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Global Existence of Large Solutions for the 3D incompressible Navier–Stokes–Poisson–Nernst–Planck Equations†

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This work is concerned with the global existence of large solutions to the three-dimensional dissipative fluid-dynamical model, which is a strongly coupled nonlinear nonlocal system characterized by the incompressible Navier–Stokes–Poisson–Nernst–Planck equations. Making full use of the algebraic structure of the system, we obtain the global existence of solutions without smallness assumptions imposed on the third component of the initial velocity field and the summation of initial densities of charged species. More precisely, we prove that there exist two positive constants c_0, C_0 such that if the initial data satisfies

$$\big(\|u_0^h\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}}+\|N_0-P_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}\big)\exp\Big\{C_0\big(\|u_0^3\|^2_{\dot{B}^{-1+\frac{3}{p}}_{p,1}}+(\|N_0+P_0\|_{\dot{B}^{-2+\frac{3}{r}}_{r,1}}+1)\exp\big\{C_0\|u_0^3\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}}\big\}+1\big)\Big\}\le c_0,$$

then the incompressible Navier–Stokes–Poisson–Nernst–Planck equations admits a unique global solution.

Keywords: Navier–Stokes equtions; Poisson–Nernst–Planck equations; global existence; large solution; Besov spaces

1. Introduction

In this paper, we study the Cauchy problem of three-dimensional (3D) incompressible Navier–Stokes–Poisson–Nernst–Planck equations:

$$\begin{cases}\partial_t u+u\cdot\nabla u-\Delta u+\nabla\pi=\Delta\phi\nabla\phi,\\ \nabla\cdot u=0,\\ \partial_t N+u\cdot\nabla N=\nabla\cdot(\nabla N-N\nabla\phi),\\ \partial_t P+u\cdot\nabla P=\nabla\cdot(\nabla P+P\nabla\phi),\\ \Delta\phi=N-P,\\ (u,N,P)|_{t=0}=(u_0,N_0,P_0),\end{cases}\tag{1.1}$$

where $(x,t)\in\mathbb{R}^3\times\mathbb{R}_+$, $u=(u^1(x,t),u^2(x,t),u^3(x,t))$ and $\pi=\pi(x,t)$ stand for the velocity field and the pressure of the incompressible fluid, respectively, $N=N(x,t)$ and $P=P(x,t)$ stand for the densities of a negatively and positively charged species, respectively, and $\phi=\phi(x,t)$ is the electrostatic potential caused by the charged species. For the sake of simplicity of presentation, we have assumed that the fluid density, viscosity, charge mobility and dielectric constant are unity.

The first two equations of (1.1) are the momentum conservation and the mass conservation equations of the incompressible flow, and the right-hand side term in the momentum equations is the Lorentz force, which exhibits $\Delta\phi\nabla\phi=\nabla\cdot\sigma$, and the electric stress σ is a rank one tensor plus a pressure, for $i,j=1,2,3$,

$$[\sigma]_{ij}=\big(\nabla\phi\otimes\nabla\phi-\frac{1}{2}|\nabla\phi|^2I\big)_{ij}=\partial_{x_i}\phi\partial_{x_j}\phi-\frac{1}{2}|\nabla\phi|^2\delta_{ij},\tag{1.2}$$

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where I is 3×3 identity matrix, δ_{ij} is the Kronecker symbol, and $\otimes$ denotes the tensor product. The electric stress σ stems from the balance of kinetic energy with electrostatic energy via the least action principle (cf. [23]). The third and fourth equations of (1.1) model the balance between diffusion and convective transport of the charged species by the flow and the electric fields, and the fifth equation of (1.1) is the Poisson equation for the electrostatic potential ϕ, where the right-hand side is the net charge density.

The system (1.1) was first introduced by Rubinstein [21], which is capable of describing electro-chemical and fluid-mechanical transport throughout the cellular environment. At the present time, modeling of electro-diffusion in electrolytes is a problem of major scientific interest, it finds that such model has a wide applications in biology (ion channels), chemistry (electro-osmosis) and pharmacology (transdermal iontophoresis), we refer the readers to see [2, 8, 9, 25] for more detailed applications of the system (1.1) in electro-hydrodynamics, and [15, 16, 17] for the computational simulations.

