The ill-posedness of the (non-)periodic travelling wave solution for the deformed continuous Heisenberg spin equation

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Abstract

Based on an equivalent derivative nonlinear Schröinger equation, some periodic and non-periodic two-parameter solutions of the deformed continuous Heisenberg spin equation are obtained. These solutions are all proved to be ill-posed by the estimates of the Fourier integral in $H^s_{\mathbb{S}^2}$ (periodic solution in $H^s_{\mathbb{S}^2}(\mathbb{T})$ and non-periodic solution in $H^s_{\mathbb{S}^2}(\mathbb{R})$ respectively). If $\alpha \neq 0$, the range of the weak ill-posedness index is $1 < s < \frac{3}{2}$ for both periodic and non-periodic solutions. However, the periodic solution admits a strong ill-posedness index in the range of $\frac{3}{2} < s < \frac{7}{2}$, whereas the range of the non-periodic solution is $1 < s < 2$. These results extend our previous work (see [38]) of the deformed continuous Heisenberg spin model to the periodic solution case and a different fractional Sobolev spaces $H^s_{\mathbb{S}^2}$.

Keywords: Heisenberg spin; soliton; ill-posedness; Fourier integral

1. Introduction

The deformed continuous Heisenberg spin (DCHS) equation is the important physical model which has received a lot of attention in the past decades. Mikhailov and Shabat[27] firstly constructed an integrable $SO(3)$ invariant integrable DCHS equation which can be written as

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx} + \alpha \mathbf{S}_x (\mathbf{S}_x)^2, \tag{1}$$

where $\wedge$ denotes the cross product of the vector $\mathbf{S}(x,t) = (S_1(x,t),\ S_2(x,t),\ S_3(x,t))$, $\mathbf{S}\cdot\mathbf{S} = 1$ and $(\mathbf{S}_x)^2 = \mathbf{S}_x \cdot \mathbf{S}_x$.

DCHS equations contain a large number of different equations which can be transformed into various different order nonlinear Schrödinger (NLS) equations. Porsezian et al.[31] firstly showed that (1) is gauge equivalent to the integrable derivative NLS equation, which can be applied to two photon self-induced transparency and ultrashot light pulse propagation in the optical fiber. Similarly, Lakshmanan et al.[10, 17, 22, 28, 30] investigated the higher order integrable DCHS equations and found they can be transform to the higher order NLS equations by associating the spin vector with the tangent to a moving curve in Euclidean space. For the higher dimensional integrable DCHS[24], how to derive the corresponding gauge equivalent NLS equations have also been presented[37].

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If $\alpha = 0$, then (1) deforms into an isotropic Heisenberg spin (IHS) equation which is an exactly integrable equation which can be regarded as a simplest case of the Landau–Lifshitz equation (LL equation)[20]. The following papers illustrate some typical progress with the LL equation. Alouges and Soyeur [1] established some necessary conditions for the existence of a global weak solution. When the spatial dimension is $n = 1$ under the periodic boundary condition setting, Guo and Huang [8] established the existence of unique smooth solutions by means of the technique of spatial differences. In $\mathbb{R}^3$, Carbou and Fabrie [4] proved the local existence and uniqueness of regular solutions and the global existence when the initial data are small in some sense. Chang, Shatah, and Uhlenbeck [5] established the existence of small-data global solutions for the cylindrical coordinates case. In normal coordinates, a global solution with small initial values also exists [2] under some special norms. In dimensions larger than three, the global existence and uniqueness of mild solutions were proved [23] under the smallness condition. Similarly, under the smallness constraint in Morrey spaces, Lin, Lai, and Wang [21] extended this result to establish an existence result for the global solution. Moreover, the solution with small initial data in critical Besov space was proved [9] to be globally well-posed in dimensions $n \geq 3$.

Inspired by the study of heat flow in harmonic maps and by the Ginzburg–Landau equation, the concentration set of the stationary weak solutions of the LL equation was estimated [7, 18, 35] to analyze the behavior of the solution at the singular point. Moreover, the singular property (and even the finite time blowup) can be proved for a special type of the solution. If the topology degree is one, the equivariant solution will blow up; its blowup rate was predicted in [26, 29, 32]. As we know, the exact solution to the LL equation provides a more intuitive way to study its dynamic behavior. We refer the reader to [6, 11, 13, 34, 36, 39, 40, 41, 42] for details.

Because (1) is a quasilinear equation, it is difficult to analyze the well-posedness of the system directly. Partial differential equations lack a theory of well-posedness. So many papers use the equivalent system of the equation (especially the equivalent nonlinear Schrödinger equation) to prove the well-posedness of the solution to the original partial differential equations, such as the Schrödinger equation with derivative

$$-\mathrm{i}W_t = \Delta W - \frac{2W^*}{1+|W|^2}\nabla W \cdot \nabla W$$

or its equivalent covariant derivative equation:

$$\left(iD_t - D_j D_j\right)\Psi_k = -\mathrm{iIm}\left(\Psi_j \Psi_j^*\right)\Psi_j.$$

Similarly, we here use an equivalent complex equation to study the deformed continuous Heisenberg spin equation. If we set the curvature κ and torsion τ are

$$\kappa = (\mathbf{S}_x \cdot \mathbf{S}_x)^{\frac{1}{2}} \quad \text{and} \quad \tau = \frac{\mathbf{S}\cdot(\mathbf{S}_x \times \mathbf{S}_{xx})}{\kappa^2}$$

respectively.

We can apply the following Hasimoto transform [31, 43, 44]:

$$Q = \kappa \exp\left(\mathrm{i}\int_{-\infty}^{x} \tau\left(t, x'\right)\mathrm{d}x'\right)$$

to transform (1) into the following nonlinear derivative Schrödinger equation (refer to [31]):

$$\mathrm{i}Q_t + Q_{xx} + \frac{1}{2}\mathrm{Q}\left|\mathrm{Q}\right|^2 - \mathrm{i}\alpha\left(|Q|^2 Q\right)_x = 0. \tag{2}$$

Equation (2) is a mix of the cubic Schrödinger equation and Alfvén equation. Without $i\alpha\left(|Q|^2Q\right)_x$, (2) is the well-known cubic Schrödinger equation, which has been studied in great detail. However, if the cubic term $\frac{1}{2}\mathrm{Q}\left|\mathrm{Q}\right|^2$ is neglected, (2) is the Alfvén equation, which comes from plasma physics [25]. For the Alfvén equation, some well-posedness results of the solution are established already. Hayashi [12] obtained the global well-posedness of it in H^1. Some similar results can be also seen in [15, 16]. Furthermore, Takaoka [33] considered the rougher data solution and showed local well-posedness in H^s with $s > \frac{1}{2}$ by an equivalent equation. If the initial condition Q_0 satisfies $\|Q_0\|_{L^2} < \sqrt{2\pi}$, Hayashi and Ozawa [14] applied the mass conservation to prove that the solution is a global one. However, a soliton-type solution to it is ill-posed on H^s $(0 < s < \frac{1}{2})$ [3].

Compared with well-posedness, the results of the ill-posedness of partial differential equations are deficient. The ill-posedness will be depend on the specific solutions. Different solutions can have different ill-posedness spaces and unequal ill-posedness indexes. This makes it difficult to obtain a general ill-posedness result. Although Bigioni and Linares proved the ill-posedness for a class of solutions as early as 2001, it remains to be determined whether solutions with general initial boundary data or the more general derivative Schrödinger equation also have ill-posedness in some special space. It is worth noting that the solution discussed in [3] is the solution in the whole space $H^s(\mathbb{R})$.

To our knowledge, there are few papers on the ill-posedness result of the DCHS equation. In the recent paper [38], we give a proof of the ill-posedness of the solution and predict an exact index range of it for the first time. As far as we know, there are no other papers to discuss the this topic and some more further work of it needs to be done. In view of this, we continue our previous work[38] to discuss the ill-posedness problem of the DCHS equation and extend the result of it to the different solutions (periodic and non-periodic solutions) and different fractional Sobolev spaces(which will be defined in the following content).

Here, we study the ill-posedness problem of (1) and (2). For $\sigma \geq 0$, J^σ is denoted by the Fourier multiplier $\xi \to (1+|\xi|^2)^{\sigma/2}$. H^σ (which includes $H^\sigma(\mathbb{R})$) and the periodic space of $H^\sigma(\mathbb{T})$ (where the period is $\mathbb{T}$) is the norm $||f||_{H^\sigma} = ||J^\sigma(f)||_{L^2}$ (L^2 is on $\mathbb{R}$ or $\mathbb{T}$) of complex fractional Sobolev spaces.

With the initial condition Q_0, the solution of equation (2) is ill-posed in $H^s(H^s(\mathbb{R})$ or $H^s(\mathbb{T}))$. And it can be classified as follows:

(I) (Weak ill-posedness) Let $\tilde{C}$ be a given constant. For any real $\delta > 0$, the solution is weakly ill-posed in H^s if and only if

$$\|Q_{c_1,\omega_1}(x,0) - Q_{c_2,\omega_2}(x,0)\|_{H^s} \leq \delta,$$
$$Q_{c_j,\omega_j}(\cdot,\ T) \in H^s, \quad \|Q_{c_1,\omega_1}(\cdot,\ T) - Q_{c_2,\omega_2}(\cdot,\ T)\|_{H^s} \geq \tilde{C}.$$

(II) (Strong ill-posedness) For any $\varepsilon > 0$ and $\delta > 0$, the solution is strongly ill-posed in H^s if and only if

$$\|Q_{c_1,\omega_1}(x,0) - Q_{c_2,\omega_2}(x,0)\|_{H^s} \leq \delta,$$
$$\|Q_{c_1,\omega_1}(\cdot,\ T) - Q_{c_2,\omega_2}(\cdot,\ T)\|_{H^s} \geq \varepsilon^{-1}$$
$$\|Q_{c_j,\omega_j}(\cdot,\ T)\|_{H^s} = \|Q_{c_j,\omega_j}(\cdot,\ 0)\|_{H^s} > \varepsilon^{-1}.$$

Taking into account that $\|\kappa\| = \|Q\|$ and $\kappa = (\mathbf{S}_x \cdot \mathbf{S}_x)^{\frac{1}{2}}$, there is an equivalent relationship between the different norms of Q and $\mathbf{S}$. So we can use the norm of Q to estimate the norm of $\mathbf{S}$. Let $F =$

$(F_1,\ F_2,\ F_3)$, and $G=(G_1,\ G_2,\ G_3)$. We use the induced distance

$$d^\sigma(F,\ G)=\left[\sum_{l=1}^{3}\|F_l-G_l\|_{H^\sigma}^2\right]^{1/2},$$

to define the vector norm in H_*^σ as follows:

$$\|F\|_{H_*^\sigma}=\left(\sum_{l=1}^{3}\|F_l\|_{H^\sigma}^2\right)^{\frac{1}{2}}.$$

At the same time, the induced norm of $H_{\mathbb{S}^2}^\sigma$ ($H_{\mathbb{S}^2}^\sigma(\mathbb{T})$ and $H_{\mathbb{S}^2}^\sigma(\mathbb{R})$) for the vector $\mathbf{S}$ ($\mathbf{S}_\infty$ is the value of $\mathbf{S}$ in $x=\infty$) is

$$\|\mathbf{S}\|_{H_{\mathbb{S}^2}^\sigma(\mathbb{T})}=\|\mathbf{S}\|_{H_*^\sigma(\mathbb{T})}.$$

and

$$\|\mathbf{S}\|_{H_{\mathbb{S}^2}^\sigma(\mathbb{R})}=\|\mathbf{S}-\mathbf{S}_\infty\|_{H_*^\sigma(\mathbb{R})},$$

respectively.

