

ARTICLE TYPE

Plankton growth dynamic driven by plankton body size in deterministic and stochastic environments

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Summary

In this paper, we establish a new phytoplankton-zooplankton model by considering the effects of plankton body size and stochastic environmental fluctuations. Mathematical theory work mainly gives the existence of boundary and positive equilibria, and shows their local as well as global stability in the deterministic model. Additionally, we explore the dynamics of V-geometric ergodicity, stochastic ultimate boundedness, stochastic permanence, persistence in the mean, stochastic extinction and the existence of a unique ergodic stationary distribution in the corresponding stochastic version. Numerical simulation work mainly reveals that plankton body size can generate great influences on the interactions between phytoplankton and zooplankton, which in turn proves the effectiveness of mathematical theory analysis. It is worth emphasizing that for the small value of phytoplankton cell size, the increase of zooplankton body size can not change the phytoplankton density or zooplankton density; for the middle value of phytoplankton cell size, the increase of zooplankton body size can decrease zooplankton density or phytoplankton density; for the large value of phytoplankton body size, the increase of zooplankton body size can increase zooplankton density but decrease phytoplankton density. Besides, it should be noted that the increase of zooplankton body size can not affect the effect of random environmental disturbance, while the increase of phytoplankton cell size can weaken its effect. These results may enrich the dynamics of phytoplankton-zooplankton models.

KEYWORDS:

Plankton body size, Stochastic environmental fluctuation, Stochastic extinction, Stationary distribution, Plankton growth dynamic

1 | INTRODUCTION

Cyanobacterial blooms caused by eutrophication of water bodies have always been a key problem in water environment control^{1,2}. These phenomena not only restrict the economic value of lakes, rivers, reservoirs and other water bodies, but also seriously threaten the safety of drinking water and human health³. In the past decades, many physical and biological processes related to the interplay of plankton in the aquatic environments have been made by different approaches. However, a clear mechanism of algal blooms is still ongoing. Hence, seeking for such a mechanism in relation to some important factors that may induce or terminate the obnoxious phenomena is of great significance.

In recent years, a large number of scholars have used mathematical models to study the interaction of plankton in the aquatic ecosystems, and have obtained a lot of dynamic mechanisms of phytoplankton growth^{4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22}. For example, Jia et al.¹⁸ indicated that the toxicity of phytoplankton and the death rate of zooplankton play a vital role in the spatial distribution of population density. Li et al.¹² suggested that refuge and toxin have a significant impact on the occurring and terminating of algal blooms in the freshwater lakes. Dai et al.⁸ demonstrated that time delay has a great influence on the nutrient-phytoplankton dynamics. Chen et al.⁴ signified that temperature and light play a key role in the dynamic mechanisms of phytoplankton growth. Zhao et al.⁶ showed that allelopathic effects can decrease the peaks of the cyclic outbreak of the harmful algal blooms. These remarkable research works strongly indicate that mathematical models can be a powerful tool to determine some key factors affecting the dynamic mechanisms of phytoplankton growth in the aquatic environments.

However, most of the mathematical model studies related to the dynamics of plankton growth^{4,5,6,14,15,16,17,13,12,18}, which work in a deterministic environment. These works have been ignored the impacts of random environmental fluctuation factors²³, such as unpredictable radiation, water temperature variations, humidity variability, infectious diseases, etc., which objectively exist in the real aquatic environments and have the potential to generate far-reaching influences on the dynamics of plankton growth or interaction. Actually, the environment in which plankton lives is always random and uncertain because the fact that most natural phenomena do not follow strictly deterministic laws, but rather oscillate randomly about some average behaviour. Hence, environmental randomness has been viewed as one of the inherent properties in population ecosystems and the population is subject to a continuous spectrum of disturbances²⁴.

In view of above viewpoints, the effects of stochastic environmental fluctuations, which can be described by the form of Gaussian white noise²⁵, have great impacts on the parameters involved in the model such as the growth rate, death rate, carrying capacity, intraspecific competition rate and so on²³. As a result, some scholars take into account the influences of stochastic environmental perturbations when studying phytoplankton-zooplankton dynamics^{26,27,28,29,30}. For example, Yu et al.²⁶ displayed that environmental fluctuations play a key role in the termination of algal blooms. Camara et al.²⁸ demonstrated that environmental stochastic noise can destroy the limit cycle attractor existing in the deterministic model. Yu et al.³¹ suggested that environmental fluctuations have a great influence on the survival of plankton. Hence, in the studies of phytoplankton-zooplankton dynamics, it is essential to consider the effects of stochastic environmental perturbations.

On the other hand, most of phytoplankton-zooplankton models or nutrient-phytoplankton models^{4,6,7,8,9,11,12,13,14,15,16,17,26,27,28,29,30,31}, it is usually assumed that the sinking rate of phytoplankton, growth rate of phytoplankton and the grazing rate of zooplankton, etc., are independent of phytoplankton cell size or zooplankton body size. In fact, the body size of aquatic organisms, especially the cell size of phytoplankton, has an important impact on ecosystem processes³². Physiologically, smaller cell size of phytoplankton is excellent competitor for scarce nutrient, while some larger cell size of phytoplankton benefit from higher maximal uptake rates and an increased ability to store excess nutrients^{33,34}. Ecologically, body size regulates zooplankton-phytoplankton trophic interactions, with biomass-specific rates of ingestion by zooplankton generally decrease as the phytoplankton cell size or zooplankton body size decreases³⁵. In particular, the cell size of phytoplankton not only defines their metabolic activity and growth rates, but also strongly influences their contributions to biogeochemical cycles via size-dependent sinking, and affects community structure and dynamics via size-dependent species interactions³⁶.

Actually, many remarkable experimental researches have shown that the effects of smaller cell size on the phytoplankton growth rate is similar to that of larger cell size on the growth rate of phytoplankton, and the peak growth rate of phytoplankton appears at the intermediate cell size^{33,37,38,39}. For example, by performing an in situ test of Raven's prediction that there is a reversal of the relationship between cell size and maximum achievable growth rate in unicellular algae at the low end of size classes,³⁸ found that a maximum in growth rate (4.8 and 3.3 divisions d^{-1} , respectively) in the 2-3 μm size class represented by coccoid *Chlorella*-like cells, with lower growth rates in both higher and lower size classes. Inspired by these facts, in this research direction, some theoretical works related to how cell size affects the phytoplankton-zooplankton dynamics have been reported recently^{40,41,42,43}. In the work of Pu et al.⁴¹, they indicated that the phytoplankton-zooplankton evolutionary dynamics are closely related to body size-dependent phytoplankton and zooplankton. Zhao et al.⁴⁰ found that the cell size of phytoplankton has a great influence on the growth and reproduction of phytoplankton. Consequently, the effects of cell size play a vital role in the phytoplankton-zooplankton dynamics.

In spite of the importance of the stochastic environmental fluctuations and the cell size of phytoplankton, there is still a lack of studies on how plankton body size affects the phytoplankton-zooplankton dynamics under the fluctuating environments. In other words, a complete understanding of the mechanisms underlying the phytoplankton growth driven by plankton body size is still largely unknown, especially in a randomly disturbed environment. Therefore, in this paper, we propose and study

a phytoplankton-zooplankton model incorporating the impacts of plankton body size and stochastic fluctuation environment, where the environmental variation is characterized by the form of white noise²⁸.

The main purpose of this paper is to study how plankton body size affects the phytoplankton-zooplankton dynamics in the deterministic and stochastic fluctuation environments. The rest of this article is organized as follows: In Section 2, we present the mathematical model and preliminaries. Section 3 is devoted to studying the dynamics of deterministic model (4). In Section 4, we are committed to investigating the dynamics of stochastic model (6). Section 5, numerical simulations are carried out to verify our analytical results. In Section 6, we give a brief discussion and the summary of the main results.

2 | MATHEMATICAL MODEL AND PRELIMINARIES

In this section, we give the mathematical model and present some preliminaries which will be used in the later paper.

2.1 | The deterministic model

Motivated by the previous works^{40,44,45}, in this subsection, we firstly present the following assumptions to our model:

(H_1) Assume that $P(t)$ is the density of phytoplankton at time t and x is its cell size and $Z(t)$ is the density of zooplankton at time t and y is its body size.

(H_2) Suppose that the phytoplankton grows exponentially in the absence of zooplankton, but is limited by its intraspecific competition^{40,41}. Hence, the term αP^2 indicates that the density dependent death of phytoplankton, which eventually leads to the reduction of phytoplankton density, where α means the intraspecific competition rate.

(H_3) Suppose that the maximum growth rate of phytoplankton depends on the cell size of phytoplankton⁴¹, namely

$$r(x) = \frac{x}{a_1 x^2 + a_2 x + a_3}, \quad (1)$$

where a_i is positive constant, $i = 1, 2, 3$. There are empirical observations^{33,37,38} indicate that the phytoplankton growth rate should be cell size dependent.

(H_4) Assume that the sinking rate of phytoplankton is proportional to the square of phytoplankton cell size x due to the cell size of phytoplankton not only defines their growth rates, but also strongly influences their contributions to bio-geochemical cycles via size-dependent sinking⁴¹, i.e.,

$$s(x) = ax^2, \quad (2)$$

where a is a positive constant which denotes the sinking rate and its value is affected by the density of the water as well as the viscosity of the water⁴¹.

(H_5) Suppose that zooplankton feeds on the phytoplankton according to Holling type I functional response and zooplankton consumption rate depends on the cell size of phytoplankton x and the body size of zooplankton y ⁴¹, which translates into energy for offspring and therefore increases the number of zooplankton populations. Namely, the term $C(x, y)PZ$, where

$$C(x, y) = \bar{C} \exp\left[-\frac{1}{\tau}(x - \kappa y)^2\right] \quad (3)$$

represents the consumption rate of zooplankton on phytoplankton, τ is consumption rate coefficient and \bar{C} is the maximum consumption rate when zooplankton encounter phytoplankton at the optimal predator-prey ratio κ ³⁹. Previous excellent work^{41,42} indicated that body size of plankton play an important role in affecting the dynamics of the interactions between phytoplankton and zooplankton species.

(H_6) Assume that the contribution of phytoplankton to zooplankton growth is proportional to the contribution of toxic substances to zooplankton death. Hence, we use the term θPZ , in proportion to $C(x, y)PZ$, to describe the distribution of toxic substances released by phytoplankton which ultimately contribute to reduce the grazing pressure of zooplankton by increasing its mortality, where θ indicates the toxin liberation rate.

(H_7) Assume that the crowding effect caused by intraspecific competition with peers of zooplankton species βZ^2 and natural death dZ result in a decrease in zooplankton species, where β represents the intraspecific competition rate and d denotes the death rate for zooplankton.

In the light of the above assumptions, a deterministic toxin producing phytoplankton-zooplankton model described by a system of ordinary differential equations can be written as follows:

$$\begin{cases} \frac{dP(t)}{dt} = r(x)P - Ps(x) - \alpha P^2 - C(x, y)PZ, \\ \frac{dZ(t)}{dt} = C(x, y)PZ - dZ - \beta Z^2 - \theta PZ, \end{cases} \quad (4)$$

subject to the initial condition $(P(0), Z(0)) \in \mathbb{R}_+^2$. All parameters in model (4) is assumed to be positive. Furthermore, for the sake of convenience, in this study, we use C_m to denote $C(x, y)$. In addition, in order to guarantee that the phytoplankton and zooplankton equations in model (4) are greater than zero and biologically reasonable, we assume that $r(x) > s(x)$ ⁴¹ and $C_m > \theta$ are established by default in the whole of this paper.

2.2 | The stochastic model

Now, we introduce environmental noise fluctuations into the deterministic model (4). As we know, the phytoplankton growth rate $r(x)$ in the phytoplankton equation and the zooplankton death rate d in the zooplankton equation are the two key parameters in affecting the phytoplankton-zooplankton dynamics. In the real situation, the growth rate $r(x)$ and the death rate d always fluctuate around an average value due to the continuous fluctuations in the environment²³. Hence, we assume that the stochastic environmental perturbation is a type of Gaussian white noise that mainly affects the growth rate of phytoplankton and the death rate of zooplankton. Actually, in the existing literature, there are many different methods to incorporate the effects of noise perturbations into ecological population systems^{46,47,48,49,50}. In this paper, by following the method introduced in^{49,50}, we model the effects of randomly varying environment in $r(x)$ and d by adopting

$$r(x) \rightarrow r(x) + \delta_1 \dot{B}_1(t), \quad -d \rightarrow -d + \delta_2 \dot{B}_2(t), \quad (5)$$

where $B_i(t)$ are independent Brownian motions with $B_i(0) = 0$, $\dot{B}_i(t)$ denotes the white noise and δ_i^2 denote the intensity of the white noise, $i = 1, 2$, then model (4) can be expressed as follow:

$$\begin{cases} dP(t) = (r(x)P - Ps(x) - \alpha P^2 - C(x, y)PZ)dt + \delta_1 P dB_1(t), \\ dZ(t) = (C(x, y)PZ - dZ - \beta Z^2 - \theta PZ)dt + \delta_2 Z dB_2(t). \end{cases} \quad (6)$$

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition (i.e., it is increasing and right continuous while \mathcal{F}_0 contains all \mathbb{P} -null sets). Moreover, let $B_i(t)$ ($i = 1, 2$) be standard Brownian motions defined on this probability space and

$$\mathbb{R}_+^n = \{(w_1, \dots, w_n) \in \mathbb{R}^n : w_i > 0, i = 1, \dots, n\}.$$

2.3 | Preliminaries

In general, consider the d -dimensional stochastic differential equation

$$dw(t) = f(w(t), t)dt + g(w(t), t)dB(t), \quad \forall t \geq t_0, \quad (7)$$

where initial value $w(t_0) = w_0 \in \mathbb{R}^d$. Denoted by $C^{2,1}(\mathbb{R}^d \times [t_0, +\infty); \mathbb{R}_+)$ the family of nonnegative function $V(w, t)$ defined on $\mathbb{R}^d \times [t_0, +\infty)$ such that they are continuously twice differentiable in w and once in t . The differential operator L of (7) is defined by⁵¹

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^d f_i(w, t) \frac{\partial}{\partial w_i} + \frac{1}{2} \sum_{i,j=1}^d [g(w, t)^T g(w, t)]_{i,j} \frac{\partial^2}{\partial w_i \partial w_j}. \quad (8)$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, +\infty); \mathbb{R}_+)$, then

$$LV(w, t) = V_t(w, t) + V_w(w, t)f(w, t) + \frac{1}{2} \text{trace}[g(w, t)^T V_{ww}(w, t)g(w, t)], \quad (9)$$

where $V_w = \frac{\partial V}{\partial w}$, $V_w = (\frac{\partial V}{\partial w_1}, \dots, \frac{\partial V}{\partial w_d})$, $V_{ww} = (\frac{\partial^2 V}{\partial w_i \partial w_j})_{d \times d}$. Based on the Itô's formula, if $w(t) \in \mathbb{R}^d$, then

$$dV(w(t), t) = LV(w(t), t)dt + V_w(w(t), t)g(w(t), t)dB(t).$$

To further our study, we need the following definitions and lemmas.

