# GROUND STATES FOR CRITICAL FRACTIONAL SCHRÖDINGER-POISSON SYSTEMS WITH VANISHING POTENTIALS

XILIN DOU AND XIAOMING HE

Abstract. This paper deals with a class of fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + V(x)u - K(x)\phi|u|^{2_s^*-3}u = a(x)f(u), & x \in \mathbb{R}^3 \\ (-\Delta)^s \phi = K(x)|u|^{2_s^*-1}, & x \in \mathbb{R}^3 \end{cases}$$

with a critical nonlocal term and multiple competing potentials, which may decay and vanish at infinity, where $s \in (\frac{3}{4}, 1)$, $2_s^* = \frac{6}{3-2s}$ is the fractional critical exponent. The problem is set on the whole space and compactness issues have to be tackled. By employing the mountain pass theorem, concentration-compactness principle and approximation method, the existence of a positive ground state solution is obtained under appropriate assumptions imposed on $V, K, a$ and $f$.

## 1. Introduction and main results

In recent years, the following nonlinear Schrödinger-Poisson systems

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

have been the object of interest for many authors. Such a system, also called Schrödinger-Maxwell equations, can be used to describe the interaction of a charged particle with the electrostatic field in quantum mechanics, where the unknowns $u$ and $\phi$ represent the wave functions associated with the particle and the electric potentials, respectively, and $V$ is an exterior potential and $K$ denotes a nonnegative density charge, and the nonlinearity $f$ simulates the interaction effect among many particles. See [3, 5] for more details on the physical background.

It is easily seen that system (1.1) can be transformed into a nonlinear Schrödinger equation with a non-local term, for example, see [5, 10]. Briefly, the Poisson equation can be solved by using the Lax-Milgram theorem. For all $u \in H^1(\mathbb{R}^3)$, the unique $\phi_{K,u} \in D^{1,2}(\mathbb{R}^3)$ is given by

$$\phi_{K,u}(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)|u(y)|^2}{|x-y|}dy,$$

which solves equation $-\Delta\phi_{K,u} = K(x)u^2$, and that, it can be substituted into the first equation of system (1.1) to obtain that

$$-\Delta u + V(x)u + K(x)\phi_{K,u}u = f(x,u), \quad x \in \mathbb{R}^3.$$

Such equation is variational, and its solutions are critical points of the corresponding energy functional $I$ defined in $H^1(\mathbb{R}^3)$.

In view of this, there has been much attention to (1.1), and many interesting works have been denoted to investigating the existence and nonexistence of positive solutions, sign-changing solutions, positive ground states solutions, multiple solutions, and semiclassical states under variant assumptions on $V, K$ and $f$, via variational methods in recent years. We refer to [2, 3, 9, 25, 29, 33, 36] and references therein.

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We note that, Azzollini and d'Avenia1 [4] firstly studied the Schrödinger-Poisson system with critical nonlocal term as follows:

$$\begin{cases} -\Delta u = \lambda u + q\phi|u|^3 u, & \text{in } B_R, \\ -\Delta\phi = |u|^5, & \text{in } B_R, \\ u = \phi = 0, & \text{on } \partial B_R, \end{cases} \tag{1.2}$$

They proved existence and nonexistence results of positive and sign changing solutions for (1.2), distinguishing the more delicate three-dimensional case from the others. Since then, some researchers began to investigate the Schrödinger-Poisson system with critical nonlocal term, but there are fewer papers devoting to these kinds of nonlocal problems in the literature. In [19, 20], Li, Li and Shi considered positive solutions to the another Schrödinger-Poisson-type system with critical growing nonlocal term

$$\begin{cases} -\Delta u + bu + q\phi|u|^3 u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

and the existence of positive solutions to (1.3) was proved by using variational method which does not require usual compactness conditions. In [22], Liu studied the following generalized Schrödinger-Poisson system with critical nonlocal term

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3 u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)|u|^5, & x \in \mathbb{R}^3. \end{cases} \tag{1.4}$$

By using the mountain pass theorem and the concentration-compactness principle, Liu obtained the existence of a positive solution for (1.4). Feng [13] studied the existence of positive solutions to (1.4) with the critical nonlinearity $f(x,u) = |u|^4 u + g(u)$, by the modified concentration-compactness principle and Nehari manifold method. Li and He [21] studied the existence and multiplicity of positive solutions for (1.4) by using the Ljusternik-Schnirelmann theory. Yin, Zhang and Shang [33] considered the existence of positive ground state solution to equation (1.4) by using the variational approach.

In the setting of the fractional Laplacian, system (1.1) becomes the fractional Schrödinger-Poisson type systems. It is a fundamental equation in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes [8,17,18]. In the fractional scenario, there are some results for fractional Schrödinger-Poisson systems available in literature. In [35], by using a perturbation approach, Zhang, do Ó and Squassina considered the existence and the asymptotical behaviors of positive solutions to the fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + V(x)u + K(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t\phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.5}$$

with $V(x) = 0$ and $K(x) = \lambda > 0$, a parameter, and a general subcritical or critical nonlinearity $f$. Teng [32] analyzed the existence of ground state solutions of (1.4) with $K(x) = 1$ and $f(x,u) = \mu|u|^{q-1}u + |u|^{2_s^*-2}u, q \in (2, 2_s^*)$, by combining Pohozaev-Nehari manifold, arguments of Brezis-Nirenberg type, the monotonicity trick and global compactness Lemma. Murcia and Siciliano [26] studied the semiclassical state of the following system

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + K(x)\phi u = f(u), & x \in \mathbb{R}^N, \\ \varepsilon^{\theta}(-\Delta)^{\alpha/2}\phi = \gamma_\alpha u^2, & x \in \mathbb{R}^N. \end{cases} \tag{1.6}$$

and established the multiplicity of positive solutions that concentrate on the minima of $V(x)$ as $\varepsilon \to 0$ by the Ljusternik-Schnirelmann category theory. For more results on multiplicity and concentration of positive solutions of (1.6), we refer to [23, 34, 37] and references therein.

We note that in the above mentioned works for the fractional Schrodinger-Poisson systems, the second Poisson equation is subcritical growth. After a bibliography review we find that there are only two papers that deals with fractional Schrödinger-Poisson system with critical nonlocal term. In [16], the second author of this paper studied the existence of ground state solution of the following fractional Schödinger-Possion equations

$$\begin{cases}(-\Delta)^s u+V(x)u-K(x)\phi|u|^{2_s^*-3}u=f(x,u), & x\in\mathbb{R}^3,\\ (-\Delta)^s\phi=K(x)|u|^{2_s^*-1}, & x\in\mathbb{R}^3.\end{cases}\tag{1.7}$$

with the nonlinearity $u^{2_s^*-1}+h(u)$, where $h$ is subcritical growth, in this case, system (1.7) is called double critical exponents case. In [14], Feng studied the existence of nonnegative solutions of (1.7), by employing the mountain pass theorem, concentration-compactness principle and approximation method, and extended the main results of [22] to the fractional Laplacian case.

When $K(x)\equiv 0$, (1.7) simplifies to the following fractional Schrödinger equation

$$(-\Delta)^s u+V(x)u=f(x,u),\quad x\in\mathbb{R}^3,\tag{1.8}$$

In [30], Secchi constructed solutions to (1.8) by variational approach in nature, and based on minimization on the Nehari manifold. do Ó, Miyagaki and Squassina [24] investigated (1.8) with critical power nonlinearity and a subcritical term $K(x)f(u)+\lambda|u|^{2_s^*-2}u$, and the involved potentials are allowed for vanishing behavior at infinity.

Motivated by the works mentioned above, the purpose of this paper is concerned with the following fractional Schödinger-Poisson system

$$\begin{cases}(-\Delta)^s u+V(x)u-K(x)\phi|u|^{2_s^*-3}u=a(x)f(u), & x\in\mathbb{R}^3\\ (-\Delta)^s\phi=K(x)|u|^{2_s^*-1}, & x\in\mathbb{R}^3.\end{cases}\tag{1.9}$$

The continuous functions $V(x)$ and $a(x)$ satisfy the following conditions:

- $(D1)$ $V(x)>0$, $a(x)>0$, $\forall x\in\mathbb{R}^3$, and $a(x)\in L^\infty(\mathbb{R}^3)$;
- $(D2)$ If $A_n\subset\mathbb{R}^3$ is a sequence of Borel sets such that $|A_n|\le C$, for all $n$ any some $C>0$, then $\lim_{x\to+\infty}\int_{A_n\cap B_r^c(0)}a(x)dx=0$, uniformly in $n\in\mathbb{N}$;
- $(D3)$ One of the following situations occurs:
$$\frac{a(x)}{V(x)}\in L^\infty(\mathbb{R}^3),\quad (D3)_1$$
or there is $p\in(2,2_s^*)$ such that
$$\frac{a(x)}{[V(x)]^{\frac{2_s^*-p}{2_s^*-2}}}\to 0,\qquad \text{as}\quad |x|\to+\infty.\quad (D3)_2$$

With respect to functions $f\in C(\mathbb{R},\mathbb{R})$, we formulate assumptions as:

- $(f_1)$ $\lim_{t\to 0^+}\frac{f(t)}{t}=0$ if $(D3)_1$ holds; or $\lim_{t\to 0^+}\frac{f(t)}{t^{p-1}}=A<+\infty$ if $(D3)_2$ holds.
- $(f_2)$ $f$ has a quasi-critical growth, that is, $\lim_{t\to+\infty}\frac{f(t)}{t^{2_s^*-1}}=0$.
- $(f_3)$ $t\to\frac{f(t)}{t^m}$ is non-decreasing in $(0,+\infty)$, for some $m\in(3-2s,2_s^*-1)$.
- $(f_4)$ $\lim_{t\to+\infty}\frac{F(t)}{t^{m+1}}=+\infty$, where $F(\tau)=\int_0^\tau f(t)dt$, and $m\in(3-2s,2_s^*-1)$ if $(D3)_1$ holds or $m\in(p-1,2_s^*-1)$ if $(D3)_2$ holds.

For the function $K(x)$, we assume that

- $(K)$ $K(x)>0$, $\forall x\in\mathbb{R}^3$, $K(x)\in L^\infty(\mathbb{R}^3)$, and there exist some constants $C,\delta>0$, and $\beta\in[3-2s,3)$, such that $|K(x)-K(x_0)|\le C|x-x_0|^\beta$ if $|x-x_0|<\delta$, where $x_0\in\mathbb{R}^3$ satisfies $K(x_0)=\max_{x\in\mathbb{R}^3}K(x)=\|K\|_\infty<+\infty$.

In order to consider the existence of positive solutions to system (1.9), in the following article, we may assume $f(u)=0$ for $u\le 0$.

Our main result can be formulated as:

**Theorem 1.1.** *Suppose the conditions $V(x)$, $K(x)$, $a(x)$ :$\mathbb{R}^3\to\mathbb{R}^+$ and $f:\mathbb{R}\to\mathbb{R}$ satisfy the conditions $(D1)-(D3)$, $(K)$ and $(f_1)-(f_4)$, then the system (1.9) has a positive ground state solution.*

**Remark 1.2.** *The hypotheses* $(D1)$*-*$(D3)$ *was introduced in [24] to study the existence of ground state solution of* (1.8) *with decaying potentials, which is original from [1], where Alves and Souto used it to treat the Schrödinger equation with vanishing potentials.*

The proof of Theorem 1.1 is variational. From the technical point of view, compared with equation (1.8), the presence of critical nonlocal term in system (1.9) makes the problem more difficulty and complicated. Firstly, we establish the geometry of mountain pass, from which we can obtain the existence of the Cerami sequence. Secondly, we check that the Cerami condition at the level $c$ is true, where $c$ is the mountain pass level associated with the energy functional of system (1.9). To this aim, through energy estimation, we shall determine the minimum threshold value below which the Cerami compactness condition is satisfied in view of the appearance of critical nonlocal term.