It is not difficult to find that if $Q\in C([0,\ T];H^\sigma)$ is the solution of (2), then the solution of (1) satisfies $S\in C([0,\ T];H_{\mathbb{S}^2}^{\sigma+1})$. Similarly, in $H_{\mathbb{S}^2}^s(\mathbb{T})$ (or $H_{\mathbb{S}^2}^s(\mathbb{R})$), two types of ill-posedness are defined as follows:

(I) The solution is weakly ill-posed in $H_{\mathbb{S}^2}^s$ if and only if

$$\|\mathbf{S}_{c_1,\omega_1}(x,0)-\mathbf{S}_{c_2,\omega_2}(x,0)\|_{H_{\mathbb{S}^2}^s}\le\delta,$$
$$\mathbf{S}_{c_j,\omega_j}(\cdot,\ T)\in H_{\mathbb{S}^2}^s,\quad \|\mathbf{S}_{c_1,\omega_1}(\cdot,\ T)-\mathbf{S}_{c_2,\omega_2}(\cdot,\ T)\|_{H_{\mathbb{S}^2}^s}\ge\tilde{C}.$$

(II) The solution is strongly ill-posed in $H_{\mathbb{S}^2}^s$ if and only if

$$\|\mathbf{S}_{c_1,\omega_1}(x,0)-\mathbf{S}_{c_2,\omega_2}(x,0)\|_{H_{\mathbb{S}^2}^s}\le\delta,$$
$$\|\mathbf{S}_{c_1,\omega_1}(\cdot,\ T)-\mathbf{S}_{c_2,\omega_2}(\cdot,\ T)\|_{H_{\mathbb{S}^2}^s}\ge\varepsilon^{-1}$$
$$\|\mathbf{S}_{c_j,\omega_j}(\cdot,\ T)\|_{H_{\mathbb{S}^2}^s},\quad \|\mathbf{S}_{c_j,\omega_j}(\cdot,\ 0)\|_{H_{\mathbb{S}^2}^s}>\varepsilon^{-1}.$$

By Fourier analysis, the following result is obtained:

Theorem 1. *There is a solution $\mathbf{S}(x,t)$ of (1) in $H_{\mathbb{S}^2}^s$, and the mapping $\mathbf{S}_0\to\mathbf{S}(t)$ is ill-posed. Specifically, if $\alpha\neq0$, there are the following two-parameter solitary wave solutions $\mathbf{S}_{c,\omega}$:*

(I) If $S_{c,\omega}$ satisfies the constrained curvature condition

$$(\mathbf{S}_x\cdot\mathbf{S}_x)^{\frac{1}{2}}=\sqrt{2}A_1\left(B_1\cos\left(\frac{1}{2}A_1\xi\right)^2+C_1\right)^{-\frac{1}{2}},$$

where $\xi=x-ct$, $A_1=\sqrt{c^2+4\,\omega}$, $B_1=4\sqrt{-\alpha^2\omega+\alpha\,c+1}$, and $C_1=-\alpha\,c-2\sqrt{-\alpha^2\omega+\alpha\,c+1}-2$, then the solution is weakly ill-posed in $H_{\mathbb{S}^2}^s(\mathbb{T})$ with $1<s<\frac{3}{2}$, and strongly ill-posed in $H_{\mathbb{S}^2}^s(\mathbb{T})$ with $\frac{3}{2}<s<\frac{5}{2}$.

(II) If $S_{c,\omega}$ satisfies the constrained curvature condition

$$(\mathbf{S}_x\cdot\mathbf{S}_x)^{\frac{1}{2}}=\sqrt{2}\left(\frac{\mathrm{e}^{A_2\xi}B_2}{A_2^2}+\frac{\mathrm{e}^{-A_2\xi}B_2}{A_2^2}+\frac{C_2}{A_2^2}\right)^{-\frac{1}{2}},$$

where $\xi=x-ct$, $A_2=\sqrt{-c^2-4\,\omega}$, $B_2=\sqrt{-\omega\,\alpha^2+\alpha\,c+1}$ and $C_2=\alpha c+2$, then the solution is weakly ill-posed in $H_{\mathbb{S}^2}^s(\mathbb{R})$ with $1<s<\frac{3}{2}$, and strongly ill-posed in $H_{\mathbb{S}^2}^s(\mathbb{R})$ with $1<s<2$.

Remark 1. In addition to (1), there are a large number of the more general DCHS models. For instance, Lakshmanan and Ganesan[19] proposed a generalised case which including linear inhomogeneities (a higher order integrable DCHS equations in the same time) as follows

$$\mathbf{S}_t = (\gamma_2 + \mu_2 x)\,\mathbf{S}\wedge\mathbf{S}_{xx} + \mu_2\mathbf{S}\wedge\mathbf{S}_x - (\gamma_1 + \mu_1 x)\,\mathbf{S}_x - \gamma\left(\mathbf{S}_{xx} + \frac{3}{2}\mathbf{S}_x^2\mathbf{S}\right)_x. \tag{3}$$

As far as we know, the well posedness and the ill-posedness problems of (3) are still open.

The H^σ norm of Q is equivalent to the $H^{\sigma+1}_{\mathbb{S}^2}$ norm of $\mathbf{S}$. In order to prove Theorem 1, we only need to prove its equivalent theorem as follows:

Theorem 2. *Let $\alpha \neq 0$ and $\xi = x - ct$; A_i, B_i and C_i $(i = 1, 2)$, as in Theorem 1. Then, (2) has a two-parameter solitary wave solution:*

$$Q_{c,\omega}(\xi, t) = e^{-\mathrm{i}\omega t}\phi(\xi)\,e^{\mathrm{i}\psi(\xi)}, \tag{4}$$

where $\phi(\xi)$ and $\psi(\xi)$ can be

$$\phi(\xi) = \sqrt{2}A_1\left(B_1\cos\left(\frac{1}{2}A_1\xi\right)^2 + C_1\right)^{-\frac{1}{2}}, \tag{5}$$

$$\psi(\xi) = \frac{3\,\alpha\,A_1}{\sqrt{(B_1 + C_1)\,C_1}}\arctan\left(\frac{C_1\tan\left(\frac{1}{2}A_1\xi\right)}{\sqrt{(B_1 + C_1)\,C_1}}\right) + \frac{1}{2}c\xi, \tag{6}$$

or

$$\phi(\xi) = \sqrt{2}\left(\frac{\mathrm{e}^{A_2\xi}B_2}{A_2^2} + \frac{\mathrm{e}^{-A_2\xi}B_2}{A_2^2} + \frac{C_2}{A_2^2}\right)^{-\frac{1}{2}}, \tag{7}$$

$$\psi(\xi) = -3\arctan\left(\frac{2\,\mathrm{e}^{A_2\xi}B_2 + C_2}{\alpha\,A_2}\right) + \frac{1}{2}c\xi. \tag{8}$$

These two different kinds of the solutions are ill-posed:

(I) Solution (4), where $\phi(\xi)$ and $\psi(\xi)$ are in the form of (5) and (6), respectively , is weakly ill-posed in $H^s(\mathbb{T})$ with $0 < s < \frac{1}{2}$, and strongly ill-posed in $H^s(\mathbb{T})$ with $\frac{1}{2} < s < \frac{3}{2}$.

(II) Solution (4), where $\phi(\xi)$ and $\psi(\xi)$ are in the form of (7) and (8), respectively, is weakly ill-posed in $H^s(\mathbb{R})$ with $0 < s < \frac{1}{2}$, and strongly ill-posed in $H^s(\mathbb{R})$ with $0 < s < 1$.

Remark 2. The Alfvén equation has a class of ill-posedness solutions in $H^s(\mathbb{R})$ $(0 < s < 1/2)$ [3]. From case (I), if the cubic term $\frac{1}{2}\mathrm{Q}\,|\mathrm{Q}|^2$ is added to the Alfvén equation, ill-posedness solutions also exist. In addition, only the weak ill-posedness of the soliton-type solution on the whole space $H^s(\mathbb{R})$ was discussed. However, theorem 2 shows that the solution to the derivative Schrödinger equation with the $\frac{1}{2}\mathrm{Q}\,|\mathrm{Q}|^2$ term can show weak and strong ill-posedness in both $H^s(\mathbb{R})$ and $H^s(\mathbb{T})$. Moreover, (3) is proved ([17]) to be geometrically as well as gauge equivalent to the generalised NLS equation with linear inhomogeneities

$$\begin{aligned} &\mathrm{i}Q_1 + \mathrm{i}\mu_1 Q + \mathrm{i}(\,\gamma_1 + \mu_1 x)\,Q_x + (\gamma_2 + \mu_2 x)\left(Q_{xx} + 2|Q|^2Q\right) \\ &\qquad + 2\mu_2\left(Q_x + Q\int_{-x}^{x}|Q|^2\,\mathrm{d}x'\right) + \mathrm{i}\gamma\left(Q_{xxx} + 6|Q|^2Q_x\right) = 0. \end{aligned} \tag{9}$$

Similar to (3), the well posedness and the ill-posedness problems of (9) are still unknown.

This paper is organized as following. In section 2, we construct the (non-) periodic travelling wave solution for the equivalent nonlinear derivative Schrödinger equation of the deformed continuous Heisenberg spin equation. In section 3, the ill-posedness of the periodic travelling wave solution is proved and the range of the ill-posedness index is presented. In section 4, we prove the weak and strong ill-posedness of the non-periodic solution (soliton solution) and estimate the ill-posedness indexes.

2. (Non-) periodic solutions to the deformed continuous Heisenberg spin equation

Under the plane wave variable $\xi = x - ct$ setting, we assume the soliton solution of (2) is as follows

$$Q_{c,\omega}(t,x) = e^{-\mathrm{i}\omega t}\phi(\xi)\, e^{i\psi(\xi)}. \tag{10}$$

By substituting (10) into (2), and separating the real and imaginary sections, we obtain

$$c\,\phi\frac{\mathrm{d}\psi}{\mathrm{d}\xi} + \alpha\,\phi^3\frac{\mathrm{d}\psi}{\mathrm{d}\xi} - \left(\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right)^2\phi + \omega\,\phi(\xi) + \frac{1}{2}\phi^3 + \frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} = 0 \tag{11}$$

and

$$-c\,\frac{\mathrm{d}\phi}{\mathrm{d}\xi} - 3\,\alpha\,\phi^2\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + 2\left(\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + \left(\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2}\right)\phi = 0 \tag{12}$$

respectively.

Note that (11)–(12) represent a system of first-order differential equations with the dependent variable ξ. To solve this system, by (12), we have

$$\psi = c_2 + \frac{c\,\xi}{2} + \int\frac{3\,\alpha\phi^4 + 4\,c_1}{4\,\phi^2}\,\mathrm{d}\xi. \tag{13}$$

Substituting (13) with (11), we have

$$16\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}\right)\phi^3 + 3\,\alpha^2\,\phi^8 + 8\,\alpha\,c\phi^6 + 8\,\phi^6 - 8\,c_1\alpha\,\phi^4 + 4\,c^2\,\phi^4 + 16\,\omega\,\phi^4 - 16\,{c_1}^2 = 0. \tag{14}$$

To solve (14), we define the auxiliary function:

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 = \sum_{j=0}^{6} h_j\phi^j, \tag{15}$$

where h_j is a undetermined function.

By (15), the second derivative of ϕ must satisfy the following equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} = \frac{1}{2}\sum_{j=1}^{6} j\,h_j\phi^{j-1}(\xi). \tag{16}$$

Substituting (15)–(16) with (14) and comparing the powers of ϕ, we have

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 = -\frac{\alpha^2}{16}\,\phi^6 + \left(-\frac{\alpha\,c}{4} - \frac{1}{4}\right)\phi^4 + \left(\frac{\alpha\,c_1}{2} - \frac{c^2}{4} - \omega\right)\phi^2 - {c_1}^2\phi^{-2}. \tag{17}$$

In (17), we study the case where $c_1 = 0$, which is

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 = -\left[\frac{\alpha^2}{16}\,\phi^4 + \left(-\frac{\alpha\,c}{4} - \frac{1}{4}\right)\phi^2 - \frac{c^2}{4} - \omega\right]\phi^2. \tag{18}$$

For the ODE (18), if $h_0 = h_1 = h_3 = h_5 = 0, h_6 < 0,\ h_4^2 - 4h_2h_6 > 0,\ h_2 > 0$, and $h_4 < 0$, then (15) has the following bell shape solution:

$$\phi(\xi) = \left\{ \frac{2h_2 sech^2 \sqrt{h_2}\,\xi}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} + h_4\right) sech^2 \sqrt{h_2}\,\xi} \right\}^{\frac{1}{2}}$$

and singular solution

$$\phi(\xi) = \left\{ \frac{2h_2 csch^2\left[\pm\sqrt{h_2}\,\xi\right]}{2\sqrt{h_4^2 - 4h_2h_6} + \left(\sqrt{h_4^2 - 4h_2h_6} - h_4\right) csch^2\left[\pm\sqrt{h_2}\,\xi\right]} \right\}^{\frac{1}{2}}.$$

By (15), we have $h_0 = h_1 = h_3 = h_5 = 0,\ h_2 = -(1/4)c^2 - \omega,\ h_4 = -(1/4)\alpha c - 1/2$ and $h_6 = -(1/16)\alpha^2$. Hence, we obtain the following theorem:

Solution 3. *Equation* (2) *has the following solution:*

$$Q = \mathrm{e}^{-\mathrm{i}\omega\, t}\mathrm{e}^{\mathrm{i}\psi(\xi)}\phi\,(\xi)\,, \tag{19}$$

where $\xi = x - ct$,

$$\psi(\xi) = \int \frac{3\,\alpha\phi^2}{4}\,\mathrm{d}\xi + \frac{c\,\xi}{2}. \tag{20}$$