Definition 2.1.⁵² (1) The population $w(t)$ is said to go to extinction if $\lim_{t \rightarrow \infty} w(t) = 0$.

(2) The population $w(t)$ is said to be weakly persistent in mean if $\lim_{t \rightarrow \infty} \sup \int_0^t w(s) ds > 0$.

(3) The population $w(t)$ is said to be strongly persistent in mean if $\lim_{t \rightarrow \infty} \inf \int_0^t w(s) ds > 0$.

Definition 2.2.⁵³ If for any $0 < \varepsilon < 1$, there is a constant ρ such that the solution $w(t) = (P(t), Z(t))$ of model (6) satisfies $\lim_{t \rightarrow \infty} \sup \mathbb{P}\{|w(t)| > \rho\} < \varepsilon$, for any initial value $P(0) > 0$ and $Z(0) > 0$, then the solution $w(t)$ is said to be stochastically ultimately bounded.

Definition 2.3.⁵³ If for any $\varepsilon \in (0, 1)$, there exists a pair of positive constants $\rho = \rho(\varepsilon)$ and $\chi = \chi(\varepsilon)$ such that for any initial value $w(0) = (P(0), Z(0))$, the solution $w(t)$ of model (6) is said to be stochastically permanent and satisfies the following properties

$$\lim_{t \rightarrow \infty} \inf \mathbb{P}\{|w(t)| \leq \rho\} \geq 1 - \varepsilon,$$

$$\lim_{t \rightarrow \infty} \inf \mathbb{P}\{|w(t)| \geq \chi\} \geq 1 - \varepsilon.$$

Lemma 2.1.⁵² Let $w(t) \in C[\Omega \times [0, \infty), (0, \infty)]$.

(1) If exists positive constant λ, η such that

$$\log w(t) \leq \lambda t - \eta \int_0^t w(s) ds + F(t), \text{ a.s.}$$

for all $t \geq 0$, where $F \in C[\Omega \times [0, \infty), \mathbb{R}]$ and $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$, then

$$\limsup_{t \rightarrow \infty} \int_0^t w(s) ds \leq \frac{\lambda}{\eta}, \text{ a.s.}$$

(2) If there exists positive constant λ, η such that

$$\log w(t) \geq \lambda t - \eta \int_0^t w(s) ds + F(t), \text{ a.s.}$$

for all $t \geq 0$, where $F \in C[\Omega \times [0, \infty), \mathbb{R}]$ and $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$, then

$$\liminf_{t \rightarrow \infty} \int_0^t w(s) ds \geq \frac{\lambda}{\eta}, \text{ a.s.}$$

We now turn our attention to the results of geometric ergodic.

Lemma 2.2.^{54,55} Let $w(t) = (P(t), Z(t))$ be the solution of model (2.6) with initial value $w_0 = (P(0), Z(0)) > 0$, and assume that the following Assumptions hold:

(S_1) (Minorization condition) For a compact set $U_1 \subset \mathbb{R}_+^2$, there exists $T, \eta > 0$ and a probability measure ν on \mathbb{R}_+^2 with $\nu(U_1) > 0$ such that

$$P_T(w_0, A) \geq \eta \nu(A), \forall w_0 \in U_1, \forall A \in \mathbb{B}(\mathbb{R}_+^2).$$

(S_2) (Lyapounov condition) There is a function $V : \mathbb{R}_+^2 \rightarrow [1, \infty)$ with $\lim_{|w(t)| \rightarrow \infty} V(w) = \infty$ and real numbers $\beta_1, \beta_2 \in (0, \infty)$ such that

$$LV(w) \leq \beta_1 - \beta_2 V(w).$$

Then the Markov process $\bar{X}(t)$ is V -geometrically ergodic: there exists a unique stationary distribution π such that for some constants $C, \lambda > 0$,

$$|\mathbb{E}g(w(t) - \pi(g))| \leq CV(w_0)e^{-\lambda t}, \forall w(0) = w_0 \in \mathbb{R}_+^2,$$

for all measurable function $g \in \mathcal{C} := \{\text{measurable } g : \mathbb{R}_+^2 \rightarrow \mathbb{R}^2 \text{ with } |g(w)| \leq V(w)\}$.

About the details of the proof of Lemma 2.2, one can see Theorem 16.0.1 in⁵⁴ or Theorem 2.5 in⁵⁵. In what follows, we present a lemma related to the existence of an ergodic stationary distribution to model (6).

Let $w(t)$ be a homogeneous Markov process in E_d (E_d denotes the d -dimensional Euclidean space) and be described by the following stochastic differential equation:

$$dw(t) = b(w)dt + \sum_{r=1}^k g_r(w)dB_r(t).$$

The diffusion matrix is defined as follows

$$A(w) = (a_{ij}(w)), \quad a_{ij}(w) = \sum_{r=1}^k g_r^i(w)g_r^j(w).$$

Lemma 2.3.⁵⁶ The Markov process has a unique ergodic stationary distribution $\mu(\cdot)$ if there exists a bounded domain $D \subset E_d$ with regular boundary Γ and

(i) there is a positive number M such that

$$\sum_{i,j=1}^d a_{i,j}(w)\xi_i\xi_j \geq M|\xi|^2, \quad w \in D, \quad \xi \in \mathbb{R}_d.$$

(ii) there exists a nonnegative function C^2 -function V such that LV is negative for any $E_d \setminus D$. Then

$$\mathbb{P}_w \left\{ \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(w(t))dt = \int_{E_d} f(w)\mu(dw) = 1 \right\}$$

for all $w \in E_d$, where $f(\cdot)$ is a function integrable with respect to measure μ .

In order to facilitate the calculations below, we define

$$X(x) = r(x) - s(x) = \frac{x - aa_1x^4 - aa_2x^3 - aa_3x^2}{a_1x^2 + a_2x + a_3} > 0,$$

and we write $X(x)$ as X for simplicity and convenience.

3 | MATHEMATICAL RESULTS

In this section, we mainly study the phytoplankton-zooplankton dynamics in the deterministic and stochastic environments.

3.1 | Mathematical analysis of the deterministic model (4)

Firstly, we present the results related to the deterministic model (4).

3.1.1 | Boundedness of positive solutions

A direct computation shows that the model (4) is continuous and Lipschitzian in \mathbb{R}_+^2 . Based on the existence and uniqueness of the solution for ordinary differential equations, there exists a unique solution of the model (4) for any positive initial condition. Further, it is easy to know that the solutions of the model (4) always exist and remain positive. In fact, using the positiveness of the solution and based on the phytoplankton equation in the model (4), we can get

$$\frac{dP(t)}{dt} \leq XP - \alpha P^2.$$

A standard comparison argument shows that

$$\limsup_{t \rightarrow \infty} P(t) \leq \frac{X}{\alpha} := m_1 > 0.$$

As a result, for any $\varepsilon > 0$, there exists a $T > 0$, such that $P(t) \leq m_1 + \varepsilon$ for $t > T$. Then from the zooplankton equation in the model (4), we have

$$\frac{dZ(t)}{dt} \leq C_m(m_1 + \varepsilon)Z - \beta Z^2.$$

Since ε is arbitrary, we get

$$\limsup_{t \rightarrow \infty} Z(t) \leq \frac{C_m m_1}{\beta} := m_2 > 0.$$

Thus, we can derive the following conclusion.

Theorem 3.1. All the solutions of the model (4) that start in \mathbb{R}_+^2 are uniformly bounded.

Proof. Define the function

$$\varrho(t) = P(t) + Z(t). \quad (10)$$

Taking the time derivative of (10) along the solution of the model (4) yields

$$\frac{d\varrho(t)}{dt} = \frac{dP(t)}{dt} + \frac{dZ(t)}{dt} = (r(x) - s(x))P - dZ - (\alpha P^2 + \beta Z^2) - \theta PZ.$$

For any $\nu > 0$, the following inequality holds

$$\frac{d\varrho(t)}{dt} + \nu\varrho(t) \leq ((\nu + r(x) - s(x))P - \alpha P^2) + ((\nu - d)Z - \beta Z^2).$$

Let $\nu = s(x) > 0$, then

$$\begin{aligned} \frac{d\varrho(t)}{dt} + s(x)\varrho(t) &\leq (r(x)P - \alpha P^2) + ((s(x) - d)Z - \beta Z^2) \\ &\leq \frac{x^2}{4\alpha(a_1 x^2 + a_2 x + a_3)^2} + \frac{(ax^2 - d)^2}{4\beta} := m_3. \end{aligned}$$

By using the theorem differential inequality⁵⁷, for all $t \geq T \geq 0$, we have

$$0 \leq \varrho(t) \leq \frac{m_3}{ax^2} - \left(\frac{m_3}{ax^2} - \varrho(T)\right)e^{-(t-T)}.$$

Hence,

$$\limsup_{t \rightarrow +\infty} (P(t) + Z(t)) \leq \frac{m_3}{ax^2}.$$

This completes the proof.

Let Δ be the set defined by

$$\Delta = \left\{ (P, Z) \in \mathbb{R}_+^2 : 0 \leq P \leq m_1, 0 \leq Z \leq m_2, 0 \leq P + Z \leq \frac{m_3}{ax^2} \right\}.$$

Then Δ is a positive invariant.

Remark 3.1. It is worth noting that for any initial value $(P(0), Z(0)) \in \mathbb{R}_+^2$, the solution $(P(t), Z(t)) \in \mathbb{R}_+^2$ eventually enters a bounded set Δ .

3.1.2 | Stability of the equilibria

In this subsection, we focus on the stability of the possible equilibria for the model (4). It is easy to know that the equilibria of the model (4) satisfies the following phytoplankton-zooplankton equations

$$\begin{aligned} r(x)P - Ps(x) - \alpha P^2 - C_m PZ &= 0, \\ C_m PZ - dZ - \beta Z^2 - \theta PZ &= 0. \end{aligned}$$

By direct calculations, we can obtain the following non-negative equilibria:

(1) $E_0 = (0, 0)$ (total extinct);

(2) $E_1 = \left(\frac{X}{\alpha}, 0\right)$ (extinct of zooplankton);

(3) $E_* = (P^*, Z^*) = \left(\frac{dC_m + \beta X}{C_m^2 + \beta\alpha - \theta C_m}, \frac{(C_m - \theta)X - ad}{C_m^2 + \beta\alpha - \theta C_m}\right)$ (coexistence of phytoplankton and zooplankton), provided $X > \frac{ad}{C_m - \theta}$.

Now, we are in the position to investigate the stability of the two boundary equilibria E_0 and E_1 , and the unique positive equilibrium E_* . Through direct calculations, the Jacobian matrix of the model (4) at equilibrium $E(P, Z)$ is

$$J_E = \begin{bmatrix} X - 2\alpha P - C_m Z & -PC_m \\ C_m Z - \theta Z & PC_m - d - 2\beta Z - P\theta \end{bmatrix}.$$

Obviously, the Jacobian matrix of the model (4) at equilibrium E_0 is

$$J_{E_0} = \begin{bmatrix} X & 0 \\ 0 & -d \end{bmatrix},$$

and the two eigenvalues of J_{E_0} are $\lambda_1 = X > 0$ and $\lambda_2 = -d < 0$. Therefore, E_0 is a saddle point.

The Jacobian matrix of the model (4) at E_1 is

$$J_{E_1} = \begin{bmatrix} -X & -\frac{C_m}{\alpha} X \\ 0 & \frac{X(C_m - \theta)}{\alpha} - d \end{bmatrix},$$

and we can get that the two eigenvalues of J_{E_1} are $\lambda_1 = -X < 0$ and $\lambda_2 = \frac{1}{\alpha}(X(C_m - \theta) - d\alpha)$. Hence, the stability of E_1 depends on the sign of λ_2 . In other words, if $X > \frac{\alpha d}{C_m - \theta}$, E_1 is saddle and is a stable node point if $X < \frac{\alpha d}{C_m - \theta}$.

The Jacobian matrix of the model (4) at E_* is

$$J_{E_*} = \begin{bmatrix} -\alpha P^* & -C_m P^* \\ (C_m - \theta)Z^* & -\beta Z^* \end{bmatrix}$$

and the characteristic equation of E_* is

$$\lambda^2 - \text{tr}(J_{E_*})\lambda + \det(J_{E_*}) = 0,$$

where $\text{tr}(J_{E_*}) = -\alpha P^* - \beta Z^* < 0$ and $\det(J_{E_*}) = \alpha\beta P^* Z^* + C_m P^* Z^*(C_m - \theta) > 0$. Based on the Routh-Hurwitz criterion, it can be known that the E_* of the model (4) is locally asymptotically stable if it exists, that is, only the condition $X > \frac{\alpha d}{C_m - \theta}$ is need.

In view of the analysis above, we have the following results.

Theorem 3.2. For the model (4), there have the following results.

- (a) E_0 is a unstable saddle.
- (b) If $X > \frac{\alpha d}{C_m - \theta}$, E_1 is a saddle point and is a stable node if $X < \frac{\alpha d}{C_m - \theta}$.
- (c) If $X > \frac{\alpha d}{C_m - \theta}$, E_* is locally asymptotically stable.

Furthermore, we consider the global stability of the zoo-plankton-free equilibrium E_1 in the model (4).

Theorem 3.3. If $X < \frac{d\alpha}{C_m - \theta}$, the boundary equilibrium point E_1 of the model (4) is globally asymptotically stable.