The structure of this paper is as follows. In Section 2, we give some variational settings and introduce some useful lemmas, which are crucial in proving Theorem 1.1; and the proof of Theorem 1.1 is completed in Section 3.

**Notation.** Throughout this paper, the Lebesgue space $L^p(\mathbb{R}^3)$ for $p > 1$ is equipped with the norm $\|u\|_p = (\int_{\mathbb{R}^3} |u|^p dx)^{\frac{1}{p}}$. The norm for the weighted Lebesgue space $L^q_a(\mathbb{R}^3)$ $(q > 1)$ is given by $\|u\|_{a,q} = (\int_{\mathbb{R}^3} a(x)|u|^q dx)^{\frac{1}{q}}$. $B_r(x)$ denotes the ball in $\mathbb{R}^N$ centered at $x$ with radius $r$. The letters $C, C_i, i = 1, 2, \cdots$, denote various positive constants whose exact values are irrelevant, and $u^{\pm} = \max\{\pm u, 0\}$.

## 2. Preliminary results

In this Section, we will give some notations and Lemmas that will be used throughout this paper. For any $s \in (0, 1)$, the homogeneous Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined by $D^{s,2}(\mathbb{R}^3) = \{u \in L^{2^*_s}(\mathbb{R}^3) : \|u\|_{D^{s,2}} < \infty\}$, where

$$\|u\|^2_{D^{s,2}} = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dxdy.$$

The fractional space $H^s(\mathbb{R}^3)$ is defined by

$$H^s(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \ \|u\|_{D^{s,2}} < \infty\right\},$$

endowed with the norm

$$\|u\| := \|u\|_{H^s} = \sqrt{|u|^2_2 + \|u\|^2_{D^{s,2}}}.$$

See [27]. We define the Sobolev space

$$E = \left\{u \in D^{s,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) u^2 dx \le +\infty\right\}$$

equipped with the norms

$$\|u\|_E := \left(\int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} u(x)|^2 + V(x)|u(x)|^2\right) dx\right)^{\frac{1}{2}}.$$

As is well-known, the Lax-Milgram theorem implies that Poisson equation $(-\Delta)^s \phi = K(x)|u|^{2^*_s - 1}$ has a unique weak solution $\phi_u \in D^{s,2}(\mathbb{R}^3)$ for $u \in E$, and $\phi_u$ can be expressed as (e.g. [16])

$$\phi_u(x) = C_s \int_{\mathbb{R}^3} \frac{K(y)|u(y)|^{2^*_s - 1}}{|x - y|^{3-2s}} dy, \tag{2.1}$$

where $C_s = \frac{\Gamma(\frac{3-2s}{2})}{2^{2s}\pi^{\frac{3}{2}}\Gamma(s)}$.

Substituting (2.1) into the first equation of (1.9), then (1.9) can be transformed into a single Schrödinger equation as follows:

$$(-\Delta)^s u + V(x)u - K(x)\phi_u |u|^{2^*_s - 3} u = a(x)f(u), \quad \forall u \in E.$$

The energy functional associated with the system (1.9) is given by

$$
\begin{aligned}
J(u) = &\frac{1}{2}\int_{\mathbb{R}^3}\left(|(-\Delta)^{\frac{s}{2}}u(x)|^2 + V(x)|u(x)|^2\right)dx \\
&- \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx - \int_{\mathbb{R}^3} a(x)f(u)dx,
\end{aligned}
\tag{2.2}
$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Under the assumption of $f$, it is easy to see that $J$ is well-defined and of class $C^1(E,\mathbb{R})$. Moreover, the Gateaux derivative of $J$ is defined as

$$
J'(u)v = \int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u(x)|^2 + V(x)|u(x)|^2)dx - \int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-3}uvdx - \int_{\mathbb{R}^3} a(x)f(u)vdx, \ \ \forall u,v,\in E.
$$

It is easy to check that the critical point $u$ of the function $J$ corresponds to the weak solution of (1.9).

Next, we recall the following embedding results.

**Lemma 2.1.** *( [30]) Let $0 < s < 1$ such that $2s < N$. Then there exists a sharp constant $S > 0$ such that for any $u \in D^{s,2}(\mathbb{R}^N)$*

$$
\|u\|_{2_s^*}^2 \leq S^{-1}\|u\|_{D^{s,2}}^2.
$$

*Moreover, the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is continuous for any $p \in [2, 2_s^*]$, and is locally compact whenever $p \in [1, 2_s^*)$.*

**Lemma 2.2.** *( [24]) Assume that conditions $(D1) - (D3)$ are satisfied. Then $E$ is compactly embedded in $L_a^q(\mathbb{R}^3)$ for $q \in (2, 2_s^*)$, if $(D3)_1$ is true; $E$ is compactly embedded in $L_a^p(\mathbb{R}^3)$, if $(D3)_2$ is true.*

**Lemma 2.3.** *( [24]) Suppose that assumptions $(D1) - (D3)$ and $(f_1) - (f_4)$ are satisfied, and $v_n$ is a sequence such that $v_n \rightharpoonup v$ in $E$. Then*

$$
\int_{\mathbb{R}^3} a(x)F(v_n)dx \to \int_{\mathbb{R}^3} a(x)F(v)dx \quad and \quad \int_{\mathbb{R}^3} a(x)f(v_n)v_ndx \to \int_{\mathbb{R}^3} a(x)f(v)vdx. \tag{2.3}
$$

**Lemma 2.4.** *( [14]) The function $\phi_u$ has the following qualities:*

*(1) If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\mathbb{R}^3)$;*
*(2) $\phi_{tu} = |t|^{2_s^*-1}\phi_u, \quad \forall t > 0, u \in H^s(\mathbb{R}^3)$;*
*(3) For each $u \in H^s(\mathbb{R}^3)$, one has $\|\phi_u\|_{D^{s,2}} \leq \|K\|_\infty S^{-\frac{1}{2}}\|u\|_{2_s^*}^{2_s^*-1}$ and*

$$
\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \leq \|K\|_\infty^2 S^{-1}\|u\|_{2_s^*}^{2(2_s^*-1)};
$$

*(4) If $u_n \to u$ in $H^s(\mathbb{R}^3)$, then, up to a subsequence, $\phi_{u_n} \to \phi_u$ in $D^{s,2}(\mathbb{R}^3)$.*

**Lemma 2.5.** *If $u_n \rightharpoonup u$ weakly in $L^{2_s^*}(\mathbb{R}^3)$, and $u_n \to u$ a.e. in $\mathbb{R}^3$, then as $n \to \infty$, we have the following conclusions:*

*(i) $|u_n|^{2_s^*-1} - |u_n - u|^{2_s^*-1} - |u|^{2_s^*-1} \to 0 \quad$ in $L^{\frac{6}{3+2s}}(\mathbb{R}^3)$;*
*(ii) $|u_n|^{2_s^*-2}u_n - |u_n - u|^{2_s^*-2}(u_n - u) - |u|^{2_s^*-2}u \to 0 \quad$ in $L^{\frac{6}{3+2s}}(\mathbb{R}^3)$;*
*(iii) $\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_{u_n-u}|u_n - u|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \to 0$;*
*(iv) $\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-3}u_n\varphi dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-3}u\varphi dx \to 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3)$.*

*Proof.* Let $v_n = u_n - u$, then $v_n \rightharpoonup 0$ in $L^{2_s^*}(\mathbb{R}^3)$. According to the intermediate value theorem, there exists $\theta \in L^\infty(\mathbb{R}^3)$ and $0 \leq \theta \leq 1$ such that

$$
\begin{aligned}
\left||v_n + u|^{2_s^*-1} - |v_n|^{2_s^*-1}\right| &= (2_s^*-1)\,|v_n + \theta u|^{2_s^*-2}\,|u| \\
&\leq (2_s^*-1)2^{2_s^*-2}\left[|v_n|^{2_s^*-2}\,|u| + |u|^{2_s^*-1}\right].
\end{aligned}
$$

For any $\varepsilon > 0$, by the Young's inequality, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\left||\upsilon_n + u|^{2_s^*-1} - |\upsilon_n|^{2_s^*-1} - |u|^{2_s^*-1}\right| &\leq (2_s^* - 1)2^{2_s^*-2}\left[|\upsilon_n|^{2_s^*-2}|u| + |u|^{2_s^*-1}\right] + |u|^{2_s^*-1} \\ &\leq \varepsilon|\upsilon_n|^{2_s^*-1} + C_\varepsilon|u|^{2_s^*-1}.\end{aligned}$$

Define the functions

$$h_n = \max\left\{\left||\upsilon_n + u|^{2_s^*-1} - |\upsilon_n|^{2_s^*-1} - |u|^{2_s^*-1}\right| - \varepsilon|\upsilon_n|^{2_s^*-1}, 0\right\},$$

which satisfies

$$h_n \to 0 \quad \text{a.e. in } \mathbb{R}^3, \quad 0 \leq h_n \leq C_\varepsilon|u|^{2_s^*-1} \in L^{\frac{6}{3+2s}}(\mathbb{R}^3).$$

Hence, using Lebesgue's dominant convergence theorem, we derive that

$$\int_{\mathbb{R}^3} h_n^{\frac{6}{3+2s}} dx \to 0.$$

From the definition of $h_n$, it follows that

$$\left||\upsilon_n + u|^{2_s^*-1} - |\upsilon_n|^{2_s^*-1} - |u|^{2_s^*-1}\right| \leq h_n + \varepsilon|\upsilon_n|^{2_s^*-1}.$$

Thus, we obtain

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}\left||\upsilon_n + u|^{2_s^*-1} - |\upsilon_n|^{2_s^*-1} - |u|^{2_s^*-1}\right|^{\frac{6}{3+2s}} dx \leq C_1\int_{\mathbb{R}^3}|h_n|^{\frac{6}{3+2s}}dx + C_2\varepsilon\int_{\mathbb{R}^3}|\upsilon_n|^{2_s^*}dx \leq C_3\varepsilon.$$

This implies that

$$\left||\upsilon_n + u|^{2_s^*-1} - |\upsilon_n|^{2_s^*-1} - |u|^{2_s^*-1}\right| \to 0 \quad \text{in } L^{\frac{6}{3+2s}}(\mathbb{R}^3),$$

and item (i) follows. The proof of item (ii) is similar to that of (i), we omit it.