(I) *If* $\alpha > 0,\ c < -2\,\alpha^{-1}$, *and* $\omega < (\alpha c + 1)/\alpha^2$, *then the equation has the following trig solution:*

$$\phi(\xi) = \left\{ \frac{2(\frac{1}{4}c^2 + \omega) sec^2 \sqrt{\frac{1}{4}c^2 + \omega}\,\xi}{\sqrt{-\alpha^2\omega + \alpha\, c + 1} - \left(\frac{1}{2}\sqrt{-\alpha^2\omega + \alpha\, c + 1} + \frac{\alpha\, c}{4} + \frac{1}{2}\right) sec^2 \sqrt{\frac{1}{4}c^2 + \omega}\,\xi} \right\}^{\frac{1}{2}} \tag{21}$$

and the singular trig solution

$$\phi(\xi) = \left\{ \frac{-2(\frac{1}{4}c^2 + \omega) csc^2 \pm \sqrt{\frac{1}{4}c^2 + \omega}\,\xi}{\sqrt{-\alpha^2\omega + \alpha\, c + 1} - \left(\frac{1}{2}\sqrt{-\alpha^2\omega + \alpha\, c + 1} - \frac{\alpha\, c}{4} - \frac{1}{2}\right) csc^2 \pm \sqrt{\frac{1}{4}c^2 + \omega}\,\xi} \right\}^{\frac{1}{2}}. \tag{22}$$

(II) *If* $\alpha > 0,\ \omega < -(1/4)\,c^2$ *and* $-2\,\alpha^{-1} < c$, *then the equation has the following bell shape solution:*

$$\phi(\xi) = \left\{ \frac{-2(\frac{1}{4}c^2 + \omega) sech^2 \sqrt{-\frac{1}{4}c^2 - \omega}\,\xi}{\sqrt{-\alpha^2\omega + \alpha\, c + 1} - \left(\frac{1}{2}\sqrt{-\alpha^2\omega + \alpha\, c + 1} - \frac{\alpha\, c}{4} - \frac{1}{2}\right) sech^2 \sqrt{-\frac{1}{4}c^2 - \omega}\,\xi} \right\}^{\frac{1}{2}} \tag{23}$$

and the singular solution

$$\phi(\xi) = \left\{ \frac{-2(\frac{1}{4}c^2 + \omega) csch^2 \pm \sqrt{-\frac{1}{4}c^2 - \omega}\,\xi}{\sqrt{-\alpha^2\omega + \alpha\, c + 1} + \left(\frac{1}{2}\sqrt{-\alpha^2\omega + \alpha\, c + 1} + \frac{\alpha\, c}{4} + \frac{1}{2}\right) csch^2 \pm \sqrt{-\frac{1}{4}c^2 - \omega}\,\xi} \right\}^{\frac{1}{2}}. \tag{24}$$

Remark 3. Solutions (21) and (22) can be converted to the same form as follows:

$$\phi(\xi) = \left\{ \frac{4(c^2 + 4\,\omega)}{4\sqrt{-\alpha^2\omega + \alpha\, c + 1}\cos^2\left(\frac{1}{2}\sqrt{c^2 + 4\,\omega}\,\xi\right) - \alpha\, c - 2\sqrt{-\alpha^2\omega + \alpha\, c + 1} - 2} \right\}^{\frac{1}{2}}$$

Similarly, (23) and (24) can be converted to the same form as follows:

$$\phi(\xi) = \left\{ \frac{-4(c^2+4\,\omega)}{4\,\cosh^2\left(\frac{1}{2}\sqrt{-c^2-4\,\omega}\,\xi\right)\sqrt{-\omega\,\alpha^2+\alpha\,c+1}+\alpha\,c-2\sqrt{-\omega\,\alpha^2+\alpha\,c+1}+2} \right\}^{\frac{1}{2}}.$$

Remark 4. The evolution of solutions (21) and (23) can be seen in Figures 1–4, which illustrate the ill-posedness of (21) and (22). By comparing the images in Figure 3 and Figure 4, it can be seen that under different parameter settings, the initial value with a sufficiently small distance at the initial time (see Figure 4) can become a solution with a sufficiently large distance at a specific time (see Figure 3).

3. Ill-posedness of the periodic solution

We prove the ill-posedness property of the solution in (21) (and at the same time for (22)). Equation (21) can be transformed to

$$\phi(\xi) = \sqrt{2}A_1\left(B_1\cos\left(\tfrac{1}{2}A_1\xi\right)^2+C_1\right)^{-\frac{1}{2}},$$

where

$$A_1 = \sqrt{c^2+4\,\omega},$$
$$B_1 = 4\sqrt{-\alpha^2\omega+\alpha\,c+1},$$
$$C_1 = -\alpha\,c-2\sqrt{-\alpha^2\omega+\alpha\,c+1}-2.$$

Then, (20) is equivalent to

$$\psi(\xi) = \frac{3\,\alpha\,A_1}{\sqrt{(B_1+C_1)C_1}}\arctan\left(\frac{C_1\tan\left(\frac{1}{2}\,A_1\xi\right)}{\sqrt{(B_1+C_1)C_1}}\right)+\tfrac{1}{2}\,c\xi.$$

Let

$$d_4 = \frac{1}{2}A_1,$$
$$d_5 = \sqrt{2}A_1$$

and

$$h^{[1]}(x) = \left(B_1\cos\left(x\right)^2+C_1\right)^{-\frac{1}{2}}.$$

Then,

$$\phi(x) = d_5h^{[1]}(d_4x),$$
$$\psi(x) = \frac{3\,\alpha\,A_1}{\sqrt{(B_1+C_1)\,C_1}}\arctan\left(\frac{C_1\tan\left(d_4\xi\right)}{\sqrt{(B_1+C_1)\,C_1}}\right)+\frac{1}{2}\,c\xi.$$

Let

$$g^{[1]}(x) = \frac{3\,\alpha\,A_1}{\sqrt{(B_1+C_1)C_1}}\arctan\left(\frac{C_1\tan(x)}{\sqrt{(B_1+C_1)C_1}}\right).$$

and

$$F^{[1]}(x) = e^{\mathrm{i}g^{[1]}(x)}h^{[1]}(x).$$

Then, by (19), we can define

$$\varphi^{[1]}_{c,\omega}(x) = Q_{c,\omega}(x,0) = d_5 e^{\mathrm{i}cx/2} F^{[1]}(d_4 x).$$

We define the Fourier transformation on the interval $\mathrm{T}_\gamma := [-\pi\gamma,\ \pi\gamma]$ as

$$\mathcal{F}_1(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\gamma}^{\pi\gamma} f(x)\ e^{-ix\xi}\, dx.$$

Let $\mathrm{Z}_\gamma = [-\gamma,\ \gamma]$. Then, $H^s(\mathrm{T}_\gamma)$ is complete in the space C^∞ with period $T\gamma$ and norm

$$\|f\|_{H^s(\mathrm{T}_\gamma)} := \|\langle\xi\rangle^s \mathcal{F}_1(f)(\xi)\|_{L^2(\mathrm{Z}_\gamma)}.$$

In the following, we study the ill-posedness of solution (21). As solution (21) is periodic in $\mathbb{R}$, its norm could be infinity in $H^s(\mathbb{R})$. Therefore, we consider its ill-posedness in one period. We first estimate its inner-product norm in an integer period (with a sufficiently large period $\lambda\mathbb{T}$), and then estimate the norm in a single period $\mathbb{T}$.

Proposition 4. *If $\alpha \neq 0$, then the Cauchy problem of equation (2) in $H^s(\mathbb{T})$ could be ill-posed, That is, $Q_0 \to Q(t)$ is not uniformly continuous. With the initial condition $Q_0 = \varphi^{[1]}_{c,\omega}(x) = d_5 e^{icx/2} F^{[1]}(d_4 x)$, the solution of equation (2) is ill-posed in $H^s(\mathbb{T})$. Specifically, we have the following:*
(I) If $0 < s < \frac{1}{2}$, then the solution is weakly ill-posed.
(II) If $\frac{1}{2} < s < \frac{3}{2}$, then the solution is strongly ill-posed.

Proof. With the scaling and time-shifting properties of the Fourier transform,

$$\mathcal{F}(\varphi^{[1]}_{c,\omega})(\xi) = \tfrac{d_5}{d_4}\mathcal{F}(F^{[1]})(\tfrac{\xi}{d_4} - \tfrac{c}{2d_4}).$$

Furthermore, we compute the norm under different initial conditions,

$$\begin{aligned}
\|\varphi^{[1]}_{c_1,\omega_1} - \varphi^{[1]}_{c_2,\omega_2}\|^2_{H^s(\mathrm{T}_{\frac{\gamma}{d_{41}}})} &= \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} \langle\xi\rangle^s |\mathcal{F}(\varphi^{[1]}_{c_1,\omega_1})(\xi) - \mathcal{F}(\varphi^{[1]}_{c_2,\omega_2})(\xi)|^2 d\xi \\
&= \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} \langle\xi\rangle^s |\frac{d_{51}}{d_{41}}\mathcal{F}(F^{[1]})(\frac{\xi}{d_{41}} - \frac{c_1}{2d_{41}}) - \frac{d_{52}}{d_{42}}\mathcal{F}(F^{[1]})(\frac{\xi}{d_{42}} - \frac{c_2}{2d_{42}})|^2 d\xi \\
&= d_{41}\int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} \langle d_{41}\eta\rangle^s |\frac{d_{51}}{d_{41}}\mathcal{F}(F^{[1]})(\eta - \frac{c_1}{2d_{41}}) - \frac{d_{52}}{d_{42}}\mathcal{F}(F^{[1]})(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}})|^2 d\eta \\
&\simeq P^{[1]}_1 + P^{[1]}_2 + P^{[1]}_3.
\end{aligned} \tag{25}$$

where

$$P^{[1]}_1 = (d_{41})^{2s+1}\int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} \langle\eta\rangle^s \frac{d^2_{51}}{d^2_{41}} |\mathcal{F}(F^{[1]})(\eta - \frac{c_1}{2d_{41}}) - \mathcal{F}(F^{[1]})(\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}})|^2 d\eta,$$

$$P^{[1]}_2 = (d_{41})^{2s+1}\int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} \langle\eta\rangle^s \frac{d^2_{51}}{d^2_{41}} |\mathcal{F}(F^{[1]})(\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}) - \mathcal{F}(F^{[1]})(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}})|^2 d\eta,$$

$$P^{[1]}_3 = (d_{41})^{2s+1}\int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} \langle\eta\rangle^s |\frac{d^2_{51}}{d^2_{41}} - \frac{d^2_{52}}{d^2_{42}}|\mathcal{F}(F^{[1]})(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}})|^2 d\eta.$$

If $N_j (j=1,\,2)$ and N are large integers, then we have the following approximation:

$$\mathrm{c_j} = -N_j \simeq -N, \quad \omega_{\mathrm{j}} = N_j^{\lambda s} - \frac{N_j^2}{4}$$

Without loss of generality, we assume that $N_1 < N_2$, and we have

$$d_{4j} = \frac{1}{2} N_j^{\frac{1}{2}\lambda s}, \quad d_{5j} = \sqrt{2} N_j^{\frac{1}{2}\lambda s},$$

$$|d_{41} - d_{42}| \simeq |N_1 - N_2|\, N^{\frac{1}{2}\lambda s - 1}.$$

With the estimation

$$(\frac{d_{51}^2}{d_{41}^2} - \frac{d_{52}^2}{d_{42}^2})^2 = 0,$$

we can compute

$$P_3^{[1]} = 0.$$

Let $\eta \simeq \frac{c_j}{2d_{4j}}$. If γ is a positive integer, let $\gamma \geq N$ and $\gamma \simeq N$. Considering the Fourier transform on the unit sphere $\eta \in B_1(N^{1-\frac{1}{2}\lambda s})$, we apply the mean-value theorem and the Cauchy-Schwarz inequality,

$$\begin{aligned}
P_1^{[1]} &\simeq N^{\frac{1}{2}\lambda s(2s+1)} \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} \langle \eta \rangle^s |\mathcal{F}(F^{[1]})(\eta - \frac{c_1}{2d_{41}}) - \mathcal{F}(F^{[1]})(\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}})|^2 d\eta \\
&\simeq N^{\frac{1}{2}\lambda s(2s+1)}\, N^{2s(1-\frac{1}{2}\lambda s)} \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\int_{\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} \left(\mathcal{F}(F^{[1]})\right)'(\zeta) d\zeta|^2 d\eta \\
&\leq N^{\frac{1}{2}\lambda s + 2s} \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} d\beta|| \int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 d\zeta| d\eta \\
&\simeq N^{\frac{1}{2}\lambda s + 2s} |1 - \frac{d_{41}}{d_{42}}| \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\eta|| \int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 d\zeta| d\eta \\
&\simeq N^{\frac{1}{2}\lambda s + 2s - 1}\, |N_1 - N_2| \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\eta|| \int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 d\zeta| d\eta.
\end{aligned} \tag{26}$$