Proof. Consider such a function: $\mathbb{R}_+^2 \rightarrow \mathbb{R}$,

$$V_1(P, Z) = \left(P - \frac{(C_m - \theta)X}{C_m\alpha} - \frac{(C_m - \theta)X}{C_m\alpha} \ln \frac{C_m\alpha P}{(C_m - \theta)X} \right) + Z.$$

The time derivative of V_1 along positive solution is

$$\begin{aligned} \frac{dV_1}{dt} &= \left(P - \frac{(C_m - \theta)X}{C_m\alpha} \right) \left(X - \frac{C_m\alpha}{C_m - \theta} P - C_m Z \right) + C_m - dZ - \beta Z^2 - \theta PZ \\ &\leq -\frac{C_m\alpha}{C_m - \theta} \left(P - \frac{(C_m - \theta)X}{C_m\alpha} \right)^2 - \beta Z^2 - \left(d - \frac{(C_m - \theta)X}{\alpha} \right) Z. \end{aligned}$$

Clearly, $\frac{dV_1}{dt} \leq 0$ if $\frac{d\alpha}{C_m - \theta} > X$, and $\frac{dV_1}{dt} = 0$ if and only if $(P, Z) = \left(\frac{X}{\alpha}, 0 \right)$. Hence Lyapunov-Lasalle's invariance principle implies the global asymptotic stability of E_1 . This completes the proof.

Remark 3.2. From Theorems 3.2 and 3.3, one can find that if the positive equilibrium E_* does not exist, the boundary equilibrium E_1 is also globally asymptotically stable. Unfortunately, model (4) has no Hopf bifurcation at the positive equilibrium point, because $\text{tr}(J_{E_*}) < 0$ and $\det(J_{E_*}) > 0$ are always hold.

In addition, we obtain the result regarding the global stability of the model (4) at the positive equilibrium E_* .

Theorem 3.4. If $\frac{d\alpha}{C_m - \theta} < X < \alpha$, the unique positive equilibrium E_* of the model (4) is globally asymptotically stable.

Proof. We present the non-existence of the limit cycles (periodic solutions) of the model (4) to examine the global stability of the positive equilibrium E_* by applying the Dulac-Bendixon criterion⁵⁸. For this purpose, consider

$$G^*(P, Z) = \frac{1 + P}{PZ},$$

and

$$\begin{aligned} F(P, Z) &= (F_1, F_2) \\ &= (r(x)P - Ps(x) - \alpha P^2 - C(x, y)PZ, C(x, y)PZ - dZ - \beta Z^2 - \theta PZ). \end{aligned}$$

Clearly, $G^* \in C^1(\mathbb{R}_+^{20})$, where \mathbb{R}_+^{20} is the interior of \mathbb{R}_+^2 . Hence, $G^*(P, Z) > 0$ in \mathbb{R}_+^{20} and we have

$$\begin{aligned} \vec{\nabla} \cdot (G^* F) &= \frac{\partial}{\partial P}(G^* F_1) + \frac{\partial}{\partial Z}(G^* F_2) \\ &= \frac{1}{Z} \frac{\partial}{\partial P} [(1 + P)(X - \alpha P - C_m)] + \frac{1}{P} \frac{\partial}{\partial Z} [(1 + P)(C_m - d - \beta Z - \theta)] \\ &= \frac{1}{Z}(X - \alpha) - \frac{2\alpha P}{Z} - \frac{C_m}{Z} - \frac{\beta}{P} - \beta < 0, \end{aligned}$$

provided $X < \alpha$. It is obvious that $\vec{\nabla} \cdot (G^* F)$ is neither equal to zero nor does it change its symbol in the interior of \mathbb{R}_+^2 . So, by Dulac-Bendixon criterion, it can be known that the model (4) has no limit cycle lying entirely in the interior of \mathbb{R}_+^2 . Because there is the only positive equilibrium E^* in the interior of \mathbb{R}_+^2 , and each of the positive solution tends to E^* . Additionally, considering the local asymptotical stability of E^* again, we can conclude that the unique positive equilibrium E_* is globally asymptotically stable if $\frac{d\alpha}{C_m - \theta} < X < \alpha$ holds. This completes the proof.

Remark 3.3. It can be seen from Theorems 3.3 and 3.4 that the existence and stability of the equilibria in the model (4) depends strongly on the parameter X .

3.2 | Mathematical analysis of the stochastic model (6)

In this section, we investigate the stochastic dynamics of model (6) including the existence and uniqueness, geometric ergodicity, boundedness and permanence, stochastic extinction, persistence in the mean and the existence of ergodic stationary distribution.

3.2.1 | Some properties of the solutions for the model (6)

We first focus on the existence and uniqueness of the positive solutions in the model (6).

Theorem 4.1. For any given initial value $(P_0, Z_0) \in \mathbb{R}_+^2$, there is a unique solution $(P(t), Z(t))$ in the model (6) on $t \geq 0$ and will remain in \mathbb{R}_+^2 with probability one.

Proof. Let $u = \ln P$ and $v = \ln Z$, then the model (6) becomes the following forms:

$$\begin{cases} du = (r(x) - s(x) - 0.5\delta_1^2 - \alpha e^u - C_m e^v)dt + \delta_1 dB_1(t), \\ dv = (C_m e^u - d - 0.5\delta_2^2 - \beta e^v - \theta e^u)dt + \delta_2 dB_2(t), \end{cases} \quad (11)$$

with initial value $(u_0, v_0) = (\ln P_0, \ln Z_0)$. Since the coefficients of model (11) are locally Lipschitz continuous, then there is a unique local solution $(u(t), v(t))$ of model (11) on the interval $[0, \tau_e)$, where τ_e denotes the explosion time. Hence, it can be got that $(P(t), Z(t)) = (e^{u(t)}, e^{v(t)})$ is a unique local positive solution for model (11) with any positive initial value (P_0, Z_0) on the interval $[0, \tau_e)$. To show the positive solution is global, we only need to verify that $\tau_e = \infty$ a.s.

On the one hand, in the light of the positivity of the solution $(P(t), Z(t))$ and based on the phytoplankton equation in the model (6), it is easy to get $dP(t) \leq [(r(x) - s(x))P(t) - \alpha P^2(t)]dt + \delta_1 P(t)dB_1(t)$. Therefore, we introduce the following auxiliary

equation:

$$\begin{cases} d\Phi_1(t) = [(r(x) - s(x))\Phi_1(t) - \alpha\Phi_1^2(t)]dt + \delta_1\Phi_1(t)dB_1(t), \\ \Phi_1(0) = P_0, \end{cases} \quad (12)$$

and derive that

$$\Phi_1(t) = \frac{e^{(r(x)-s(x)-\frac{1}{2}\delta_1^2)t} + \delta_1 B_1(t)}{\frac{1}{P_0} + \alpha \int_0^t e^{(r(x)-s(x)-\frac{1}{2}\delta_1^2)s + \delta_1 B_1(s)} ds},$$

which is the unique solution of (12). Based on the comparison theorem for stochastic differential equation⁵⁹, we can get

$$P(t) \leq \Phi_1(t), \quad t \in [0, \tau_e) \text{ a.s.} \quad (13)$$

Making use of (13) to the zooplankton equation in the model (6), one can get that $dZ(t) \leq [(C_m\Phi_1 - d)Z(t) - \beta Z^2(t)]dt + \delta_2 Z(t)dB_2(t)$. Considering the following auxiliary equation:

$$\begin{cases} d\Phi_2(t) = [(C_m\Phi_1(t) - d)\Phi_2(t) - \beta\Phi_2^2(t)]dt + \delta_2\Phi_2(t)dB_2(t), \\ \Phi_2(0) = Z_0, \end{cases} \quad (14)$$

then

$$\Phi_2(t) = \frac{e^{(C_m\Phi_1 - d - \frac{1}{2}\delta_2^2)t + \delta_2 B_2(t)}}{\frac{1}{Z_0} + \beta \int_0^t e^{(C_m\Phi_1 - d - \frac{1}{2}\delta_2^2)s + \delta_2 B_2(s)} ds}$$

is the unique solution of (14). According to comparison theorem for stochastic differential equation⁵⁹, we can obtain

$$Z(t) \leq \Phi_2(t), \quad t \in [0, \tau_e) \text{ a.s.} \quad (15)$$

On the other hand, from the phytoplankton equation in the model (6) and by means of (15), we have $dP(t) \geq [(r(x) - s(x) - C_m\Phi_2)P(t) - \alpha P^2(t)]dt + \delta_1 P(t)dB_1(t)$. Taking into account the auxiliary equation:

$$\begin{cases} d\Phi_3(t) = [(r(x) - s(x) - C_m\Phi_2(t))\Phi_3(t) - \alpha\Phi_3^2(t)]dt + \delta_1\Phi_3(t)dB_1(t), \\ \Phi_3(0) = P_0, \end{cases} \quad (16)$$

then

$$\Phi_3(t) = \frac{e^{(r(x)-s(x)-C_m\Phi_2-\frac{1}{2}\delta_1^2)t + \delta_1 B_1(t)}}{\frac{1}{P_0} + \alpha \int_0^t e^{(r(x)-s(x)-C_m\Phi_2-\frac{1}{2}\delta_1^2)s + \delta_1 B_1(s)} ds}$$

is the unique solution of (16). From the comparison theorem for stochastic differential equation⁵⁹, we can get

$$P(t) \geq \Phi_3(t), \quad t \in [0, \tau_e) \text{ a.s.} \quad (17)$$

Making use of (13) and (17) into the zooplankton equation in the model (6), one can derive that $dZ(t) \geq [(C_m\Phi_3 - d - \theta\Phi_1)Z(t) - \beta Z^2(t)]dt + \delta Z(t)dB_4(t)$. Similarly, we have that

$$\Phi_4(t) = \frac{e^{(C_m\Phi_3 - \theta\Phi_1 - d - \frac{1}{2}\delta_2^2)t + \delta_2 B_2(t)}}{\frac{1}{Z_0} + \beta \int_0^t e^{(C_m\Phi_3 - \theta\Phi_1 - d - \frac{1}{2}\delta_2^2)s + \delta_2 B_2(s)} ds}$$

is the unique solution of the following equation:

$$\begin{cases} d\Phi_4(t) = [(C_m\Phi_3(t) - d - \theta\Phi_1(t))\Phi_4(t) - \beta\Phi_4^2(t)]dt + \delta_2\Phi_4(t)dB_2(t), \\ \Phi_4(0) = Z_0, \end{cases}$$

and

$$Z(t) \geq \Phi_4(t), \quad t \in [0, \tau_e) \text{ a.s.} \quad (18)$$

From (13), (15), (17) and (18), we can get that

$$\begin{cases} \Phi_3(t) \leq P(t) \leq \Phi_1(t), \quad t \geq 0 \text{ a.s.} \\ \Phi_4(t) \leq Z(t) \leq \Phi_2(t), \quad t \geq 0 \text{ a.s.} \end{cases} \quad (19)$$

Based on the previous work in the study⁶⁰, it can be known that $\Phi_i(t)(i = 1, 2, 3, 4)$ will not be explored at any finite time, and by virtue of comparison theorems for stochastic differential equations⁵⁹, we can derive that $(P(t), Z(t))$ will globally exist. This completes the proof.

Remark 4.1. Theorem 4.1 indicates that all the solutions of the model (6) will remain in \mathbb{R}_+^2 , which is more different from that in the deterministic model (4). From Remark 3.1, one can know that every trajectory is eventually staying in a compact set $\Delta \subset \mathbb{R}_+^2$.

Next, we present the result regarding the V -geometric ergodicity of the Markov process $w(t) = (P(t), Z(t))$ for the model (6).

Theorem 4.2. Markov process of $w(t) = (P(t), Z(t))$ of the model (6) with initial value $w_0 = (P(0), Z(0)) \in \mathbb{R}_+^2$ is V -geometrically ergodic.

Proof. Let $Q = P + Z$, and define

$$V(w(t)) = Q + \frac{1}{Q}$$

for $w(t) = (P(t), Z(t)) \in \mathbb{R}_+^2$. It follows that $V(w(t)) \rightarrow \infty$ as $|w(t)| \rightarrow \infty$. Making use of Itô's formula and adopting $h = \min\{s(x), d\}$, we have

$$\begin{aligned} LV &= r(x)P - Ps(x) - \alpha P^2 - dZ - \beta Z^2 - \theta PZ \\ &+ \frac{Ps(x) + \alpha P^2 + dZ + \beta Z^2 + \theta PZ - r(x)P}{Q^2} + \frac{\delta_1 P^2 + \delta_2 Z^2}{Q^3} \\ &\leq -hQ - \frac{r(x) - (d + s(x) + \delta_1^2 + \delta_2^2)}{Q} - \beta Z^2 - \theta PZ + \alpha + \beta + 0.5\theta + r(x) \\ &\leq -hQ - \frac{r(x) - (d + s(x) + \delta_1^2 + \delta_2^2)}{Q} + \alpha + \beta + 0.5\theta + r(x) \\ &\leq -\bar{h}(Q + \frac{1}{Q}) + l = l - \bar{h}V, \end{aligned} \tag{20}$$

where $l = \alpha + \beta + 0.5\theta + r(x)$ and $\bar{h} = \min\{h, r(x) - (d + s(x) + \delta_1^2 + \delta_2^2)\}$. Thus (S_2) (Lyapunov condition) in Lemma 2.2 holds.

Since the model (6) is uniformly elliptic, Proposition 11.1 in⁶¹ guarantees the existence of a function $P : \mathbb{R}_+ \times \mathbb{R}_+^2 \rightarrow (0, \infty)$ such that P is jointly continuous, $P_t(w_0, Y)$ is strictly positive for all (t, w_0, Y) , and such that for all measure sets A

$$P_t(w_0, A) = \int_A P_t(w_0, Y) dY.$$

It follows that for any $\epsilon > 0$, there exists a positive constant $a = a(\epsilon, t) > 0$ so that $\inf\{(w_0, Y) : w_0, Y \in \mathbb{R}_+^2, |w_0|, |Y| \leq \epsilon\} \geq a$. Assumption (S_2) (Minorization condition) follows immediately this, since for any measurable set A

$$\begin{aligned} P_t(w, A) &= \int_A P_t(w_0, Y) dY \geq a \text{Leb}(A \cap \mathcal{B}_\omega(0)) \\ &= a \text{Leb}(\mathcal{B}_\omega(0)) v(A), \end{aligned}$$

where Leb is Lebesgue measure and

$$v(A) = \text{Leb}(A \cap \mathcal{B}_\omega(0)) / \text{Leb}(\mathcal{B}_\omega(0)).$$

Thus (S_1) in Lemma 2.2 holds. This completes the proof.

Theorem 4.3. The solutions of the model (6) are stochastically ultimately bounded and permanent for any initial value $w_0 = (P_0, Z_0) \in \mathbb{R}_+^2$.

