equations, Shallow Water equations, Blow-up criterion, Degenerate Viscosities, Regular solutions.

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1 Introduction

The time evolution of the density and the velocity of a general viscous isentropic compressible fluid occupying a domain $\Omega \subset \mathrm{R}^{\mathrm{N}}$ is governed by the compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \quad (t,x) \in (0,\infty)\times\mathbb{R}^N, \\ (\rho u)_t + \mathrm{div}(\rho u\otimes u) + \nabla P = \mathrm{div}\mathbb{T}, \quad (t,x)\in(0,\infty)\times\mathbb{R}^N, \\ (\rho,u)\mid_{t=0} = (\rho_0(x), u_0(x)), \qquad x\in\mathbb{R}^N, \\ (\rho,u)\to(0,0) \ \ as \ \ |x|\to\infty, \qquad t>0. \end{cases} \tag{1.1}$$

where ρ is the density, and $u = (u^1,\cdots,u^N)^\top \in \mathbb{R}^N (N=2,3)$, is the velocity of the fluid. We assume that the pressure P satisfies

$$P = A\rho^\gamma, \quad \gamma > 1, \tag{1.2}$$

where A is a positive constant, γ is the adiabatic exponent. $\mathbb{T}$ denotes the viscosity stress tensor with the following form

$$\mathbb{T} = \mu(\rho)\mathcal{D}(u) + \lambda(\rho)\mathrm{div}u\mathbb{I}_2. \tag{1.3}$$

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Here $\mathbb{I}_2$ is the $N \times N$ identity matrix, $\mu(\rho) = \alpha\rho$ is the shear viscosity coefficient, and $\lambda(\rho) = \beta\rho$ is the bulk viscosity coefficient, α and β are both constants satisfying

$$\alpha > 0, \quad \alpha + \beta \geq 0. \tag{1.4}$$

We will establish the corresponding results on (1.1) - (1.4), then reveal the applications to various shallow water models.

There are huge amounts of literature on the large time existence and behavior of solutions to the system (1.1). When the viscosity coefficients were assumed to be fixed positive numbers, Kazhikhov and Shelukhin [30] established the first existence result on the compressible Navier-Stokes equations in one dimensional space. Due to the difficulty from the vacuum, the initial density should be bounded away from zero in their work. It has been extended by [37] and Hoff [26] for the discontinuous initial data. For the multidimensional case, Matsumura and Nishida [36] first established the global existence with the small initial data. To remove the difficulty from the vacuum, Lions in [34] introduced the concept of renormalized solutions to establish the global existence of weak solutions for $\gamma > \frac{9}{5}$ concerning large initial data that may vanish, and then Feireisl et al.[20] and Feireisl [18] extended the existence result to $\gamma > \frac{3}{2}$.

The problem becomes even more challenging when the viscosity coefficients depend on the density. Indeed, the system (1.1) is highly degenerated at the vacuum because when the density vanishes, the velocity cannot even be defined. The vacuum makes estimates of the gradient on the velocity field very difficult. To deal with this difficulty, Bresch [4] developed a new mathematical entropy for $\lambda(\rho)$ and $\mu(\rho)$ satisfying $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$, see also in [1], [3] and [6] for the case with an additional quadratic friction term $r\rho|u|u$. This new entropy offers a nice estimate $\mu'(\rho)\frac{\nabla\rho}{\sqrt{\rho}} \in L^\infty([0,T]; L^2(R^2))$ provided that $\mu'(\rho_0)\frac{\nabla\rho_0}{\sqrt{\rho_0}} \in L^2(R^2)$. For spherically symmetric case, see [17, 16, 24]. For our problem, such an entropy exists when $\beta = 0$. Unfortunately, BD-entropy does not apply to many interesting models, including the one in (1.11) and those bounds are not enough to treat the compressible Navier-Stokes equations without additional control on the vacuum. In fact, we cannot expect too much regularity results of Lions's weak solutions since the result of Xin[40], who proved that there is no global smooth solution to the Cauchy problem if the initial density with compact support. So more and more people are beginning to investigate the mechanism for possible breakdown of smooth solutions.

There are several works [13, 19, 28] trying to establish blowup criteria for the strong (smooth) solutions to the barotropic compressible Navier-Stokes equations (1.3). When the viscosity coefficients were assumed to be fixed positive numbers, In particular, it is proved in [19] for three dimensions, if $7\mu > 9\lambda$, then

$$\lim_{T\to\bar{T}}\left(\sup_{0\le t\le T}\|\rho\|_{L^\infty} + \int_0^T (\|\rho\|_{W^{1,q_0}} + \|\nabla\rho\|_{L^2}^4)dt\right) = +\infty,$$

where $\bar{T} < \infty$ is the maximal time of existence of a strong solution and $q_0 > 3$ is a constant. Recently, Huang and Xin [27] succeeded in removing the crucial condition above and established the blowup criterion

$$\lim_{T\to T^*}\int_0^T \|\mathcal{D}(u)\|_{L^\infty}dt = \infty.$$

When the viscosity coefficients depend on the density, several blowup criteria are given to the local-in-time strong solution in [10]. Recently, Xin Zhong [39] proved the strong solution to the problem exists globally if the gradient of velocity satisfies $\|\nabla u\|_{L^2(0,T;L^\infty)} < \infty$. However, all the results mentioned above on the blow-up of the solutions of compressible flows are for viscosities with a uniformly positive lower bound, i.e., both $\mu(\rho) \geq \underline{\mu}$ and $\lambda(\rho) \geq \underline{\lambda}$, where $\underline{\lambda}, \underline{\mu}$ are positive constants. Therefore, a natural question arises: when viscosities without a uniformly positive lower bound, how we can describe the mechanism of blow up to ensure the global existence of strong (or classical) solutions to the system (1.1)? So, the aim of this paper is to establish the blowup criterion results for the system (1.1) and shallow water equations with degenerate viscosities.

Before stating the main result, we explain the notations and conventions used throughout this paper. We denote

$$\int f dx = \int_{R^N} f dx$$

For $1 \leq r \leq \infty$, we denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$L^r = L^r(\mathrm{R}^N), \qquad D^{k,r} = \{u \in L^1_{loc}(\mathrm{R}^{\mathrm{N}}) \big| \, \|\nabla^k u\|_{L^r} < \infty\},$$
$$\|u\|_{D^{k,r}} \triangleq \|\nabla^k u\|_{L^r}, \quad W^{k,r} = L^r \cap D^{k,r}, \quad H^k = W^{k,2}, \quad D^k = D^{k,2}.$$

A detailed study of homogeneous Sobolev space can be found in [21].

By introducing a proper notion of solution class, Li et.al.[33] proved the local-in-time well-posedness of this class of smooth solutions and established a Beale-Kato-Majda type blow-up criterion[29]. The definition of the solution class and main result are as follows:

Definition 1.1. (Regular solution to Cauchy problem (1.1). Let $T > 0$ be a finite constant. A solution (ρ, u) to Cauchy problem (1.1) is called a regular solution in $[0, T] \times \mathbb{R}^2$ if (ρ, u) satisfies

$$
\begin{aligned}
&(A) \quad \rho > 0, \rho \in C^1([0,T] \times \mathbb{R}^2), \rho^{\frac{\gamma-1}{2}} \in C([0,T]; H^3), (\rho^{\frac{\gamma-1}{2}})_t \in C([0,T]; H^2);\\
&(B) \quad \nabla\rho/\rho \in C([0,T]; L^6 \cap D^1 \cap D^2), (\nabla\rho/\rho)_t \in C([0,T]; H^1);\\
&(C) \quad u \in C([0,T]; H^3) \cap L^2([0,T]; H^4), u_t \in C([0,T]; H^1) \cap L^2([0,T]; D^2)\\
&(D) \quad \lim_{|x|\to\infty} (u_t + u \cdot \nabla u + Lu) = \lim_{|x|\to\infty} \Big((\nabla\rho/\rho) \cdot Q(u) \Big), \quad for \quad t > 0.
\end{aligned}
$$