The mathematical analysis of the system (1.1) was initiated by Jerome [12], where the author established a local existence–uniqueness theory of the system (1.1) in the Kato's analytical semigroup framework. Since the right-hand side term $\Delta\phi\nabla\phi$ has a nice algebraic structure (1.2), it can be regarded that $\nabla\phi$ plays the same role as the velocity field u. Based on this observation, by using the Hardy–Littlewood–Sobolev inequality and the Sobolev embeddings $W^{1,\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ and $\dot{B}^{-1+\frac{3}{q}}_{q,1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ with $1 \le q \le p \le \infty$, Zhao–Deng–Cui [28, 29] established local well-posedness and global well-posedness with small initial data of the system (1.1) in critical Lebesgue spaces and Besov spaces under the heat semigroup framework. For more analytical results concerning about the global existence of (large) weak solutions and (small) mild solutions, convergence rate estimates to stationary solutions of time-dependent solutions and other related topics we refer the readers to see [5, 10, 13, 14, 22, 24, 26, 27, 30] and references therein.

In order to give a better explanation of our main results, we let $v := N - P$, $w := N + P$, and the system (1.1) turns into

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u + \nabla\pi = -v\nabla(-\Delta)^{-1}v, \\ \nabla\cdot u = 0, \\ \partial_t v + u\cdot\nabla v = \nabla\cdot(\nabla v + w\nabla(-\Delta)^{-1}v), \\ \partial_t w + u\cdot\nabla w = \nabla\cdot(\nabla w + v\nabla(-\Delta)^{-1}v), \\ (u,v,w)|_{t=0} = (u_0, v_0, w_0), \end{cases} \tag{1.3}$$

where $v_0 = N_0 - P_0$ and $w_0 = N_0 + P_0$. Moreover, let $u := (u^h, u^3)$, where $u^h := (u^1, u^2)$ and u^3 denote the horizontal components and vertical component of the velocity field u, respectively. Using the divergence free condition $\nabla\cdot u = 0$, it is easy to verify that the vertical component u^3 satisfies the following equation:

$$\partial_t u^3 - \Delta u^3 = -\operatorname{div}_h(u^h u^3) + 2u^3 \operatorname{div}_h u^h - \partial_3\pi - v\partial_3(-\Delta)^{-1}v, \tag{1.4}$$

which reveals that the equation on the vertical component u^3 is actually a linear equation with coefficients depending on the horizontal components u^h and the density function v. Based on this observation, the authors in [32] proved that under the conditions $1 < q \le p < 6$ and $\frac{1}{p} + \frac{1}{q} \ge \frac{2}{3}$, there exist two positive constants c_0 and C_0 such that if the initial data (u_0, v_0, w_0) satisfies

$$\big(\|u_0^h\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + \|v_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}} + \|w_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}\big) \exp\Big\{C_0(\|u_0^3\|^2_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + 1)\Big\} \le c_0, \tag{1.5}$$

then system (1.3) admits a unique global solution. On the other hand, observing that the fourth equation of (1.3) is a linear equation for w with coefficients depending on the velocity field u and the density function v, which may suggest us that we do not need to impose any smallness condition on initial data w_0 to ensure global existence of solutions. Indeed, Ma [18] showed that under the conditions $1 \le p < \infty$, $1 \le q < 6$, $q \le 2p$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{3} < \frac{1}{p} + \frac{1}{q}$, there exist two positive constants c_0 and C_0 such that if the initial data (u_0, v_0, w_0) satisfies

$$\big(\|u_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + \|v_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}\big) \exp\Big\{C_0\|w_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}\Big\} \le c_0, \tag{1.6}$$

then system (1.3) still has a unique global solution.

Motivated by the above global existence results in [18, 32], in this paper, we aim at relaxing the smallness conditions imposed on the vertical component of initial velocity field and the summation of the initial densities of charged species such that system (1.3) still has a unique global solution. Considering the algebraic structures of the nonlinear terms in (1.3), by [11], the nonlinear term $v\nabla(-\Delta)^{-1}v$ has a nice symmetric structure as

$$\begin{aligned} \partial_{x_i} v\partial_{x_i}(-\Delta)^{-1}v =& \frac{1}{2}\partial_{x_i}(-\Delta)\{((-\Delta)^{-1}v)(\partial_{x_i}(-\Delta)^{-1}v)\} \\ &+ \partial_{x_i}\nabla\cdot\{((-\Delta)^{-1}v)(\nabla\partial_{x_i}(-\Delta)^{-1}v)\} \\ &+ \frac{1}{2}\partial^2_{x_i}\{((-\Delta)^{-1}v)v\}. \end{aligned} \tag{1.7}$$