Proof. Define

$$V(t) = Q(t) + \frac{1}{Q(t)},$$

where $Q(t) = P(t) + Z(t)$. Applying Itô's formula and using (20), we have

$$\begin{aligned} E(e^{ht}V(t)) &= E[V(0)] + E\left[\int_0^t e^{\bar{h}s}(\bar{h}V(s) + LV(s))ds\right] \\ &\leq E[V(0)] + lE\left[\int_0^t e^{\bar{h}s}ds\right] \\ &= E[V(0)] + \frac{l}{h}(e^{\bar{h}t} - 1). \end{aligned}$$

It follows that

$$E[V(t)] \leq e^{-\bar{h}t}E[V(0)] + \frac{l}{h}(1 - e^{-\bar{h}t}) \leq E[V(0)] + \frac{l}{h} := G.$$

We chose constant ϑ sufficiently large such that $\frac{G}{\vartheta} < 1$. By Chebyshev's inequality,

$$\mathbb{P}\left\{Q + \frac{1}{Q} > \vartheta\right\} \leq \frac{1}{\vartheta}E\left[Q + \frac{1}{Q}\right] \leq \frac{G}{\vartheta} := \varepsilon,$$

which implies

$$1 - \varepsilon \leq \mathbb{P}\left\{Q + \frac{1}{Q} \leq \vartheta\right\} \leq \mathbb{P}\left\{\frac{1}{\vartheta} \leq Q \leq \vartheta\right\}.$$

Noting that $Q^2 \leq 2|w|^2 \leq 2Q^2$, we get

$$\mathbb{P}\left\{\frac{1}{\sqrt{2\vartheta}} \leq \frac{Q}{\sqrt{2}} \leq |w| \leq Q \leq \vartheta\right\} \geq 1 - \varepsilon.$$

According to Definition 2.2 and Definition 2.3, the model (6) is stochastically ultimately bounded and permanent. This ends the proof.

3.2.2 | Stochastic extinction and persistence in the mean of the model (6)

In this subsection, we mainly study the stochastic extinction and persistence in the mean of each population, which respectively determine that the plankton will die out or survive in the future. First of all, we show the result related to the stochastic extinction for phytoplankton and zooplankton in the model (6).

Theorem 4.4. If $X < \frac{1}{2}\delta_1^2$, then for any given initial value $(P_0, Z_0) \in \mathbb{R}_+^2$, the solutions of the model (6) obey

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln P(t)}{t} &\leq X - \frac{1}{2}\delta_1^2 < 0 \quad a.s., \\ \limsup_{t \rightarrow \infty} \frac{\ln Z(t)}{t} &\leq -(d + \frac{1}{2}\delta_2^2) < 0 \quad a.s., \end{aligned}$$

namely, both phytoplankton and zooplankton of the model (6) go to extinction with probability one.

Proof. Making use of Itô's formula to model (6) yields

$$d \ln P(t) = \left(X - \frac{1}{2}\delta_1^2 - \alpha P - C_m Z\right)dt + \delta_1 dB_1(t), \quad (21)$$

$$d \ln Z(t) = \left((C_m - \theta)P - (d + \frac{1}{2}\delta_2^2) - \beta Z\right)dt + \delta_2 dB_2(t). \quad (22)$$

Integrating (21) from 0 to t and dividing by t on both sides, we can derive

$$\frac{\ln P(t)}{t} \leq X - \frac{1}{2}\delta_1^2 + \frac{\delta_1 B_1(t)}{t} + \frac{\ln P(0)}{t}. \quad (23)$$

Based on the strong law of large numbers for martingale⁵², we can get

$$\lim_{t \rightarrow \infty} \left(\frac{\delta_1 B_1(t)}{t} + \frac{\ln P(0)}{t} \right) = 0 \text{ a.s.} \quad (24)$$

Taking the limit on both sides of (23) and using (24), and if $X < \frac{1}{2}\delta_1^2$, we have

$$\limsup_{t \rightarrow \infty} \frac{\ln P(t)}{t} \leq X - \frac{1}{2}\delta_1^2 < 0 \text{ a.s.} \quad (25)$$

Let $\lambda = X - \frac{1}{2}\delta_1^2 < 0$, in this case, (25) indicates that for any $\varepsilon_1 > 0$, there is a $T(\omega_1) > 0$ such that $\frac{\ln P(t)}{t} \leq \lambda + \varepsilon_1$ for all $t \leq T(\omega_1)$. Hence, we have

$$P(t) \leq e^{\lambda t + \varepsilon_1}. \quad (26)$$

From the equation (22), we can derive

$$d \ln Z(t) \leq [(C_m - \theta)P - (d + \frac{1}{2}\delta_2^2)]dt + \delta_2 dB_2(t). \quad (27)$$

Integrating (27) from 0 to t and dividing by t on both sides, then from (26), we have

$$\frac{\ln Z(t)}{t} \leq -(d + \frac{1}{2}\delta_2^2) + \frac{1}{t} \int_0^t (C_m - \theta)e^{\lambda s + \varepsilon_1} ds + \frac{\delta_2(t)}{t} + \frac{\ln Z(0)}{t}.$$

According to the strong law of large numbers for martingale⁵¹ and the arbitrariness of ε_1 , we get $\lim_{t \rightarrow \infty} \left(\frac{\int_0^t e^{\lambda s + \varepsilon_1} ds}{t} + \frac{\delta_2 B_2(t)}{t} + \frac{\ln Z(0)}{t} \right) = 0$ a.s., so we have

$$\limsup_{t \rightarrow \infty} \frac{\ln Z(t)}{t} \leq -(d + \frac{1}{2}\delta_2^2) < 0 \text{ a.s.}$$

That is to say, if phytoplankton on which zooplankton lives is extinct, the zooplankton will eventually go to extinction with probability one. This ends the proof.

Remark 4.2. Based on Theorem 4.4, it can be known that if phytoplankton of the model (6) goes to extinction, the zooplankton will be extinct. It is natural to have such a question: what will limit the abundance of phytoplankton if zooplankton becomes extinct? In fact, when $\mathbb{P}\{\lim_{t \rightarrow \infty} Z(t) = 0\} = 1$, then from phytoplankton equation in the model (6), we can get the following limiting equation:

$$dP(t) = (XP - \alpha P^2)dt + \delta_1 P dB_1(t).$$

The distribution of $P(t)$ converges to a stationary distribution which has a density function⁶²

$$\mathbb{P}(P) = \frac{\left(\frac{2\alpha}{\delta_1^2}\right)^{\frac{2X}{\delta_1^2} - 1}}{\Gamma\left(\frac{2X}{\delta_1^2} - 1\right)} P^{\frac{2X}{\delta_1^2} - 2} e^{-\frac{2\alpha P}{\delta_1^2}},$$

for $\delta_1^2 < 2X$, where $\Gamma(\mu) = \int_0^\infty t^{\mu-1} e^{-t} dt$. Hence, the mean of density $\mathbb{P}(P)$ is $E(P) = \frac{X}{\alpha} - \frac{\delta_1^2}{2\alpha}$. It is easy to verified that

$$\lim_{\delta_1 \rightarrow 0} E(P) = \lim_{\delta_1 \rightarrow 0} \left(\frac{X}{\alpha} - \frac{\delta_1^2}{2\alpha} \right) = \frac{X}{\alpha},$$

which implies that $P(t)$ will tend to $\frac{X}{\alpha}$. Namely, the dynamics of zooplankton-free situation for the stochastic model (6) is the same as the result provided in Theorem 3.3 for the deterministic model (4).

Next, we show the result concerning stochastic persistent in the mean of each population in the model (6).

Theorem 4.5. Let $X > \frac{\Psi}{C_m - \theta}$, for any given initial value $(P_0, Z_0) \in \mathbb{R}_+^2$, if one of the following conditions holds:

$$(i) \alpha\beta > C_m(C_m - \theta) \text{ and } X > \frac{\alpha^2\beta(d + 0.5\delta_2^2)}{(C_m - \theta)(\alpha\beta - C_m(C_m - \theta))} + \frac{0.5\delta_1^2\alpha\beta - C_m\Psi}{\alpha\beta - C_m(C_m - \theta)};$$

$$(ii) \alpha\beta < C_m(C_m - \theta) \text{ and } X < \frac{\alpha^2\beta(d + 0.5\delta_2^2)}{(C_m - \theta)(\alpha\beta - C_m(C_m - \theta))} + \frac{0.5\delta_1^2\alpha\beta - C_m\Psi}{\alpha\beta - C_m(C_m - \theta)},$$

then the solution $(P(t), Z(t))$ of the model (6) obeys

$$\lim_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t P(s) ds \leq \frac{X - 0.5\delta_1^2}{\alpha} \text{ a.s.},$$

$$\lim_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t Z(s) ds \leq \frac{(C_m - \theta)X - \Psi}{\alpha\beta} \text{ a.s.},$$

$$\lim_{t \rightarrow \infty} \inf \frac{1}{t} \int_0^t P(s) ds \geq \frac{1}{\alpha^2\beta} [X(\alpha\beta - C_m(C_m - \theta)) - 0.5\delta_1^2\alpha\beta + C_m\Psi] \text{ a.s.},$$

$$\lim_{t \rightarrow \infty} \inf \frac{1}{t} \int_0^t Z(s) ds \geq \frac{1}{\alpha^2\beta^2} [(C_m - \theta)(X(\alpha\beta - C_m(C_m - \theta)) - 0.5\delta_1^2\alpha\beta + C_m\Psi) - \alpha^2\beta(d + 0.5\delta_2^2)] \text{ a.s.},$$

where $\Psi = 0.5\delta_1^2(C_m - \theta) + \alpha d + 0.5\alpha\delta_2^2$. That is to say, both phytoplankton and zooplankton of the model (6) are persistent in the mean.

Proof. From (21), we have

$$\frac{\ln P(t)}{t} \leq X - 0.5\delta_1^2 - \frac{\int_0^t \alpha P(s) ds}{t} + \frac{\delta_1 B_1(t)}{t} + \frac{\ln P(0)}{t}.$$

If $X > 0.5\delta_1^2$, based on (24) and by virtue of Lemma 2.1, we can derive

$$\lim_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t P(s) ds \leq \frac{X - 0.5\delta_1^2}{\alpha} \text{ a.s.}, \quad (28)$$

which implies that the phytoplankton of the model (6) is persistent in the mean.

From (22) and (28), we also have

$$\frac{\ln Z(t)}{t} \leq (C_m - \theta) \lim_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t P(s) ds - (d + 0.5\delta_2^2) - \frac{\beta \int_0^t Z(s) ds}{t} + \frac{\delta_2 B_2(t)}{t} + \frac{\ln Z(0)}{t}.$$

According to the strongly law of numbers for martingales⁵¹, we get

$$\lim_{t \rightarrow \infty} \left(\frac{\delta_2 B_2(t)}{t} + \frac{\ln Z(0)}{t} \right) = 0 \text{ a.s.} \quad (29)$$

For the sake of convenience, we define $\Psi = 0.5\delta_1^2(C_m - \theta) + \alpha d + 0.5\alpha\delta_2^2 > 0$. Let $X > \frac{\Psi}{C_m - \theta}$, by means of (29) and Lemma 2.1, we obtain

$$\lim_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t Z(s) ds \leq \frac{(C_m - \theta)X - \Psi}{\alpha\beta} \text{ a.s.} \quad (30)$$

On the other hand, according to the equations (21) and (30), we have

$$\begin{aligned} \frac{\ln P(t)}{t} &\geq X - 0.5\delta_1^2 - C_m \lim_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t Z(s) ds - \frac{\alpha \int_0^t P(s) ds}{t} + \frac{\delta_1 B_1(t)}{t} + \frac{\ln P(0)}{t} \\ &= X - 0.5\delta_1^2 - \frac{C_m(C_m - \theta)X - \Psi C_m}{\alpha\beta} - \frac{\alpha \int_0^t P(s) ds}{t} + \frac{\delta_1 B_1(t)}{t} + \frac{\ln P(0)}{t}. \end{aligned}$$

Define a function $f(X) = X - 0.5\delta_1^2 - \frac{C_m(C_m - \theta)X - \Psi C_m}{\alpha\beta}$, then if we have $f(X) > 0$, it is necessary to ensure $(\alpha\beta - C_m(C_m - \theta))X > 0.5\delta_1^2\alpha\beta - C_m\Psi$. Consequently, let $\alpha\beta > C_m(C_m - \theta)$, we have $f(X) > 0$ if $X > \frac{0.5\delta_1^2\alpha\beta - \Psi C_m}{\alpha\beta - C_m(C_m - \theta)}$; let $\alpha\beta < C_m(C_m - \theta)$, we have $f(X) > 0$ if $X < \frac{0.5\delta_1^2\alpha\beta - \Psi C_m}{\alpha\beta - C_m(C_m - \theta)}$.