Now, we show item (iii). Note that for every $w \in D^{s,2}(\mathbb{R}^3)$,

$$\begin{aligned}|\langle\phi_{u_n} - \phi_{v_n} - \phi_u, w\rangle| &= \left|\int_{\mathbb{R}^3} K(x)w(|u_n|^{2_s^*-1} - |v_n|^{2_s^*-1} - |u|^{2_s^*-1})\right| \\ &\leq \|K\|_\infty\|w\|_{2_s^*}\||u_n|^{2_s^*-1} - |v_n|^{2_s^*-1} - |u|^{2_s^*-1}\|_{2_s^*/(2_s^*-1)}\end{aligned},$$

then we get

$$\phi_{u_n} - \phi_{u_n-u} - \phi_u \to 0 \quad \text{in} \quad D^{s,2}(\mathbb{R}^3).$$

Using $u_n - u \rightharpoonup 0$ in $L^{2_s^*}(\mathbb{R}^3)$ and $u_n - u \to 0$ a.e. $x \in \mathbb{R}^3$, and Lemma 2.4, we have $\phi_{u_n-u} \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^3)$. Therefore, as $n \to \infty$,

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_{v_n}|v_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \\ &= \int_{\mathbb{R}^3} K(x)[\phi_{u_n} - \phi_{v_n} - \phi_u]|u_n|^{2_s^*-1}dx + \int_{\mathbb{R}^3} K(x)\phi_{v_n}[|u_n|^{2_s^*-1} - |u_n - u|^{2_s^*-1} - |u|^{2_s^*-1}]dx \\ &\quad + \int_{\mathbb{R}^3} K(x)\phi_{v_n}|u|^{2_s^*-1}dx + \int_{\mathbb{R}^3} K(x)\phi_u[|u_n|^{2_s^*-1} - |u|^{2_s^*-1}]dx \\ &\to 0.\end{aligned}$$

$u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^3)$, implies $u_n \rightharpoonup u$ in $L^{2_s^*}(\mathbb{R}^3)$. Then, as $K \in L^\infty(\mathbb{R}^3)$, we see that

$$\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)|u|^{2_s^*-3}u\varphi dx \to 0 \quad \text{as} \quad n \to \infty. \tag{2.4}$$

Since $u_n \to u$ a.e. in $\mathbb{R}^3$ and $K \in L^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} |K(x)\phi_{u_n}[|u_n|^{2_s^*-3}u_n - |u|^{2_s^*-3}u]|^{\frac{2_s^*}{2_s^*-1}} dx$$
$$\leq C\left( \|\phi_{u_n}\|_{2_s^*}^{\frac{2_s^*}{2_s^*-1}} \|u_n\|_{2_s^*}^{\frac{2_s^*(2_s^*-2)}{2_s^*-1}} + \|\phi_{u_n}\|_{2_s^*}^{\frac{2_s^*}{2_s^*-1}} \|u\|_{2_s^*}^{\frac{2_s^*(2_s^*-2)}{2_s^*-1}} \right) \leq C,$$

we have $K(x)\phi_{u_n}[|u_n|^{2_s^*-3}u_n - |u|^{2_s^*-3}u] \rightharpoonup 0$ in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$ as $n \to \infty$. Thus, we have

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}[|u_n|^{2_s^*-3}u_n - |u|^{2_s^*-3}u]\varphi dx \to 0 \quad \text{as} \quad n \to \infty \tag{2.5}$$

which together with (2.4)-(2.5) ensures

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-3}u_n\varphi dx \to \int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-3}u\varphi dx \quad \text{as} \quad n \to \infty.$$

This proves item (iv). □

**Lemma 2.6.** *Under conditions $(D1)-(D3)$ and $(f_1)-(f_4)$, the functional $J(u)$ satisfies the mountain pass geometry.*

*Proof.* By (2.2), we directly get that $J(0) = 0$. Fixed $u_0 \in E$, for any $t > 0$,

$$J(tu_0) = \frac{t^2}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u_0|^2 + V(x)u_0^2)dx - \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx$$
$$- \int_{\mathbb{R}^3} a(x)F(x, tu_0)dx.$$

Using $(f_2)$, we see that $\lim_{t\to\infty} J(tu_0) = -\infty$. Hence, there exists $t_0 > 0$ large enough such that $J(t_0u_0) < 0$. Set $e = t_0u_0 \in E$, then $J(e) < 0$.

Multiplying $\phi_u$ on the both sides of equation $(-\Delta)^s\phi_u = K(x)|u|^{2_s^*-1}$ and integrating on $\mathbb{R}^3$, we get $\|\phi_u\|_{D^{s,2}}^2 = \int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*-1}dx$. By the continuous embedding $E \hookrightarrow D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ and Lemma 2.1, one has

$$\|u\|_{2_s^*}^2 \leq S^{-1}\|u\|_{D^{s,2}}^2 \leq S^{-1}\|u\|_E^2. \tag{2.6}$$

Therefore,

$$\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \leq \|K\|_\infty \|\phi_u\|_{2_s^*} \|u\|_{2_s^*}^{2_s^*-1} \leq \|K\|_\infty S^{-\frac{1}{2}}\|\phi_u\|_{D^{s,2}}\|u\|_{2_s^*}^{2_s^*-1}.$$

Thus,

$$\|\phi_u\|_{D^{s,2}} \leq \|K\|_\infty S^{-\frac{1}{2}}\|u\|_{2_s^*}^{2_s^*-1},$$

and so,

$$\int_{\mathbb{R}^3} K(x)\phi_u|u|^{2_s^*-1}dx \leq \|K\|_\infty^2 S^{-1}\|u\|_{2_s^*}^{2(2_s^*-1)} \leq \|K\|_\infty^2 S^{-2_s^*}\|u\|_E^{2(2_s^*-1)}. \tag{2.7}$$

*Case 1: $(D3)_1$ holds.* In this case, by $(f_1)$ and $(f_2)$, for any $\varepsilon > 0$ small enough, there is $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|t|^{2_s^*-1} \quad \text{and} \quad |F(t)| \leq \varepsilon|t|^2 + C_\varepsilon|t|^{2_s^*}, \quad \forall t \in \mathbb{R}. \tag{2.8}$$

Combining (2.6), (2.7), (2.8) and (2.2), we infer that

$$J(u) \geq \frac{1}{2}\|u\|_E^2 - \frac{1}{2(2_s^*-1)}\|K\|_\infty^2 S^{-2_s^*}\|u\|_E^{2(2_s^*-1)} - \int_{\mathbb{R}^3} a(x)[\varepsilon|u|^2 + C_\varepsilon|u|^{2_s^*}]dx.$$

By virtue of

$$\int_{\mathbb{R}^3} a(x)|u|^2dx \leq \left\|\frac{a}{V}\right\|_\infty \int_{\mathbb{R}^3} V(x)|u|^2dx \leq \left\|\frac{a}{V}\right\|_\infty \|u\|_E^2, \tag{2.9}$$

and

$$\int_{\mathbb{R}^3} a(x)|u|^{2_s^*} dx \le \|a\|_\infty \int_{\mathbb{R}^3} |u|^{2_s^*} dx \le \|a\|_\infty S^{-\frac{2_s^*}{2}} \|u\|_E^{2_s^*}, \tag{2.10}$$

we can obtain that

$$J(u) \ge \frac{1}{2}\|u\|_E^2 - \frac{1}{2(2_s^*-1)} \|K\|_\infty^2 S^{-2_s^*} \|u\|_E^{2(2_s^*-1)} - \varepsilon \left\|\frac{a}{V}\right\|_\infty \|u\|_E^2 - C_\varepsilon \|a\|_\infty S^{-\frac{2_s^*}{2}} \|u\|_E^{2_s^*}.$$

Take $\varepsilon$ small enough such that $\varepsilon \left\|\frac{a}{V}\right\|_\infty \le \frac{1}{4}$, then

$$J(u) \ge \frac{1}{4}\|u\|_E^2 \left(1 - C_1 \|u\|_E^{2(2_s^*-2)} - C_2 \|u\|_E^{2_s^*-2}\right).$$

Thus, there exists a small enough $\rho > 0$ such that if $\|u\|_E = \rho$, there holds $J(u) \ge a > 0$.

*Case 2:* $(D3)_2$ *holds.* It follows from $(f_1)$ and $(f_2)$ that for any sufficiently small $\varepsilon > 0$ there are $L_{1\varepsilon} > 0$ and $L_{2\varepsilon} > 0$ such that, for all $t \in \mathbb{R}$

$$|f(t)| \le L_{1\varepsilon} |t|^{p-1} + L_{2\varepsilon} |t|^{2_s^*-1} \quad \text{and} \quad |F(t)| \le L_{1\varepsilon} |t|^p + L_{2\varepsilon} |t|^{2_s^*}. \tag{2.11}$$

Combining (2.6), (2.7), and (2.11) and (2.2), we get

$$J(u) \ge \frac{1}{2}\|u\|_E^2 - \frac{1}{2(2_s^*-1)} \|K\|_\infty^2 S^{-2_s^*} \|u\|_E^{2(2_s^*-1)} - \int_{\mathbb{R}^3} a(x)\left[L_{1\varepsilon}|u|^p + L_{2\varepsilon}|u|^{2_s^*}\right] dx. \tag{2.12}$$

Now, it is sufficient to estimate the value of this integral $\int_{\mathbb{R}^3} a(x)|u|^p dx$. To this aim, we use the method due to Alves and Souto in [1] to show the desired estimation.

Let $h(t) = V(x)t^{2-p} + t^{2_s^*-p}$ for $t > 0$ and $p \in (2, 2_s^*)$. Then solving $h'(t) = 0$, we get the root

$$t_0 = \left[\frac{p-2}{2_s^*-p} V(x)\right]^{\frac{3-2s}{4s}} > 0.$$

Using it together with $g''(t_0) = t_0^{-p}(p-2)V(x)(2_s^*-2) > 0$, We know that $g(t)$ reaches the minimum at $t_0$. Hence,

$$h(t) \ge h(t_0) = \frac{(2_s^*-2)(p-2)^{\frac{2-p}{2_s^*-2}}}{(2_s^*-p)^{\frac{2_s^*-p}{2_s^*-2}}} [V(x)]^{\frac{2_s^*-p}{2_s^*-2}}.$$

Condition $(D3)_2$ implies that for each $\varepsilon > 0$ small enough, there exists $\mathbb{R} > 0$ large enough such that

$$a(x) \le \varepsilon \frac{(2_s^*-2)(p-2)^{\frac{2-p}{2_s^*-2}}}{(2_s^*-p)^{\frac{2_s^*-p}{2_s^*-2}}} [V(x)]^{\frac{2_s^*-p}{2_s^*-2}} \le \varepsilon[V(x)|u|^{2-p} + |u|^{2_s^*-p}], \quad |x| > R. \tag{2.13}$$

Consequently,

$$\begin{aligned} \int_{\mathbb{R}^3} a(x)|u|^p dx &= \int_{|x|\le R} a(x)|u|^p dx + \int_{|x|>R} a(x)|u|^p dx \\ &\le \|a\|_\infty \int_{|x|\le R} |u|^p dx + \int_{|x|>R} \varepsilon[V(x)|u|^{2-p} + |u|^{2_s^*-p}]|u|^p dx \\ &\le \|a\|_\infty \left(\int_{|x|\le R} |u|^{2_s^*} dx\right)^{\frac{p}{2_s^*}} \left(\int_{|x|\le R} 1 dx\right)^{\frac{2_s^*-p}{2_s^*}} + \varepsilon\left[\|u\|_E^2 + \|u\|_{2_s^*}^{2_s^*}\right] \\ &\le \|a\|_\infty |B_R(0)|^{\frac{2_s^*-p}{2_s^*}} \|u\|_{2_s^*}^p + \varepsilon\|u\|_E^2 + \varepsilon\|u\|_{2_s^*}^{2_s^*}. \end{aligned} \tag{2.14}$$