This enables us to treat the equation of w in a weaker Besov space $\dot{B}^{-2+\frac{3}{r}}_{r,1}$ with $1 \le r < \infty$. However, the nonlinear coupled term $w\nabla(-\Delta)^{-1}v$ has lack of such a symmetric structure, which can not exhibit such a good expression as (1.7) and prevents us to obtain good estimates for the equation v in a weaker Besov spaces. These observations essentially indicate that the difference of charged densities v plays more important role than the summation of charged densities w in mathematical analysis of the system (1.3). Based on these careful observations, by using analytical methods in [18, 32], we intend to prove the global existence of solutions under the assumptions that the horizontal components of the velocity field and the difference of initial densities are small while the vertical component of the velocity field and the total initial densities could be chosen suitable large. Moreover, we consider the functional space of solutions of the system (1.3) with initial data v_0 and w_0 belonging to the different low regularity Besov spaces with different regularity and integral indices, which can indicate more specific coupling relations between the difference and the summation of negatively and positively charged densities.

Now we state our main results as follows.

Theorem 1.1 *Let p, q, r be three positive numbers such that $1 \le p, q, r < \infty$, and satisfy the following conditions:*

$$\frac{1}{q} - \frac{1}{p} \ge -\min\{\frac{1}{3}, \frac{1}{2p}\}, \quad \max\{\frac{1}{q} - \frac{1}{r}, \frac{1}{r} - \frac{1}{q}\} < \frac{1}{3} < \min\{\frac{1}{p} + \frac{1}{q}, \frac{1}{p} + \frac{1}{r}, \frac{1}{q} + \frac{1}{r}\}.$$

Then for any $u_0 \in \dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $v_0 \in \dot{B}^{-2+\frac{3}{q}}_{q,1}(\mathbb{R}^3)$ and $w_0 \in \dot{B}^{-2+\frac{3}{r}}_{r,1}(\mathbb{R}^3)$, there exists $T > 0$ such that the system (1.3) admits a unique solution (u, v, w) on $[0, T]$ satisfying

$$\begin{cases} u \in C([0,T], \dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)) \cap \mathcal{L}^\infty(0,T; \dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)) \cap L^1(0,T; \dot{B}^{1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)), \\ v \in C([0,T], \dot{B}^{-2+\frac{3}{q}}_{q,1}(\mathbb{R}^3)) \cap \mathcal{L}^\infty(0,T; \dot{B}^{-2+\frac{3}{q}}_{q,1}(\mathbb{R}^3)) \cap L^1(0,T; \dot{B}^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)), \\ w \in C([0,T], \dot{B}^{-2+\frac{3}{r}}_{r,1}(\mathbb{R}^3)) \cap \mathcal{L}^\infty(0,T; \dot{B}^{-2+\frac{3}{r}}_{r,1}(\mathbb{R}^3)) \cap L^1(0,T; \dot{B}^{\frac{3}{r}}_{r,1}(\mathbb{R}^3)). \end{cases} \tag{1.8}$$

Besides, there exists a positive constant ε such that if the initial data satisfies

$$\|(u_0, v_0, w_0)\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1} \times \dot{B}^{-2+\frac{3}{q}}_{q,1} \times \dot{B}^{-2+\frac{3}{r}}_{r,1}} \le \varepsilon,$$

then the above assertion holds for $T = \infty$, i.e., the solution (u, v, w) is global.

Theorem 1.2 *Let p, q, r be three positive numbers such that $1 < p < 6$, $1 < q, r < \infty$, and satisfy the following conditions:*

$$\frac{1}{q} - \frac{1}{p} \ge -\min\{\frac{1}{3}, \frac{1}{2p}\}, \quad \max\{\frac{1}{q} - \frac{1}{r}, \frac{1}{r} - \frac{1}{q}\} < \frac{1}{3} < \min\{\frac{1}{p} + \frac{1}{q}, \frac{1}{p} + \frac{1}{r}, \frac{1}{q} + \frac{1}{r}\}.$$