Then, by using Lemma 2.1 and applying (29), we derive

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t P(s) ds \geq \frac{1}{\alpha^2 \beta} [X(\alpha\beta - C_m(C_m - \theta)) - 0.5\delta_1^2 \alpha\beta + C_m \Psi] \text{ a.s.}, \quad (31)$$

provided

$$\alpha\beta > C_m(C_m - \theta) \quad \text{and} \quad X > \frac{0.5\delta_1^2 \alpha\beta - \Psi C_m}{\alpha\beta - C_m(C_m - \theta)}$$

or

$$\alpha\beta < C_m(C_m - \theta) \quad \text{and} \quad X < \frac{0.5\delta_1^2 \alpha\beta - \Psi C_m}{\alpha\beta - C_m(C_m - \theta)}.$$

On the basis of (22) and by means of (31), we have

$$\frac{\ln Z(t)}{t} \geq (C_m - \theta) \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t P(s) ds - (d + \frac{1}{2}\delta_2^2) - \frac{\beta \int_0^t Z(s) ds}{t} + \frac{\delta_2 B_2(t)}{t} + \frac{\ln Z(0)}{t}.$$

For the above inequality, similarly, if $\alpha\beta > C_m(C_m - \theta)$ and

$$X > \frac{\alpha^2 \beta (d + 0.5\delta_2^2)}{(C_m - \theta)(\alpha\beta - C_m(C_m - \theta))} + \frac{0.5\delta_1^2 \alpha\beta - C_m \Psi}{\alpha\beta - C_m(C_m - \theta)},$$

or if $\alpha\beta < C_m(C_m - \theta)$ and

$$X < \frac{\alpha^2 \beta (d + 0.5\delta_2^2)}{(C_m - \theta)(\alpha\beta - C_m(C_m - \theta))} + \frac{0.5\delta_1^2 \alpha\beta - C_m \Psi}{\alpha\beta - C_m(C_m - \theta)},$$

we have

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t Z(s) ds \geq \frac{1}{\alpha^2 \beta^2} [(C_m - \theta)(X(\alpha\beta - C_m(C_m - \theta)) - 0.5\delta_1^2 \alpha\beta + C_m \Psi) - \alpha^2 \beta (d + 0.5\delta_2^2)] \text{ a.s.} \quad (32)$$

Because $\frac{\Psi}{C_m - \theta} > 0.5\delta_1^2 > 0$, so it is easy to find that if $X > \frac{\Psi}{C_m - \theta}$, then (28) and (30) hold. Furthermore, when $\alpha\beta > C_m(C_m - \theta)$, we have

$$\frac{\alpha^2 \beta (d + 0.5\delta_2^2)}{(C_m - \theta)(\alpha\beta - C_m(C_m - \theta))} + \frac{0.5\delta_1^2 \alpha\beta - C_m \Psi}{\alpha\beta - C_m(C_m - \theta)} > \frac{0.5\delta_1^2 \alpha\beta - C_m \Psi}{\alpha\beta - C_m(C_m - \theta)};$$

when $\alpha\beta < C_m(C_m - \theta)$, we have

$$\frac{\alpha^2 \beta (d + 0.5\delta_2^2)}{(C_m - \theta)(\alpha\beta - C_m(C_m - \theta))} + \frac{0.5\delta_1^2 \alpha\beta - C_m \Psi}{\alpha\beta - C_m(C_m - \theta)} < \frac{0.5\delta_1^2 \alpha\beta - C_m \Psi}{\alpha\beta - C_m(C_m - \theta)}.$$

Therefore, if $\alpha\beta > C_m(C_m - \theta)$ and

$$X > \frac{\alpha^2 \beta (d + 0.5\delta_2^2)}{(C_m - \theta)(\alpha\beta - C_m(C_m - \theta))} + \frac{0.5\delta_1^2 \alpha\beta - C_m \Psi}{\alpha\beta - C_m(C_m - \theta)},$$

or if $\alpha\beta < C_m(C_m - \theta)$ and

$$X < \frac{\alpha^2 \beta (d + 0.5\delta_2^2)}{(C_m - \theta)(\alpha\beta - C_m(C_m - \theta))} + \frac{0.5\delta_1^2 \alpha\beta - C_m \Psi}{\alpha\beta - C_m(C_m - \theta)},$$

we can get that (31) and (32) hold simultaneously. So, the above Theorem 4.5 comes true. This completes the proofs.

3.2.3 | Stationary distribution and ergodicity

In this subsection, we show that there exists a uniqueness and ergodicity of stationary distribution for the model (6), which provides a better description of permanence for the plankton. For the sake of simplicity, set $\bar{\lambda} = 0.5\delta_1^2 + \delta_2^2 + 2(d + \beta + \theta) + \alpha + C_m - X$.

Theorem 4.6. If $X < 0.5\delta_1^2 + \delta_2^2 + 2(d + \beta + \theta) + \alpha + C_m$, for any initial value $(P_0, Z_0) \in \mathbb{R}_+^2$, the model (6) has a unique stationary distribution and it has ergodic property.

Proof. Let $\zeta(t) = \ln P(t)$ and $\eta(t) = \ln Z(t)$ for $t \geq 0$, we have

$$\begin{cases} d\zeta(t) = (X - 0.5\delta_1^2 - \alpha e^\zeta - C_m e^\eta)dt + \delta_1 dB_1(t), \\ d\eta(t) = (C_m e^\zeta - d - 0.5\delta_2^2 - \beta e^\eta - \theta e^\zeta)dt + \delta_2 dB_2(t). \end{cases} \quad (33)$$

Based on the excellent work in⁶³, the ergodic property of the model (6) is equivalent to that of system (33). Define a C^2 -function $V : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ by

$$V(\zeta, \eta) = M[\zeta + \eta + e^\eta] + \frac{(e^\zeta + e^\eta)^{\vartheta+1}}{\vartheta+1} - V(\zeta_0, \eta_0) = MV_1(\zeta, \eta) + V_2(\zeta, \eta),$$

where $V_1(\zeta, \eta) = \zeta + \eta + e^\eta$, $V_2(\zeta, \eta) = \frac{(e^\zeta + e^\eta)^{\vartheta+1}}{\vartheta+1} - V(\zeta_0, \eta_0)$, $M = \frac{2}{\bar{\lambda}} \max\{2, \sup_{(\zeta, \eta) \in \mathbb{R}_+^2} [-0.25\alpha e^{(\vartheta+2)\zeta} - 0.25\beta e^{(\vartheta+2)\eta} + H]\}$, $H = \sup_{(\zeta, \eta) \in \mathbb{R}_+^2} \{-0.5\alpha e^{(\vartheta+2)\zeta} - 0.5\beta e^{(\vartheta+2)\eta} - d e^{(\vartheta+1)\eta} + X(e^\zeta + e^\eta)^\vartheta e^\zeta + 0.5\vartheta\delta_1^2 e^{(\vartheta+1)\zeta} + 0.5\vartheta\delta_2^2 e^{(\vartheta+1)\eta}\}$, ϑ is a constant satisfying $0 < \vartheta < 1$ and $V(\zeta_0, \eta_0)$ is the minimum value of $V(\zeta, \eta)$ at point (ζ_0, η_0) , and $\bar{\lambda} > 0$ which is equivalent to $X < 0.5\delta_1^2 + \delta_2^2 + 2(d + \beta + \theta) + \alpha + C_m$. An application of Itô's formula to $V_1(\zeta, \eta)$, yields

$$\begin{aligned} \mathcal{L}V_1(\zeta, \eta) &= X - 0.5\delta_1^2 - \alpha e^\zeta - C_m e^\eta + C_m e^\zeta - d - 0.5\delta_2^2 - \beta e^\eta \\ &\quad - \theta e^\zeta + C_m e^{\zeta+\eta} - d e^\eta - 0.5\delta_2^2 e^\eta - \beta e^{2\eta} - \theta e^{\zeta+\eta} \\ &\leq X - 0.5\delta_1^2 - d - 0.5\delta_2^2 - \alpha(1 + \zeta) - C_m(1 + \eta) - \beta(1 + \eta) - \theta(1 + \zeta) \\ &\quad - d(1 + \eta) - 0.5\delta_2^2(1 + \eta) - \beta(1 + 2\eta) - \theta(1 + \zeta + \eta) + C_m e^\zeta + C_m e^{\zeta+\eta} \\ &\leq X - 0.5\delta_1^2 - \delta_2^2 - 2d - 2\beta - 2\theta - \alpha - C_m + 2C_m e^{\zeta+\eta} = -\bar{\lambda} + 2C_m e^{\zeta+\eta}. \end{aligned} \quad (34)$$

Similarly, we have

$$\begin{aligned} \mathcal{L}V_2(\zeta, \eta) &= (e^\zeta + e^\eta)^\vartheta X e^\zeta - \alpha e^{2\zeta} - d e^\eta - \beta e^{2\eta} - \theta e^{\zeta+\eta} + 0.5\vartheta(e^\zeta + e^\eta)^{\vartheta-1}(e^{2\zeta}\delta_1^2 + e^{2\eta}\delta_2^2) \\ &\leq X(e^\zeta + e^\eta)^\vartheta e^\zeta - \alpha e^{\vartheta+2}\zeta - d e^{(\vartheta+1)\eta} - \beta e^{(\vartheta+2)\eta} + 0.5\vartheta\delta_1^2 e^{(\vartheta+1)\zeta} + 0.5\vartheta\delta_2^2 e^{(\vartheta+1)\eta} \\ &= -0.5\alpha e^{(\vartheta+2)\zeta} - 0.5\beta e^{(\vartheta+2)\eta} + H. \end{aligned} \quad (35)$$

It follows from (34) and (35) that

$$\mathcal{L}V(\zeta, \eta) \leq -M\bar{\lambda} + 2MC_m e^{\zeta+\eta} - 0.5\alpha e^{(\vartheta+2)\zeta} - 0.5\beta e^{(\vartheta+2)\eta} + H.$$

Define a bounded closed set

$$D = \{(\zeta, \eta) : |\zeta| \leq \ln \epsilon^{-1}, |\eta| \leq \ln \epsilon^{-1}, (\zeta, \eta) \in \mathbb{R}_+^2\},$$

where $0 < \epsilon < 1$ is a sufficiently small number. In the set $D^C = \mathbb{R}_+^2 \setminus D$, we choose the sufficiently small ϵ to satisfy the following conditions

$$0 < \epsilon < \frac{0.125\bar{\lambda}}{C_m}, \quad (36)$$

$$0 < \epsilon < \frac{0.125\beta}{MC_m}, \quad (37)$$

$$0 < \epsilon < \frac{0.125\alpha}{MC_m}, \quad (38)$$

$$-M\bar{\lambda} - 0.25\alpha \frac{1}{\epsilon^{\vartheta+2}} + H_1 \leq -1, \quad (39)$$

$$-M\bar{\lambda} - 0.25\beta \frac{1}{\epsilon^{\vartheta+2}} + H_2 \leq -1, \quad (40)$$

where

$$H_1 = \sup_{(\zeta, \eta) \in \mathbb{R}_+^2} \{2MC_m e^{\zeta+\eta} - 0.25\alpha e^{(\vartheta+2)\zeta} - 0.5\beta e^{(\vartheta+2)\eta} + H\},$$

and

$$H_2 = \sup_{(\zeta, \eta) \in \mathbb{R}_+^2} \{2MC_m e^{\zeta+\eta} - 0.25\beta e^{(\theta+2)\eta} - 0.5\alpha e^{(\theta+2)\zeta} + H\}.$$

For convenience, we divide D^C into four domains,

$$D_\epsilon^1 = \{(\zeta, \eta) \in \mathbb{R}_+^2 : -\infty < \zeta \leq \ln \epsilon\},$$

$$D_\epsilon^2 = \{(\zeta, \eta) \in \mathbb{R}_+^2 : -\infty < \eta \leq \ln \epsilon\},$$

$$D_\epsilon^3 = \{(\zeta, \eta) \in \mathbb{R}_+^2 : \zeta \geq \ln \epsilon^{-1}\},$$

$$D_\epsilon^4 = \{(\zeta, \eta) \in \mathbb{R}_+^2 : \eta \geq \ln \epsilon^{-1}\}.$$

Clearly, $D^c = D_\epsilon^1 \cup D_\epsilon^2 \cup D_\epsilon^3 \cup D_\epsilon^4$. Next, we will verify $\mathcal{L}V(\zeta, \eta) \leq -1$ on D^C , which is equivalent to demonstrating it on the four cases above.

Case 1. On domain D_ϵ^1 , owing to $-\infty < \zeta \leq \ln \epsilon$ and then $e^{\zeta+\eta} \leq \epsilon e^\eta \leq \epsilon(1 + e^{(\theta+2)\eta})$, we have

$$\begin{aligned} \mathcal{L}V(\zeta, \eta) &\leq -0.25M\bar{\lambda} + (2MC_m\epsilon - 0.25M\bar{\lambda}) + (2MC_m\epsilon - 0.25\beta)e^{(\theta+2)\eta} - 0.25\alpha e^{(\theta+2)\zeta} \\ &\quad + \{-0.5M\bar{\lambda} - 0.25\alpha e^{(\theta+2)\zeta} - 0.25\beta e^{(\theta+2)\eta} + H\} \\ &\leq -0.25M\bar{\lambda} + (2MC_m\epsilon - 0.25M\bar{\lambda}) + (2MC_m\epsilon - 0.25\beta)e^{(\theta+2)\eta} - 0.25\alpha e^{(\theta+2)\zeta} \\ &\quad + \{-0.5M\bar{\lambda} + \sup_{(\zeta, \eta) \in \mathbb{R}_+^2} [-0.25\alpha e^{(\theta+2)\zeta} - 0.25\beta e^{(\theta+2)\eta} + H]\}. \end{aligned}$$

Combined with the definition of M , (36) and (37), we get

$$\mathcal{L}V(\zeta, \eta) \leq -0.25M\bar{\lambda} - 0.25\alpha e^{(\theta+2)\zeta} \leq -0.25M\bar{\lambda} \leq -1.$$

Case 2. For any $(\zeta, \eta) \in D_\epsilon^2$, similarly, owing to $e^{\zeta+\eta} \leq \epsilon e^\zeta \leq \epsilon(1 + e^{(\theta+2)\zeta})$, we get

$$\begin{aligned} \mathcal{L}V(\zeta, \eta) &\leq -0.25M\lambda + (2MC_m\epsilon - 0.25M\lambda) + (2MC_m\epsilon - 0.25\alpha)e^{(\theta+2)\zeta} - 0.25\beta e^{(\theta+2)\eta} \\ &\quad + \{-0.5M\bar{\lambda} - 0.25\alpha e^{(\theta+2)\zeta} - 0.25\beta e^{(\theta+2)\eta} + H\} \\ &\leq -0.25M\bar{\lambda} + (2MC_m\epsilon - 0.25M\bar{\lambda}) + (2MC_m\epsilon - 0.25\beta)e^{(\theta+2)\eta} \\ &\quad - 0.25\beta e^{(\theta+2)\zeta} + \{-0.5M\bar{\lambda} + \sup_{(\zeta, \eta) \in \mathbb{R}_+^2} [-0.25\alpha e^{(\theta+2)\zeta} - 0.25\beta e^{(\theta+2)\eta} + H]\}. \end{aligned}$$

Combined with the definition of M , (36) and (38), we get

$$\mathcal{L}V(\zeta, \eta) \leq -0.25M\bar{\lambda} - 0.25\beta e^{(\theta+2)\zeta} \leq -0.25M\bar{\lambda} \leq -1.$$

Case 3. On D_ϵ^3 , we have

$$\begin{aligned} \mathcal{L}V(\zeta, \eta) &\leq -M\bar{\lambda} + 2MC_m e^{\zeta+\eta} - 0.5\alpha e^{(\theta+2)\zeta} - 0.5\beta e^{(\theta+2)\eta} + H \\ &\leq -M\bar{\lambda} - 0.25\alpha \frac{1}{\epsilon^{\theta+2}} + H_1, \end{aligned}$$

which implies $\mathcal{L}V(\zeta, \eta) \leq -1$ in view of (39).