By the Fubini theorem,

$$\begin{aligned}
&\int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\eta|| \int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 d\zeta| d\eta \\
= \;& \int_0^{\frac{\gamma}{d_{41}}} \eta \int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 d\zeta d\eta - \int_{-\frac{\gamma}{d_{41}}}^0 \eta \int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 d\zeta d\eta \\
= \;& \int_{\frac{-c_1}{2d_{41}}}^{\frac{\gamma}{d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 \int_{\zeta + \frac{c_1}{2d_{41}}}^{(\zeta + \frac{c_1}{2d_{41}})\frac{d_{42}}{d_{41}}} \eta d\eta d\zeta - \int_{-\frac{\gamma}{d_{41}}}^{\frac{-c_1}{2d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 \int_{(\zeta + \frac{c_1}{2d_{41}})\frac{d_{42}}{d_{41}}}^{\zeta + \frac{c_1}{2d_{41}}} \eta d\eta d\zeta \\
= \;& \frac{1}{2} \int_{-\frac{c_1}{2d_{41}}}^{\frac{\gamma}{d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 (\zeta + \frac{c_1}{2d_{41}})^2 \left[(\frac{d_{42}}{d_{41}})^2 - 1\right] d\zeta \\
& -\frac{1}{2} \int_{-\frac{\gamma}{d_{41}}}^{-\frac{c1}{2d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 (\zeta + \frac{c_1}{2d_{41}})^2 \left[1 - (\frac{d_{42}}{d_{41}})^2\right] d\zeta \\
= \;& \frac{1}{2} \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 (\zeta + \frac{c_1}{2d_{41}})^2 \left[(\frac{d_{42}}{d_{41}})^2 - 1\right] d\zeta.
\end{aligned} \tag{27}$$

Note that

$$\frac{{d_{41}}^2 - {d_{42}}^2}{{d_{41}}^2} = \frac{N_1^{\lambda s} - N_2^{\lambda s}}{N_1^{\lambda s}} \simeq \frac{(N_1 - N_2)N^{\lambda s-1}}{N^{\lambda s}} = \frac{N_1 - N_2}{N}.$$

Then, by (26) and (27), we have

$$\begin{aligned}
{P_1}^{[1]} &\leq N^{\frac{1}{2}\lambda s+2s-1}|N_1 - N_2| \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2(\zeta + \frac{c_1}{2d_{41}})^2[1 - (\frac{d_{42}}{d_{41}})^2]d\zeta \\
&\simeq N^{\frac{1}{2}\lambda s+2s-2}|N_1 - N_2|^2 \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2(\zeta + \frac{c_1}{2d_{41}})^2 d\zeta \\
&\simeq N^{\frac{1}{2}\lambda s+2s-2}|N_1 - N_2|^2 N^{2(1-\frac{1}{2}\lambda s)} \left\|\left(\mathcal{F}(F^{[1]})\right)'(\zeta)\right\|^2_{L^2(\mathrm{Z}_{\frac{\gamma}{d_{41}}})} \\
&= N^{-\frac{1}{2}\lambda s+2s}|N_1 - N_2|^2 \left\|\left(\mathcal{F}(F^{[1]})\right)'(\zeta)\right\|^2_{L^2(\mathrm{Z}_{\frac{\gamma}{d_{41}}})}.
\end{aligned} \tag{28}$$

Similar to the computation of P_1,

$$\begin{aligned}
{P_2}^{[1]} &\simeq (d_{41})^{2s+1}\left(\frac{d_{42}}{d_{41}}\right)^{2s+1} \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\eta|^{2s}|\mathcal{F}(F^{[1]})(\eta - \frac{c_1}{2d_{41}}) - \mathcal{F}(F^{[1]})(\eta - \frac{c_2}{2d_{42}})|^2 d\eta \\
&\simeq N^{\frac{1}{2}\lambda s(2s+1)}\, N^{2s(1-\frac{1}{2}\lambda s)} \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\int_{\eta-\frac{c_2}{2d_{42}}}^{\eta-\frac{c_1}{2d_{41}}} \left(\mathcal{F}(F^{[1]})\right)'(\zeta)d\zeta|^2 d\eta \\
&\leqslant N^{\frac{1}{2}\lambda s+2s} \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\int_{\eta-\frac{c_2}{2d_{42}}}^{\eta-\frac{c_1}{2d_{41}}} d\zeta|| \int_{\eta-\frac{c_2}{2d_{42}}}^{\eta-\frac{c_1}{2d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 d\zeta| d\eta \\
&\simeq N^{\frac{1}{2}\lambda s+2s}|\frac{c_1}{2d_{41}} - \frac{c_2}{2d_{42}}| \int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} \int_{\eta-\frac{c_2}{2d_{42}}}^{\eta-\frac{c_1}{2d_{41}}} |\left(\mathcal{F}(F^{[1]})\right)'(\zeta)|^2 d\zeta d\eta \\
&\simeq N^{\frac{1}{2}\lambda s+2s}|\frac{c_1}{2d_{41}} - \frac{c_2}{2d_{42}}|^2 \left\|\left(\mathcal{F}(F^{[1]})\right)'\right\|^2_{L^2(\mathrm{Z}_{\frac{\gamma}{d_{41}}})} \\
&\simeq N^{-\frac{1}{2}\lambda s+2s}|N_2 - N_1|^2 \left\|\left(\mathcal{F}(F^{[1]})\right)'\right\|^2_{L^2(\mathrm{Z}_{\frac{\gamma}{d_{41}}})}.
\end{aligned} \tag{29}$$

Note that

$$\left\|\left(\mathcal{F}(F^{[1]})\right)'\right\|^2_{L^2(\mathbb{R})} = \left\|xh^{[1]}(x)\right\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{+\infty} \frac{2\,x^2}{B_1\cos(2\,x) + B_1 + 2\,C_1} dx = +\infty,$$

Here, the estimate of the supreme (28) and (29) of $P_1^{[1]}$ goes to infinity because $|F|$ is a function with π as its period.

Note that with the Plancherel theorem, we have the estimate of F,

$$
\begin{aligned}
\left\|\mathcal{F}(F^{[1]})\right\|^2_{L^2(\mathrm{Z}_{\frac{\gamma}{d_{41}}})} &= \left\|h^{[1]}(x)\right\|^2_{L^2(\mathrm{T}_{\frac{\gamma}{d_{41}}})} \\
&= \int_{-\frac{\gamma\pi}{d_{41}}}^{\frac{\gamma\pi}{d_{41}}} \frac{2\ dx}{B_1 \cos(2\,x) + B_1 + 2\,C_1} \\
&= \int_0^{\frac{\gamma\pi}{d_{41}}} \frac{2\ dx}{B_1 \cos(2\,x) + B_1 + 2\,C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{2\ dx}{B_1 \cos(2\,x) + B_1 + 2\,C_1} \\
&\leq \int_0^{\frac{\gamma\pi}{d_{41}}} \frac{2\ dx}{B_1 + 2\,C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{dx}{C_1} \\
&= \frac{2\,\gamma\pi}{(B_1 + 2\,C_1)\,d_{41}} + \frac{\gamma\pi}{C_1\,d_{41}} \\
&\simeq \frac{\gamma}{d_{41}} N^{-1}.
\end{aligned} \tag{30}
$$

Similarly,

$$
\begin{aligned}
\left\|\left(\mathcal{F}(F^{[1]})\right)'\right\|^2_{L^2(\mathrm{Z}_{\frac{\gamma\pi}{d_{41}}})} &= \left\|x h^{[1]}(x)\right\|^2_{L^2(\mathrm{T}_{\frac{\gamma\pi}{d_{41}}})} \\
&= \int_{-\frac{\gamma\pi}{d_{41}}}^{\frac{\gamma\pi}{d_{41}}} \frac{2\,x^2\ dx}{B_1 \cos(2\,x) + B_1 + 2\,C_1} \\
&= \int_0^{\frac{\gamma\pi}{d_{41}}} \frac{2\,x^2\ dx}{B_1 \cos(2\,x) + B_1 + 2\,C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{2\,x^2\ dx}{B_1 \cos(2\,x) + B_1 + 2\,C_1} \\
&\leq \int_0^{\frac{\gamma\pi}{d_{41}}} \frac{2\,x^2\ dx}{B_1 + 2\,C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{x^2\ dx}{C_1} \\
&= \frac{2\,\gamma^3\pi^3}{3\ (B_1 + 2\,C_1)\,{d_{41}}^3} + \frac{\gamma^3\pi^3}{3\,C_1 {d_{41}}^3} \\
&\simeq \frac{\gamma^3}{{d_{41}}^3} N^{-1}.
\end{aligned} \tag{31}
$$

Similarly, we can estimate the lower bound of $(\mathcal{F}(F))'$,

$$
\begin{aligned}
\left\|\left(\mathcal{F}(F^{[1]})\right)'\right\|^2_{L^2(\mathrm{Z}_{\frac{\gamma}{d_{41}}})} &= \int_0^{\frac{\gamma\pi}{d_{41}}} \frac{2\,x^2\ dx}{B_1 \cos(2\,x) + B_1 + 2\,C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{2\,x^2\ dx}{B_1 \cos(2\,x) + B_1 + 2\,C_1} \\
&\geq \int_0^{\frac{\gamma\pi}{d_{41}}} \frac{x^2\ dx}{B_1 + C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^{0} \frac{x^2\ dx}{B_1 + 2\,C_1} \\
&= \frac{\gamma^3\pi^3}{3\ (B_1 + C_1)\,{d_{41}}^3} + \frac{\gamma^3\pi^3}{3\ (B_1 + 2\,C_1)\,{d_{41}}^3} \\
&\simeq \frac{\gamma^3}{{d_{41}}^3} N^{-1},
\end{aligned}
$$

Equations (31) and (32) indicate that $(\mathcal{F}(F))'$ is in a scale of $\frac{\gamma^3}{{d_{41}}^3} N^{-1}$. Combining (25), (28), (29), (31), and (32), we have the estimate in $H^s(\mathrm{T}_{\frac{\gamma}{d_{41}}})$,

$$\|\varphi^{[1]}_{c_1,\omega_1} - \varphi^{[1]}_{c_2,\omega_2}\|^2_{H^s(\mathrm{T}_{\frac{\gamma}{d_{41}}})} \leq N^{-\frac{1}{2}\lambda s+2s-1}|N_2 - N_1|^2 \frac{\gamma^3}{{d_{41}}^3}.$$

We integrate the function in one period and choose the period as follows:

$$T_{\frac{1}{d_{41}}} = \left[-\frac{\pi}{d_{41}},\ \frac{\pi}{d_{41}}\right], \quad \mathrm{Z}_{\frac{1}{d_{41}}} = \left[-\frac{1}{d_{41}},\ \frac{1}{d_{41}}\right].$$

Then, we have the estimate in $H^s(\mathrm{T}_{\frac{1}{d_{41}}})$,

$$\|\varphi^{[1]}_{c_1,\omega_1} - \varphi^{[1]}_{c_2,\omega_2}\|^2_{H^s(\mathrm{T}_{\frac{1}{d_{41}}})} \leq N^{-2\lambda s+2s+1}|N_2 - N_1|^2.$$

If $-2\lambda s + 2s + 1 < 0$, let $b = |-2\lambda s + 2s + 1|$. Then, we can control the distance between solitons

$$N_2 - N_1 = \delta N^{\varepsilon s}, \tag{32}$$

to control the norm

$$\|\varphi^{[1]}_{c_1,\omega_1} - \varphi^{[1]}_{c_2,\omega_2}\|^2_{H^s(\mathrm{T}_{\frac{1}{d_{41}}})} \leq \tilde{C}\delta^2 N^{2\varepsilon s-b} \leq \tilde{C}\delta^2,$$

where $\tilde{C} > 0$ is a constant, and ε and δ are any real values larger than 0.