Combining (2.6), (2.10), (2.14) with (2.12), we infer that

$$
\begin{aligned}
J(u) \geq & \frac{1}{2}\|u\|_E^2-\frac{1}{2(2_s^*-1)}\|K\|_{\infty}^2 S^{-2_s^*}\|u\|_E^{2(2_s^*-1)} \\
& -L_{1\varepsilon}\int_{\mathbb{R}^3} a(x)|u|^p dx-L_{2\varepsilon}\int_{\mathbb{R}^3} a(x)|u|^{2_s^*} dx \\
\geq & \frac{1}{2}\|u\|_E^2-\frac{1}{2(2_s^*-1)}\|K\|_{\infty}^2 S^{-2_s^*}\|u\|_E^{2(2_s^*-1)} \\
& -L_{1\varepsilon}\left[\|a\|_{\infty}|B_R(0)|^{\frac{2_s^*-p}{2_s^*}}\|u\|_{2_s^*}^p+\varepsilon\|u\|_E^2+\varepsilon\|u\|_{2_s^*}^{2_s^*}\right]-L_{2\varepsilon}\|a\|_{\infty} S^{-2_s^*}\|u\|_E^{2_s^*} \\
\geq & \frac{1}{2}\|u\|_E^2-\frac{1}{2(2_s^*-1)}\|K\|_{\infty}^2 S^{-2_s^*}\|u\|_E^{2(2_s^*-1)}-L_{1\varepsilon}\|a\|_{\infty}|B_R(0)|^{\frac{2_s^*-p}{2_s^*}} S^{-\frac{p}{2}}\|u\|_E^p \\
& -\varepsilon L_{1\varepsilon}\|u\|_E^2-\varepsilon L_{1\varepsilon} S^{-\frac{2_s^*}{2}}\|u\|_E^{2_s^*}-L_{2\varepsilon}\|a\|_{\infty} S^{-\frac{2_s^*}{2}}\|u\|_E^{2_s^*}.
\end{aligned}
$$

Since $\varepsilon$ can be small enough, we may choose $\varepsilon L_{1\varepsilon} \leq \frac{1}{4}$, to get

$$
J(u)=\frac{1}{4}\|u\|_E^2-C_1\|u\|_E^{2(2_s^*-1)}-C_2\|u\|_E^p-C_3\|u\|_E^{2_s^*}, \quad p \in (2, 2_s^*).
$$

Consequently, there exists $\rho>0$ small enough such that $J(u) \geq \alpha>0$ when $\|u\|_E=\rho$. The proof is completed. □

**Remark 2.7.** *By Lemma 2.6, we can obtain a sequence $\{u_n\} \subset E$ such that*

$$
J(u_n) \rightarrow c \quad \text{and} \quad (1+\|u_n\|_E)\left\|J'(u_n)\right\| \rightarrow 0,
$$

*where $c=\inf_{\gamma \in \Gamma} \max_{t \in[0,1]} J(\gamma(t))>0$ with*

$$
\gamma=\{\gamma \in C([0,1], E) \,|\, \gamma(0)=0 \quad \text{and} \quad J(\gamma(1))<0\}.
$$

*$\{u_n\}$ is called the Cerami sequence for $J$ at the level $c$.*

**Lemma 2.8.** *Under conditions $(D1)-(D3)$ and $(f_1)-(f_4)$, then $u_n$, the Cerami sequence for $J$ at the level $c$ is bounded in $E$.*

*Proof.* From $(f_3)$ and $(D1)$, we can derive that $a(x)f(u_n)u_n \geq (m+1)a(x)F(u_n)$, $m \in (3-2s, 2_s^*-1)$. Since $\{u_n\}$ is a Cerami sequence for $J$ at the level $c$, we have for $m \in (3-2s, 2_s^*-1)$,

$$
\begin{aligned}
c+1 & \geq J(u_n)-\frac{1}{m+1}\left\langle J'(u_n), u_n\right\rangle \\
& =\frac{m-1}{2(m+1)}\|u_n\|_E^2+\frac{2(2_s^*-1)-1-m}{2(2_s^*-1)(m+1)} \int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)|u_n|^{2_s^*} dx \\
& \quad +\frac{1}{m+1} \int_{\mathbb{R}_3} a(x)\left[f(u_n)u_n-(m+1)F(u_n)\right] dx \\
& \geq \frac{m-1}{2(m+1)}\|u_n\|_E^2,
\end{aligned}
$$

which completes the proof. □

**Lemma 2.9.** *Under conditions $(D1)-(D3)$ and $(f_1)-(f_4)$, and let $u_n \subset E$ be a Cerami sequence for $J$ at the level $c$, and $u_n \rightharpoonup u$ in $E$, where $c \in \left(0, \frac{2s}{3+2s}\|K\|_{\infty}^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}\right)$. Then $u \not\equiv 0$.*

*Proof.* Suppose by contradiction that, $u \equiv 0$. From (2.3) we get

$$
c+o_n(1)=J(u_n)=\frac{1}{2}\|u_n\|_E^2-\frac{1}{2(2_s^*-1)} \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1} dx+o_n(1).
$$

and

$$o(1) = J'(u_n)u_n = \|u_n\|_E^2 - \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx + o_n(1).$$

Then

$$c = \frac{1}{2}\lim_{n\to\infty}\|u_n\|_E^2 - \lim_{n\to\infty}\frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx. \tag{2.15}$$

and

$$0 = \lim_{n\to\infty}\|u_n\|_E^2 - \lim_{n\to\infty}\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx.$$

Let

$$\lim_{n\to\infty}\|u_n\|_E^2 = \lim_{n\to\infty}\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^{2_s^*-1}dx = l. \tag{2.16}$$

By (2.16) and (2.15), we have $c = \frac{2s}{3+2s}l$. Moreover, from (2.7) and (2.16) we deduce that

$$l \le \|K\|_\infty^2 S^{-2_s^*} l^{2_s^*-1}.$$

Therefore, either $l = 0$ or $l \ge \|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}$. If $l \ge \|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}$, then $c \ge \frac{2s}{3+2s}\|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}$, which is a contradiction. If $l = 0$, then we have $c = 0$, which contradicts to the assumption $c > 0$. The proof is completed. □

In the next part, we estimate the interval value of $c$. To this purpose, we let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with support in $B_1(0)$ such that $\eta \in [0,1]$ on $B_1(0)$. We define

$$u_\varepsilon = \eta(x)U_\varepsilon(x), u^* = \frac{u_\varepsilon}{\|u_\varepsilon\|_{L^{2_s^*}(\mathbb{R}^3)}},$$

where

$$U_\varepsilon(x) = \frac{\kappa\varepsilon^{\frac{3-2s}{2}}}{(\varepsilon^2 + |x-x_0|^2)^{\frac{3-2s}{2}}}, \tag{2.17}$$

solves the equation $(-\Delta)^s u = u^{2_s^*-1}, x \in \mathbb{R}^3$, $\kappa$ is a normalization constant, see [31]. The following estimates can be deduced from the standard arguments [32],

$$\|(-\Delta)^{\frac{s}{2}}u_\varepsilon\|_2^2 = S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}), \tag{2.18}$$

$$\|u_\varepsilon\|_{2_s^*}^{2_s^*} = S_s^{\frac{3}{2s}} + O(\varepsilon^3), \tag{2.19}$$

$$\|Vu_\varepsilon\|_2^2 = O(\varepsilon^{3-2s}), \tag{2.20}$$

and

$$\int_{\mathbb{R}^3}|u_\varepsilon|^p dx = \begin{cases} O(\varepsilon^{\frac{3(2-p)+2sp}{2}}) & \text{if} \quad p > \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{3(2-p)+2sp}{2}}|\log\varepsilon|) & \text{if} \quad p = \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{(3-2s)p}{2}}) & \text{if} \quad p < \frac{3}{3-2s}. \end{cases} \tag{2.21}$$

**Lemma 2.10.** *Assume $V(x)$, $a(x)$ and $K(x)$ satisfy the assumptions $(D1)-(D3)$ and $(K)$, respectively. Let*

$$h(t) = \frac{t^2}{2}\left(\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u_\varepsilon\right|^2 dx + \int_{\mathbb{R}^3}V(x)|u_\varepsilon|^2dx\right) - \frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3}K(x)\phi_{u_\varepsilon}|u_\varepsilon|^{(2_s^*-1)}dx,$$

*Then for any $t \ge 0$,*

$$h(t) \le \frac{2s}{3+2s}\|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}).$$

*Proof.* Set $h(t)=\frac{t^2}{2}a-\frac{t^{2(2_s^*-1)}}{2(2_s^*-1)}b$, where

$$a=\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2dx+\int_{\mathbb{R}^3}V(x)|u_\varepsilon|^2dx \quad\text{and}\quad b=\int_{\mathbb{R}^3}K(x)\phi_{u_\varepsilon}|u_\varepsilon|^{(2_s^*-1)}dx.$$

Solving $h'(t)=0$, the root of the equation is $t_0=(\frac{a}{b})^{\frac{1}{2(2_s^*-2)}}$. Combining $h'(t_0)=0$ with $h''(t_0)=(4-22_s^*)a<0$, we have that $h(t)$ reaches the maximum at $t_0$. Then,

$$\max_{t\geq 0}h(t)=h(t_0)=\frac{(2_s^*-2)a^{\frac{3+2s}{4s}}}{2(2_s^*-1)b^{\frac{3-2s}{4s}}}=\frac{(2_s^*-2)\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2dx\right)^{\frac{3+2s}{4s}}}{2(2_s^*-1)\left(\int_{\mathbb{R}^3}K(x)\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx\right)^{\frac{3-2s}{4s}}}.$$

On the other hand,

$$\int_{\mathbb{R}^3}K(x)|u_\varepsilon|^{2_s^*}dx=K(x_0)\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx-\left(\int_{\mathbb{R}^3}[K(x_0)-K(x)]|u_\varepsilon|^{2_s^*}dx\right)^{\frac{3-2s}{4s}}. \tag{2.22}$$

Once that $(K)$ is true, we have

$$\int_{\mathbb{R}^3}[K(x_0)-K(x)]|u_\varepsilon|^{2_s^*}dx\leq C\int_{|x-x_0|<\delta}|x-x_0|^\beta|u_\varepsilon|^{2_s^*}dx+2\|K\|_\infty\int_{|x-x_0|\geq\delta}|u_\varepsilon|^{2_s^*}dx=I_1+I_2.$$

Substituting (2.19) into $I_1$ and $I_2$, respectively, we deduce that

$$\begin{aligned}I_1&\leq C\int_{|x-x_0|<\delta}|x-x_0|^\beta|U_\varepsilon|^{2_s^*}dx=C\int_{|x-x_0|<\delta}\frac{\kappa^{2_s^*}|x-x_0|^\beta\varepsilon^3}{(\varepsilon^2+|x-x_0|^2)^3}dx\\&=C\kappa^{\frac{6}{3-2s}}\int_0^\delta\frac{r^{2+\beta}\varepsilon^3}{(\varepsilon^2+r^2)^3}dr\leq C\varepsilon^\beta\int_0^{+\infty}(1+r^2)^{\frac{\beta}{2}-\frac{5}{2}}d(1+r^2)=C\varepsilon^\beta,\end{aligned}$$

and

$$\begin{aligned}I_2&=2\|K\|_\infty\int_{|x-x_0|\geq\delta}|u_\varepsilon|^{2_s^*}dx\leq C\int_{|x-x_0|\geq\delta}\frac{\kappa^{2_s^*}\varepsilon^3}{(\varepsilon^2+|x-x_0|^2)^3}dx\\&\leq C\varepsilon^3\int_\delta^{+\infty}\frac{r^2}{(\varepsilon^2+r^2)^3}dr\leq C\varepsilon^3\int_\delta^{+\infty}r^{-4}dr=C\varepsilon^3.\end{aligned}$$