Theorem 1.1. *If the initial data (ρ_0, u_0) satisfy the regularity conditions*

$$\rho_0 > 0, \ \ (\rho_0^{\frac{\gamma-1}{2}}, u_0) \in H^3, \ \ \nabla\rho_0/\rho_0 \in L^6 \cap D^1 \cap D^2. \tag{1.5}$$

Then there exists a time $T_* > 0$ and a unique regular solution (ρ, u) to be Cauchy problem 1.1, satisfying

$$
\begin{gathered}
\rho^{\frac{\gamma-1}{2}} \in C([0,T_*]; H^3), \quad (\rho^{\frac{\gamma-1}{2}})_t \in C([0,T_*]; H^2),\\
\nabla\rho/\rho \in C([0,T_*]; L^6 \cap D^1 \cap D^2), \quad (\nabla\rho/\rho)_t \in C([0,T_*]; H^1),\\
u \in C([0,T_*]; H^3) \cap L^2([0,T_*]; H^4), \quad u_t \in C([0,T_*]; H^1) \cap L^2([0,T]; D^2),\\
u_{tt} \in L^2([0,T_*]; L^2), \quad t^{\frac{1}{2}} u \in L^\infty([0,T_*]; D^4),\\
t^{\frac{1}{2}} u_t \in L^\infty([0,T_*]; D^2) \cap L^2([0,T_*]; D^3), \quad t^{\frac{1}{2}} u_{tt} \in L^\infty([0,T_*]; L^2) \cap L^2([0,T_*]; D^1).
\end{gathered}
\tag{1.6}
$$

Moreover, if $1 < \gamma \leq 3$, then $\rho(x,t) \in C^1([0,T_*] \times \mathbb{R}^2)$. If $\bar{T} < \infty$ is the maximal existence time, then we have both

$$\lim_{T\to\bar{T}} \left(\sup_{0\leq t\leq T} \left\| \frac{\nabla\rho}{\rho}(\cdot,t) \right\|_{L^6(\mathbb{R}^2)} + \int_0^T \|\mathcal{D}(u)(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} dt \right) = +\infty \tag{1.7}$$

and

$$\limsup_{T\to\bar{T}} \int_0^T \|\mathcal{D}(u)(\cdot,t)\|_{L^\infty\cap D^{1,6}(\mathbb{R}^2)} dt = +\infty. \tag{1.8}$$

Our main result can be stated as follows:

Theorem 1.2. *Let (ρ, u) be a regular solution of the Cauchy problem* (1.1) *on* $[0,T]\times\mathbb{R}^N$. *Assume that the initial data* (ρ_0, u_0) *satisfying*

$$\rho_0 > 0,\ \ (\rho_0^{\frac{\gamma-1}{2}}, u_0) \in H^3,\ \ \nabla\rho_0/\rho_0 \in L^2\cap D^1\cap D^2.$$

If $\bar{T} < \infty$ *is the maximal existence time, then we have*

$$\limsup_{T\to\bar{T}} \int_0^T \|\nabla u(\cdot,t)\|^2_{L^\infty(\mathbb{R}^N)} dt = +\infty, \tag{1.9}$$

where $N = 2, 3$.

We now comment on the analysis of this paper. Comparing the blow-up results established in [33], the main work in this paper is getting the estimate of $\|\nabla\rho/\rho\|_{L^\infty(0,T;L^6)}$. Besides the high degeneracy in momentum equations, viscosity coefficients vanish as the density allows vacuum which preventing us from using a similar method proposed by [10, 33, 39] et al. Form the observation of the usual energy estimate and the mathematical entropy, as well as careful computation, we can see that the key step in proving Theorem 1.2 is to derive the $L^\infty(0,T;L^2)$-estimate on $\nabla\rho/\rho$ which is essentially equivalent to $\nabla\rho$ when the initial density has a uniformly positive lower bound. When $\inf\rho_0 = 0$, some of the new difficulties are arise due to the appearance of vacuum. In order to overcome these difficulties, we will study from a mathematical view point. Inspired by Bresch [4] and Guo [25], we found that the boundedness of the temporal integral of the super-norm in space of the deformation tensor guarantee estimate on $\nabla\rho/\rho$ and the velocity u. In the process of the proof, we can see that the estimates on gradient of the velocity u can be derived from $\|\nabla\rho/\rho\|_{L^\infty(0,T;L^2)}$, see Lemma 3.3 and Lemma 3.4. So, the estimate on $\|\nabla\rho/\rho\|_{L^\infty(0,T;L^6)}$ was also obtained, see Lemma 3.5. Thus a estimate by using the effective stress tensor will lead to a prior estimates on the $L^\infty(0,T;L^6)$ estimate on $\nabla\rho/\rho$. The method we use is motivated by [2]. The detail of the proof of Theorem 1.2 is given in Sect. 3.

Remark 1.1. *For $N = 2$ and $\mu(\rho) = \lambda(\rho) = \rho$ in (1.1), the system is called the shallow water equations, see also [34]. In fact, the shallow water systems, can also be modeled by another way. In lakes and sea, the depth of the water is much smaller than the two other dimensions. So the flows which govern the currents can be tentatively averaged to obtain a simpler two-dimensional set of equations; these are known as Saint-Venant's shallow water equations. The Cauchy problem of shallow water equations reads as:*

$$\begin{cases} h_t + \mathrm{div}(hU) = 0, \\ (hU)_t + \mathrm{div}(hU\otimes U) + \nabla h^2 = \nu(h,U), \\ (h,U)\,|_{t=0} = (h_0(x), U_0(x)), & x\in\mathbb{R}^2, \\ (h,U)\to(0,0)\ \ as|x|\to\infty, & t>0. \end{cases} \tag{1.10}$$

Here h, $U = (U^1, U^2)^\top \in \mathbb{R}^2$ and $\nu(h,U)$ denote the height of the free surface, the mean horizontal velocity of the fluid and the viscous term, respectively. There are various models (or approximations) which are possible for viscous terms like for instance $\mathrm{div}(h\mathcal{D}(U)), h\Delta U, \mathrm{div}(h\nabla U), \Delta(hU)$ *(see [5, 34]), where $\mathcal{D}(U) = \frac{1}{2}(\nabla U + \nabla^\top U)$. Particularly, the case for $\nu = \mathrm{div}(h\nabla D(U))$ is corresponding to the well-known viscous Saint-Venant model [23] and later, Gent [22] suggests $\nu = \mathrm{div}(h\mathcal{D}(U))$. Recently, by a more trivial computation, Marche [35] and Bresch-Noble[7, 8] suggest that*

$$\nu(h,U) = \mathrm{div}(2h\mathcal{D}(U) + 2h\mathrm{div}U\mathbb{I}_2). \tag{1.11}$$

The shallow water equation (1.10) *turns to isentropic compressible Navier-Stokes equations with viscosities depending on density when replaces h with ρ. Clearly, when $\gamma = 2$,* (1.1) - (1.4) *include most of shallow water models mentioned above.*

Remark 1.2. *For $N = 3$, the local-in-time existence of this class of smooth solutions for Cauchy problem (1.1) can also be established by a similar way in Li et.al.[33] which we omit here for simplicity.*

Remark 1.3. *For $\mu(\rho) = \rho^{\alpha}, \lambda(\rho) = \rho^{\beta}$ in system (1.1), the blow-up result* (1.9) *are also hold for Cauchy problem (1.1) which can also be established by a similar way in this paper, maybe some proper change also needed.*

2 Preliminaries

In this section, we present some known facts and importent inequalities we will use in our proof. The first one is the following well-known Gagliardo-Nirenberg inequality which will be used later frequently(see [31]).