Similarly, we can estimate the lower bound of $\mathcal{F}(F^{[1]})$,

$$\begin{aligned}\left\|\mathcal{F}(F^{[1]})\right\|^2_{L^2(\mathrm{Z}_{\frac{\gamma\pi}{d_{41}}})} &= \int_0^{\frac{\gamma\pi}{d_{41}}} \frac{2\ dx}{B_1\cos(2\,x) + B_1 + 2\,C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^0 \frac{2\ dx}{B_1\cos(2\,x) + B_1 + 2\,C_1} \\ &\geq \int_0^{\frac{\gamma\pi}{d_{41}}} \frac{dx}{B_1 + C_1} + \int_{-\frac{\gamma\pi}{d_{41}}}^0 \frac{dx}{B_1 + 2\,C_1} \\ &\simeq \frac{\gamma}{d_{41}} N^{-1}.\end{aligned} \tag{33}$$

Combining (30) and (33), we then compute the norm of $\varphi^{[1]}_{c,\omega}$ in $H^s(\mathrm{T}_{\frac{\gamma}{d_{41}}})$,

$$\begin{aligned}\|\varphi^{[1]}_{c,\omega}\|^2_{H^s(\mathrm{T}_{\frac{\gamma}{d_{41}}})} &\simeq d_4^{2s+1}\frac{d_5^2}{d_4^2}\int_{\mathrm{Z}_{\frac{\gamma}{d_{41}}}} |\eta|^{2s}|\mathcal{F}(F^{[1]})(\eta - \frac{c}{2d_4})|^2 d\eta \\ &\simeq N^{(\frac{1}{2}\lambda s)(2s+1)} N^{2s(1-\frac{1}{2}\lambda s)}\|h^{[1]}(x)\|^2_{L^2(\mathrm{T}_{\frac{\gamma}{d_{41}}})} \\ &\simeq N^{\frac{1}{2}\,s\lambda+2\,s-1}\frac{\gamma}{d_{41}},\end{aligned} \tag{34}$$

We then have the estimate of the norm of $\varphi^{[1]}_{c,\omega}$ in $H^s(\mathrm{T}_{\frac{1}{d_{41}}})$

$$\|\varphi^{[1]}_{c,\omega}\|^2_{H^s(\mathrm{T}_{\frac{1}{d_{41}}})} \simeq N^{2\,s-1}. \tag{35}$$

Assume that the solution is $Q_{c_j,\omega_j}(x,\ T)$ at $t = T$. By the translational invariant of the traveling wave solution and (35), we have

$$\|Q_{c_j,\omega_j}(x,\ T)\|^2_{H^s(\mathrm{T}_{\frac{1}{d_{41}}})} = \|\varphi^{[1]}_{c_j,\omega_j}\|^2_{H^s(\mathrm{T}_{\frac{1}{d_{41}}})} \simeq N^{2\,s-1}.$$

On the other hand,

$$Q_{c_j,\omega_j}(x,\ T)=e^{-i\omega_j T}e^{i\psi(x-c_jT)}d_{5j}h^{[1]}(d_{4j}(x-c_j)T).$$

Restricting $Q_{c_j,\omega_j}(x,T)$ on the sphere $B_{(d_{4j})^{-1}}(Tc_j)$, we can choose c_j and ω_j to determine the phase. Then, combining (30) and (33), we have

$$\begin{aligned}\|Q_{c_1,\omega_1}(x,\ T)-Q_{c_2,\omega_2}(x,\ T)\|^2_{L^2(\mathrm{T}_{\frac{\gamma}{d_{41}}})} &\simeq \|Q_{c_1,\omega_1}(x,\ T)\|^2_{L^2(\mathrm{T}_{\frac{\gamma}{d_{41}}})}+\|Q_{c_2,\omega_2}(x,\ T)\|^2_{L^2(\mathrm{T}_{\frac{\gamma}{d_{41}}})}\\ &\simeq {d_{5j}}^2\int_{-\frac{\gamma\pi}{d_{41}}}^{\frac{\gamma\pi}{d_{41}}} {h^{[1]}}^2(d_{4j}(x-c_jT))dx\\ &\simeq \frac{1}{d_{4j}}{d_{5j}}^2\|h^{[1]}(x)\|^2_{L^2(\mathrm{T}_{\frac{\gamma}{d_{41}}})}\\ &\simeq N^{\frac{1}{2}\lambda s-1}\frac{\gamma}{d_{41}},\end{aligned}$$

So

$$\begin{aligned}\|Q_{c_1,\omega_1}(x,\ T)-Q_{c_2,\omega_2}(x,\ T)\|^2_{H^s(\mathrm{T}_{\frac{\gamma}{d_{41}}})} &= \int_{-\frac{\gamma}{d_{41}}}^{\frac{\gamma}{d_{41}}}(1+|\mu|^2)^s|\hat{Q}_{c_1,\omega_1}(\mu)-\hat{Q}_{c_2,\omega_2}(\mu)|^2d\mu\\ &\geq N^{2s}\|Q_{c_1,\omega_1}(x,\ T)-Q_{c_2,\omega_2}(x,\ T)\|^2_{L^2(\mathrm{T}_{\frac{\gamma}{d_{41}}})}\\ &\simeq N^{2s+\frac{1}{2}\lambda s-1}\frac{\gamma}{d_{41}}\end{aligned}$$

Furthermore,

$$\|Q_{c_1,\omega_1}(x,\ T)-Q_{c_2,\omega_2}(x,\ T)\|^2_{H^s(\mathrm{T}_{\frac{1}{d_{41}}})}\geq N^{2s-1}$$

In the following, we study the separability of the wave packet. We choose c_1 and c_2 such that the wave has separability with $N^{-\frac{1}{2}\lambda s}$. In fact, we only need to choose N such that $N^{\frac{1}{2}\lambda s+\varepsilon s}\gg\ (T\delta)^{-1}$, and by the phase distance $N_1-N_2=\delta N^{\varepsilon s}$ of (32), we have

$$T(c_2-c_1)=T(N_2-N_1)\gg\max(\frac{1}{d_{41}},\ \frac{1}{d_{42}})\simeq N^{-\frac{1}{2}\lambda s}.$$

As a summary,

$$\|Q_{c_1,\omega_1}(x,0)-Q_{c_2,\omega_2}(x,0)\|^2_{H^s(\mathrm{T}_{\frac{1}{d_{41}}})}\leq\delta,$$

$$Q_{c_j,\omega_j}(x,\ T)\in H^s(\mathrm{T}_{\frac{1}{d_{41}}}),\quad \|Q_{c_1,\omega_1}(x,\ T)-Q_{c_2,\omega_2}(x,\ T)\|^2_{H^s(\mathrm{T}_{\frac{1}{d_{41}}})}\geq\varepsilon.$$

Then this force $\lambda>0$ and $s>0$ to be

$$\left\{(\lambda,\ s)\,|\,1-\frac{1}{2}s\lambda>0,\ -2\lambda s+2s+1<0,\ 2\,s-1<0\right\}.$$

We then have the range of the weak ill-posedness index s:

$$0<s<\frac{1}{2}.$$

Similarly, if we solve the system

$$\left\{(\lambda,\ s)\,|\,1-\frac{1}{2}s\lambda>0,\ -2\lambda s+2s+1<0,\ 2\,s-1>0\right\},$$

we obtain the strong ill-posedness index s as follows:

$$\frac{1}{2}<s<\frac{3}{2}.$$

4. Ill-posedness of the non-periodic solution

In this section, we study the ill-posedness of (23) (and, at the same time, for (24)). In $\mathbb{R}$, equation (23) is not periodic, and has limited energy in $H^s(\mathbb{R})$. Therefore, we can apply the Fourier transform in the whole real line. We define the Fourier transform of the Lebesgue integrable function $f:\mathbb{R}\to\mathbb{C}$ in $\mathbb{R}$ as

$$\mathcal{F}_2(f)(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\ e^{-ix\xi}\,dx$$

where ξ is any real number.

Denote $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$. Then, the Sobolev space $H^s(\mathbb{R})$ is a complete C^∞ functional space with norm

$$\|f\|_{H^s(\mathbb{R})} := \|\langle\xi\rangle^s \mathcal{F}_2(f)(\xi)\|_{L^2(\mathbb{R})}.$$

Note that

$$sech(x)=2\,\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)^{-1},$$

Then, the solution in (23) can be transformed to

$$\phi(\xi)=\sqrt{2}\left(\frac{\mathrm{e}^{-\sqrt{-c^2-4\,\omega}x}\sqrt{-\omega\,\alpha^2+\alpha\,c+1}+\mathrm{e}^{\sqrt{-c^2-4\,\omega}x}\sqrt{-\omega\,\alpha^2+\alpha\,c+1}+\alpha\,c+2}{-c^2-4\,\omega}\right)^{-\frac{1}{2}}.$$

Let

$$A_2=\sqrt{-c^2-4\,\omega},\quad B_2=\sqrt{-\omega\,\alpha^2+\alpha\,c+1},\quad C_2=\alpha c+2.$$

Then, (23) is transformed to

$$\phi(\xi)=\sqrt{2}\left(\tfrac{\mathrm{e}^{A_2\xi}B_2}{A_2^2}+\tfrac{\mathrm{e}^{-A_2\xi}B_2}{A_2^2}+\tfrac{C_2}{A_2^2}\right)^{-\frac{1}{2}}.$$

At the same time, (20) is in the form of

$$\psi(\xi)=-3\ \arctan\left(\tfrac{2\,\mathrm{e}^{A_2\xi}B_2+C_2}{\alpha\,A_2}\right)+\tfrac{1}{2}\,c\xi.$$

Let

$$d_4=A_2,$$

$$d_5=\sqrt{2}\sqrt{\frac{B_2}{A_2^2}},$$

and

$$h^{[2]}(x)=\left(\mathrm{e}^{A_2x}+\mathrm{e}^{-A_2x}+\tfrac{C_2}{B_2}\right)^{-\frac{1}{2}}.$$

Then,

$$\phi(x)=d_5h^{[2]}(d_4x).$$

Let

$$g^{[2]}(x)=-3\ \arctan\left(\tfrac{2\,\mathrm{e}^{A_2x}B_2+C_2}{\alpha\,A_2}\right),$$

and

$$F^{[2]}(x) = e^{\mathrm{i}g^{[2]}(x)} h^{[2]}(x).$$

By (19), we define

$$\varphi^{[2]}_{c,\omega}(x) = Q_{c,\omega}(x,0) = d_5 e^{\mathrm{i}cx/2} F^{[2]}(d_4 x),$$

The transformation formula in $\mathbb{R}$ is

$$\mathcal{F}_2(\varphi^{[2]}_{c,\omega})(\xi) = \tfrac{d_5}{d_4}\mathcal{F}_2(F^{[2]})(\tfrac{\xi}{d_4} - \tfrac{c}{2d_4}).$$

Proposition 5. *If $\alpha \neq 0$, then the Cauchy problem of (2) could be ill-posed in $H^s(\mathrm{T}_\lambda)$. That is, $Q_0 \to Q(t)$ is not uniformly continuous. With the initial condition $Q_0 = \varphi^{[2]}_{c,\omega}(x) = d_5 e^{\mathrm{i}cx/2} F^{[2]}(d_4 x)$, the solution of (2) could be ill-posed in $H^s(\mathbb{R})$. Moreover, we have the following:*

(I) If $0 < s < \frac{1}{2}$, then the solution is weakly ill-posed.

(II) If $0 < s < 1$, then the solution is strongly ill-posed.

Proof. Similar to (25), we have

$$\begin{aligned} \|\varphi^{[2]}_{c_1,\omega_1} - \varphi^{[2]}_{c_2,\omega_2}\|^2_{H^s(\mathbb{R})} &= \int_{\mathbb{R}} \langle\xi\rangle^s |\mathcal{F}_2(\varphi^{[2]}_{c_1,\omega_1})(\xi) - \mathcal{F}_2(\varphi^{[2]}_{c_2,\omega_2})(\xi)|^2 d\xi \\ &\simeq P^{[2]}_1 + P^{[2]}_2 + P^{[2]}_3. \end{aligned} \tag{36}$$

where

$$P^{[2]}_1 = (d_{41})^{2s+1}\int_{\mathbb{R}} \langle\eta\rangle^s \frac{d_{51}^2}{d_{41}^2} |\mathcal{F}_2(F^{[2]})(\eta - \frac{c_1}{2d_{41}}) - \mathcal{F}_2(F^{[2]})(\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}})|^2 d\eta,$$

$$P^{[2]}_2 = (d_{41})^{2s+1}\int_{\mathbb{R}} \langle\eta\rangle^s \frac{d_{51}^2}{d_{41}^2} |\mathcal{F}_2(F^{[2]})(\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}) - \mathcal{F}_2(F^{[2]})(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}})|^2 d\eta,$$

$$P^{[2]}_3 = (d_{41})^{2s+1}\int_{\mathbb{R}} \langle\eta\rangle^s |\frac{d_{51}^2}{d_{41}^2} - \frac{d_{52}^2}{d_{42}^2}|\mathcal{F}_2(F^{[2]})(\eta\frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}})|^2 d\eta.$$

Assuming $\eta \simeq \frac{c_j}{2d_{4j}}$, we perform the Fourier transformation on the unit sphere $\eta \in B_1(N^{1-\frac{1}{2}\lambda s})$. We can estimate $P^{[2]}_1$, as with (26), to obtain

$$\begin{aligned} P^{[2]}_1 &\simeq (d_{41})^{2s+1}\frac{d_{51}^2}{d_{41}^2} N^{2s(1-\frac{1}{2}\lambda s)} \int_{\mathbb{R}} |\int_{\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} \mathcal{F}'_2(F^{[2]})(\beta) d\beta|^2 d\eta \\ &\leq (d_{41})^{2s+1}\frac{d_{51}^2}{d_{41}^2} N^{2s(1-\frac{1}{2}\lambda s)} \int_{\mathbb{R}} |\int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} d\alpha|| \int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\mathcal{F}'_2(F^{[2]})(\beta)|^2 d\beta| d\eta \\ &\simeq (d_{41})^{2s+1}\frac{d_{51}^2}{d_{41}^2} N^{2s(1-\frac{1}{2}\lambda s)} |1 - \frac{d_{41}}{d_{42}}| \int_{\mathbb{R}} |\eta|| \int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\mathcal{F}'_2(F^{[2]})(\beta)|^2 d\beta| d\eta, \end{aligned} \tag{37}$$

Denote

$$I^{[2]}_1 = \int_0^\infty \eta \int_{\eta\frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\hat{L}'(\beta)|^2 d\beta d\eta.$$