Then for any $u_0 = (u_0^h, u_0^3) \in \dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $v_0 \in \dot{B}^{-2+\frac{3}{q}}_{q,1}(\mathbb{R}^3)$ and $w_0 \in \dot{B}^{-2+\frac{3}{r}}_{r,1}(\mathbb{R}^3)$, there exist two positive constants c_0 and C_0 such that if the initial data (u_0, v_0, w_0) satisfies

$$\big(\|u_0^h\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + \|v_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}\big) \exp\Big\{C_0\big(\|u_0^3\|^2_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + (\|w_0\|_{\dot{B}^{-2+\frac{3}{r}}_{r,1}} + 1)\exp\big\{C_0\|u_0^3\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}}\big\} + 1\big)\Big\} \le c_0, \tag{1.9}$$

then the system (1.3) admits a unique global solution (u, v, w) satisfying (1.8)

Remark 1.1 The initial condition (1.9) exhibits that the initial data u_0^3 and w_0 can be taken large as long as we take the initial data u_0^h and v_0 small enough compared with the size of u_0^3 and w_0, which we can still get the global existence of solutions to the system (1.3). Back to the original system (1.1), the condition (1.9) is equivalent to the following condition:

$$\big(\|u_0^h\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + \|N_0 - P_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}\big) \exp\Big\{C_0\big(\|u_0^3\|^2_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + (\|N_0 + P_0\|_{\dot{B}^{-2+\frac{3}{r}}_{r,1}} + 1)\exp\big\{C_0\|u_0^3\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}}\big\} + 1\big)\Big\} \le c_0, \tag{1.10}$$

thus Theorem 1.2 implies global existence of solutions for the system (1.1) with only requiring the horizontal components of the initial velocity field and the difference of initial negatively and positively charged densities are small enough.

Remark 1.2 The specific coupled relation between v and w was indicated by the condition $\max\{\frac{1}{q} - \frac{1}{r}, \frac{1}{r} - \frac{1}{q}\} < \frac{1}{3} < \frac{1}{q} + \frac{1}{r}$, which tells us that the regularity of solution v or w can be taken beyond the regularity index $-\frac{3}{2}$, but one can not take both of them less than $-\frac{3}{2}$ at the same time. Indeed, the regularity of v can be taken much weaker as long as the regularity of w is not that much weaker, i.e., q can be taken large enough as long as we take r closing to 3 such that the condition $\max\{\frac{1}{q} - \frac{1}{r}, \frac{1}{r} - \frac{1}{q}\} < \frac{1}{3} < \frac{1}{q} + \frac{1}{r}$ holds. Hence, Theorem 1.2 can be regarded as an extension of global existence results in [28, 32, 18], where the global existence of solutions with small initial data was proved in critical Besov spaces with the same regularity and integral indices for v and w, and the regularity index must less than $-\frac{3}{2}$.

This paper is organized as follows. In section 2, we first introduce definitions of the homogeneous Besov spaces and the Chemin–Lerner mixed time-space spaces based on the Littlewood–Paley dyadic decomposition theory, then review some known bilinear estimates which used frequently in the proofs of Theorems 1.1 and 1.2. In Section 3, we first establish two crucial nonlinear estimates of the pressure π, then derive the desired estimates of u^h and u^3 by using the weighted Chemin–Lerner type norm; while in Section 4, we derive the desired estimates of v and w. Finally in the last section, we complete the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

Throughout this paper, C and C_i $(i = 1, 2, \cdots)$ stand for the generic harmless constants. For brevity, we shall use the notation $f \lesssim g$ instead of $f \leq Cg$, and $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$. For any Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $f \in \mathcal{X}$ and $g \in \mathcal{Y}$, we write $\|(f, g)\|_{\mathcal{X}\times\mathcal{Y}} := \|f\|_{\mathcal{X}} + \|g\|_{\mathcal{Y}}$. For all $T > 0$ and $\rho \in [1, \infty]$, we denote by $C([0, T], \mathcal{X})$ the set of continuous functions on $[0, T]$ with values in $\mathcal{X}$, and denote by $L^\rho(0, T; \mathcal{X})$ the set of measurable functions on $[0, T]$ with values in $\mathcal{X}$ such that $t \to \|f(t)\|_{\mathcal{X}}$ belongs to $L^\rho(0, T)$.