Lemma 2.1. *Let $r \in (1, +\infty)$ and $h \in W^{1,p}(\mathbb{R}^2) \cap L^r(\mathbb{R}^2)$. Then*

$$\|h\|_{L^q(\mathbb{R}^2)} \leq C\|\nabla h\|^{\theta}_{L^p(\mathbb{R}^2)}\|h\|^{1-\theta}_{L^r(\mathbb{R}^2)},$$

where $\theta = \Big(\frac{1}{r} - \frac{1}{q}\Big)\Big(\frac{1}{r} - \frac{1}{p} + \frac{1}{2}\Big)^{-1}$. If $p < 2$, then $q \in [r, \frac{2p}{2-p}]$ when $r < \frac{2p}{2-p}$; and $q \in [\frac{2p}{2-p}, r]$ when $r > \frac{2p}{2-p}$. If $p = 2$, then $q \in [r, \infty)$. If $p > 2$, then $q \in [r, \infty)$.

Some common versions of this inequality can be written as

$$\begin{aligned} &\|f\|_{L^3(\mathbb{R}^2)} \leq C\|f\|^{\frac{2}{3}}_{L^2(\mathbb{R}^2)}\|\nabla f\|^{\frac{1}{3}}_{L^2(\mathbb{R}^2)}, \qquad \|f\|_{L^6(\mathbb{R}^2)} \leq C\|f\|^{\frac{1}{3}}_{L^2(\mathbb{R}^2)}\|\nabla f\|^{\frac{2}{3}}_{L^2(\mathbb{R}^2)}, \\ &\|f\|_{L^\infty(\mathbb{R}^2)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^2)}, \qquad\qquad \|f\|_{L^\infty(\mathbb{R}^2)} \leq C\|f\|^{\frac{1}{2}}_{L^6(\mathbb{R}^2)}\|\nabla f\|^{\frac{1}{2}}_{L^3(\mathbb{R}^2)}. \end{aligned} \tag{2.1}$$

which may be used in our following proof.

Lemma 2.2. *For $p \in [2, 6], q \in (1, \infty)$, and $r \in (3, \infty)$, there exists some generic constant $C > 0$ which may depend on q, r such that for $f \in H^1(\mathrm{R}^3)$, we have*

$$\|f\|^p_{L^p(\mathbb{R}^3)} \leq C\|f\|^{(6-p)/2}_{L^2(\mathbb{R}^3)}\|\nabla f\|^{(3p-6)/2}_{L^2(\mathbb{R}^3)},$$

Next, the following well-known Gronwall's inequality plays an important role in proving our estimates on regular solutions.

Lemma 2.3. *Suppose that h and r are integrable on (a, b) and nonnegative a.e. in (a, b). Further assume that $y \in C[a, b]$, $y' \in L^1(a, b)$, and*

$$y'(t) \leq h(t) + r(t)y(t) \quad for \quad a.e. \quad t \in (a, b).$$

Then

$$y(t) \leq \Big[y(a) + \int_a^t h(s)\exp\Big(-\int_a^s r(\tau)ds\Big)\Big]\exp\Big(\int_a^t r(s)ds\Big). \quad t \in [a, b]$$

For the proof of this lemma, we refer [38], pp. $[12 - 13]$, to readers.

Due to harmonic analysis, we have the following regularity estimate result for Lamó operator. For problem

$$-\alpha\Delta u - (\alpha + \beta)\nabla \mathrm{div} u = Lu = F, \quad u \to 0 \ \ as \ \ |x| \to \infty, \tag{2.2}$$

we have

Lemma 2.4. *If $u \in D^{1,q}(\mathbb{R}^2)$ with $1 < q < +\infty$ is a weak solution to problem* (2.2)*, then*

$$\|u\|_{D^{k+2,q}} \leq C\|F\|_{D^{k,q}},$$

where C depends on α, β and q.

The proof can be obtained via the classical estimates from harmonic analysis.

3 Proof of Theorem1.2

Let (ρ, u) be the unique regular solution to the Cauchy problem (1.1) with the maximal existence time $\bar{T}$. To prove Theorem 1.2, we assume otherwise that

$$\lim_{T\to\bar{T}} \int_0^T \|\nabla u(\cdot,t)\|^2_{L^\infty(\mathbb{R}^2)} dt = C_0 < +\infty. \tag{3.1}$$

From the definition of regular solutions, we know that, for $\phi = \rho^{\frac{\gamma-1}{2}}$, (ϕ, u) satisfies

$$\begin{cases} \phi_t + u\cdot\nabla\phi + \frac{\gamma-1}{2}\phi\mathrm{div}u = 0, \\ (\nabla\ln\rho)_t + \nabla(u\cdot(\nabla\ln\rho)) + \nabla\mathrm{div}u = 0, \\ u_t + u\cdot\nabla u + 2\theta\phi\nabla\phi + Lu = \nabla\rho/\rho\cdot Q(u). \end{cases} \tag{3.2}$$

where $\theta = \frac{a\gamma}{\gamma-1}$, $Q(u) = \nabla u + \mathrm{div}u\mathbb{I}_N, N = 2, 3$.

Without loss of generality, we will command $\alpha = \beta = 1$ in (1.3) and (1.4) in our calculations. By assumptions (3.1) and (3.2), we first show that density ρ is uniformly bounded. The proof presented as following can be seen from [12] and [33]. For the convenience of readers we will give here.

Lemma 3.1. *Let (ρ, u) be the unique regular solution to the Cauchy problem* (1.1) *on* $[0, T)$ *satisfying* (3.2)*. Then*

$$\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^n)} + \|\phi\|_{L^\infty([0,T];L^q(\mathbb{R}^n)} \le C, \quad 0 \le T \le \bar{T}, \tag{3.3}$$

where $C > 0$ depends on C_0, constant $q \in [2, \infty]$ and T.

Proof. First, it is obvious that ϕ can be represented by

$$\phi(t,x) = \phi_0(W(0,t,x)) + \exp\Big(-\frac{\gamma-1}{2}\int_0^t \mathrm{div}(s,(W(s,t,x))ds\Big), \tag{3.4}$$

where $W \in C^1([0,T]\times[0,T]\times\mathbb{R}^n))$ is the solution to the initial value problem

$$\begin{cases} \frac{d}{dt}W(t,s,x) = u(t, W(t,s,x)), \quad 0 \le T \le \bar{T}, \\ W(s,s,x) = x, \quad 0 \le s \le T, x \in \mathbb{R}^2. \end{cases} \tag{3.5}$$

Then it is clear that $\|\phi\|_{L^\infty}([0,T]\times\mathbb{R}^n) \le |\phi_0|\exp(CC_0)$.

Next, multiplying $(3.2)_1$ by 2ϕ and integrating over $\mathbb{R}^n$, we get

$$\frac{d}{dt}\|\phi\|^2_{L^2} \le C\|\mathrm{div}u\|_\infty\|\phi\|^2_{L^2}. \tag{3.6}$$

From (1.7) and Gronwall's inequality, we immediately obtain the desired conclusions. □

The key estimates on $\nabla\ln\rho$ and ∇u will be given in the following lemma.

Lemma 3.2. *Let (ρ, u) be the unique regular solution to the Cauchy problem* (1.1) *on* $[0, T)$ *satisfying 3.1. Then*

$$\sup_{0\le t\le T}\Big(\|\nabla\ln\rho\|^2_{L^2} + \|u\|^2_{L^2}\Big) + \int_0^T \|\nabla u\|^2_{L^2}dt + \int_0^T \|\nabla\phi\|^2_{L^2}dt \le C, \quad 0 \le T \le \bar{T}. \tag{3.7}$$

where C only depends on C_0 and $\bar{T}$.