Case 4. When $(\zeta, \eta) \in D_\epsilon^4$,

$$\begin{aligned} \mathcal{L}V(\zeta, \eta) &\leq -M\bar{\lambda} + 2MC_m e^{\zeta+\eta} - 0.5\alpha e^{(\theta+2)\zeta} - 0.5\beta e^{(\theta+2)\eta} + H \\ &\leq -M\bar{\lambda} - 0.25\beta \frac{1}{\epsilon^{\theta+2}} + H_2. \end{aligned}$$

By virtue of (40), we have $\mathcal{L}V(\zeta, \eta) \leq -1$ on D_ϵ^4 . Consequently, we can conclude that

$$\mathcal{L}V(\zeta, \eta) \leq -1 \quad \text{for all } (\zeta, \eta) \in D^C.$$

Therefore, the condition of (ii) in Lemma 2.3 is satisfied.

On the other hand, one can see that there is

$$M_0 = \min_{(P, Z) \in Q_\delta} \{\delta_1^2 P^2, \delta_2 Z^2\},$$

such that

$$\begin{aligned} \sum_{i,j=1}^2 a_{ij}(P, Z)\xi_i\xi_j &= \begin{pmatrix} \delta_1 P \xi_1 & \delta_2 Z \xi_2 \end{pmatrix} \begin{pmatrix} \delta_1 P \xi_1 \\ \delta_2 Z \xi_2 \end{pmatrix} \\ &= \delta_1^2 P^2 \xi_1^2 + \delta_2^2 Z^2 \xi_2^2 \geq M_0 \|\xi\|^2, \end{aligned}$$

for any $(P, Z) \in \mathcal{D} \subset \mathbb{R}_+^2$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}_+^2$. Namely, the condition (i) in Lemma 2.3 is also satisfied, indicating that model (6) is ergodic and has a unique stationary distribution. This completes the proof.

4 | NUMERICAL SIMULATIONS

In this section, based on the previous theoretical works, we further explore the impacts of plankton body size on the resulting dynamics of the deterministic model (4) as well as its corresponding stochastic model (6). As an example, we choose such a set of parameters: $a_1 = 0.02$, $a_2 = 0.03$, $a_3 = 0.04$, $a = 0.1$, $\bar{C} = 0.6$, $\tau = 2$, $\kappa = 0.5$, $\alpha = 0.1$, $\beta = 0.2$, $\theta = 0.1$ and $d = 0.4$, where some of them are taken from⁴¹. In the following examples, we use the above parameters by default and only change the corresponding key parameters declared in the following simulations.

4.1 | Effects of plankton body size on the dynamics of the deterministic model (4)

To first see the effects of plankton body size on the plankton density in model (4), we only vary phytoplankton cell size x and zooplankton body size y , the results are shown in Fig.1 . From Fig.1 (a), it can be found that the equilibrium level of phytoplankton density increases as the phytoplankton cell size increases. However, for the small value of phytoplankton cell size, the increase of zooplankton body size can not change the equilibrium level of phytoplankton density; for the middle and large values of phytoplankton cell size, the equilibrium level of phytoplankton density decreases as the zooplankton body increases. Fig.1 (b) clearly confirms this result.

Additionally, with increase of phytoplankton cell size, the equilibrium level of zooplankton density first increases and finally decreases, as displayed in Fig.1 (c). Interestingly, for the small value of phytoplankton cell size, the increase of zooplankton body size has no influences on the equilibrium level of zooplankton density; for the middle value of phytoplankton cell size, the zooplankton density decreases as zooplankton body size increases; for the large value of phytoplankton body size, the zooplankton density increases as the zooplankton body size increases, which is further demonstrated by Fig.1 (d).

Actually, by comparing Fig.1 (b) and Fig.1 (d), it can be concluded that if phytoplankton cell size is small, the increase of zooplankton body size does not affect the density of phytoplankton and zooplankton, which further shown in Figs.2 (a) and 2 (b); if the phytoplankton cell size is middle level, the increase of zooplankton body size can reduce simultaneously the density of phytoplankton and zooplankton (see Figs.2 (c) and 2 (d)); if the phytoplankton cell size is large, the increase of zooplankton body size can increase the density of zooplankton, but decrease the density of phytoplankton (see Figs.2 (e) and 2 (f)).

Nevertheless, the analysis shows that the very small phytoplankton cell size can cause the extinction of phytoplankton and zooplankton, as shown in Figs.3 (a) and 3 (b) (see the red line). Also, it is observed that the very large phytoplankton cell size can result in the extinction of zooplankton, but can make phytoplankton reach a peak, as displayed in Figs.3 (a) and 3 (b) (see the blue line). Furthermore, in spite of the very small zooplankton body size can not give rise to the extinction of phytoplankton and zooplankton, while the very large zooplankton body size can lead zooplankton to go extinct, which are demonstrated by Figs.3 (c) and 3 (d).

By obtaining $y = 0.2$, Fig.4 (a) shows that the increase of phytoplankton cell size can lead to the change of the stability of the positive equilibrium from global asymptotic stability to local asymptotic stability, but the stability is not disturbed. However, the increase of toxin release rate of phytoplankton can result in the dynamics of model (4) change from stable positive equilibrium E_* to stable boundary equilibrium E_1 . In other words, the increase of toxin released by phytoplankton can cause the extinction of zooplankton. By taking $x = 0.01$, it is found that the increase of zooplankton body size can not change the stability of the

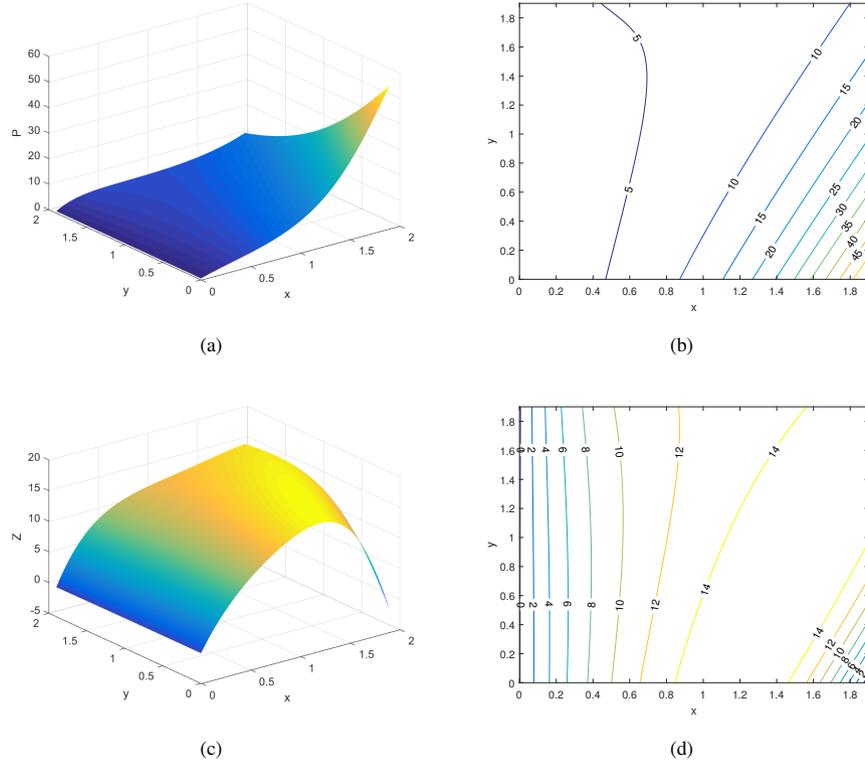


FIGURE 1 (a)The relationships between phytoplankton density and phytoplankton cell size x and zooplankton body size y . (c)The relationships between zooplankton density and phytoplankton cell size x and zooplankton body size y . (b) Contour plot of phytoplankton density with x and y . (d) Contour plot of zooplankton density with x and y .

equilibrium E_* or E_1 , which is displayed in Fig.4 (b). By comparison, it can be concluded that the increase of phytoplankton cell size or zooplankton body size or toxin released by phytoplankton can not change the stability dynamics of model (4), but the increase of phytoplankton toxin can make the stability transition of model (4) (i.e., the stable positive equilibrium point becomes the stable boundary equilibrium point). Additionally, for region III (see Fig.4), we obtain $(x, y) = (0.001, 0.02)$, which satisfies the parameter condition of Theorem 3.3. By direct calculation, we get that $E_1 = (0.252, 0)$ which is globally asymptotically stable, as shown in Fig.4 (c). In region I, we adopt $(x, y) = (0.1, 0.2)$, based on Theorem 3.4, similarly, we get that the positive equilibrium $E_* = (2.20, 3.50)$ is globally asymptotically stable, which is further demonstrated by Fig.4 (d).

By fixing $y = 0.2$, and varying the value of x , Table 1 shows that the level of phytoplankton and zooplankton densities increases as the phytoplankton cell size increases. However, the plankton density is almost unchanged with the increase of zooplankton body size, which is displayed in Table 2 (In this case, we take $x = 0.1$ and only vary the value of y). In other

TABLE 1 The level of plankton density in model (4) with different phytoplankton cell size.

Phytoplankton cell size	Level of phytoplankton density	Level of zooplankton density
$x=0.05$	1.50	1.76
$x=0.15$	2.84	5.09
$x=0.20$	3.45	6.57
$x=0.25$	4.04	7.95
$x=0.30$	4.61	9.24
$x=0.35$	5.17	10.46
$x=0.40$	5.74	11.60

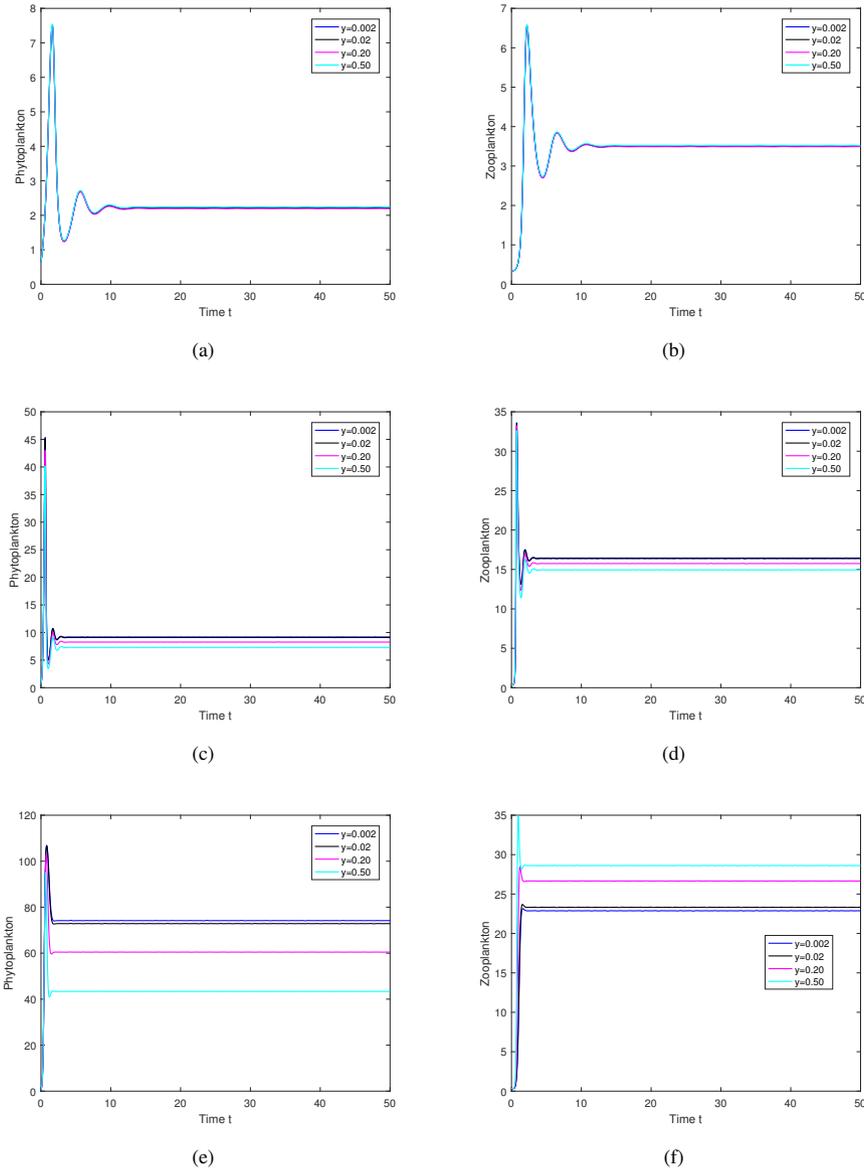


FIGURE 2 Effects of zooplankton body size on the plankton density in model (4). (a)-(b) for the small phytoplankton cell size $x = 0.1$. (c)-(d) for the middle phytoplankton cell size $x = 0.6$. (e)-(f) for the large phytoplankton cell size $x = 1.6$.

words, the increase of phytoplankton cell size is conducive to the survival of plankton, but the increase of zooplankton body size has almost no effect on the survival of plankton, which are demonstrated by Figs.5 (a) and 5 (b), respectively.

4.2 | Effects of plankton body size on the dynamics of the stochastic model (6)

In this subsection, we investigate the impacts of plankton body size on the stochastic dynamics of the interactions between phytoplankton and zooplankton in the model (6). By using the method mentioned in Higham⁶⁴, model (6) can be rewritten as following discretization equations:

$$\begin{cases} P_{j+1} = P_j + (r(x)P_j - s(x)P_j - \alpha P_j^2 - C(x, y)P_j Z_j) \Delta t + \delta_1 P_j \xi_j \sqrt{\Delta t} + \frac{\delta_1^2}{2} P_j (\xi_j^2 - 1) \Delta t, \\ Z_{j+1} = Z_j + (C(x, y)P_j Z_j - d Z_j - \beta Z_j^2 - \theta P_j Z_j) \Delta t + \delta_2 Z_j \eta_j \sqrt{\Delta t} + \frac{\delta_2^2}{2} Z_j (\eta_j^2 - 1) \Delta t, \end{cases}$$

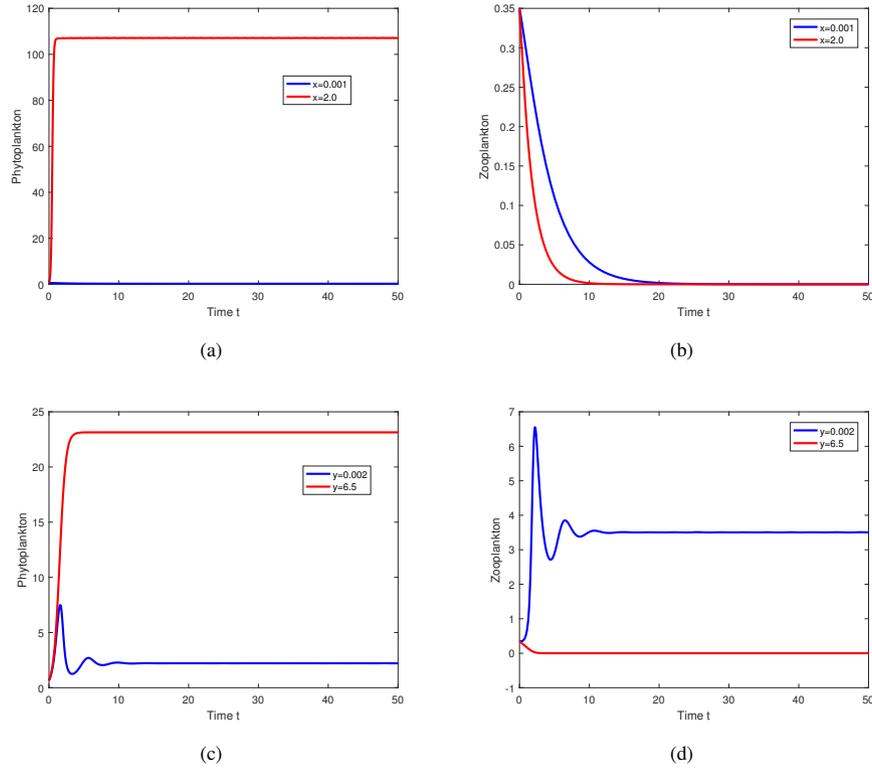


FIGURE 3 Effects of plankton body size on the survival of plankton in model (4). (a)-(b) for fixing $y = 0.2$ and varying x . (c)-(d) for fixing $x = 0.1$. and varying y

where ξ_j and η_j are two independent Gaussian random variable $N(0, 1)$ for $j = 1, 2, \dots, n$.