By Fubini's theorem, we can change the order of integration,

$$\begin{aligned} I^{[2]}_1 &= \int_{\frac{-c_1}{2d_{41}}}^{\infty} |\mathcal{F}'_2(F^{[2]})(\beta)|^2 \int_{\beta + \frac{c_1}{2d_{41}}}^{(\beta + \frac{c_1}{2d_{41}})\frac{d_{42}}{d_{41}}} \eta d\eta d\beta \\ &= \frac{1}{2}\int_{-\frac{c_1}{2d_{41}}}^{\infty} |\mathcal{F}'_2(F^{[2]})(\beta)|^2 (\beta + \frac{c_1}{2d_{41}})^2 \left[(\frac{d_{42}}{d_{41}})^2 - 1\right] d\beta. \end{aligned} \tag{38}$$

Moreover, we set

$$I_2^{[2]} = \int_{-\infty}^{0} \eta \int_{\eta \frac{d_{41}}{d_{42}} - \frac{c1}{2d_{41}}}^{\eta - \frac{c1}{2d_{41}}} |\mathcal{F}'_2(F^{[2]})(\beta)|^2 d\beta d\eta.$$

Similarly,

$$I_2^{[2]} = \frac{1}{2} \int_{-\infty}^{-\frac{c1}{2d_{41}}} |\mathcal{F}'_2(F^{[2]})(\beta)|^2 (\beta + \frac{c_1}{2d_{41}})^2 \left[1 - (\frac{d_{42}}{d_{41}})^2\right] d\beta. \tag{39}$$

Let $N_j (j = 1, 2)$ and N be large positive integers with the following relation:

$$\mathrm{c_j} = N_j \simeq N, \quad \omega_{\mathrm{j}} = -N_j^{\lambda s} - \frac{N_j^2}{4}$$

$$d_{4j} = 2N_j^{\frac{1}{2}\lambda s}, \quad d_{5j} = \left(\frac{\sqrt{\alpha^2 (N_j^{\lambda s} + N_j^2/4) + \alpha N_j + 1}}{2N_j^{\lambda s}} \right)^{-\frac{1}{2}} \simeq N_j^{-\frac{1}{2} + \frac{1}{2}\lambda s}, \quad \alpha \neq 0.$$

Assume that $N_1 < N_2$. Then, we have

$$|d_{41} - d_{42}| \simeq |N_1 - N_2| \, N^{\frac{1}{2}\lambda s - 1}.$$

Similarly,

$$\frac{{d_{41}}^2 - {d_{42}}^2}{{d_{41}}^2} = \frac{4N_1^{\lambda s} - 4N_2^{\lambda s}}{4N_1^{\lambda s}} \simeq \frac{(N_1 - N_2) N^{\lambda s - 1}}{N^{\lambda s}} = \frac{N_1 - N_2}{N}$$

and

$$\frac{c_1}{2d_{41}} - \frac{c_2}{2d_{42}} = \frac{N_1}{2N_1^{\frac{1}{2}\lambda s}} - \frac{N_2}{2N_2^{\frac{1}{2}\lambda s}} \simeq \frac{N_1 - N_2}{N}.$$

Combining (37), (38), and (39), we obtain

$$\begin{aligned} P_1^{[2]} &\leq N^{\frac{1}{2}\lambda\, s - 2 + 2\, s} \, |N_1 - N_2| \, (I_1^{[2]} - I_2^{[2]}) \\ &\simeq N^{\frac{1}{2}\lambda\, s - 2 + 2\, s} \, |N_1 - N_2| \left| 1 - (\frac{d_{42}}{d_{41}})^2 \right| \int_{\mathbb{R}} |\mathcal{F}'_2(F^{[2]})(\beta)|^2 (\beta + \frac{c_1}{2d_{41}})^2 d\beta \\ &\simeq N^{\frac{1}{2}\lambda\, s - 2 + 2\, s} \, |N_1 - N_2| \left| 1 - (\frac{d_{42}}{d_{41}})^2 \right| N^{2(1 - \frac{1}{2}\lambda s)} \left\| \mathcal{F}'_2(F^{[2]}) \right\|_{L^2}^2 \\ &= N^{-\frac{1}{2}\lambda\, s - 1 + 2\, s} \, (N_1 - N_2)^2 \left\| \mathcal{F}'_2(F^{[2]}) \right\|_{L^2}^2, \end{aligned} \tag{40}$$

and, similarly,

$$\begin{aligned} P_2^{[2]} &\simeq (d_{41})^{2s+1} \frac{d_{51}^2}{d_{41}^2} \left(\frac{d_{42}}{d_{41}} \right)^{2s+1} \int_{\mathbb{R}} |\eta|^{2s} |\mathcal{F}_2(F^{[2]})(\eta - \frac{c_1}{2d_{41}}) - \mathcal{F}_2(F^{[2]})(\eta - \frac{c_2}{2d_{42}})|^2 d\eta \\ &\simeq \frac{d_{51}^2 {d_{42}}^{2s+1}}{d_{41}^2} N^{2s(1 - \frac{1}{2}\lambda s)} \int_{\mathbb{R}} | \int_{\eta - \frac{c_2}{2d_{42}}}^{\eta - \frac{c_1}{2d_{41}}} \mathcal{F}'_2(F^{[2]})(\alpha) d\alpha|^2 d\eta \\ &\leq \frac{d_{51}^2 {d_{42}}^{2s+1}}{d_{41}^2} | \frac{c_1}{2d_{41}} - \frac{c_2}{2d_{42}} | N^{2s(1 - \frac{1}{2}\lambda s)} \int_{\mathbb{R}} \int_{\eta - \frac{c_2}{2d_{42}}}^{\eta - \frac{c_1}{2d_{41}}} |\mathcal{F}'_2(F^{[2]})(\alpha)|^2 d\alpha d\eta \\ &\simeq \frac{d_{51}^2 {d_{42}}^{2s+1}}{d_{41}^2} | \frac{c_1}{2d_{41}} - \frac{c_2}{2d_{42}} |^2 N^{2s(1 - \frac{1}{2}\lambda s)} \left\| \mathcal{F}'_2(F^{[2]}) \right\|_{L^2}^2 \\ &\simeq N^{\frac{1}{2}\,\lambda\, s - 3 + 2\, s} |N_2 - N_1|^2 \left\| \mathcal{F}'_2(F^{[2]}) \right\|_{L^2}^2. \end{aligned} \tag{41}$$

We compute

$$\frac{{d_{51}}^2}{{d_{41}}^2} - \frac{{d_{52}}^2}{{d_{42}}^2} \simeq \left(\frac{N_1^{-\frac{1}{2}+\frac{1}{2}\lambda s}}{N_1^{\frac{1}{2}\lambda s}}\right)^2 - \left(\frac{N_2^{-\frac{1}{2}+\frac{1}{2}\lambda s}}{N_2^{\frac{1}{2}\lambda s}}\right)^2 \simeq (N_1 - N_2)\, N^{-2},$$

and so,

$$\begin{aligned} P_3^{[2]} &\simeq (d_{41})^{2s+1}(\frac{d_{51}^2}{d_{41}^2} - \frac{d_{52}^2}{d_{42}^2})^2 \int_{\mathbb{R}} |\eta|^{2s} |\mathcal{F}_2(F^{[2]})(\eta \frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}})|^2 d\eta \\ &\simeq (d_{41})^{2s+1}(\frac{d_{51}^2}{d_{41}^2} - \frac{d_{52}^2}{d_{42}^2})^2 \left(\frac{d_{42}}{d_{41}}\right)^{2s+1} \int_{\mathbb{R}} |\eta|^{2s} |\mathcal{F}_2(F^{[2]})(\eta - \frac{c_2}{2d_{42}})|^2 d\eta \\ &\simeq N^{\lambda\, s^2 + \frac{1}{2}\,\lambda\, s - 4} |N_1 - N_2|^2 \left(\frac{c_2}{2d_{42}}\right)^{2s} \left\|\mathcal{F}_2(F^{[2]})\right\|^2 \\ &\simeq N^{\frac{1}{2}\lambda\, s - 4 + 2\, s}(N_1 - N_2)^2 \left\|F^{[2]}\right\|_{L^2}^2. \end{aligned} \tag{42}$$

Let $K = C_2/B_2$, where $\tilde{C}$ is a constant. Thus, we have

$$\begin{aligned} \left\|F^{[2]}\right\|_{L^2}^2 &= \left\|h^{[2]}(x)\right\|_{L^2}^2 \\ &= \int_{-\infty}^{+\infty} \frac{dx}{\mathrm{e}^x + \mathrm{e}^{-x} + K} \\ &\le \int_0^{+\infty} \frac{dx}{\mathrm{e}^x} + \int_{-\infty}^0 \frac{dx}{\mathrm{e}^{-x}} \\ &\le \tilde{C}. \end{aligned} \tag{43}$$

Similarly, we have

$$\begin{aligned} \left\|\mathcal{F}'{}_2(F^{[2]})\right\|^2 &= \left\|x h^{[2]}(x)\right\|^2 \\ &= \int_{-\infty}^{+\infty} \frac{x^2 dx}{\mathrm{e}^x + \mathrm{e}^{-x} + K} \\ &\le \int_0^{+\infty} \frac{x^2 dx}{\mathrm{e}^x} + \int_{-\infty}^0 \frac{x^2 dx}{\mathrm{e}^{-x}} \\ &\le \tilde{C}, \end{aligned} \tag{44}$$

If $1 - \frac{1}{2}\lambda s > 0$, substituting (44) with (40) and (41),

$$P_1^{[2]} \le \tilde{C} N^{-\frac{1}{2}\lambda\, s - 1 + 2\, s} \left(N_1 - N_2\right)^2,$$

$$P_2^{[2]} \le \tilde{C} N^{\frac{1}{2}\,\lambda\, s - 3 + 2\, s} \left(N_1 - N_2\right)^2.$$

Combining (42) and (43), the following holds:

$$P_3^{[2]} \le \tilde{C} N^{\frac{1}{2}\lambda\, s - 4 + 2\, s}(N_1 - N_2)^2,$$

If $-\frac{1}{2}\lambda\, s - 1 + 2\, s < 0$, $\frac{1}{2}\,\lambda\, s - 3 + 2\, s < 0$, and $\frac{1}{2}\lambda\, s - 4 + 2\, s < 0$, (36) satisfies

$$\|\varphi_{c_1,\omega_1}^{[2]} - \varphi_{c_2,\omega_2}^{[2]}\|_{H^s}^2 \le \frac{\tilde{C}(N_1 - N_2)^2}{N^b},$$

where

$$b = Min\left\{|-\frac{1}{2}\lambda\, s-1+2\, s|,\ |\frac{1}{2}\,\lambda\, s-3+2\, s|,\ |\frac{1}{2}\lambda\, s-4+2\, s|\right\}.$$

Then, we can control the distance between solitons

$$N_2 - N_1 = \delta N^{\varepsilon s}$$

to control the norm

$$\|\varphi^{[2]}_{c_1,\omega_1} - \varphi^{[2]}_{c_2,\omega_2}\|^2_{H^s} \le \tilde{C}\delta^2 N^{2\varepsilon s - b} \le \tilde{C}\delta^2. \tag{45}$$

According to (43), there is an upper bound of $\left\|F^{[2]}\right\|^2_{L^2}$. Moreover, the lower bound of it can be estimated as follows:

$$\begin{aligned}
\left\|F^{[2]}\right\|^2_{L^2} &= \left\|h^{[2]}(x)\right\|^2_{L^2} \\
&= \int_{-\infty}^{+\infty} \frac{dx}{\mathrm{e}^x + \mathrm{e}^{-x} + K} \\
&= \begin{cases} \frac{2}{\sqrt{K^2-4}}\operatorname{arctanh}(\frac{K}{\sqrt{K^2-4}}), \quad K>2 \\ 1, \quad K=2 \\ \frac{1}{\sqrt{-K^2+4}}\left(-2\,\arctan\left(\frac{K}{\sqrt{-K^2+4}}\right)+\pi\right), \quad 0\le K<2 \end{cases} \\
&\ge \tilde{C}(\alpha) > 0,
\end{aligned}$$

where $\tilde{C}(\alpha)$ is a constant that depends on α.