Proof. Firstly, from $(1.1)_1$, we get

$$\begin{aligned}
&\frac{d}{dt}\int |\ln\rho| dx \\
&= -2\int \mathrm{div}u \ln\rho dx + \int \mathrm{div}u(\ln\rho)^2 dx \\
&\leq C(|\mathrm{div}(u)|_{L^\infty}^2 + 1)\int |\ln\rho|^2 dx + \frac{\alpha}{4}\int |\nabla u|^2 dx,
\end{aligned} \tag{3.8}$$

and multiply $(3.2)_1$ by $\nabla\ln\rho$, integrate over $\mathbb{R}^N$, yields

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla\ln\rho| dx + 2\int \nabla(\mathrm{div}u)\nabla\ln\rho dx \\
&= -2\int \partial_j \ln\rho\partial_j u_i\partial_i \ln\rho dx + \int u\cdot\nabla|\nabla\ln\rho|^2 dx \\
&\leq C\|\mathcal{D}(u)\|_{L^\infty}\int |\nabla\ln\rho|^2 dx,
\end{aligned} \tag{3.9}$$

where we have use the fact

$$\int \partial_j \ln\rho\partial_j u_i\partial_i \ln\rho dx = \int \mathcal{D}(u) : \nabla\ln\rho\otimes\nabla\ln\rho dx.$$

Secondly, multiplying $(1.1)_2$ by u and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |u|^2 dx + \int |\nabla u|^2 dx + 2\int |\mathrm{div}u|^2 dx \\
&= \int(-u\cdot\nabla u\cdot u - \theta\nabla\phi^2\cdot u + \nabla\ln\rho\cdot Q(u)\cdot u)dx \\
&\equiv: L_1 + L_2 + L_3.
\end{aligned}$$

The right-hand side terms can be estimated as follows

$$L_1 = -\int u\cdot\nabla u\cdot u dx \leq C\|\mathrm{div}u\|_{L^\infty}\|u\|_{L^2}^2,$$

$$L_2 = \int \theta\nabla\phi^2\cdot u dx \leq C\|\phi_2\|^2\|\mathrm{div}u\|_{L^\infty} \leq C\|\mathrm{div}u\|_{L^\infty},$$

$$\begin{aligned}
L_3 = \int \psi\cdot Q(u)\cdot u dx &\leq (\|\nabla\ln\rho\|_{L^2}^2 + \|u\|_{L^2}^2)|Q(u)|_\infty \\
&\leq \|\mathcal{D}(u)\|_{L^\infty}(\|u\|_{L^2}^2 + \|\nabla\ln\rho\|_{L^2}^2).
\end{aligned}$$

so we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|L^{2^2} \\
&\leq C(\|\mathcal{D}(u)\|_{L^\infty} + 1)(\|u\|_{L^2}^2 + \|\nabla\ln\rho\|_{L^2}^2) + C\|\mathrm{div}u\|_{L^\infty}.
\end{aligned} \tag{3.10}$$

Thirdly, multiplying $(1.1)_2$ by $\frac{\nabla\ln\rho}{\rho}$, integrating over $\mathbb{R}^N$ and using Sobolev inequality, we have

$$\int \rho(u_t + u\cdot\nabla u)\frac{\nabla\ln\rho}{\rho}dx + \int \nabla P\cdot\frac{\nabla\ln\rho}{\rho}dx - \int \mathrm{div}(\rho\mathcal{D}(u))\frac{\nabla\ln\rho}{\rho}dx - \int \nabla(\rho\mathrm{div}u)\frac{\nabla\ln\rho}{\rho}dx = 0$$

which denoted by $\sum_{i=4}^{7} L_i = 0$. And we have the fact that

$$
\begin{aligned}
L_4 &= \int \rho(u_t + u \cdot \nabla u)\frac{\nabla \ln \rho}{\rho} dx \\
&= \int (u_t \nabla \ln \rho + u \cdot \nabla u \cdot \nabla \ln \rho) dx \\
&= \frac{d}{dt}\int u \nabla \ln \rho dx - \int u (\nabla \ln \rho)_t dx - \int \partial_i (u_j \partial_j u_i) \ln \rho dx \\
&= \frac{d}{dt}\int u \nabla \ln \rho dx + \int \mathrm{div} u \frac{\mathrm{div}(\rho u)}{\rho} dx - \int \partial_i u_j \partial_j u_i \ln \rho - \int u_j \partial_i \partial_j u_i \ln \rho dx \\
&= \frac{d}{dt}\int u \nabla \ln \rho dx + \int (\mathrm{div} u)^2 dx - \int \partial_i u_j \partial_j u_i ln\rho + \int (\mathrm{div} u)^2 \ln \rho + u_j \mathrm{div} u \partial_j \ln \rho dx,
\end{aligned}
$$

$$
L_5 = \int \nabla P \cdot \frac{\nabla \ln \rho}{\rho} dx = \frac{4\theta}{(\gamma - 1)} \int |\nabla(\rho^{\frac{\gamma-1}{2}})|^2 dx,
$$

$$
\begin{aligned}
L_6 &= -\int \mathrm{div}(\rho \mathcal{D}(u)) \frac{\nabla \ln \rho}{\rho} dx = \frac{1}{2}\int \rho(\partial_j u_i + \partial_i u_j)\partial_j \Big(\frac{\partial_i \ln \rho}{\rho}\Big) \\
&= \frac{1}{2}\int (\partial_j u_i + \partial_i u_j)\partial_j \partial_i \ln \rho dx - \frac{1}{2}\int (\partial_j u_i + \partial_i u_j)\partial_i \ln \rho \partial_j \ln \rho dx \\
&= \frac{1}{2}\int (\partial_j u_i \partial_i \partial_j \ln \rho + \partial_i u_j \partial_j \partial_i \ln \rho) dx - \frac{1}{2}\int (\partial_j u_i + \partial_i u_j)\partial_i \ln \rho \partial_j \ln \rho dx \\
&= -\int \nabla \mathrm{div} u \nabla \ln \rho dx - \frac{1}{2}\int (\partial_j u_i + \partial_i u_j)\partial_i \ln \rho \partial_j \ln \rho dx,
\end{aligned}
$$

and

$$
\begin{aligned}
L_7 &= -\int \nabla(\rho \mathrm{div} u)\frac{\nabla \ln \rho}{\rho} dx = \int \rho \partial_i u_i \partial_j \Big(\frac{\partial_j \ln \rho}{\rho}\Big) dx \\
&= \int \partial_i u_i \partial_j u_j \ln \rho dx - \int \partial_i u_i \partial_j \ln \rho \partial_j \ln \rho dx \\
&= -\int \nabla \mathrm{div} u \cdot \nabla \ln \rho dx - \int \mathrm{div} u |\nabla \ln \rho|^2 dx.
\end{aligned}
$$

So those equations above and using Young's inequality, Sobolev inequality yield

$$
\begin{aligned}
&\frac{d}{dt}\int u \nabla \ln \rho dx + \int (\mathrm{div} u)^2 + \frac{4\theta}{(\gamma - 1)}\int |\nabla(\rho^{\frac{\gamma-1}{2}})|^2 dx - 2\int \nabla \mathrm{div} u \cdot \nabla \ln \rho dx \\
&= -2\int u \mathrm{div} u \nabla \ln \rho dx + \int \partial_i u_j \partial_j u_i \ln \rho dx - \int (\mathrm{div} u)^2 \ln \rho dx \\
&\quad + \int \mathrm{div} u |\nabla \ln \rho|^2 dx + \int \mathcal{D}(u) : \nabla \ln \rho \otimes \nabla \ln \rho dx \\
&\le 2\|\mathrm{div} u\|_\infty \int (|u|^2 + |\nabla \ln \rho|^2) dx + \|\mathcal{D}(u)\|_\infty \int |\nabla \ln \rho|^2 dx \\
&\quad + \frac{1}{4}\|\mathcal{D}(u)\|_\infty \Big(\frac{\alpha}{\|\mathcal{D}(u)\|_\infty}\int |\nabla u|^2 dx + C\|\mathcal{D}(u)\|_\infty \int (\ln \rho)^2 dx\Big) \\
&\le C\|\mathcal{D}u\|_\infty^2 \int \Big((\ln \rho)^2 + |\nabla \ln \rho|^2 + |u|^2\Big) dx + \frac{\alpha}{4}\int |\nabla u|^2 dx
\end{aligned}
\tag{3.11}
$$