Based on Theorem 4.4 and Theorem 4.5, the analysis indicates that the plane region is divided into three regions I, II and III by $\delta_1 - X$ parameters, where I indicates that both phytoplankton and zooplankton are extinct, II signifies that phytoplankton is persistent while zooplankton is extinct, and III denotes that both phytoplankton and zooplankton are persistent, which is shown in Fig.6 (a). By fixing $(x, y) = (0.1, 0.2)$, and adopting $(\delta_1, \delta_2) = (1.21, 1.22)$, $(\delta_1, \delta_2) = (0.21, 1.22)$ and $(\delta_1, \delta_2) = (0.21, 0.22)$, the corresponding results are displayed in subgraph (i), (ii) and (iii) in Fig.6 (b), respectively. It is worth noting that the only difference between these subgraphs in the Fig.6 (b) is that the noise intensities are different. Consequently, one can get that the large stochastic environmental fluctuations can cause the extinction of phytoplankton and zooplankton in the model (6).

By fixing noise intensities $(\delta_1, \delta_2) = (0.25, 0.22)$, on the one hand, we obtain $x = 0.1$ and only vary zooplankton body size $y = 0.2, 1.2, 3.5, 6.5$; on the other hand, we adopt $y = 0.2$ and only change phytoplankton cell size $x = 0.001, 0.11, 1.11, 2.11$, it is found that the results of these two cases are similar to those in Fig.3, as shown in Fig.7. By comparing Fig.3 and Fig.7, it

TABLE 2 The level of plankton density in model (4) with different zooplankton body size.

Zooplankton body size	Level of phytoplankton density	Level of zooplankton density
$y=0.05$	2.207	3.498
$y=0.15$	2.197	3.491
$y=0.20$	2.196	3.490
$y=0.25$	2.197	3.491
$y=0.30$	2.200	3.494
$y=0.35$	2.206	3.498
$y=0.40$	2.215	3.505

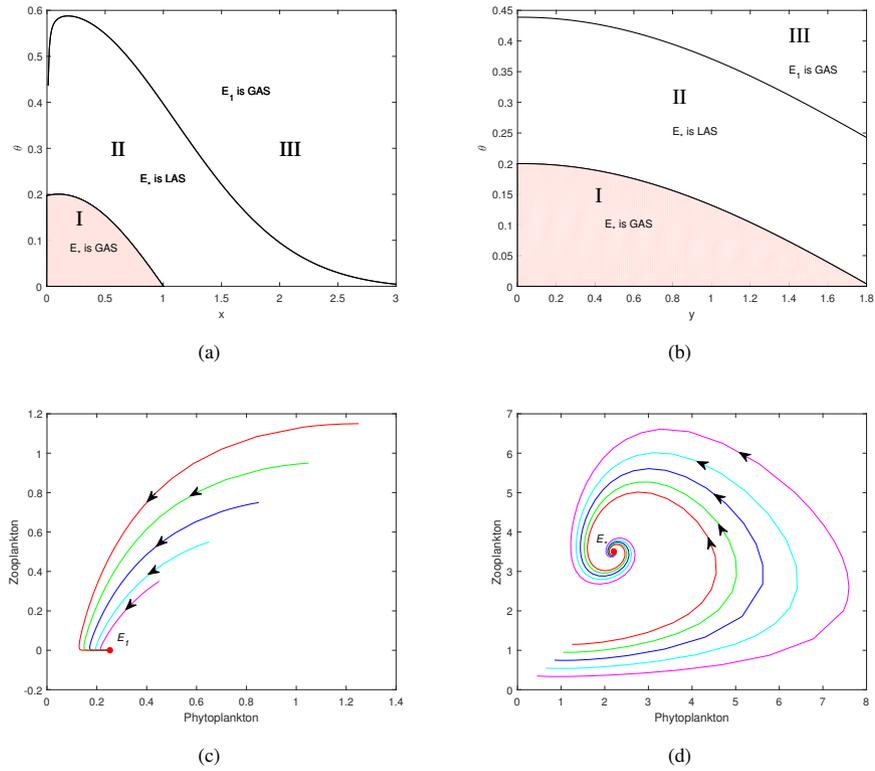


FIGURE 4 Effects of phytoplankton cell size on the stability of equilibria in the model (4), where *I* indicates that E_* is globally asymptotically stable (GAS), *II* means that E_* is locally asymptotically stable (LAS) and E_1 is unstable, and *III* denotes that E_1 is globally asymptotically stable (GAS). (a) Two-parameter bifurcation diagram of model (4) with θ and x . (b) Two-parameter bifurcation diagram of model (4) with θ and y . (c) Phase diagram of E_1 . (d) Phase diagram of E_* .

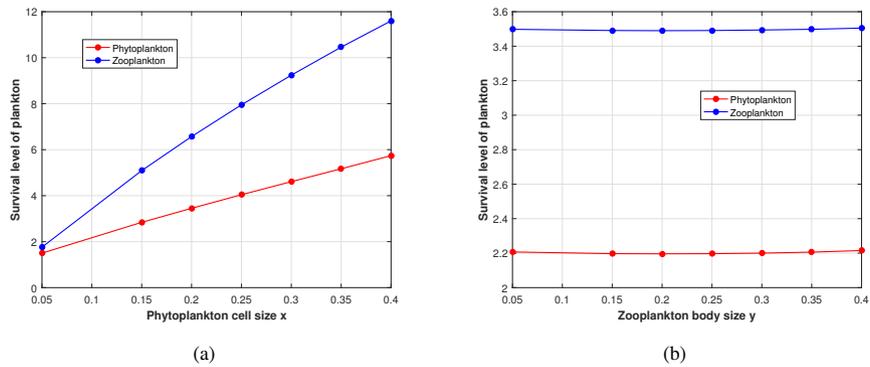


FIGURE 5 Effects of plankton body size on the survival level of plankton in model (4). a for the effects of phytoplankton cell size. b for the effects of zooplankton body size.

can be asserted that whether in a deterministic environment or a random disturbance environment, the very small phytoplankton cell size can lead to the extinction of phytoplankton and zooplankton, while the very small zooplankton body size can maintain the sustainable survival of phytoplankton and zooplankton. Additionally, the very large zooplankton body size or phytoplankton cell size can lead to the extinction of zooplankton, but can not affect the long-term survival of phytoplankton.

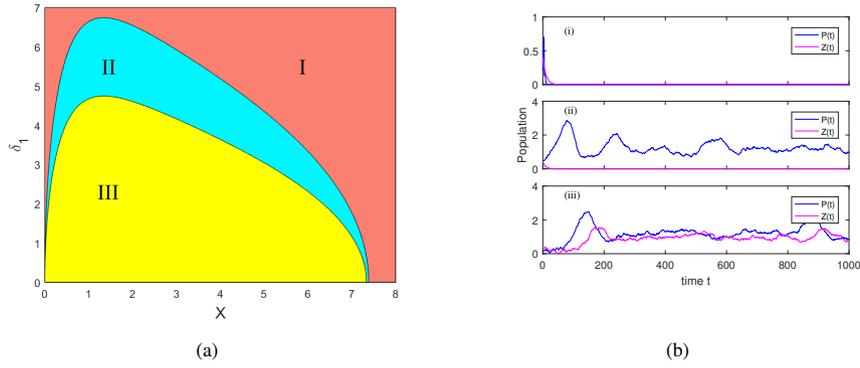


FIGURE 6 (a) Bifurcation diagram of the model (6) with respect to δ_1 and X , where I denotes that both phytoplankton and zooplankton are extinct, II indicates that phytoplankton is persistent but zooplankton dies out and III signifies that both phytoplankton and zooplankton are persistent. (b) The solutions of the model (6) with $(x, y) = (0.1, 0.2)$, where (i) for $(\delta_1, \delta_2) = (1.21, 1.22)$, (ii) for $(\delta_1, \delta_2) = (0.21, 1.22)$ and (iii) for $(\delta_1, \delta_2) = (0.21, 0.22)$.

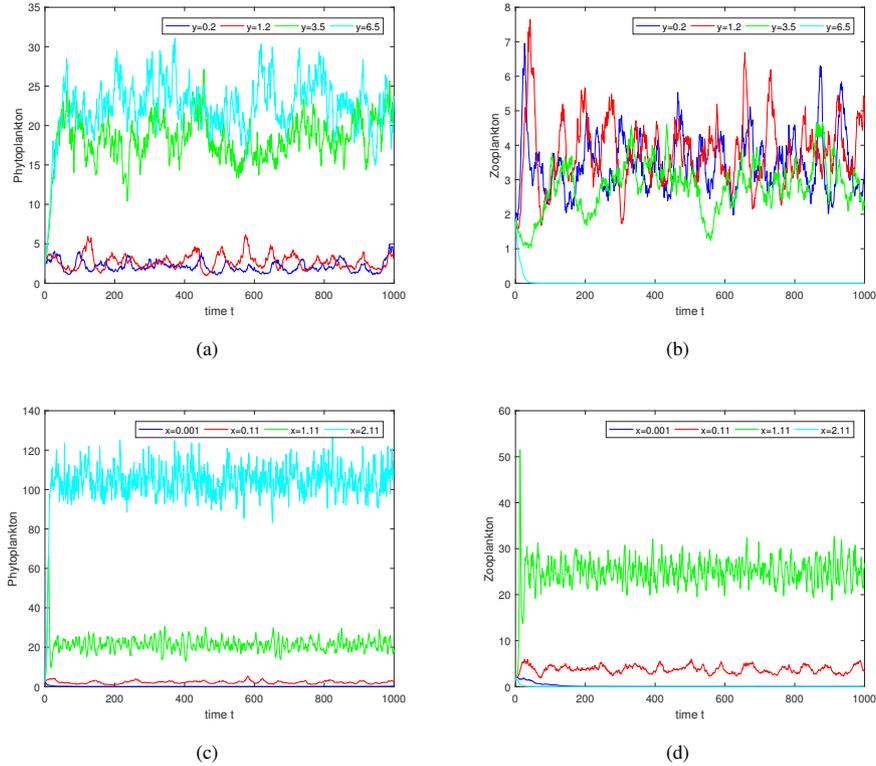


FIGURE 7 Effects of plankton body size on the survival of plankton in model (6) with $(\delta_1, \delta_2) = (0.25, 0.22)$. (a)-(b) for the effects of zooplankton body size and $x = 0.1$. (c)-(d) for the effects of phytoplankton cell size and $y = 0.2$.

By fixing $(x, y) = (0.04, 0.2)$, we obtain $(\delta_1, \delta_2) = (0.11, 0.11)$, $(0.15, 0.15)$ and $(0.18, 0.18)$, which satisfy the parameter condition $\bar{\lambda} > 0$ of Theorem 4.6. Based on Theorem 4.6, we repeat the simulation 10000 times and never observe any extinction scenario up to $t = 1000$, showing the stationary distribution of $P(t)$ and $Z(t)$ at time $t = 1000$ for the stochastic model (6), which is confirmed by the histograms in Fig.8. It can be found from Figs.8 (a), 8 (b) and 8 (c) or Figs.8 (d), 8 (e) and 8 (f)

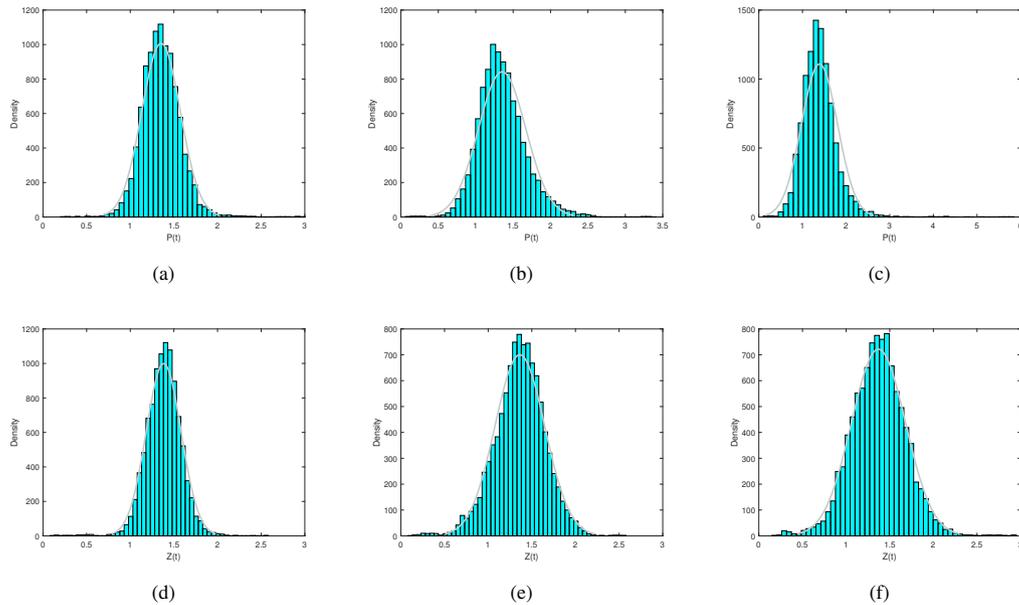


FIGURE 8 Effects of noise intensity on the stationary distribution of phytoplankton and zooplankton in model (6), where the grey curve represents the probability density function of plankton. (a) and (d) for $(\delta_1, \delta_2) = (0.11, 0.11)$. (b) and (e) for $(\delta_1, \delta_2) = (0.15, 0.15)$. (c) and (f) for $(\delta_1, \delta_2) = (0.18, 0.18)$.

that when the white noise intensity is small, the stationary distributions appear closer to normal distribution and they are positively skewed when the white noise intensity is large. Obviously, the stochastic model (6) has a unique stationary distribution. By comparison, we find that the increase of white noise can cause a positive shift in the stationary distribution of plankton in model (6).