From $\beta\in[3-2s,3)$, we have that

$$\int_{\mathbb{R}^3}[K(x_0)-K(x)]|u_\varepsilon|^{2_s^*}dx\leq C\varepsilon^\beta. \tag{2.23}$$

Hence, by (2.22) and (2.23) we get

$$\int_{\mathbb{R}^3}K(x)|u_\varepsilon|^{2_s^*}dx\geq\|K\|_\infty\|u_\varepsilon\|_{2_s^*}^{2_s^*}-C\varepsilon^\beta=\|K\|_\infty S^{\frac{3}{2s}}+O(\varepsilon^\beta). \tag{2.24}$$

From (2.18), (2.24) and $\beta\in[3-2s,3)$, one has

$$\begin{aligned}\int_{\mathbb{R}^3}K(x)|u_\varepsilon|^{2_s^*}&=\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}\phi_{u_\varepsilon}(-\Delta)^{\frac{s}{2}}|u_\varepsilon|dx\\&\leq\frac{1}{2\|K\|_\infty}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\phi_{u_\varepsilon}|^2+\frac{\|K\|_\infty}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}|u_\varepsilon||^2dx\\&\leq\frac{1}{2\|K\|_\infty}\int_{\mathbb{R}^3}K(x)\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx+\frac{\|K\|_\infty}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2dx,\end{aligned}$$

which shows that

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx &\geq 2\,\|K\|_\infty \int_{\mathbb{R}^3} K(x)|u_\varepsilon|^{2_s^*}dx - \|K\|_\infty^2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2dx \\ &\geq 2\,\|K\|_\infty^2\, S^{\frac{3}{2s}} + O(\varepsilon^\beta) - \|K\|_\infty^2\, S^{\frac{3}{2s}} - O(\varepsilon^{3-2s}) \\ &= \|K\|_\infty^2\, S_s^{\frac{3}{2s}} - O(\varepsilon^{3-2s}).\end{aligned}$$

Therefore, we obtain

$$\max_{t\geq 0} h(t) \leq \frac{2s}{3+2s}\frac{\left(S^{\frac{3}{2s}} + O(\varepsilon^{3-2s})\right)^{\frac{3+2s}{4s}}}{\left(\|K\|_\infty^2\, S_s^{\frac{3}{2s}} - O(\varepsilon^{3-2s})\right)^{\frac{3-2s}{4s}}} \leq \frac{2s}{3+2s}\,\|K\|_\infty^{-\frac{3-2s}{2s}}\, S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}),$$

which implies the assertion. □

**Lemma 2.11.** *Assume $V(x)$, $a(x)$, $K(x)$ and $f(u)$ satisfy the assumptions $(D1)-(D3)$ and $(f_1)-(f_4)$, respectively. Then,*

$$c \in \left(0, \frac{2s}{3+2s}\,\|K\|_\infty^{-\frac{3-2s}{2s}}\, S^{\frac{3}{2s}}\right).$$

*Proof.* According to the mountain pass geometry, there is $\rho > 0$ small enough such that $J(u) \geq a > 0$ when $\|u\|_E = \rho$. Thus, $c \geq a > 0$. Since $u_\varepsilon \geq 0$ and $u_\varepsilon \not\equiv 0$, from $(f_2)$, it follows that $\lim_{t\to+\infty} J(tu_\varepsilon) = -\infty$. Combining this with $J(0) = 0$ and $J(u) > 0$ when $\|u\|_E = \rho$ for some $\rho > 0$, there is $t_\varepsilon > 0$ such that $J(t_\varepsilon u_\varepsilon) = \max_{t\geq 0} J(tu_\varepsilon)$.

**Claim 1:** $c \leq \max_{t\geq 0} J(tu_\varepsilon) = J(t_\varepsilon u_\varepsilon)$. Indeed, it follows from Lemma 2.6 that there is $t_0 > 0$ large enough such that $J(t_0u_\varepsilon) < 0$. Define $\gamma_0(t) = tt_0u_\varepsilon$. Obviously, $\gamma_0(t) \in \Gamma$. Hence,

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \leq \max_{t\in[0,1]} J(tt_0u_\varepsilon) = \max_{t\in[0,t_0]} J(tu_\varepsilon) \leq \max_{t\geq o} J(tu_\varepsilon) = J(t_\varepsilon u_\varepsilon).$$

Claim 1 implies that

$$J(t_\varepsilon u_\varepsilon) > 0 \qquad \text{and} \qquad \frac{d}{dt}J(tu_\varepsilon)|_{t=t_\varepsilon} = 0,$$

that is,

$$\begin{aligned}\frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2 + V(x)u_\varepsilon^2)dx - \frac{t_\varepsilon^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}(x)|u_\varepsilon|^{2_s^*-1}dx \\ - \int_{\mathbb{R}^3} a(x)F(x, t_\varepsilon u_\varepsilon)dx > 0.\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}t_\varepsilon \int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2 + V(x)u_\varepsilon^2)dx - t_\varepsilon^{22_s^*-3}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}(x)|u_\varepsilon|^{2_s^*-1}dx \\ - \int_{\mathbb{R}^3} a(x)f(x, t_\varepsilon u_\varepsilon)u_\varepsilon dx = 0.\end{aligned} \tag{2.26}$$

**Claim 2:** There exist some $\varepsilon_2 > 0$ and $t_2 > 0$ such that $t_\varepsilon \le t_2$ for each $\varepsilon \in (0, \varepsilon_2)$. We first consider $(D3)_1$ holds. Inserting (2.8) and (2.9) into (2.25), we obtain

$$\begin{aligned}
&\frac{t_\varepsilon^{2(2_s^*-1)}}{2(2_s^*-1)} \int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}(x)|u_\varepsilon|^{2_s^*-1}dx \\
&\le \frac{t_\varepsilon^2}{2}\|u\|_E^2 + \int_{\mathbb{R}^3} a(x)F(x, t_\varepsilon u_\varepsilon)dx \\
&\le \frac{t_\varepsilon^2}{2}\|u\|_E^2 + \varepsilon t_\varepsilon^2 \int_{\mathbb{R}^3} a(x)|u_\varepsilon|^2 dx + C_\varepsilon t_\varepsilon^{2_s^*} \int_{\mathbb{R}^3} a(x)|u_\varepsilon|^{2_s^*}dx \\
&\le \frac{t_\varepsilon^2}{2}\|u\|_E^2 + \varepsilon t_\varepsilon^2 \left\|\frac{a}{V}\right\|_\infty \|u_\varepsilon\|_E^2 + C_\varepsilon t_\varepsilon^{2_s^*}\|a\|_\infty \|u_\varepsilon\|_{2_s^*}^{2_s^*}.
\end{aligned}$$

Because $\varepsilon$ is small enough, we can choose $\varepsilon\left\|\frac{a}{V}\right\|_\infty \le \frac{1}{2}$, verifying

$$\frac{t_\varepsilon^{2(2_s^*-2)}}{2(2_s^*-1)} \int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}(x)|u_\varepsilon|^{2_s^*-1}dx \le \|u_\varepsilon\|_E^2 + C_\varepsilon t_\varepsilon^{2_s^*-2}\|a\|_\infty \|u_\varepsilon\|_{2_s^*}^{2_s^*}.$$

Therefore, there exist $\varepsilon_{21} > 0$ and $t_{21} > 0$ such that $t_\varepsilon \le t_{21}$ for $\varepsilon \in (0, \varepsilon_{21})$.

Now we suppose that $(D3)_2$ holds. Combining (2.11), (2.14) and (2.25), we have

$$\begin{aligned}
&\frac{t_\varepsilon^{2(2_s^*-1)}}{2(2_s^*-1)} \int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}(x)|u_\varepsilon|^{2_s^*-1}dx \\
&\le \frac{t_\varepsilon^2}{2}\|u\|_E^2 + L_{1\varepsilon}t_\varepsilon^p \int_{\mathbb{R}^3} a(x)|u_\varepsilon|^p dx + L_{2\varepsilon}t_\varepsilon^{2_s^*}\int_{\mathbb{R}^3} a(x)|u_\varepsilon|^{2_s^*}dx \\
&\le \frac{t_\varepsilon^2}{2}\|u\|_E^2 + L_{1\varepsilon}t_\varepsilon^p\left[\|a\|_\infty |B_R(0)|^{\frac{2_s^*-p}{2_s^*}}\|u_\varepsilon\|_{2_s^*}^p + \varepsilon\|u_\varepsilon\|_E^2 + \varepsilon\|u_\varepsilon\|_{2_s^*}^{2_s^*}\right] \\
&\quad + L_{2\varepsilon}t_\varepsilon^{2_s^*}\|a\|_\infty S^{-\frac{2_s^*}{2}}\|u\|_E^{2_s^*} \\
&= \frac{t_\varepsilon^2}{2}\|u\|_E^2 + L_{1\varepsilon}t_\varepsilon^p\|a\|_\infty |B_R(0)|^{\frac{2_s^*-p}{2_s^*}}\|u_\varepsilon\|_{2_s^*}^p + L_{1\varepsilon}t_\varepsilon^p\varepsilon\|u_\varepsilon\|_E^2 + L_{1\varepsilon}t_\varepsilon^p\varepsilon\|u_\varepsilon\|_{2_s^*}^{2_s^*} \\
&\quad + L_{2\varepsilon}t_\varepsilon^{2_s^*}\|a\|_\infty S^{-\frac{2_s^*}{2}}\|u\|_E^{2_s^*}.
\end{aligned}$$

Therefore, there are $\varepsilon_{22} > 0$ and $t_{22} > 0$ such that $t_\varepsilon < t_{22}$ for $\varepsilon \in (0, \varepsilon_{22})$. Let $\varepsilon_2 = \min\{\varepsilon_{21}, \varepsilon_{22}\}$ and $t_2 = \min\{t_{21}, t_{22}\}$. Then $t_\varepsilon \le t_2$ for $\varepsilon \in (0, \varepsilon_2)$.

**Claim 3:** There exist some $\varepsilon_1 > 0$ and $t_1 > 0$ such that $t_\varepsilon \ge t_1$ for every $\varepsilon \in (0, \varepsilon_1)$.
We first consider $(D3)_1$ holds. From (2.8), (2.9), and (2.26), we have

$$\begin{aligned}
\|u_\varepsilon\|_E^2 &\le t_\varepsilon^{2(2_s^*-2)}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}(x)|u_\varepsilon|^{2_s^*-1}dx + \varepsilon\int_{\mathbb{R}^3} a(x)|u_\varepsilon|^2dx + C_\varepsilon t_\varepsilon^{2_s^*-2}\int_{\mathbb{R}^3} a(x)|u_\varepsilon|^{2_s^*}dx \\
&\le t_\varepsilon^{2(2_s^*-2)}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}(x)|u_\varepsilon|^{2_s^*-1}dx + \varepsilon\left\|\frac{a}{V}\right\|_\infty\|u\|_E^2 + C_\varepsilon t_\varepsilon^{2_s^*-2}\|a\|_\infty\|u_\varepsilon\|_{2_s^*}^{2_s^*}.
\end{aligned}$$

Take $\varepsilon$ sufficiently small, such that $\varepsilon\left\|\frac{a}{V}\right\|_\infty \le \frac{1}{2}$. So,

$$\|u_\varepsilon\|_E^2 \le t_\varepsilon^{2(2_s^*-2)}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}(x)|u_\varepsilon|^{2_s^*-1}dx + \varepsilon\int_{\mathbb{R}^3} a(x)|u_\varepsilon|^2dx + C_\varepsilon t_\varepsilon^{2_s^*-2}\|a\|_\infty\|u_\varepsilon\|_{2_s^*}^{2_s^*},$$

which implies that there exist $\varepsilon_{11} > 0$ and $t_{11} > 0$ such that $t_\varepsilon \ge t_{11}$ for $\varepsilon \in (0, \varepsilon_{11})$.