2.1. Littlewood–Paley dyadic decomposition and Besov spaces

Let us briefly recall the Littlewood–Paley dyadic decomposition theory and the stationary/time dependent Besov spaces for convenience. More details may be found in Chap. 2 and Chap. 3 in the book [1]. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions, and $\mathcal{S}'(\mathbb{R}^3)$ the space of tempered distributions. Choose a smooth radial non-increasing funciton χ with $\operatorname{Supp}\chi \subset B(0, \frac{4}{3})$ and $\chi \equiv 1$ on $B(0, \frac{3}{4})$. Set $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$. It is not difficult to check that φ is supported in the shell $\{\xi \in \mathbb{R}^3,\ \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, and

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R}^3\backslash\{0\}.$$

Let $h = \mathcal{F}^{-1}\varphi$. Then for any $f \in \mathcal{S}'(\mathbb{R}^3)$, the homogeneous dyadic blocks Δ_j $(j \in \mathbb{Z})$ are defined by

$$\Delta_j f(x) := \varphi(2^{-j}D)f(x) = 2^{3j}\int_{\mathbb{R}^3} h(2^j y)f(x-y)dy.$$

Let $\mathcal{S}'_h(\mathbb{R}^3)$ be the space of tempered distribution $f \in \mathcal{S}'(\mathbb{R}^3)$ such that

$$\lim_{j\to-\infty} S_j f(x) = 0,$$

where $S_j f$ $(j \in \mathbb{Z})$ stands for the low frequency cut-off defined by $S_j f := \chi(2^{-j}D)f$. Then one has the unit decomposition for any tempered distribution $f \in \mathcal{S}'_h(\mathbb{R}^3)$:

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f. \tag{2.1}$$

The above homogeneous dyadic block Δ_j and the partial summation operator S_j satisfy the following quasi-orthogonal properties: for any $f, g \in \mathcal{S}'(\mathbb{R}^3)$, one has

$$\Delta_i\Delta_j f \equiv 0 \quad \text{if} \quad |i-j| \geq 2 \quad \text{and} \quad \Delta_i(S_{j-1}f\Delta_j g) \equiv 0 \quad \text{if} \quad |i-j| \geq 5. \tag{2.2}$$

Moreover, using Bony's homogeneous paraproduct decomposition (cf. [3]), one can formally split the product of two temperate distributions f and g as follows:

$$fg = T_f g + T_g f + R(f, g), \tag{2.3}$$

where the paraproduct between f and g is defined by

$$T_f g := \sum_{j\in\mathbb{Z}} S_{j-1}f\Delta_j g = \sum_{j\in\mathbb{Z}}\sum_{k\leq j-2} \Delta_k f\Delta_j g,$$

and the remaining term is defined by

$$R(f, g) := \sum_{j\in\mathbb{Z}} \Delta_j f\widetilde{\Delta}_j g \quad \text{and} \quad \widetilde{\Delta}_j := \Delta_{j-1} + \Delta_j + \Delta_{j+1}.$$

Based on those dyadic blocks, the homogeneous Besov spaces can be defined as follows:

Definition 2.1 *For any $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$ and $f \in \mathcal{S}'(\mathbb{R}^3)$, we set*

$$\|f\|_{\dot{B}^s_{p,r}} := \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{srj}\|\Delta_j f\|^r_{L^p}\right)^{\frac{1}{r}} & \text{for } 1 \leq r < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\Delta_j f\|_{L^p} & \text{for } r = \infty, \end{cases}$$

and the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined by

- *For $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define*

$$\dot{B}^s_{p,r}(\mathbb{R}^3) := \left\{f \in \mathcal{S}'_h(\mathbb{R}^3) : \ \|f\|_{\dot{B}^s_{p,r}} < \infty\right\}.$$

- *If $k \in \mathbb{N}$ and $\frac{3}{p} + k \le s < \frac{3}{p} + k + 1$ (or $s = \frac{3}{p} + k + 1$ if $r = 1$), then $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined as the subset of distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\partial^\beta f \in \dot{B}^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta| = k$.*

Remark 2.1 Let $s \in \mathbb{R}$, $1 \le p, r \le \infty$, and $f \in \mathcal{S}'_h(\mathbb{R}^3)$. Then $u \in \dot{B}^s_{p,r}(\mathbb{R}^3)$ if and only if there exists $\{d_{j,r}\}_{j\in\mathbb{Z}}$ such that $d_{j,r} \ge 0$, $\|d_{j,r}\|_{\ell^r} = 1$ and

$$\|\Delta_j u\|_{L^p} \lesssim d_{j,r} 2^{-js} \|u\|_{\dot{B}^s_{p,r}} \quad \text{for all} \quad j \in \mathbb{Z}.$$

The fundamental idea in the proofs of Theorems 1.1 and 1.2 is to localize system (1.3) through the Littlewood–Paley dyadic decomposition, so we need the following definition of the Chemin–Lerner mixed time-space spaces, which was first introduced by Chemin–Lerner [4].