Finally, adding (3.8), (3.9), (3.10) and (3.11) yields

$$\begin{aligned}
&\frac{d}{dt}\int(|u+\nabla\ln\rho|^2+(\ln\rho)^2)dx+\int[(\nabla u)^2+(\mathrm{div}u)^2]dx+\frac{4\gamma}{(\gamma-1)^2}\int|\nabla\phi|^2dx\\
&\le C\|\mathcal{D}(u)\|_\infty^2\int\Big((\ln\rho)^2+|\nabla\ln\rho|^2+|u|^2\Big)dx.
\end{aligned}$$

According to Gronwall's inequality, we complete the proof. □

Lemma 3.3. *Let* (ρ,u) *be the unique regular solution to the Cauchy problem* (1.1) *on* $[0,T)$ *satisfying* (1.7)*. Then*

$$\sup_{0\le t\le T}\|\nabla u(t)\|_{L^2}^2+\sup_{0\le t\le T}\|\nabla\phi(t)\|_{L^2}^2+\int_0^T(\|\nabla^2u\|_{L^2}^2+\|\dot{u}\|_{L^2}^2)dt\le C,$$

where, $\dot{u}=u_t+u\cdot\nabla u$*,* C *only depends on* C_0 *and* $\bar{T}$*.*

Proof. Firstly, multiplying $(3.2)_3$ by $-Lu-\theta\nabla\phi^2$ and integrate over $\mathbb{R}^N$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\alpha|\nabla u|_2^2+(\alpha+\beta)|\mathrm{div}u|_2^2\Big)dx+\int(-Lu-\theta\nabla\phi^2)^2dx\\
=\;&-\alpha\int(u\cdot\nabla)\cdot\omega'dx+(\alpha+\beta)\int(u\cdot\nabla)\cdot\nabla\mathrm{div}udx\\
&-\theta\int(u\cdot\nabla)\cdot\nabla\phi^2dx-\theta\int u_t\cdot\nabla\phi^2dx+\theta\int\Big(\nabla\ln\rho\cdot Q(u)\Big)\cdot\nabla\phi^2dx\\
&-\int\Big(\nabla\ln\rho\cdot Q(u)\Big)\cdot(\alpha\Delta u+(\alpha+\beta)\nabla\mathrm{div}u)dx\equiv:\sum_{i=8}^{11}L_i.
\end{aligned}\tag{3.12}$$

where we have used the following fact:

$$-\Delta u+\nabla\mathrm{div}u=(\partial_{x_2}\omega-\partial_{x_1}\omega)^\top=\omega',\omega=u^2_{x_1}-u^1_{x_2},\quad N=2;$$

$$-\Delta u+\nabla\mathrm{div}u=\nabla\times\omega=\omega',\quad\omega=\nabla\times u,\quad N=3.$$

From the standard elliptic estimate, we have

$$\begin{aligned}
&\|\nabla^2u\|_{L^2}^2-C\|\theta\nabla\phi^2\|_{L^2}^2\\
&\le C\|\alpha\Delta u+(\alpha+\beta)\nabla\mathrm{div}u\|_{L^2}^2-C\|\theta\nabla\phi^2\|_{L^2}^2\\
&\le C\|\alpha\Delta u+(\alpha+\beta)\nabla\mathrm{div}u-\theta\nabla\phi^2\|_{L^2}^2.
\end{aligned}\tag{3.13}$$

Now we estimate the right-hand side of (3.12) term by term. According to

$$\frac{1}{2}\nabla(|u|^2)-u\cdot\nabla u=(u^{(2)}\omega,-u^{(1)}\omega)^{top}=\omega'',$$

and Hölder's inequality, Gagliardo-Nirenberg inequality and Young's inequality, we obtain

$$\begin{aligned}
|L_8|&=\alpha\Big|\int(u\cdot\nabla)\cdot\omega'dx\Big|=\alpha\Big|\int(\frac{1}{2}\nabla|u|^2-\omega'')\cdot\omega'dx\Big|\\
&=\alpha\Big|\int-\omega''\cdot\omega'dx\Big|=\frac{\alpha}{2}\Big|\int u^{(2)}\alpha_{x_2}\omega^2+u^{(1)}\alpha_{x_1}\omega^2dx\Big|\\
&=\frac{\alpha}{2}\Big|\int\omega^2\mathrm{div}udx\Big|\le C\|\mathrm{div}u\|_\infty\|\nabla u\|_2^2
\end{aligned}$$

$$
\begin{aligned}
|L_9| &= \theta\Big|\int (u\cdot\nabla)\cdot\nabla\phi^2)^2 dx\Big| \\
&= \theta\Big| -\int \nabla u : (\nabla u)^\top \phi^2 dx - \int \phi^2 u\cdot\nabla \mathrm{div} u dx\Big| \\
&= \theta\Big| -\int \nabla u : (\nabla u)^\top \phi^2 dx + \int (\mathrm{div} u)^2\phi^2 dx + \int u\cdot\nabla\phi^2 \mathrm{div} u dx\Big| \\
&\le C\|\nabla u\|_{L^2}^2 + C\|\mathrm{div} u\|_{L^\infty}\|u\|_2\|\nabla\phi^2\|_{L^2} \\
&\le C(\|\nabla u\|_{L^2}^2 + \|\mathrm{div} u\|_{L^\infty} + \|\mathrm{div} u\|_{L^\infty}\|\nabla\phi^2\|_{L^2}^2),
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
L_{10} &= -\theta\int u_t\cdot\nabla\phi^2)^2 dx = \theta\frac{d}{dt}\int \phi^2 \mathrm{div} u dx - \theta\int (\phi^2)_t \mathrm{div} u dx \\
&= \theta\frac{d}{dt}\int \phi^2 \mathrm{div} u dx + \theta\int u\cdot\phi^2\int dx + \theta(\gamma-1)\int \phi^2(\mathrm{div} u)^2 dx \\
&\le \theta\frac{d}{dt}\int \phi^2 \mathrm{div} u dx + C\|\nabla u\|_{L^2}^2 + C\|\mathrm{div} u\|_{L^\infty}\|\nabla\phi^2\|_{L^2}^2,
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
L_{11} &= -\int \Big(\nabla\ln\rho\cdot Q(u)\Big)\cdot(\alpha\Delta u + (\alpha+\beta)\nabla\mathrm{div} u)dx \\
&\le C\|\nabla\ln\rho\|_2\|\nabla^2 u\|_{L^2}\|\mathcal{D}(u)\|_{L^\infty} \le C(\epsilon)(\|\nabla\ln u)\|_{L^2}^2 + \epsilon\|\nabla^2 u\|_{L^2}^2,
\end{aligned}
$$

where $\epsilon > 0$ is a sufficiently small constant. Combining (3.12)-(3.15), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\Big(\alpha|\nabla u|^2 + (\alpha+\beta)|\mathrm{div} u|^2 - \theta\phi^2\mathrm{div} u\Big)dx \\
&\le C((\|\nabla u\|_{L^2}^2 + \|\nabla\phi^2\|_{L^2}^2)(\|\mathrm{div} u\|_{L^\infty} + 1) + \|\mathrm{div} u\|_{L^\infty}).
\end{aligned}
\tag{3.16}
$$

Secondly, applying ∇ to $(3.2)_1$ and multiplying by $(\nabla\phi)^\top$, we have

$$
\begin{aligned}
&(|\nabla\phi|^2)_t + \mathrm{div}(|\nabla\phi|^2 u) + (\gamma-2)|\nabla\phi|^2\mathrm{div} u \\
&= -2(\nabla\phi)^\top\nabla u(\nabla\phi) - (\gamma-1)\phi\nabla\phi\cdot\nabla\mathrm{div} u \\
&= -2(\nabla\phi)^\top\mathcal{D}(u)(\nabla\phi) - (\gamma-1)\phi\nabla\phi\cdot\nabla\mathrm{div} u.
\end{aligned}
\tag{3.17}
$$