Now, suppose that $(D3)_2$ is true. From (2.11), (2.14) and (2.26), we see that

$$\begin{aligned} t_\varepsilon \|u_\varepsilon\|_E^2 &= t_\varepsilon^{22_s^*-3}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}(x)|u_\varepsilon|^{2_s^*-1}dx + \int_{\mathbb{R}^3} a(x)f(x,t_\varepsilon u_\varepsilon)u_\varepsilon dx \\ &\le t_\varepsilon^{22_s^*-3}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}(x)|u_\varepsilon|^{2_s^*-1}dx + L_{1\varepsilon}t_\varepsilon^{p-1}\int_{\mathbb{R}^3} a(x)|u_\varepsilon|^p dx + L_{2\varepsilon}t_\varepsilon^{2_s^*-1}\int_{\mathbb{R}^3} a(x)|u_\varepsilon|^{2_s^*}dx \\ &\le t_\varepsilon^{22_s^*-3}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*}dx + L_{1\varepsilon}t_\varepsilon^{p-1}\|a\|_\infty |B_R(0)|^{\frac{2_s^*-p}{2_s^*}}\|u_\varepsilon\|_{2_s^*}^p + L_{1\varepsilon}t_\varepsilon^{p-1}\varepsilon\|u_\varepsilon\|_E^2 \\ &\quad + L_{1\varepsilon}t_\varepsilon^{p-1}\varepsilon\|u_\varepsilon\|_{2_s^*}^{2_s^*} + L_{2\varepsilon}t_\varepsilon^{2_s^*-1}\|a\|_\infty\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx, \end{aligned}$$

combining with Claim 2, we infer that

$$\begin{aligned} \|u_\varepsilon\|_E^2 &\le t_\varepsilon^{2(2_s^*-2)}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*}dx + L_{1\varepsilon}t_\varepsilon^{p-2}\|a\|_\infty |B_R(0)|^{\frac{2_s^*-p}{2_s^*}}\|u_\varepsilon\|_{2_s^*}^p + L_{1\varepsilon}t_2^{p-2}\varepsilon\|u_\varepsilon\|_E^2 \\ &\quad + L_{1\varepsilon}t_\varepsilon^{p-2}\varepsilon\|u_\varepsilon\|_{2_s^*}^{2_s^*} + L_{2\varepsilon}t_\varepsilon^{2_s^*-2}\|a\|_\infty\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx. \end{aligned}$$

Similarly, choosing $\varepsilon L_{1\varepsilon}t_2^{p-2} \le \frac{1}{2}$, one has

$$\begin{aligned} \frac{1}{2}\|u_\varepsilon\|_E^2 &\le t_\varepsilon^{2(2_s^*-2)}\int_{\mathbb{R}^3} K(x)\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*}dx + L_{1\varepsilon}t_\varepsilon^{p-2}\|a\|_\infty |B_R(0)|^{\frac{2_s^*-p}{2_s^*}}\|u_\varepsilon\|_{2_s^*}^p \\ &\quad + L_{1\varepsilon}t_\varepsilon^{p-2}\varepsilon\|u_\varepsilon\|_{2_s^*}^{2_s^*} + L_{2\varepsilon}t_\varepsilon^{2_s^*-2}\|a\|_\infty\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx. \end{aligned}$$

showing that there are $\varepsilon_{12} > 0$ and $t_{12} > 0$ such that $t_\varepsilon \ge t_{12}$ for $\varepsilon \in (0, \varepsilon_{12})$.

Let $\varepsilon_1 = \min\{\varepsilon_{11}, \varepsilon_{12}\}$ and $t_1 = \max\{t_{11}, t_{12}\}$.Then $t_\varepsilon \ge t_1$ for $\varepsilon \in (0, \varepsilon_1)$. Denote by $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, then for each $\varepsilon \in (0, \varepsilon_0)$, we have $0 < t_1 < t_\varepsilon \le t_2$.

The rest of the proof is to estimate the upper bound of $J(t_\varepsilon u_\varepsilon)$. By Lemma 2.9 we get

$$\begin{aligned} J(t_\varepsilon u_\varepsilon) &= h(t_\varepsilon) - \int_{\mathbb{R}^3} a(x)F(x,t_\varepsilon u_\varepsilon)dx \\ &\le \frac{2s}{3+2s}\|K\|_\infty^{-\frac{3-2s}{2s}} S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) - \int_{\mathbb{R}^3} a(x)F(x,t_\varepsilon u_\varepsilon)dx. \end{aligned} \tag{2.27}$$

When case $(D3)_1$ occurs, we have from $(f_1)$ that, for given $\varepsilon_1 > 0$ small enough, there is $\delta_1 > 0$ small enough such that $|f(t)| \le \varepsilon_1 t$, $0 < t < \delta_1$; and by $(f_2)$, given $\varepsilon_2 > 0$ small enough, there is $M_2 > 0$ large enough such that $|f(t)| \le \varepsilon_2 t^{2_s^*-1}$, $t > M_2$. Since $f(t) \in C[\delta_1, M_2]$, we see that $|f(t)| \le \frac{\bar{A}}{\delta_1^{2_s^*-1}}t^{2_s^*-1}$ for some $\bar{A} > 0$. Consequently, for every $t > 0$, we have

$$|f(t)| \le \varepsilon_1 t + \left(\varepsilon_2 + \frac{\bar{A}}{\delta^{2_s^*-1}}\right)t^{2_s^*-1} \quad \text{and} \quad |F(t)| \le \varepsilon_1 t^2 + \left(\varepsilon_2 + \frac{\bar{A}}{\delta_1^{2_s^*-1}}\right)t^{2_s^*}.$$

On the other hand, from $(f_4)$, it follows that given $X > 0$ large enough, there exists $M_3 > 0$ large enough such that $|F(t)| \ge Xt^{m+1}$ with $m \in (3-2s, 2_s^*-1)$, $t > M_3$. So,

$$F(t) \ge -\varepsilon_1 t^2, \quad \text{if } 0 < t < \delta_1; \quad F(t) \ge Xt^2, \quad \text{if } t > M_3,$$

and

$$F(t) \ge -\varepsilon_1 t^2 + \left(\varepsilon_2 + \frac{\bar{A}}{\delta^{2_s^*-1}}\right)M_3^{2_s^*-2}t^2, \text{ if } \delta_1 \le t \le M_3.$$

Therefore, the exists $M > 0$ such that for $t > 0$,

$$F(t) \ge -Mt^2. \tag{2.28}$$

Again, by $(f_4)$, we have that for $t > M_3$,

$$F(t) \geq \frac{3X2^{\frac{(3-2s)(m+1)-2}{2}}}{a_0\kappa^{m+1}t_1^{m+1}\pi}t^{m+1}, \tag{2.29}$$

where $a_0 = \min_{x\in B_1(x_0)} a(x)$. Set $\varepsilon_3 = \min\left\{\varepsilon_0, \frac{(\kappa t_1)^{\frac{2}{3-2s}}}{2M_3^{\frac{2}{3-2s}}}, 1\right\}$, then $B_\varepsilon(x_0) \subset B_1(x_0)$ for each $\varepsilon \in (0, \varepsilon_3)$.

For $x \in B_\varepsilon(x_0)$,

$$t_\varepsilon u_\varepsilon = t_\varepsilon \eta(x) U_\varepsilon(x) = t_\varepsilon \frac{\kappa\varepsilon^{\frac{3-2s}{2}}}{(\varepsilon^2 + |x-x_0|^2)^{\frac{3-2s}{2}}} > t_1 \frac{\kappa\varepsilon^{\frac{3-2s}{2}}}{(2\varepsilon^2)^{\frac{3-2s}{2}}} > M_3.$$

Then, from (2.28) and (2.29), it follows that

$$\begin{aligned}
\int_{\mathbb{R}^3} a(x)F(t_\varepsilon u_\varepsilon)dx &= \int_{B_\varepsilon(x_0)} a(x)F(t_\varepsilon u_\varepsilon)dx + \int_{\mathbb{R}^3\setminus B_\varepsilon(x_0)} a(x)F(t_\varepsilon u_\varepsilon)dx \\
&\geq \int_{B_\varepsilon(x_0)} a(x)\frac{3X2^{\frac{(3-2s)(m+1)-2}{2}}}{a_0\kappa^{m+1}t_1^{m+1}\pi}(t_\varepsilon u_\varepsilon)^{m+1}dx + \int_{\mathbb{R}^3\setminus B_\varepsilon(x_0)} a(x)\left[-M(t_\varepsilon u_\varepsilon)^2\right]dx \\
&\geq \frac{3X2^{\frac{(3-2s)(m+1)-2}{2}}}{\kappa^{m+1}\pi}\int_{B_\varepsilon(x_0)} (u_\varepsilon)^{m+1}\,dx - Mt_2^2\left\|\frac{a}{V}\right\|_\infty \|Vu_\varepsilon\|_2^2 \\
&\geq \frac{3X2^{\frac{(3-2s)(m+1)-2}{2}}}{\kappa^{m+1}\pi}\int_{B_\varepsilon(x_0)} \frac{\kappa^{m+1}\varepsilon^{\frac{(3-2s)(m+1)}{2}}}{(\varepsilon^2+|x-x_0|^2)^{\frac{(3-2s)(m+1)}{2}}} - Mt_2^2\left\|\frac{a}{V}\right\|_\infty \|Vu_\varepsilon\|_2^2 \\
&\geq X\varepsilon^{3-\frac{(3-2s)(m+1)}{2}} - Mt_2^2\left\|\frac{a}{V}\right\|_\infty \|Vu_\varepsilon\|_2^2.
\end{aligned} \tag{2.30}$$

Therefore, by (2.27), (2.30) and (2.20), we have

$$\begin{aligned}
J(t_\varepsilon u_\varepsilon) &\leq \frac{2s}{3+2s}\|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) - \left(X\varepsilon^{3-\frac{(3-2s)(m+1)}{2}} - Mt_2^2\left\|\frac{a}{V}\right\|_\infty \|Vu_\varepsilon\|_2^2\right) \\
&\leq \frac{2s}{3+2s}\|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) - X\varepsilon^{3-\frac{(3-2s)(m+1)}{2}}.
\end{aligned}$$

Since $m \in (3-2s, 2_s^*-1)$, then

$$0 < 3 - \frac{(3-2s)(m+1)}{2} < -2s^2 + 7s - 3. \tag{2.31}$$

Simple calculation shows that, for $s \in (0,1)$,

$$3 - 2s > -2s^2 + 7s - 3, \tag{2.32}$$

then for $\varepsilon$ small enough, we derive that

$$J(t_\varepsilon u_\varepsilon) < \frac{2s}{3+2s}\|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}.$$

When case $(D3)_2$ occurs, by a similar argument as that of (2.27), there is $M > 0$ such that for $t > 0$,

$$F(t) \geq -Mt^p.$$

Then for $\varepsilon \in (0, \varepsilon_3)$, we have

$$\int_{\mathbb{R}^3} a(x)F(t_\varepsilon u_\varepsilon)dx \geq X\varepsilon^{3-\frac{(3-2s)(m+1)}{2}} - Mt_2^p\left\|\frac{a}{V}\right\|_\infty \|Vu_\varepsilon\|_p^p.$$

Hence, by (2.21), we get

$$\begin{aligned} J(t_\varepsilon u_\varepsilon) &\le \frac{2s}{3+2s}\|K\|_\infty^{-\frac{4s}{3-2s}} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) - \left(X\varepsilon^{3-\frac{(3-2s)(m+1)}{2}} - Mt_2^p \left\|\frac{a}{V}\right\|_\infty \|Vu_\varepsilon\|_p^p\right) \\ &= \frac{2s}{3+2s}\|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) - X\varepsilon^{3-\frac{(3-2s)(m+1)}{2}} + \begin{cases} O(\varepsilon^{\frac{3(2-p)+2sp}{2}}), & p > \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{3(2-p)+2sp}{2}}|\log\varepsilon|), & p = \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{3-2s}{2}p}), & p < \frac{3}{3-2s}. \end{cases} \end{aligned}$$

When $2 < p < \frac{3}{3-2s}$, $O(\varepsilon^{3-2s}) + O(\varepsilon^{\frac{3-2s}{2}p}) = O(\varepsilon^{3-2s})$. Since $X > 0$ can be large enough, let $X = \varepsilon^{-\frac{4s-(3-2s)p}{2}}$, combining with $3 - \frac{(3-2s)(m+1)}{2} - \frac{4s-(3-2s)p}{2} < 3-2s$ for $m+1 > p$, we have $O(\varepsilon^{3-2s}) - X\varepsilon^{3-\frac{(3-2s)(m+1)}{2}} < 0$ for $\varepsilon$ sufficiently small.