So, we have

$$\begin{aligned}
\|\varphi_{c,\omega}\|^2_{H^s} &\simeq d_4^{2s+1}\frac{d_5^2}{d_4^2}\int_{-\infty}^{+\infty} |\eta|^{2s}|\mathcal{F}_2(F^{[2]})(\eta - \frac{c}{2d_4})|^2 d\eta \\
&\simeq N^{\frac{1}{2}\lambda s(2s+1)} N^{-1} N^{2s(1-\frac{1}{2}\lambda s)}\|F^{[2]}\|^2_{L^2} \\
&\simeq N^{\frac{1}{2}\lambda\, s + 2\, s - 1}.
\end{aligned}$$

In the following deduction, we consider the time-dependent solution (i.e., the solution $Q^{[2]}_{c_j,\omega_j}(x,\ T)$ at $t=T$). As we know, the solitary wave satisfies the translational invariance property. Thus, we have

$$\|Q^{[2]}_{c_j,\omega_j}(\cdot,\ T)\|^2_{H^s} = \|\varphi_{c_j,\omega_j}\|^2_{H^s} \simeq N^{\frac{1}{2}\lambda\, s+2\, s-1}.$$

By

$$\|Q^{[2]}_{c_1,\omega_1}(\cdot,\ T) - Q^{[2]}_{c_2,\omega_2}(\cdot,\ T)\|^2_{H^s} = \int_{\mathbb{R}} (1+|\mu|^2)^s |\hat{Q}^{[2]}_{c_1,\omega_1}(\mu) - \hat{Q}^{[2]}_{c_2,\omega_2}(\mu)|^2 d\mu,$$

we have

$$\|Q^{[2]}_{c_1,\omega_1}(\cdot,\ T) - Q^{[2]}_{c_2,\omega_2}(\cdot,\ T)\|^2_{H^s} \ge N^{2s}\|Q^{[2]}_{c_1,\omega_1}(\cdot,\ T) - Q^{[2]}_{c_2,\omega_2}(\cdot,\ T)\|^2_{L^2}. \tag{46}$$

In addition, we noted that

$$Q^{[2]}_{c_j,\omega_j}(x,\ T) = e^{-i\omega_j T} e^{i\psi(x-c_jT)} d_{5j} h^{[2]}(d_{4j}(x-c_j)T),$$

so we restrict $Q^{[2]}_{c_j,\omega_j}(T)$ on the sphere $B_{(d_{6j})^{-1}}(Tc_j)$. At the same time, different values of c_j and ω_j can be used to avoid the superposition of peaks, such that

$$\begin{aligned}
\|Q^{[2]}_{c_1,\omega_1}(\cdot,\ T) - Q^{[2]}_{c_2,\omega_2}(\cdot,\ T)\|^2_{L^2} &\simeq \|Q^{[2]}_{c_1,\omega_1}(\cdot,\ T)\|^2_{L^2} + \|Q^{[2]}_{c_2,\omega_2}(\cdot,\ T)\|^2_{L^2} \\
&\simeq {d_{5j}}^2 \int {h^{[2]}}^2(d_{4j}(x-c_j)T)dx \\
&\simeq \frac{1}{d_{4j}}{d_{5j}}^2\|F^{[2]}\|^2_{L^2} \\
&\simeq N^{\frac{1}{2}\lambda s - 1}.
\end{aligned} \tag{47}$$

By (46) and (47), we have

$$\|Q^{[2]}_{c_1,\omega_1}(\cdot,\ T)-Q^{[2]}_{c_2,\omega_2}(\cdot,\ T)\|^2_{H^s}\geq N^{\frac{1}{2}\lambda s+2s-1}.$$

We now turn to the possibility of the dispersion property for the soliton solution. If $Q^{[2]}_{c_j,\omega_j}(T)$ is on $B_{(d_{4j})^{-1}}(Tc_j)(j=1,2)$, we can choose c_1 and c_2 to ensure the dispersion on the scale of $N^{-\frac{1}{2}\lambda s}$. Specifically, we use N to ensure $N^{\frac{1}{2}\lambda s+\varepsilon s}\gg (T\delta)^{-1}$. At the same time, we keep $N_1-N_2=\delta N^{\varepsilon s}$, which is also used in (45). Then, we have

$$T(c_2-c_1)=T(N_2-N_1)>>\max(\frac{1}{d_{41}},\ \frac{1}{d_{42}})\simeq N^{-\frac{1}{2}\lambda s}.$$

Based on the above analysis, in order to obtain weak ill-posedness, λ and s ($s>0$) must fall into the following set:

$$\left\{(\lambda,s)\,|\,1-\frac{1}{2}\,\lambda\,s>0,\ -\frac{1}{2}\,\lambda\,s+2\,s<1,\ \frac{1}{2}\,\lambda\,s+2\,s<1,\ \frac{1}{2}\,\lambda\,s+2\,s<3,\ \frac{1}{2}\,\lambda\,s+2\,s<4\right\},$$

which shows that the ill-posedness index is

$$0<s<\frac{1}{2}.$$

Similarly, in order to obtain the strong ill-posedness of the solution, the set of λ and s is as follows:

$$\left\{(\lambda,s)\,|\,1-\frac{1}{2}\,\lambda\,s>0,\ -\frac{1}{2}\,\lambda\,s+2\,s<1,\ \frac{1}{2}\,\lambda\,s+2\,s>1,\ \frac{1}{2}\,\lambda\,s+2\,s<3,\ \frac{1}{2}\,\lambda\,s+2\,s<4\right\},$$

which indicates that the range of s is

$$0<s<1.$$

Proof of Theorem 2. With Proposition 4 and Proposition 5, Theorem 2 is now proved.

Proof of Theorem 1. If $f(x)$ is square-integrable in T_γ, the Plancherel theorem states that

$$\int_{\mathrm{T}_\gamma}|f(x)|^2dx=\int_{\mathrm{Z}_\gamma}|\mathcal{F}(f)(\xi)|^2d\xi$$

which can be used to estimate the relationship between Q and $\mathbf{S}$ as follows:

$$\begin{aligned}\|Q\|^2_{H^s(\mathrm{T}_\gamma)}&=\int_{\mathrm{Z}_\gamma}\langle\xi\rangle^s|\mathcal{F}_1(Q)(\xi)|^2d\xi\\&\simeq\int_{\mathrm{Z}_\gamma}\langle\xi\rangle^s\big(|S_{1x}|^2+|S_{2x}|^2+|S_{3x}|^2\big)d\xi\\&\simeq\int_{\mathrm{Z}_\gamma}\langle\xi\rangle^s\left(|\mathcal{F}_1(|S_{1x}|)(\xi)|^2+|\mathcal{F}_1(|S_{2x}|)(\xi)|^2+|\mathcal{F}_1(|S_{3x}|)(\xi)|^2\right)d\xi\\&\simeq\int_{\mathrm{Z}_\gamma}\langle\xi\rangle^s|\xi|^2\left(|\mathcal{F}_1(|S_1|)(\xi)|^2+|\mathcal{F}_1(|S_2|)(\xi)|^2+|\mathcal{F}_1(|S_3|)(\xi)|^2\right)d\xi\\&\simeq\|\mathbf{S}\|^2_{H^{s+1}_{\mathrm{S}^2}(\mathrm{T}_\gamma)}.\end{aligned}$$

Hence, we have

$$\|Q\|^2_{H^s(\mathrm{T}_\gamma)}\simeq\|\mathbf{S}\|^2_{H^{s+1}_{\mathrm{S}^2}(\mathrm{T}_\gamma)}. \tag{48}$$

$\mathbf{S}_j$ ($j = 1, 2$) falls on the sphere and $\mathbf{S}_j \cdot \mathbf{S}_j = 1$. Furthermore, the components of the vector $\mathbf{S}_j$ are non-intersecting traveling wave solutions. Hence, it may be assumed that $S_{1,i_x} \simeq S_{2,i_x}$ ($i = 1, 2, 3$). Then, we have

$$\begin{aligned}
&\|Q_1 - Q_2\|^2_{H^s(\mathrm{T}_\gamma)} \\
&= \int_{\mathrm{Z}_\gamma} \langle\xi\rangle^s |\mathcal{F}_1(Q_1)(\xi) - \mathcal{F}_1(Q_2)(\xi)|^2 d\xi \\
&\simeq \int_{\mathbb{R}} \langle\xi\rangle^s \left(|\mathcal{F}_1(Q_1)(\xi)|^2 + |\mathcal{F}_1(Q_2)(\xi)|^2 - 2|\mathcal{F}_1(Q_1)(\xi)||\mathcal{F}_1(Q_2)(\xi)|\right) d\xi \\
&\simeq \int_{\mathbb{R}} \langle\xi\rangle^s \left(|Q_1|^2 + |Q_2|^2 - 2|Q_1||Q_2|\right) d\xi \\
&\simeq \int_{\mathbb{R}} \langle\xi\rangle^s \left(||S_{1,1_x} - S_{2,1_x}|^\wedge|^2 + ||S_{1,2_x} - S_{2,2_x}|^\wedge|^2 + ||S_{3,2_x} - S_{3,2_x}|^\wedge|^2\right) d\xi \\
&\simeq \int_{\mathbb{R}} \langle\xi\rangle^s |\xi|^2 \left(|\mathcal{F}_1(|S_{1,1} - S_{2,1}|)(\xi)|^2 + |\mathcal{F}_1(|S_{1,2} - S_{2,2}|)(\xi)|^2 + |\mathcal{F}_1(|S_{1,3} - S_{2,3}|)(\xi)|^2\right) d\xi \\
&\simeq \|\mathbf{S}_1 - \mathbf{S}_2\|^2_{H^{s+1}_{\mathbb{S}^2}(\mathrm{T}_\gamma)}
\end{aligned}$$

which indicates

$$\|\mathbf{S}_1 - \mathbf{S}_2\|^2_{H^{s+1}_{\mathbb{S}^2}(\mathrm{T}_\gamma)} \simeq \|Q_1 - Q_2\|^2_{H^s(\mathrm{T}_\gamma)}. \tag{49}$$

Similar to (48) and (49), the non-periodic case admits the following isometric isomorphism relationship:

$$\|Q\|^2_{H^s(\mathbb{R})} \simeq \|\mathbf{S}\|^2_{H^{s+1}_{\mathbb{S}^2}(\mathbb{R})}, \qquad \|\mathbf{S}_1 - \mathbf{S}_2\|^2_{H^{s+1}_{\mathbb{S}^2}(\mathbb{R})} \simeq \|Q_1 - Q_2\|^2_{H^s(\mathbb{R})}. \tag{50}$$

With the equivalence relationship (48), (49), (50), and Theorem 2, we finish the proof of Theorem 1.

5. Conclusions

In this paper, we studied two different types of two-parameter solitary wave solutions to the deformed continuous Heisenberg spin equation (1). By the derivative Schröinger equation, we constructed these solutions and analyzed their ill-posedness in periodic space $H^s_{\mathbb{S}^2}(\mathbb{T})$ and non-periodic space $H^s_{\mathbb{S}^2}(\mathbb{R})$. Although different spaces were used to measure the two solutions, the range of the weak ill-posedness index was the same: $1 < s < \frac{3}{2}$. It is interesting to see that $s = \frac{1}{2}$ is a critical index that determines the ill-posedness. However, the strong ill-posedness index differed. Following the same analysis in weak cases, we obtained a result whereby periodic and non-periodic solutions cannot be well-posed in a bounded subset of $H^s_{\mathbb{S}^2}$ with the indexes $\frac{3}{2} < s < \frac{5}{2}$ and $1 < s < 2$, respectively.

In both weak and strong cases, it is worth pointing out that we only discussed the $\alpha \neq 0$ setting. Whether the $\lim_{\alpha\to 0}$ case will also keep the same range of the ill-posedness index remains unknown, and further results need to be explored in our future work. Moreover, as the solution in this paper assumes that $c_1 = c_2 = 0$, another interesting topic is the ill-posedness property in the case where $c_1,\ c_2 \neq 0$.