Definition 2.2 *For $0 < T \le \infty$, $s \in \mathbb{R}$ and $1 \le p, r, \rho \le \infty$. We define the mixed time-space $\mathcal{L}^\rho(0, T; \dot{B}^s_{p,r}(\mathbb{R}^3))$ as the completion of $\mathcal{C}([0, T]; \mathcal{S}(\mathbb{R}^3))$ by the norm*

$$\|f\|_{\mathcal{L}^\rho_T(\dot{B}^s_{p,r})} := \left(\sum_{j\in\mathbb{Z}} 2^{srj} \left(\int_0^T \|\Delta_j f(\cdot, t)\|^\rho_{L^p} dt \right)^{\frac{r}{\rho}} \right)^{\frac{1}{r}} < \infty$$

and with the standard modification for $\rho = \infty$ or $r = \infty$.

Remark 2.2 According to the Minkowski inequality, it holds that

$$\|f\|_{\mathcal{L}^\rho_T(\dot{B}^s_{p,r})} \le \|f\|_{L^\rho_T(\dot{B}^s_{p,r})} \quad \text{if} \quad \rho \le r; \quad \|f\|_{L^\rho_T(\dot{B}^s_{p,r})} \le \|f\|_{\mathcal{L}^\rho_T(\dot{B}^s_{p,r})} \quad \text{if} \quad r \le \rho.$$

In particular, for $\rho = r = 1$, one has

$$\|f\|_{\mathcal{L}^1_T(\dot{B}^s_{p,1})} \approx \|f\|_{L^1_T(\dot{B}^s_{p,1})}.$$

In order to prove Theorem 1.2, we need to introduce the following important weighted Chemin–Lerner type norm (see [19, 20]).

Definition 2.3 *Let $f(t) \in L^1_{loc}(\mathbb{R}_+)$, $f(t) \ge 0$. For $1 \le p, r, \rho \le \infty$, we define*

$$\|u\|_{\mathcal{L}^\rho_{T,f}(\dot{B}^s_{p,r})} := \left(\sum_{j\in\mathbb{Z}} 2^{srj} \left(\int_0^T f(t) \|\Delta_j u(\cdot, t)\|^\rho_{L^p} dt \right)^{\frac{r}{\rho}} \right)^{\frac{1}{r}} < \infty$$

and with the standard modification for $\rho = \infty$ or $r = \infty$.

2.2. Analytical tools in Besov spaces

Let us recall the classical Bernstein inequality (see Lemma 2.1 in [1]).

Lemma 2.4 *Let $\mathcal{B}$ be a ball, and $\mathcal{C}$ a ring in $\mathbb{R}^3$. There exists a constant C such that for any positive real number λ, any nonnegative integer k and any couple of real numbers (a, b) with $1 \le a \le b \le \infty$, we have*

$$\operatorname{supp} \mathcal{F}(f) \subset \lambda\mathcal{B} \quad \Rightarrow \quad \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \le C^{k+1} \lambda^{k+3(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a}, \tag{2.4}$$

$$\operatorname{supp} \mathcal{F}(f) \subset \lambda\mathcal{C} \quad \Rightarrow \quad C^{-1-k}\lambda^k \|f\|_{L^a} \le \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \le C^{1+k}\lambda^k \|f\|_{L^a}. \tag{2.5}$$

More generally, for any smooth homogeneous function σ of degree m on $\mathbb{R}^3/\{0\}$ and $1 \le a \le \infty$, it holds that

$$\operatorname{supp} \mathcal{F}(f) \subset \lambda\mathcal{C} \quad \Rightarrow \quad \|\sigma(D) f\|_{L^a} \lesssim \lambda^k \|f\|_{L^a}. \tag{2.6}$$

An obvious consequence of (2.5) and (2.6) is that $\|\partial^\alpha f\|_{\dot{B}^s_{p,r}} \approx \|f\|_{\dot{B}^{s+k}_{p,r}}$ with multi-index $|\alpha| = k$ and $k \in \mathbb{N}$. Moreover, the following lower bound for the integral involving the Laplacian $-\Delta$, which can be regarded as a nonlinear generalization of (2.5), will also be used, for details, see Lemma 8 in [7].