By fixing zooplankton body size $y = 0.2$ and noise intensity $(\delta_1, \delta_2) = (0.11, 0.12)$, we only vary $x = 0.03, 0.13, 0.43$. The solutions of the deterministic model (4) and its corresponding stochastic model (6) are shown in Fig.9 . In Fig.9 (a), the solutions of the stochastic model (6) are strongly perturbed around the positive equilibrium of the deterministic model (4) when $x = 0.03$. In the case of $x = 0.13$, however, the irregularity of random variation for plankton is weakened and the amplitude of fluctuation for plankton is decreased, and the solutions of model (6) are closer to the positive equilibrium of the model (4), as shown in Fig.9 (c). In the case of $x = 0.43$, this phenomenon becomes more obvious, which is verified by Fig.9 (e). By comparison, it can be asserted that the increase of phytoplankton cell size can weaken the effects of random environmental disturbances. Figs.9 (b), 9 (d) and 9 (f) further clearly confirm this conclusion.

By fixing phytoplankton cell size $x = 0.04$ and noise intensity $(\delta_1, \delta_2) = (0.11, 0.12)$, we only change $y = 0.1, 0.2, 0.3$. However, results show that the increase of zooplankton body size can not cause the irregularity of random variation and the range of stochastic fluctuation for plankton in model (6) to increase or decrease, as shown in Figs.10 (a),10 (c) and 10 (e). Therefore, the increase of zooplankton body size can not change the effects of random environmental disturbance on the plankton in model (6), which is further demonstrated by Figs. 10 (b), 10 (d) and 10 (f).

5 | CONCLUDING REMARKS

In this paper, we investigate a new phytoplankton-zooplankton model (PZ model) in the deterministic and stochastic fluctuation environments, where the growth rate and the sinking rate of phytoplankton are assumed to depend on phytoplankton cell size, and the maximum consumption rate of zooplankton is assumed to depend on both phytoplankton cell size and zooplankton body size. The stochastic environmental fluctuation is assumed to be a type of white noise that mainly influences the growth rate of phytoplankton and the death rate of zooplankton. This is a well-established way of introducing stochastic environmental noise into population models⁴⁹. As far as we know, few people have studied phytoplankton-zooplankton dynamics, taking into account plankton body size and random environmental disturbance. In other words, the dynamic mechanisms underlying the growth of

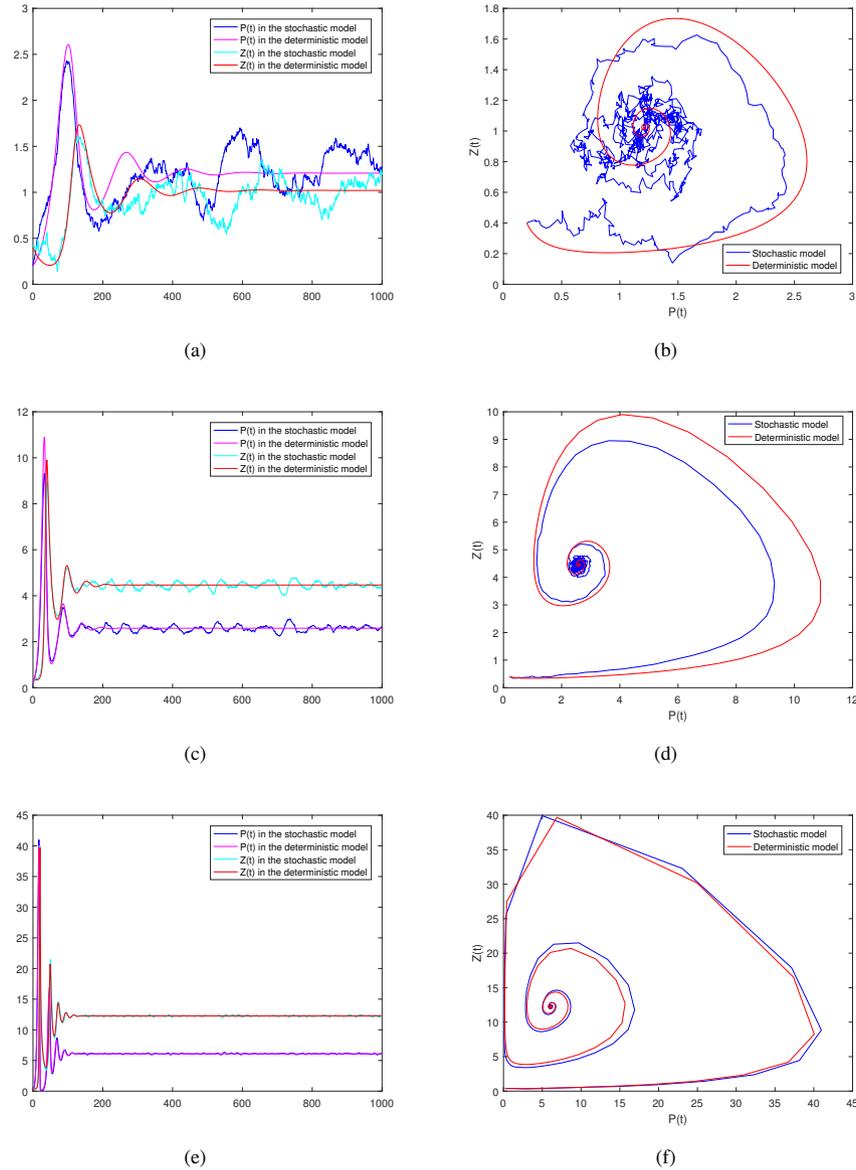


FIGURE 9 Effects of phytoplankton cell size on the phytoplankton-zooplankton dynamics with $\gamma = 0.2$. (a),(c) and (e) The solutions in stochastic model (6) and its corresponding deterministic model (4) with different phytoplankton cell size. (b),(d) and (f) The phase diagram of phytoplankton and zooplankton in stochastic model (6) and its corresponding deterministic model (4). (a)-(b) for $x = 0.03$. (c)-(d) for $x = 0.13$. (e)-(f) for $x = 0.43$.

phytoplankton related to plankton body size and stochastic environmental fluctuation remain largely unknown. Consequently, the main purpose of this paper is to investigate how plankton body size affects the phytoplankton-zooplankton dynamics in the stochastic fluctuation environments.

Mathematically, in the case of PZ model without environmental noise, we prove the existence of boundary and positive equilibria, and give the sufficient conditions guaranteeing the local and global stability of these equilibria. In the case of PZ model with environmental noise, we provide the stochastic dynamics including the existence and uniqueness, V-geometric ergodicity, stochastic ultimate boundedness and stochastic permanence, and provide the sufficient conditions for the stochastic extinction and persistence in the mean, as well as for the existence of a unique ergodic stationary distribution of phytoplankton and zooplankton.

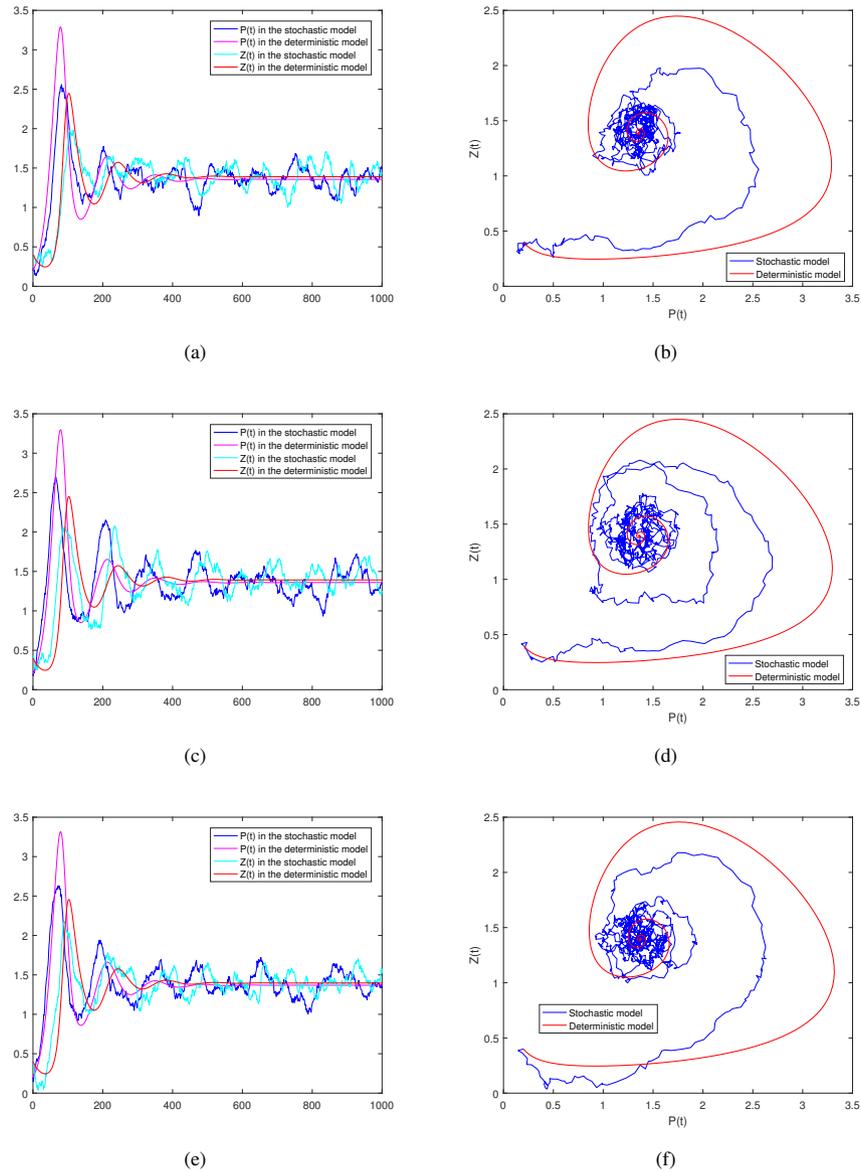


FIGURE 10 Effects of zooplankton body size on the phytoplankton-zooplankton dynamics with $x = 0.04$. (a),(c) and (e) The solutions in stochastic model (6) and its corresponding deterministic model (4) with different zooplankton body size. (b),(d) and (f) The phase diagram of phytoplankton and zooplankton in stochastic model (6) and its corresponding deterministic model (4). (a)-(b) for $y = 0.1$. (c)-(d) for $y = 0.2$. (e)-(f) for $y = 0.3$.

Ecologically, via numerical simulations, we find that when phytoplankton cell size is small, the increase of zooplankton body size has no impacts on the plankton density in the deterministic environment (see Figs.2 (a) and 2 (b)), while it can increase the phytoplankton density and decrease zooplankton density in the stochastic environment (Figs.7 (a) and 7 (b)), which is a new result that has not been obtained in^{65,66}. Furthermore, it is found that when phytoplankton cell size is intermediate-level, the increase of zooplankton body size can reduce the density of phytoplankton and zooplankton at the same time (see Figs.2 (c) and 2 (d)). This result is consistent with one of the conclusions in⁶⁶. Moreover, it is shown that when phytoplankton cell size is large, the increase of zooplankton body size can increase the density of zooplankton, but decrease the density of phytoplankton (see Figs.2 (e) and 2 (f)). Recently, Liao⁶⁷ studied a phytoplankton-zooplankton model with plankton body size and stochastic environmental fluctuations, and Zhao et al.⁴⁰ investigated a phytoplankton-zooplankton model with plankton body size and

time delay, however, they did not explore the effect of zooplankton body size on plankton growth under different cell size of phytoplankton. In contrast, the result can be used as a supplement to this studies^{67,40}.

However, whether in a deterministic environment or a random disturbance environment, the very small phytoplankton cell size can lead to the extinction of phytoplankton and zooplankton, while the very small zooplankton body size can maintain the sustainable survival of phytoplankton and zooplankton (see Figs.3 and 7). Additionally, the very large zooplankton body size or phytoplankton cell size can lead to the extinction of zooplankton, but can not affect the long-term survival of phytoplankton (see Figs.3 (a) and 3 (b), Figs.7 (a) and 7 (b)).

Actually, the large environmental noise is capable to cause the extinction of plankton (see Fig.6 (b)), the increase of phytoplankton cell size has the ability to decrease the irregularity of random variation and the amplitude of random fluctuation for plankton, but the increase of zooplankton body size has almost no impacts on the effects of random environmental disturbance (see Figs. 9 and 10). In other words, the increase of phytoplankton cell size can weaken the effects of random environmental disturbances, while the increase of zooplankton body size can not, which indicates that the cell size of phytoplankton is more sensitive to the changes of external environments than the body size of zooplankton, and the phytoplankton with large cell size may have stronger survival ability than the phytoplankton with smaller cell size in the stochastic fluctuation environments. This result supports the conclusion obtained in Branco et al.⁶⁸ that phytoplankton will evolve towards large cell size, but it is opposite to the results obtained in Jiang et al.⁴³ that phytoplankton cell size will evolve towards small cell size.

In spite of some interesting results have been achieved in this paper, there are some interesting topics deserve further investigation. For example, in this paper, our model is autonomous, it is of interest to investigate the non-autonomous case and study other important properties, such as seasonal variation, individual life cycle, etc. In addition, it is also significant to introduce the colored noise, such as continuous-time Markov chain, into the model (6). These will leave our future works.

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Author contributions

Tiancai Liao: Conceptualization, Formal analysis, Methodology, Resources, Validation, Visualization, Writing- original draft.

Financial disclosure

We declare that we have no financial and personal relationships with others or improper organization that will affect our work, no professional or other personal interests of any nature, or where any products, services or companies may be deemed to affect, or review, manuscripts.

Conflict of interest

There is no conflict of interest in this work.

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