When $p = \frac{3}{3-2s}$, $O(\varepsilon^{3-2s}) + O(\varepsilon^{\frac{2(2-p)+2sp}{2}}|\log\varepsilon|) = O(\varepsilon^{3-2s})$. Because $X > 0$ can be large enough, choose $X = \varepsilon^{-(2s-3/2)}$, using $3 - \frac{(3-2s)(m+1)}{2} - \left(2s - \frac{3}{2}\right) < 3-2s$ for $m+1 > \frac{3}{3-2s}$, we get $O(\varepsilon^{3-2s}) - X\varepsilon^{3-\frac{(3-2s)(m+1)}{2}} < 0$ for $\varepsilon$ sufficiently small.

When $\frac{3}{3-2s} < p < \frac{4s}{3-2s}$, $O(\varepsilon^{3-2s} + O(\varepsilon^{\frac{3(2-p)+2sp}{2}}) = O(\varepsilon^{3-2s})$. Because $X > 0$ is large enough, let $X = \varepsilon^{-(2s-3/2)}$, combining with $3 - \frac{(3-2s)(m+1)}{2} - \left(2s - \frac{3}{2}\right) < 3-2s$ for $m+1 > p > \frac{3}{3-2s}$, we infer $O(\varepsilon^{3-2s}) - X\varepsilon^{3-\frac{(3-2s)(m+1)}{2}} < 0$ for $\varepsilon$ sufficiently small.

when $\frac{4s}{3-2s} \le p < 2_s^*$, $O(\varepsilon^{3-2s} + O(\varepsilon^{\frac{3(2-p)+2sp}{2}}) = O(\varepsilon^{\frac{3(2-p)+2sp}{2}})$. Since $3 - \frac{(3-2s)(m+1)}{2} - \frac{3(2-p)+2sp}{2} < 0$ for $m+1 > p$, we derive $O(\varepsilon^{\frac{3(2-p)+2sp}{2}}) - X\varepsilon^{3-\frac{(3-2s)(m+1)}{2}} < 0$ for $\varepsilon$ sufficiently small.

From the four cases above, we have,

$$J(t_\varepsilon u_\varepsilon) < \frac{2s}{3+2s}\|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}.$$

Combining $J(t_\varepsilon u_\varepsilon) < \frac{2s}{3+2s}\|K\|_\infty^{-\frac{4s}{3-2s}} S^{\frac{3}{2s}}$ with Claim 1, we have that

$$c < J(t_\varepsilon u_\varepsilon) < \frac{2s}{3+2s}\|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}.$$

As a result, Claim 1, we derive that

$$0 < c \le J(t_\varepsilon u_\varepsilon) < \frac{2s}{3+2s}\|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}},$$

which completes the proof. □

## 3. Proof of Theorem 1.1

In this section, we will prove that system (1.9) has a positive ground state solution. In order to obtain the desired result, we need to apply the following theorem A.I due to Berestycki and Loins [7], which is crucial in the prof of the main result.

**Lemma 3.1.** *( [7]) Let $P$ and $Q$ be two continuous functions from $\mathbb{R} \to \mathbb{R}$ and satisfy the following:*

$(I_1)$ $\frac{P(t)}{Q(t)} \to 0$ *as* $|t| \to +\infty$.

*Suppose $u_n$ is a sequence of measurable functions from $\mathbb{R}^N \to \mathbb{R}$ such that*

$(I_2)$ $\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))|dx < +\infty$;

*and*

$(I_3)$ $P(u_n(x)) \to v(x)$ *a.e. in* $\mathbb{R}^n$ *as* $n \to \infty$.

*Then for any bounded Borel set $B$, one has*

$$\int_B |P(u_n(x) - v(x)|dx \to 0 \text{ as } n \to \infty.$$

*If one further assumes that*

$(I_4)$ $\frac{P(t)}{Q(t)} \to 0$ *as* $|t| \to +\infty$;
*and*

$(I_5)$ $u_n(x) \to 0$ *as* $|x| \to +\infty$, *uniform consideration of* $n$,

*then* $P(u_n)$ *converges to* $v$ *in* $L^1(\mathbb{R}^n)$ *as* $n \to \infty$.

**Lemma 3.2.** *Under conditions* $(D1)-(D3)$ *and* $(f_1)-(f_4)$, *then, there is* $u \in E$ *such that* $J'(u) = 0$.

*Proof.* Lemma 2.6 implies that there is a sequence $u_n \subset E$ such that $J(u_n) \to c$ and $(1+\|u_n\|_E)\,\|J'(u_n)\| \to 0$ as $n \to \infty$.

From Lemma 2.8, we see that $u_n$ is bound in $E$. Then, up to a subsequence, exist a $u \in E$ such that $u_n \rightharpoonup u$ in $E$. Because $u_n \rightharpoonup u$, for each $w \in C_0^\infty(\mathbb{R}^3)$, that

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} w + V(x) u_n w dx \to \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} w + V(x) u w dx.$$

From Lemma 2.6, we see that

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n} |u_n|^{2_s^*-3} u_n w dx \to \int_{\mathbb{R}^3} K(x)\phi_u |u|^{2_s^*-3} u w dx.$$

Now, we prove that

$$\int_{\mathbb{R}^3} a(x) f(u_n) w dx \to \int_{\mathbb{R}^3} a(x) f(u) w dx. \tag{3.1}$$

For this aim, we use some ideas from Alves and Souto [1]. When $(D3)_1$ holds, repeating the similar arguments used in Lemma 2.11, we have for any $t > 0$ and $q \in (2, 2_s^*)$,

$$|f(t)| \le \varepsilon\,|t| + \varepsilon\,|t|^{2_s^*-1} + C_\varepsilon\,|t|^{q-1}\,.$$

Consequently,

$$\begin{aligned}
\int_{\mathbb{R}^3} a(x) f(u_n) w dx &\le \int_{\mathbb{R}^3} a(x)\left[\varepsilon|u_n| + \varepsilon|u_n|^{2_s^*-1} + C_\varepsilon |u_n|^{q-1}\right]|w|dx \\
&\le \varepsilon \left\|\frac{a}{V}\right\|_\infty \int_{\mathbb{R}^3} V(x)|u_n||w|dx + \varepsilon\,\|a\|_\infty \int_{\mathbb{R}_3} |u_n|^{2_s^*-1}|w|dx + C_\varepsilon \int_{\mathbb{R}^3} a(x)|u_n|^{q-1}|w|dx \\
&\le \varepsilon C\left(\int_{\mathbb{R}^3} V(x)|u_n||w|dx + \int_{\mathbb{R}_3} |u_n|^{2_s^*-1}|w|dx\right) + C_\varepsilon \int_{\mathbb{R}^3} a(x)|u_n|^{q-1}|w|dx.
\end{aligned}$$

By Hölder inequality, we get

$$\int_{\mathbb{R}^3} V(x)|u_n||w|dx \le \left[\int_{\mathbb{R}^3} V(x)|u_n|^2 dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^3} V(x)|w|^2 dx\right]^{\frac{1}{2}},$$

$$\int_{\mathbb{R}^3} |u_n|^{2_s^*-1}|w|dx \le \left(\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx\right)^{\frac{2_s^*-1}{2_s^*}} \left(\int_{\mathbb{R}^3} |w|^{2_s^*} dx\right)^{\frac{1}{2_s^*}},$$

$$\begin{aligned}
\int_{\mathbb{R}^3} a(x)|u_n|^{q-1}|w|dx &\le \left[\int_{\mathbb{R}^3} a(x)|u_n|^q dx\right]^{\frac{q-1}{q}} \left[\int_{\mathbb{R}^3} a(x)|w|^q dx\right]^{\frac{1}{q}} \\
&\le \left[\int_{\mathbb{R}^3} a(x)|u_n|^q dx\right]^{\frac{q-1}{q}} \|a\|_\infty^{\frac{1}{q}} \left[\int_{suppw} |w|^q dx\right]^{\frac{1}{q}} \\
&\le C\left[\int_{\mathbb{R}^3} a(x)|u_n|^q dx\right]^{\frac{q-1}{q}}.
\end{aligned}$$

In view of $u_n \rightharpoonup u$ in $E \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ and $w \in C_0^\infty(\mathbb{R}^3)$, we see that $\int_{\mathbb{R}^3} V(x)|u_n||w|dx$ and $\int_{\mathbb{R}^3} |u_n|^{2_s^*-1}|w|dx$ are both bounded. By Lemma 2.2 and $u_n \rightharpoonup u$ in $E$, we know that $\int_{\mathbb{R}^3} a(x)|u_n|^q dx \to \int_{\mathbb{R}^3} a(x)|u|^q dx$, and there exists $r > 0$ large enough such that $\int_{B_r^c(0)} a(x)|u_n|^q dx \le \varepsilon$. Thus, we have

$$\left| \int_{B_r^c(0)} a(x) f(u_n)\omega dx \right| \le \varepsilon C \cdot 2C_1 + C_\varepsilon \varepsilon \le (2C_1+1)C\varepsilon. \tag{3.2}$$

When $(D3)_2$ holds, by $(f_1)$, we have that for given $\varepsilon > 0$ small, there exists $t_1 > 0$ small enough such that $|f(t)| \le C_\varepsilon |t|^{p-1}$, $0 < t < t_1$. Hence, from (2.13), it follows that $0 < |u_n| < t_1$, and so

$$|a(x)f(u_n)w| \le \varepsilon C_\varepsilon \left[ V(x)|u_n| + |u_n|^{2_s^*-1} \right] |w|, \ \ |x| > r. \tag{3.3}$$

On the other hand, by $(f_2)$, for given $\varepsilon > 0$ small, there exists $t_2 > 0$ large enough such that $|f(t)| \le \varepsilon |t|^{2_s^*-1}$, $t > s_2$. Thus, if $|u_n| > t_2$ holds, we get

$$|a(x)f(u_n)w| \le \|a\|_\infty \varepsilon |u_n|^{2_s^*-1}|w|. \tag{3.4}$$

From equations (3.3) and (3.4), we assume $0 < |u_n| < t_1$ or $|u_n| > t_2$,

$$|a(x)f(u_n)w| \le \varepsilon C \left[ V(x)|u_n| + |u_n|^{2_s^*-1} \right] |w|, \ \ |x| > r.$$