Lemma 2.5 *Suppose that $\operatorname{supp} \mathcal{F}(f) \subset \{\xi \in \mathbb{R}^3 : \quad K_1 2^j \le |\xi| \le K_2 2^j\}$ for some $K_1, K_2 > 0$ and $j \in \mathbb{Z}$. Then there exists a constant κ so that for all $1 < p < \infty$, we have*

$$-\int_{\mathbb{R}^3} \Delta f |f|^{p-2} f dx = (p-1) \int_{\mathbb{R}^3} |\nabla f|^2 |f|^{p-2} dx \ge \kappa 2^{2j} \|f\|^p_{L^p}, \tag{2.7}$$

where κ is a constant depending only on p, K_1 and K_2.

The following basic properties of Besov spaces are often used (see [1]).

Lemma 2.6 *The following properties hold:*

i) *Completeness:* $\dot{B}^s_{p,r}(\mathbb{R}^3)$ *is a Banach space whenever* $|s|<\frac{3}{p}$ *or* $s=\frac{3}{p}$ *and* $r=1$.
ii) *Derivatives: There exists a universal constant* C *such that*

$$C^{-1}\|u\|_{\dot{B}^s_{p,r}}\leq\|\nabla u\|_{\dot{B}^{s-1}_{p,r}}\leq C\|u\|_{\dot{B}^s_{p,r}}.$$

iii) *Fractional derivative: Let* $\Lambda=\sqrt{-\Delta}$ *and* $\sigma\in\mathbb{R}$. *Then the operator* Λ^σ *is an isomorphism from* $\dot{B}^s_{p,r}(\mathbb{R}^3)$ *to* $\dot{B}^{s-\sigma}_{p,r}(\mathbb{R}^3)$.
iv) *Imbedding: For* $1\leq p_1\leq p_2\leq\infty$ *and* $1\leq r_1\leq r_2\leq\infty$, *we have the continuous imbedding* $\dot{B}^s_{p_1,r_1}(\mathbb{R}^3)\hookrightarrow\dot{B}^{s-3(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}^3)$.
v) *Interpolation: For* $s_1,s_2\in\mathbb{R}$ *such that* $s_1<s_2$ *and* $\theta\in(0,1)$, *there exists a constant* C *such that*

$$\|u\|_{\dot{B}^{s_1\theta+s_2(1-\theta)}_{p,r}}\leq C\|u\|^\theta_{\dot{B}^{s_1}_{p,r}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}.$$

The following crucial estimates for the product of two functions in the homogeneous Besov spaces are also used frequently throughout this paper (see Lemma 5.3 in [30]).

Lemma 2.7 *Let* $1\leq p_1,p_2\leq\infty$, $s_1\leq\frac{3}{p_1}$, $s_2\leq\min\{\frac{3}{p_1},\frac{3}{p_2}\}$, *and* $s_1+s_2>3\max\{0,\frac{1}{p_1}+\frac{1}{p_2}-1\}$. *Assume that* $f\in\dot{B}^{s_1}_{p_1,1}(\mathbb{R}^3)$, $g\in\dot{B}^{s_2}_{p_2,1}(\mathbb{R}^3)$. *Then we have* $fg\in\dot{B}^{s_1+s_2-\frac{3}{p_1}}_{p_2,1}(\mathbb{R}^3)$, *and the following inequality holds:*

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{3}{p_1}}_{p_2,1}}\lesssim\|f\|_{\dot{B}^{s_1}_{p_1,1}}\|g\|_{\dot{B}^{s_2}_{p_2,1}}. \tag{2.8}$$

2.3. Bilinear estimates

In this subsection, we recall the following bilinear estimates which are crucial steps to the proof of Theorem 1.1, for the detailed proofs of these bilinear estimates, we refer the readers to see [30, 31, 32]. Here and in the sequel we denote $(d_j)_{j\in\mathbb{Z}}$ a generic element of $l^1(\mathbb{Z})$ such that $d_j\geq0$ and $\sum_{j\in\mathbb{Z}}d_j=1$.