Integrating (3.17) over $\mathbb{R}^N$, we get

$$
\frac{d}{dt}\|\nabla\phi\|_{L^2}^2 \le C(\epsilon)(\|\mathcal{D}(u)\|_{L^\infty} + 1)\|\nabla\phi\|_{L^2}^2 + \epsilon\|\nabla^2 u\|_{L^2}^2.
\tag{3.18}
$$

Adding (3.18) to (3.16), from Gronwall's inequality we immediately obtain

$$
\|\nabla u(t)\|_{L^2}^2 + \|\nabla\phi(t)\|_{L^2}^2 + \int_0^t \|\nabla^2 u(s)\|_{L^2}^2 dt \le C, \quad 0\le t\le T.
$$

Finally, due to $\dot{u} = -Lu - 2\theta\phi\nabla\phi + \nabla\ln\rho\cdot Q(u)$, we deduce that

$$
\int_0^t \|\dot{u}\|_{L^2}^2 dt \le C\int_0^t (\|\nabla^2 u\|_{L^2}^2 + \|\phi\|_{L^\infty}^2\|\nabla\phi\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\|\nabla\ln\rho\|_{L^2}^2)dt \le C.
$$

□

Next, we proceed to improve the regularity of ϕ, $\nabla\ln\rho$ and u. To this end, we first drive some bounds on derivatives of u based on the above estimates.

Lemma 3.4. *Let* (ρ, u) *be the unique regular solution to the Cauchy problem* (1.1) *on* $[0, \bar{T})$ *satisfying* (3.1). *Then*

$$\begin{aligned}&\sup_{0\le t\le T}\|\dot{u}(t)\|_{L^2}^2+\sup_{0\le t\le T}\|\nabla^2u(t)\|_{L^2}+\int_0^T\|\nabla\dot{u}\|_{L^2}^2dt\\&\le C\|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}^2+\|\dot{u}\|_{L^2}^2\|\nabla\ln\rho\|_{L^6}^3+\|\nabla^2u\|_{L^2}^2+C\|\nabla u\|_{L^\infty}+C,\end{aligned}\tag{3.19}$$

where C only depends on C_0 *and* $\bar{T}$.

Proof. Using $Lu = -u_t - u\cdot\nabla u - 2\theta\phi\nabla\phi + \nabla\ln\rho\cdot Q(u)$ and Lemma2.4, we have

$$\|\nabla^2u\|_{l^2}\le C(\|\dot{u}\|_{L^2}+\|\phi\|_{L^\infty}\|\nabla\phi\|_{L^2}+\|\nabla\ln\rho\|_{L^2}\|\mathcal{D}(u)\|_{L^\infty}),\tag{3.20}$$

from Youngs inequality, which implies that

$$\|\nabla^2u\|_{L^2}\le C(\|\dot{u}\|_{L^2}+\|\nabla\phi\|_{L^2}+\|\nabla u\|_{L^2})\le C(1+\|u_t\|_{L^2}),\tag{3.21}$$

Next, operating $\dot{u}^j[\partial/\partial t+\mathrm{div}(u\cdot)]\frac{1}{\rho}$ to $(3.2)_3$, and integrating the resulting equation over $\mathbb{R}^3$, one obtains

$$\begin{aligned}&\left(\frac{1}{2}\int|\dot{u}|^2dx\right)_t-\frac{1}{2}\int|\dot{u}|^2\mathrm{div}u\,dx\\&=-\int\dot{u}^j[\partial_j(\rho^{\gamma-1})_t+\mathrm{div}(u\partial_j(\rho^{\gamma-1})]dx\\&\quad+\int\dot{u}^j\{[\frac{1}{\rho}\mathrm{div}(\rho\nabla u^j)]_t+\mathrm{div}(u\frac{1}{\rho}\mathrm{div}(\rho\nabla u^j))\}dx\\&\quad+\int\dot{u}^j\{[\frac{1}{\rho}\partial_j(\rho\mathrm{div}u)]_t+\mathrm{div}(u\frac{1}{\rho}\partial_j(\rho\mathrm{div}u))\}dx\\&\triangleq\sum_{i=1}^3M_i.\end{aligned}\tag{3.22}$$

Now we calculate the right of (3.22) term by term. Integration by parts, using the equation $(1.10)_1$ and Hölder inequality, M_1 yieds that

$$\begin{aligned}M_1&=-\int\dot{u}^j[\partial_j(\rho^{\gamma-1})_t+\mathrm{div}(u\partial_j(\rho^{\gamma-1})]dx\\&=\int\dot{u}^j[(\gamma-2)\partial_j(\rho^{\gamma-1})\mathrm{div}u+(\gamma-1)\rho^{\gamma-1}\partial_j\mathrm{div}u+\partial_ju^i\partial_i(\rho^{\gamma-1})]dx\\&\le C\|\mathrm{div}u\|_{L^\infty}\|\dot{u}\|_{L^2}\|\nabla\phi\|_{L^2}+C\|\nabla\dot{u}\|_{L^2}\|\nabla u\|_{L^2}\end{aligned}\tag{3.23}$$

where we have used the fact that

$$\partial_j(P-P(\tilde{\rho}))_t+\mathrm{div}\{u\partial_j(P-P(\tilde{\rho}))\}+(\gamma-1)\partial_jP\mathrm{div}u+\partial_ju^i\partial_iP+\gamma P\partial_j\mathrm{div}u=0.$$

For M_2 and M_3, integration by parts and recalling the mass equation $(1.10)_1$, then using Young's inequality leads to

$$\begin{aligned}M_2&=\int\dot{u}^j\left\{[\frac{1}{\rho}\mathrm{div}(\rho\nabla u^j)]_t+\mathrm{div}(u\frac{1}{\rho}\mathrm{div}(\rho\nabla u^j))\right\}dx\\&=\int\dot{u}^j\left\{[\frac{1}{\rho}\partial_i\rho\partial_iu^j+\partial_i\partial_iu^j]_t+\partial_k[u^k\partial_i\partial_iu^j+u^k\frac{1}{\rho}\partial_i\rho\partial_iu^j]\right\}dx\\&=\int\dot{u}^j\left[(\frac{1}{\rho}\partial_i\rho\partial_iu^j)_t+\partial_k(u^k\frac{1}{\rho}\partial_i\rho\partial_iu^j)\right]dx+\int\dot{u}^j\left[(\partial_i\partial_iu^j)_t+\partial_k(u^k\partial_i\partial_iu^j)\right]dx\\&\triangleq I_1+I_2.\end{aligned}$$