Define $A_n = \left\{ x \in \mathbb{R}^3 : \ t_1 \le |u_n| \le t_2 \right\}$. Then

$$\begin{aligned} \left| \int_{B_r^c(0)} a(x) f(u_n) w dx \right| &\le \int_{B_r^c(0)\cap A_n^c} |a(x)f(u_n)w| dx + \int_{B_r^c(0)\cap A_n} |a(x)f(u_n)w| dx \\ &\le \varepsilon C \int_{B_r^c(0)\cap A_n^c} \left[ V(x)|u_n| + |u_n|^{2_s^*-1} \right] |w| dx + \int_{B_r^c(0)\cap A_n} |a(x)f(u_n)w| dx \\ &= I_3 + I_4. \end{aligned}$$

$$\begin{aligned} I_3 &\le \varepsilon C \int_{\mathbb{R}^3} V(x)|u_n||w| dx + \varepsilon C \int_{\mathbb{R}^3} |u_n|^{2_s^*-1}|w| dx \\ &\le \left( \int_{\mathbb{R}^3} V(x)|u_n|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} V(x)|w|^2 dx \right)^{\frac{1}{2}} + \varepsilon C \left( \int_{\mathbb{R}^3} |u_n|^{\frac{6}{3-2s}} dx \right)^{\frac{3+2s}{6}} \left( \int_{\mathbb{R}^3} |w|^{2_s^*} dx \right)^{\frac{1}{2_s^*}}. \end{aligned}$$

From the boundedness of $\{u_n\}$ in $E \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ and $w \in C_0^\infty(\mathbb{R}^3)$, we have $I_3 \le 2\varepsilon C C_1$.

Similarly, by (2.11), one has

$$I_4 \le \int_{suppw \cap B_r^c(0)\cap A_n} a(x) \left[ L_{1\varepsilon}|u_n|^{p-1} + L_{2\varepsilon}|u_n|^{2_s^*-1} \right] |w| dx \le C \int_{B_r^c(0)\cap A_n} a(x) dx.$$

Therefore,

$$\left| \int_{B_r^c(0)} a(x) f(u_n) w dx \right| \le 2\varepsilon C C_1 + C \int_{B_r^c(0)\cap A_n} a(x) dx.$$

From $u_n \rightharpoonup u$ in $E \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$, it follows that $\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \le C$, and so, $\int_{A_n} |u_n|^{2_s^*} dx \le C$.

From $A_n = \left\{ x \in \mathbb{R}^3 : \ t_1 \le |u_n| \le t_2 \right\}$, we know $t_1^{2_s^*}|A_n| \le \int_{A_n} |u_n|^{2_s^*} dx$, we have $|A_n| \le \frac{C}{t_1^{2_s^*}}$. Thus, condition $(D2)$ implies that $\lim_{r\to+\infty} \int_{A_n \cap B_r^c(0)} a(x) dx = 0$, uniformly in $n \in \mathbb{N}$, then $\int_{A_n \cap B_r^c(0)} a(x) dx \le \frac{\varepsilon}{C}$ for $r$ large enough. Hence,

$$\left| \int_{B_r^c(0)} a(x) f(u_n) w dx \right| \le (2CC_1 + 1)\, \varepsilon. \tag{3.5}$$

By (3.2) and (3.5), we get

$$\left| \int_{B_r^c(0)} a(x) f(u_n) w dx \right| \le C\varepsilon. \tag{3.6}$$

Next, we show that $\int_{B_r(0)} a(x) f(u_n) w dx \to \int_{B_r(0)} a(x) f(u) w dx$ as $n \to +\infty$.

From Lemma 3.1, we can take $P(t)$ and $Q(t)$ by $f(t)w$ and $t^{2_s^*-1}w$, respectively. Condition $(f_2)$ implies that $(I_1)$ holds. Because $u_n \rightharpoonup u$ in $E \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ and $w \in C_0^\infty(\mathbb{R}^3)$, then

$$\sup_n \int_{\mathbb{R}^3} |u_n^{2_s^*-1} w| dx \le \sup_n \left( \int_{\mathbb{R}^3} |u_n|^{\frac{6}{3-2s}} dx \right)^{\frac{3+2s}{6}} \left( \int_{suppw} |w|^{2_s^*} dx \right)^{\frac{1}{2_s^*}} < +\infty.$$

From $u_n \rightharpoonup u$ in $E$, $f \in C^1$, we see that $f(u_n)w \to f(u)\omega$ a.e. in $\mathbb{R}^3$ as $n \to +\infty$. Hence, for bounded Borel set $B_r(0)$, one has

$$\int_{B_r(0)} |f(u_n)w - f(u)\omega| dx \to 0, \quad as\ n \to +\infty,$$

implying

$$\left| \int_{B_r(0)} [a(x)f(u_n)w - a(x)f(u)w] \, dx \right| \to 0,$$

that is,

$$\int_{B_r(0)} a(x)f(u_n)w \to \int_{B_r(0)} a(x)f(u)w. \tag{3.7}$$

Combining (3.6) and (3.7), we have as $n \to +\infty$,

$$\int_{\mathbb{R}^3} a(x)f(u_n)w \to \int_{\mathbb{R}^3} a(x)f(u)w.$$

Thus, $J'(u_n)w \to \langle J'(u), w \rangle$ for every $w \in C_0^\infty(\mathbb{R}^3)$, implying $J'(u) = 0$, which completes the proof. □

**Proof of Theorem 1.1**. By Lemma 3.2, we see that $(u, \phi_u)$ is a weak solution of the system (1.9). Next, we show that $(u, \phi_u)$ is a ground state solution. For this aim, we need to prove $J(u) = c$. Because $u \not\equiv 0$, $f(u)u \ge (m+1)F(u)$ and $m \in (3-2s, 2_s^* - 1)$, then

$$\begin{aligned} J(u) &= J(u) - \frac{1}{m+1} J'(u)u \\ &= \frac{m-1}{2(m+1)} \|u\|_E^2 + \frac{2(2_s^*-1) - 1 - m}{2(2_s^*-1)(m+1)} \int_{\mathbb{R}^3} K(x)\phi_u(x)|u|^{2_s^*} dx \\ &\quad + \frac{1}{m+1} \int_{\mathbb{R}_3} a(x) \left[ f(u)u - (m+1)F(u) \right] dx \\ &\ge \frac{m-1}{2(m+1)} \|u\|_E^2 \\ &> 0. \end{aligned}$$

Denote by $v_n = u_n - u$. Using Brezis-Lieb lemma [7] and Lemmas 2.3, 2.5, we have

$$c + o_n(1) > J(u_n) - J(u) = \frac{1}{2} \|v_n\|_E^2 - \frac{1}{2(2_s^*-1)} \int_{\mathbb{R}^3} K(x)\phi_{v_n}(x)|v_n|^{2_s^*-1} dx + o_n(1).$$

and

$$o_n(1) = J'(u_n)u_n - J'(u)u = \|v_n\|_E^2 - \int_{\mathbb{R}^3} K(x)\phi_{v_n}(x)|v_n|^{2_s^*-1} dx + o_n(1).$$

Hence,

$$c \ge \frac{1}{2} \lim_{n\to\infty} \|v_n\|_E^2 - \lim_{n\to\infty} \frac{1}{2(2_s^*-1)} \int_{\mathbb{R}^3} K(x)\phi_{v_n}(x)|v_n|^{2_s^*-1} dx. \tag{3.8}$$

and

$$0 = \lim_{n\to\infty} \|v_n\|_E^2 - \lim_{n\to\infty} \int_{\mathbb{R}^3} K(x)\phi_{v_n}(x)|v_n|^{2_s^*-1} dx.$$

We assume

$$\lim_{n\to\infty} \|v_n\|_E^2 = \lim_{n\to\infty} \int_{\mathbb{R}^3} K(x)\phi_{v_n}(x)|v_n|^{2_s^*-1} dx = l,$$

By (3.8), we get $c \geq \frac{2s}{3+2s} l$. But from (2.7) we have

$$\int_{\mathbb{R}^3} K(x)\phi_{v_n}|v_n|^{2_s^*-1}dx \leq \|K\|_\infty^2 S_s^{-2_s^*} \|v_n\|_E^{2(2_s^*-1)}.$$

The results generated such that

$$l \leq \|K\|_\infty^2 S^{-2_s^*} l^{2_s^*-1}.$$

Therefore, either $l = 0$ or $l \geq \|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}$. If $l \geq \|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}$, then $c \geq \frac{2s}{3+2s} \|K\|_\infty^{-\frac{3-2s}{2s}} S^{\frac{3}{2s}}$, which leads to the contradiction. Thus, $l = 0$, and we get $\lim_{n\to\infty} \|v_n\|_E^2 = 0$, which implies that $v_n \to 0$ in $E$. This means that $u_n \to u$ in $E$ as $n \to \infty$. So we can get that $J(u_n) \to J(u)$.

We only need to prove that the ground state solution is positive. Put $u^+ = \max\{u, 0\}$ the positive part of $u$. We note that all the calculations above can be repeated word by word, replacing $J$ with the functional

$$\begin{aligned} J^+(u) = &\frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2dx \\ &- \frac{1}{2(2_s^*-1)}\int_{\Re^3} \phi_{u^+}|u^+|^{2_s^*-1}dx - \int_{\mathbb{R}^3} a(x)F(u)dx. \end{aligned} \tag{3.9}$$

Using $u^- = \min\{u, 0\}$ as a test function in (3.9), in view of $(J^+)'(u)u^- = 0$, and $(a-b)(a^- - b^-) \geq |a^- - b^-|^2$, we conclude that

$$\begin{aligned} \|u^-\|_{D^{s,2}}^2 \leq &\iint_{\mathbb{R}^6} \frac{(u(x)-u(y))((u^-(x)-u^-(y))}{|x-y|^{3+2s}}dxdy + \int_{\mathbb{R}^3} V(x)|u^-|^2dx \\ = &0. \end{aligned}$$

Thus, $u^- = 0$ and $u \geq 0$ is a solution of (1.9). By Lemma 6.1 [28], and by Theorem 3.4 [12] we can see that $u \in L^\infty(\mathbb{R}^3) \cap C^{0,\alpha}(\mathbb{R}^3)$ for some $\alpha \in (0,1)$. Next we only need to prove that the solution $u$ is positive. Otherwise, if $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^3$, then $(-\Delta)^s u(x_0) = 0$ and by the definition of $(-\Delta)^s$, we have [27]:

$$(-\Delta)^s u(x_0) = -\frac{C_s}{2}\int_{\mathbb{R}^3} \frac{u(x_0+y)+u(x_0-y)-2u(x_0)}{|y|^{3+2s}}dy.$$

Hence, $\int_{\mathbb{R}^3} \frac{u(x_0+y)+u(x_0-y)}{|y|^{3+2s}}dy = 0$, which means that $u \equiv 0$, a contradiction. Thus, $u(x) > 0$ in $\mathbb{R}^3$. This completes the proof. $\square$

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(X. Dou) College of Science, Minzu University of China
Beijing 100081, China
*E-mail address*: xldou816@163.com

(X. He) College of Science, Minzu University of China
Beijing 100081, China
*E-mail address*: xmhe923@muc.edu.cn