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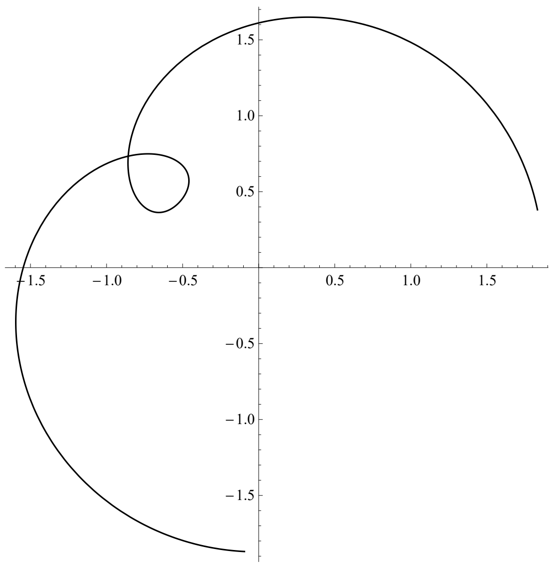
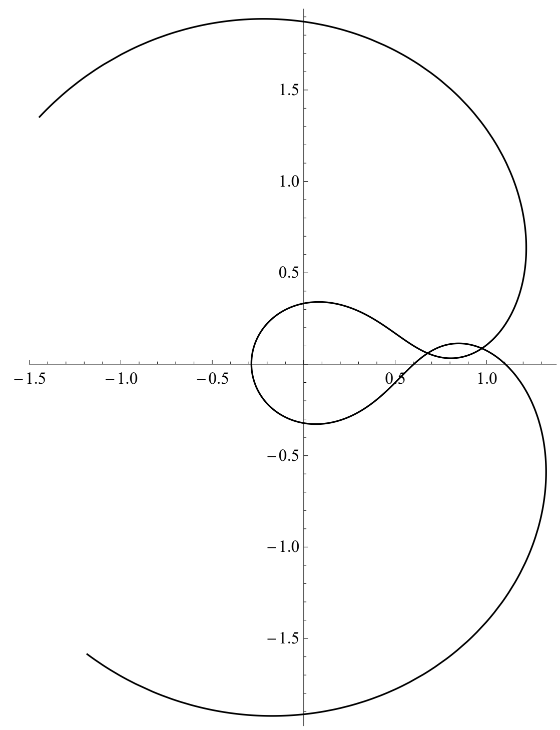
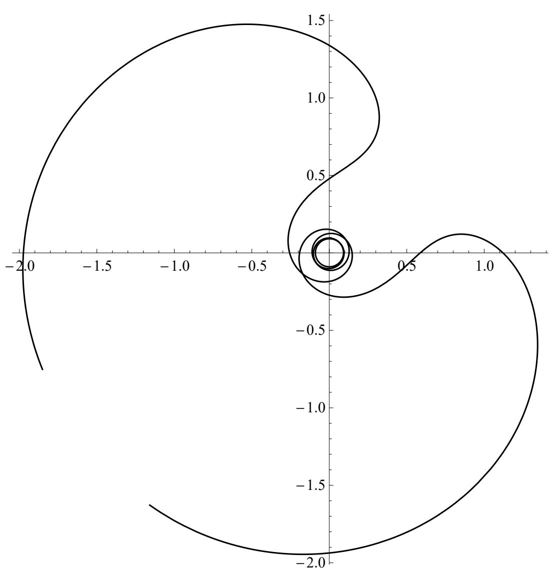


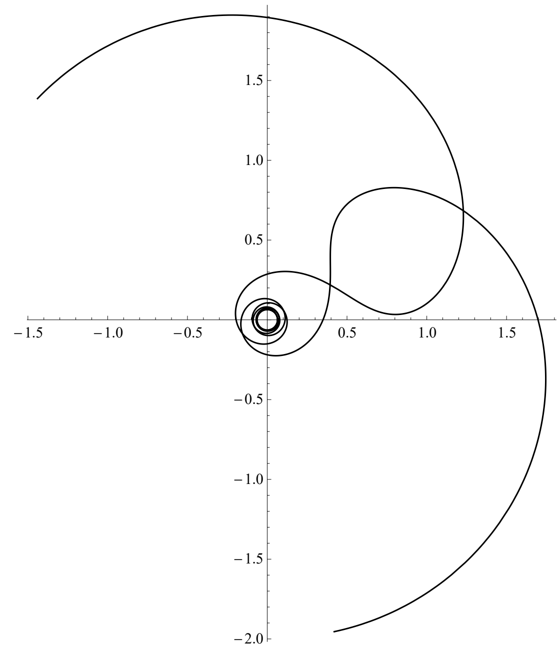
(a) α=10, c=-10, ω=-15, $x \in [10.4967, -9.50327]$



(b) α=50,c=-50,ω =-575, $x \in [-50.2221, -49.7779]$

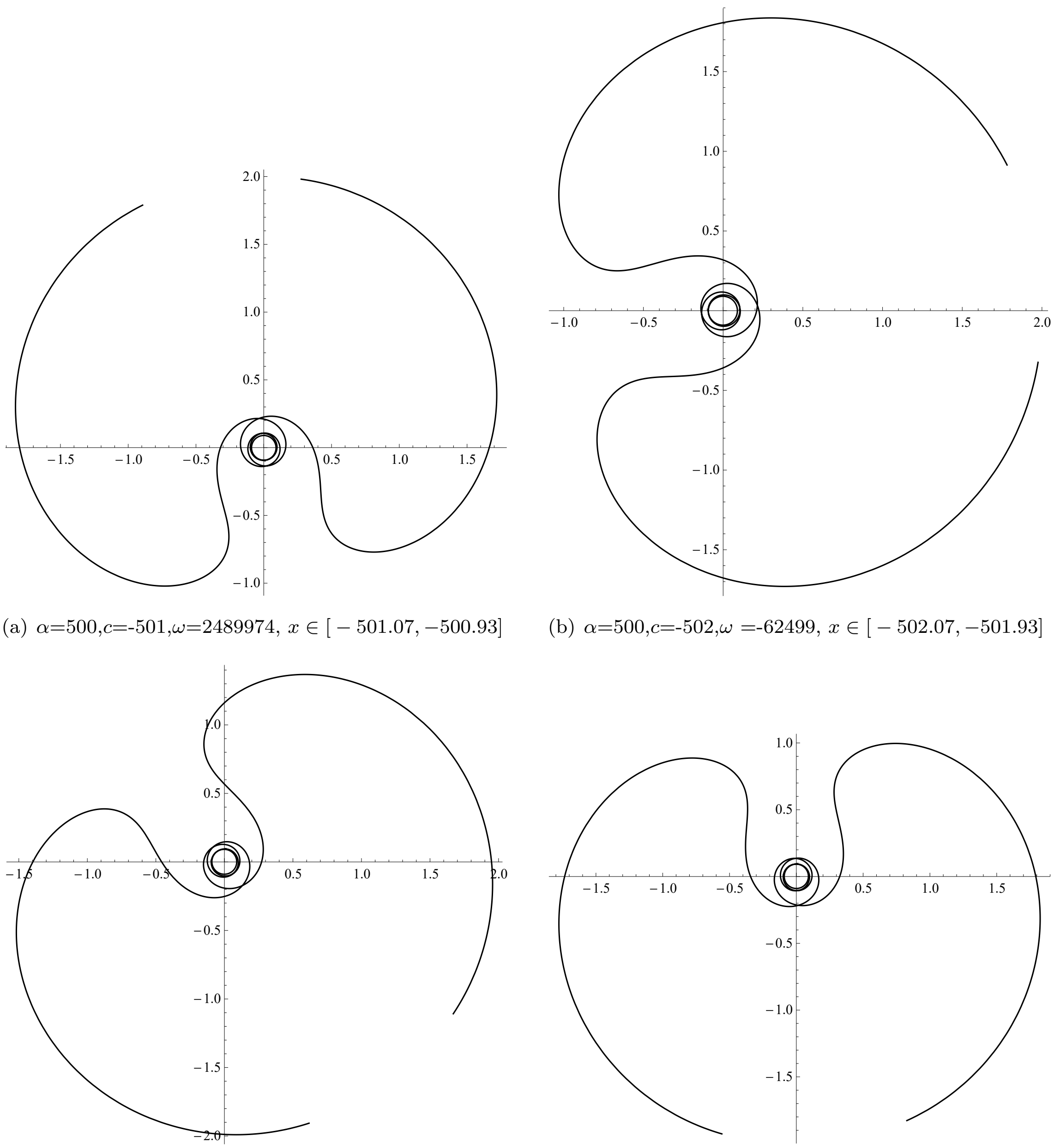


(c) α=500,c=-500,ω=-62000, $x \in [-500.07, -499.93]$



(d) α=1000, c =-1000, ω=-249000, $x \in [-1000.05, -999.95]$

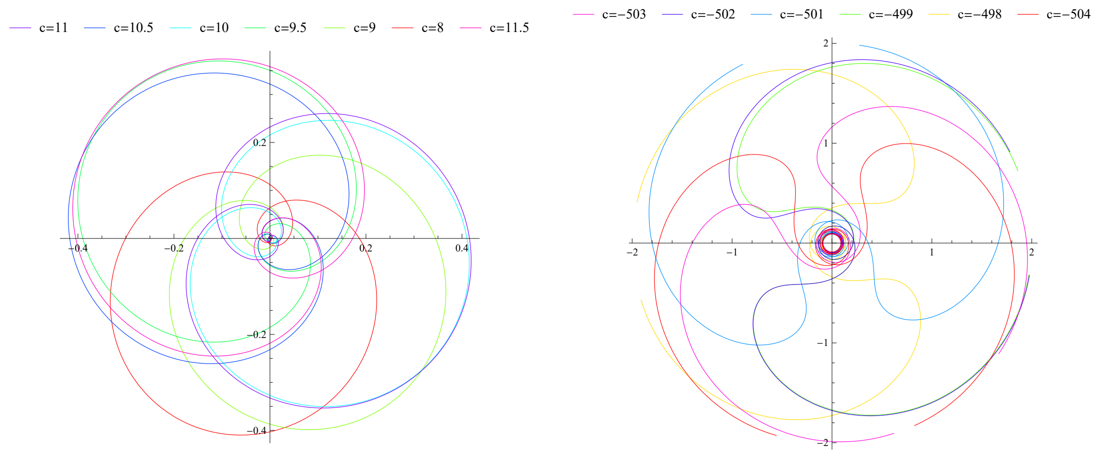
Figure 1: Complex plane image of the solution when $\phi(\xi)$ takes (21), t=1, and $\omega = c - c^2/4$. In order to ensure that the image of periodic solution is drawn continuously, t and x shall meet $-\frac{\pi}{2} \leqslant \frac{1}{2}\sqrt{c^2 + 4\omega_1}(x - ct) \leqslant \frac{\pi}{2}$. It is observed from the figure that the complex plane image of the solution is axisymmetric. Moreover, with the synchronous increase of α and C, the number of times the complex plane images of the solution around the coordinates origin are intertwined with each other will increase.



(a) α=500,c=-501,ω=2489974, $x \in [-501.07, -500.93]$ (b) α=500,c=-502,ω =-62499, $x \in [-502.07, -501.93]$

(c) α=500,c=-503,ω=2509974, $x \in [-503.07, -502.93]$ (d) α=500,c=-504,ω =-63000, $x \in [-504.07, -503.93]$

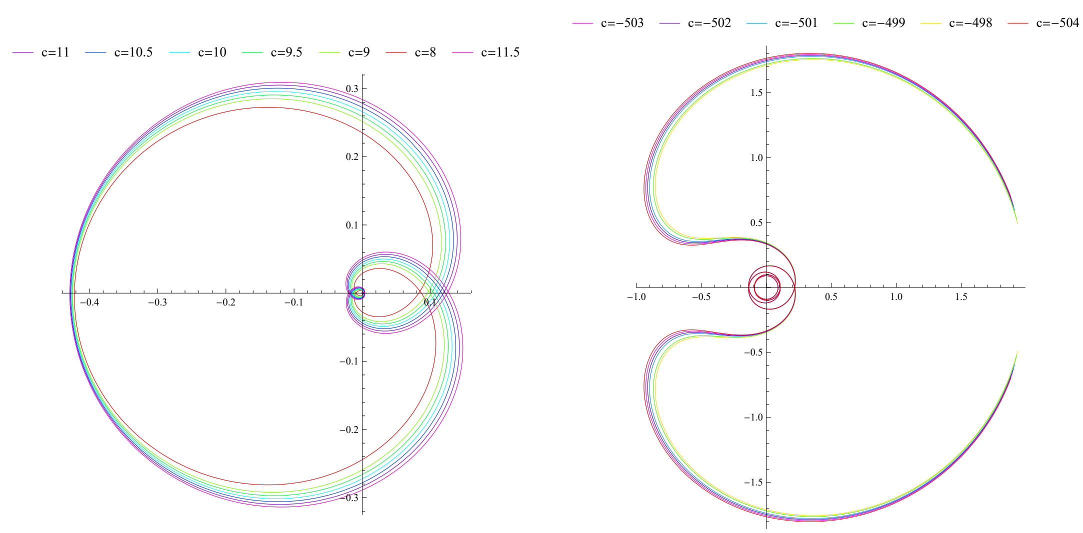
Figure 2: Complex plane image of the solution for different but similar c when $\phi(\xi)$ takes (21), t=1, and $\omega = c - c^2/4$. It can be seen that when α is fixed and $\omega = c - c^2/4$, if c changes slightly, the solution will rotate around the coordinates origin.



(a) Complex plane image of the solution for different c when $\phi(\xi)$ takes (23), t=1, α=10, $\omega = c - c^2/4$, and $x \in [-15, 15]$.

(b) Complex plane image of the solution for different but similar c when $\phi(\xi)$ takes (23), t=1, and $\omega = c - c^2/4$.

Figure 3: Comparison of the non-periodic solution and periodic solution when $t = 1$. It can be seen that when c increases, the complex plane image of the solution accelerates to rotate clockwise, and the heart-shaped ring in the middle also increases.



(a) Complex plane image of the solution for different c when $\phi(\xi)$ takes (23), t=1, α=0, $\omega = c - c^2/4$, and $x \in [-15, 15]$.

(b) Complex plane image of the solution for different but similar c when $\phi(\xi)$ takes (23), t=0, and $\omega = c - c^2/4$

Figure 4: Comparison between the non-periodic solution and periodic solution when $t = 0$. It can be seen that the shape of the complex plane image of the solution is almost the same when Cchanges slightly, for both the periodic and non-periodic solutions.

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