We computing I_1 and I_2, respectively, as following:

$$
\begin{aligned}
I_1 &= \int \dot{u}^j \left[(\frac{1}{\rho}\partial_i\rho\partial_i u^j)_t + \partial_k(u^k\frac{1}{\rho}\partial_i\rho\partial_i u^j) \right] dx \\
&= \int \dot{u}^j \left[(\partial_i \ln\rho\partial_i u^j)_t + \partial_k(u^k\partial_i \ln\rho\partial_i u^j) \right] dx \\
&= \int \dot{u}^j \left[(\partial_i \ln\rho)_t\partial_i u^j + u^k\partial_k\partial_i \ln\rho\partial_i u^j \right] dx \\
&\quad + \int \dot{u}^j \left[\partial_i \ln\rho(\partial_i u^j)_t + \partial_i \ln\rho\partial_k(u^k\partial_i u^j) \right] dx \\
&= \int \dot{u}^j\partial_i u^j \left[(\partial_i \ln\rho)_t + \partial_i(u^k\partial_k \ln\rho) - \partial_i u^k\partial_k \ln\rho \right] dx \\
&\quad + \int \dot{u}^j\partial_i \ln\rho \left[(\partial_i u^j)_t + \partial_i(u^k\partial_k u^j) - \partial_i u^k\partial_k u^j + \partial_k u^k\partial_i u^j \right] dx \\
&= -\int \dot{u}^j\partial_i u^j(\partial_i \mathrm{div} u + \partial_i u^k\partial_k \ln\rho) dx \\
&\quad + \int \dot{u}^j\partial_i \ln\rho \left[\partial_i\dot{u}^j - \partial_i u^k\partial_k u^j + \partial_k u^k\partial_i u^j \right] dx \\
&\le C\|\dot{u}\|_{L^2}^2\|\nabla\rho\|_{L^6}^3 + \frac{1}{8}\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2\|\dot{u}\|_{L^2}\|\nabla\ln\rho\|_{L^2} + \|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}\|\nabla^2 u\|_{L^2},
\end{aligned}
$$

where we use the fact that, when $N = 2$,

$$
\begin{aligned}
\|\dot{u}\|_{L^3}\|\nabla\ln\rho\|_{L^6}\|\nabla\dot{u}\|_{L^2} &\le \|\dot{u}\|_{L^2}^{\frac{2}{3}}\|\nabla\dot{u}\|_{L^2}^{\frac{4}{3}}\|\nabla\ln\rho\|_{L^6} \\
&\le C\|\dot{u}\|_{L^2}^2\|\nabla\rho\|_{L^6}^3 + \frac{1}{8}\|\nabla\dot{u}\|_{L^2}^2 \text{£}
\end{aligned}
$$

when $N = 3$,

$$
\begin{aligned}
\|\dot{u}\|_{L^4}\|\nabla\ln\rho\|_{L^4}\|\nabla\dot{u}\|_{L^2} &\le \|\dot{u}\|_{L^2}^{\frac{1}{4}}\|\nabla\dot{u}\|_{L^2}^{\frac{7}{4}}\|\nabla\ln\rho\|_{L^2}^{\frac{1}{4}}\|\nabla\ln\rho\|_{L^6}^{\frac{3}{4}} \\
&\le C\|\dot{u}\|_{L^2}^2\|\nabla\rho\|_{L^6}^3 + \frac{1}{8}\|\nabla\dot{u}\|_{L^2}^2.
\end{aligned}
$$

For I_2, integration by parts, we have

$$
\begin{aligned}
I_2 &= \int \dot{u}^j \left[(\partial_i\partial_i u^j)_t + \partial_k(u^k\partial_i\partial_i u^j) \right] dx \\
&= \int \dot{u}^j \left[(\partial_i\partial_i u^j)_t + \partial_k u^k\partial_i\partial_i u^j + u^k\partial_i\partial_i\partial_k u^j) \right] dx \\
&= \int \dot{u}^j \left[(\partial_i\partial_i u^j)_t + \partial_k u^k\partial_i\partial_i u^j + \partial_i(u^k\partial_i\partial_k u^j) - \partial_i u^k\partial_i\partial_k u^j \right] dx \\
&= \int \dot{u}^j \left[(\partial_i\partial_i u^j)_t + \partial_k u^k\partial_i\partial_i u^j + \partial_i\partial_i(u^k\partial_k u^j) - \partial_i(\partial_i u^k\partial_k u^j) - \partial_i u^k\partial_i\partial_k u^j \right] dx \\
&= \int \dot{u}^j \left[\partial_i\partial_i\dot{u}^j + \partial_k u^k\partial_i\partial_i u^j - \partial_i(\partial_i u^k\partial_k u^j) - \partial_i u^k\partial_i\partial_k u^j \right] dx \\
&= -\int \nabla\dot{u}^2 dx + \int \partial_i\dot{u}^j\partial_i u^k\partial_k u^j dx - \int \partial_i\dot{u}^j\partial_i u^k\partial_k u^j \\
&\le -\int |\nabla\dot{u}|^2 dx + \|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}\|\nabla u\|_{L^2}.
\end{aligned}
$$

Thus, we have

$$M_2 \leq -\int |\nabla\dot{u}|^2 dx + \|\dot{u}\|_{L^3}\|\nabla \ln\rho\|_{L^6}\|\nabla\dot{u}\|_{L^2} + \|\nabla u\|_{L^\infty}^2\|\dot{u}\|_{L^2}\|\nabla\rho\|_{L^2}$$
$$+ \|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}\|\nabla u\|_{L^2}.$$

Similarly,

$$M_3 \leq -\int (\mathrm{div}\dot{u})^2 dx + \|\dot{u}\|_{L^3}\|\nabla \ln\rho\|_{L^6}\|\nabla\dot{u}\|_{L^2} + \|\nabla u\|_{L^\infty}^2\|\dot{u}\|_{L^2}\|\nabla\rho\|_{L^2}$$
$$+ \|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}\|\nabla u\|_{L^2}.$$

Thus, we have

$$\begin{aligned}
&\left(\frac{1}{2}\int |\dot{u}|^2 dx\right)_t + \int |\nabla\dot{u}|^2 dx + \int (\mathrm{div}\dot{u})^2 dx \\
&\leq C\|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}^2 + C\|\nabla\dot{u}\|_{L^2}\|\nabla u\|_{L^2} + C\|\dot{u}\|_{L^2}^2\|\nabla\rho\|_{L^6}^3 + \frac{1}{8}\|\nabla\dot{u}\|_{L^2}^2 \\
&\quad + \|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}\|\nabla^2 u\|_{L^2} + C\|\nabla u\|_{L^\infty} \\
&\leq C\|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}^2 + \frac{1}{4}\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + \|\dot{u}\|_{L^2}^2\|\nabla \ln\rho\|_{L^6}^3 \\
&\quad + \|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^\infty} + C,
\end{aligned}$$

which imply

$$\begin{aligned}
&\left(\frac{1}{2}\int |\dot{u}|^2 dx\right)_t + \int |\nabla\dot{u}|^2 dx + \int (\mathrm{div}\dot{u})^2 dx \\
&\leq C\|\nabla u\|_{L^\infty}\|\dot{u}\|_{L^2}^2 + \|\dot{u}\|_{L^2}^2\|\nabla \ln\rho\|_{L^6}^3 + \|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^\infty} + C.
\end{aligned} \tag{3.24}$$

Thus, we complete the proof of this Lemma.

□

Lemma 3.5. *Let* (ρ, u) *be the unique regular solution to the Cauchy problem* (1.1) *on* $[0, \bar{T})$ *satisfying* (3.1). *Then*

$$\sup_{0\leq t\leq T}(\|\dot{u}\|_{L^2} + \|\nabla \ln\rho\|_{L^6}) + \int_0^T (\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla^2 u\|_{L^6}^2)dt < C.$$

Proof. Firstly, apply ∇ to $(1.1)_1$, and mulptiply the result equation by $6|\nabla\ln\rho|^4\nabla\ln\rho$, we have

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla\ln\rho|^6 dx \\
&= -6\int |\nabla\ln\rho|^4(\partial_j \ln\rho\,\partial_j u_i \partial_i \ln\rho + u_i\partial_i|\nabla\ln\rho|^2)dx \\
&\qquad -6\int \partial_j\partial_i u_i |\nabla\ln\rho|^4 \partial_j \ln\rho dx \\
&\leq C\|\mathcal{D}(u)\|_{L^\infty}\int |\nabla\ln\rho|^6 dx + \|\nabla^2 u\|_{L^6}\|\nabla\ln\rho\|_{L^6}^5.
\end{aligned}$$

That is

$$\frac{d}{dt}\|\nabla\ln\rho\|_{L^6}^6 \leq C\|\mathcal{D}(u)\|_{L^\infty}\|\nabla\ln\rho\|_{L^6}^6 + \|\nabla^2 u\|_{L^6}\|\nabla\ln\rho\|_{L^6}^5. \tag{3.25}$$

When $n=2$, according to $Lu = \dot{u} - 2\theta\phi\nabla\phi + \nabla\ln\rho\cdot Q(u)$, we have

$$\begin{aligned}
\|\nabla^2 u\|_{L^6} &\le C(\|\dot{u}\|_{L^6} + \|u\cdot\nabla u\|_{L^6} + \|\phi\nabla\phi\|_{L^6} + \|\nabla\ln\rho\cdot Q(u)\|_{L^6}) \\
&\le C(\|\dot{u}\|_{L^2}^{\frac{1}{3}}\|\nabla\dot{u}\|_{L^2}^{\frac{2}{3}} + \|\nabla\phi\|_6 + \|\nabla\ln\rho\|_{L^6}\|\mathcal{D}(u)\|_{L^\infty}),
\end{aligned} \tag{3.26}$$

Adding (3.26) to (3.25), yields

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\ln\rho\|_{L^6}^3 \\
&\le C(1+\|\mathcal{D}(u)\|_{L^\infty})\|\nabla\ln\rho\|_{L^6}^3 + C\|\dot{u}\|_{L^2}^{\frac{1}{3}}\|\nabla\dot{u}\|_{L^2}^{\frac{2}{3}}\|\nabla\ln\rho\|_{L^6}^2 \\
&\le C(1+\|\mathcal{D}(u)\|_{L^\infty})\|\nabla\ln\rho\|_{L^6}^3 + \frac{1}{4}\|\nabla\dot{u}\|_{L^2}^2 + C\|\dot{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\ln\rho\|_{L^6}^3.
\end{aligned} \tag{3.27}$$

Adding (3.27) to (3.24), yields

$$\begin{aligned}
&\left(\int|\dot{u}|^2dx + \|\nabla\ln\rho\|_{L^6}^3\right)_t + \int|\nabla\dot{u}|^2dx + \int(\mathrm{div}\dot{u})^2dx \\
&\le C(\|\nabla u\|_{L^\infty} + \|\dot{u}\|_{L^2}^2 + 1)(\|\dot{u}\|_{L^2}^2 + \|\nabla\ln\rho\|_{L^6}^3) + C\|\nabla u\|_{L^\infty} + C.
\end{aligned} \tag{3.28}$$

When $n=3$, similarly,

$$\begin{aligned}
\|\nabla^2 u\|_{L^6} &\le C(\|\dot{u}\|_{L^6} + \|u\cdot\nabla u\|_{L^6} + \|\phi\nabla\phi\|_{L^6} + \|\nabla\ln\rho\cdot Q(u)\|_{L^6}) \\
&\le C(\|\nabla\dot{u}\|_{L^2} + \|\nabla\phi\|_6 + \|\nabla\ln\rho\|_{L^6}\|\mathcal{D}(u)\|_{L^\infty}).
\end{aligned} \tag{3.29}$$

Adding (3.29) to (3.25), we have

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\ln\rho\|_{L^6}^2 \\
&\le C(1+\|\mathcal{D}(u)\|_{L^\infty})\|\nabla\ln\rho\|_{L^6}^2 + C\|\nabla\dot{u}\|_{L^2}\|\nabla\ln\rho\|_{L^6} \\
&\le C(1+\|\mathcal{D}(u)\|_{L^\infty})\|\nabla\ln\rho\|_{L^6}^3 + \frac{1}{4}\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla\ln\rho\|_{L^6}^2.
\end{aligned} \tag{3.30}$$

Adding (3.30) and (3.24) to get

$$\begin{aligned}
&\left(\int|\dot{u}|^2dx + \|\nabla\ln\rho\|_{L^6}^2\right)_t + \int|\nabla\dot{u}|^2dx + \int(\mathrm{div}\dot{u})^2dx \\
&\le C(\|\nabla u\|_{L^\infty} + \|\dot{u}\|_{L^2}^2 + 1)(\|\dot{u}\|_{L^2}^2 + \|\nabla\ln\rho\|_{L^6}^2) + C\|\nabla u\|_{L^\infty} + C.
\end{aligned} \tag{3.31}$$

From the momentum equation $(3.2)_3$ we have

$$\|\dot{u}(\tau)\|_{L^2} \le C(\|\phi\|_{L^\infty}\|\nabla\phi\|_{L^2} + \|\nabla^2 u\|_{L^2} + \|\nabla\ln\rho\|_{L^2}\|\nabla u\|_{L^2}), \tag{3.32}$$

which combine (1.1), we have

$$\lim_{\tau\to 0}\|\dot{u}(\tau)\|_{L^2} \le C(\|\phi_0\|_{L^\infty}\|\nabla\phi_0\|_{L^2} + \|\nabla^2 u_0\|_{L^2} + \|\nabla\ln\rho_0\|_{L^2}\|\nabla u_0\|_{L^2}) \le C_0.$$

In (3.31), let $\tau\to 0$, by Gronwall's inequality and Lemma 3.4, we have

$$\sup_{0\le t\le T}(\|\dot{u}\|_{L^2} + \|\nabla\ln\rho(t)\|_{L^6}) + \int_0^T\|\nabla\dot{u}\|_{L^2}^2ds < C.$$

At last, by (3.26) and Lemma 3.4, we have

$$\int_0^T\|\nabla^2 u\|_{L^6}^2ds \le C\int_0^T(1+\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla\ln\rho\|_{L^6}^2)ds \le C.$$

Thus, we complete the proof of this Lemma. □

We should point out that (3.2) is actually a parabolic-hyperbolic system. Then by a standard way, we can get other higher order estimates for the regularity of the regular solutions. We will give the following results.

Lemma 3.6. *Let* (ρ, u) *be the unique regular solution to the Cauchy problem* (1.1) *on* $[0, \bar{T})$ *satisfying* (3.1). *Then*

$$\sup_{0\le t\le T}(\|\nabla^3\phi(t)\|_{L^3}^2 + |\nabla \ln\rho(t)\|_{H^2}^2 + \|\phi_t(t)\|_{L^2}^2 + t\|\nabla^4 u(t)\|_{L^2}^2) < C,$$

$$\sup_{0\le t\le T}(\|(\nabla \ln\rho(t))_t\|_{H^1}^2 + \|u(t)\|_{H^3}^2 + \|u_t(t)\|_{H^1}^2) + \int_0^T \|\nabla^4 u\|_{L^2}^2 dt < C,$$

$$\sup_{0\le t\le T}(t\|\nabla^2 u_t(t)\|_{L^2}^2 + t\|u_{tt}(t)\|_{L^2}^2) + \int_0^T (t\|\nabla^3 u_t\|_{L^3}^2 + t\|\nabla u_{tt}\|_{L^2}^2)dt < C.$$

where $0 \le T \le \bar{T}$*, and* C *only depends on* C_0 *and* T.

Proof. The method of this proof is similar to [33]. We omit the details here. □

Now, we will complete the proof of Theorem 1.2 for the sake of lemmas above. If (ρ, u) exists up to the time $\bar{T} > 0$,withe the maximal time $\bar{T} < \infty$. From Lemmas 5.1 − 5.6 that we can extend the regular solution $(\rho^{\frac{\gamma-1}{2}}, \nabla\rho/\rho, u)$ beyond $t \ge \bar{T}$. In fact, in view of lemmas above, the functions $(\rho, u)|_{t=\bar{T}} = (\rho^{\frac{\gamma-1}{2}}, \nabla\rho/\rho, u)_{t=\bar{T}} = \lim_{t\leftarrow\bar{T}}$ satisfy the conditions imposed on the initial data (1.5). That is, $(\rho, u)|_{t=\bar{T}}$ satisfy (1.5) too. Therefore, we can take $(\rho, u)|_{t=\bar{T}}$ as the initial data and apply the local existence Theorem 1.1 to extend the local regular solutions beyond $\bar{T}$. This contradicts the assumption on $\bar{T}$.

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