Lemma 2.8 *Let* $1\leq p<\infty$. *Then we have*

$$\|\Delta_j(u\cdot\nabla u)\|_{L^1_t(L^p)}\lesssim 2^j\|\Delta_j(u\otimes u)\|_{L^1_t(L^p)}\lesssim d_j2^{(1-\frac{3}{p})j}\|u\|_{\mathcal{L}^\infty_T(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\|u\|_{L^1_T(\dot{B}^{1+\frac{3}{p}}_{p,1})},$$

which implies that

$$\|u\cdot\nabla u\|_{L^1_T(\dot{B}^{-1+\frac{3}{p}}_{p,1})}=\|\nabla\cdot(u\otimes u)\|_{L^1_T(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\lesssim\|u\|_{\mathcal{L}^\infty_T(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\|u\|_{L^1_T(\dot{B}^{1+\frac{3}{p}}_{p,1})}. \tag{2.9}$$

Lemma 2.9 *Let* $1\leq p,q<\infty$ *and* $\frac{1}{q}-\frac{1}{p}\geq-\min\{\frac{1}{3},\frac{1}{2p}\}$. *Then we have*

$$\|\Delta_j(v\nabla(-\Delta)^{-1}v)\|_{L^1_t(L^p)}\lesssim d_j2^{(1-\frac{3}{p})j}\|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})},$$

which implies that

$$\|(v\nabla(-\Delta)^{-1}v)\|_{L^1_t(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\lesssim\|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}. \tag{2.10}$$

Lemma 2.10 *Let* $1\leq p,q<\infty$ *and* $\frac{1}{p}+\frac{1}{q}>\frac{1}{3}$. *Then we have*

$$\|\Delta_j(u\cdot\nabla v)\|_{L^1_t(L^q)}\lesssim d_j2^{(2-\frac{3}{q})j}\big(\|u\|_{\mathcal{L}^\infty_T(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}+\|u\|_{L^1_T(\dot{B}^{1+\frac{3}{p}}_{p,1})}\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}\big),$$

which implies that

$$\|u\cdot\nabla v\|_{L^1_T(\dot{B}^{-2+\frac{3}{q}}_{q,1})}\lesssim\|u\|_{\mathcal{L}^\infty_T(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}+\|u\|_{L^1_T(\dot{B}^{1+\frac{3}{p}}_{p,1})}\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}. \tag{2.11}$$

Lemma 2.11 *Let* $1\leq q,r<\infty$ *and* $\frac{1}{q}-\frac{1}{r}<\frac{1}{3}<\frac{1}{q}+\frac{1}{r}$. *Then we have*

$$\|\Delta_j(w\nabla(-\Delta)^{-1}v)\|_{L^1_t(L^q)}\lesssim d_j2^{(1-\frac{3}{q})j}\big(\|w\|_{\mathcal{L}^\infty_T(\dot{B}^{-2+\frac{3}{r}}_{r,1})}\|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}+\|w\|_{L^1_T(\dot{B}^{\frac{3}{r}}_{r,1})}\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}\big),$$

which implies that

$$\|w\nabla(-\Delta)^{-1}v\|_{L^1_T(\dot{B}^{-1+\frac{3}{q}}_{q,1})}\lesssim\|w\|_{\mathcal{L}^\infty_T(\dot{B}^{-2+\frac{3}{r}}_{r,1})}\|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}+\|w\|_{L^1_T(\dot{B}^{\frac{3}{r}}_{r,1})}\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}. \tag{2.12}$$

Lemma 2.12 *Let $1 \le p, r < \infty$ and $\frac{1}{p} + \frac{1}{r} > \frac{1}{3}$. Then we have*

$$\|\Delta_j(u \cdot \nabla w)\|_{L^1_t(L^r)} \lesssim d_j 2^{(2-\frac{3}{r})j}\big(\|u\|_{\mathcal{L}^\infty_T(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\|w\|_{L^1_t(\dot{B}^{\frac{3}{r}}_{r,1})} + \|u\|_{L^1_T(\dot{B}^{1+\frac{3}{p}}_{p,1})}\|w\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{r}}_{r,1})}\big),$$

which implies that

$$\|u \cdot \nabla v\|_{L^1_T(\dot{B}^{-2+\frac{3}{q}}_{q,1})} \lesssim \|u\|_{\mathcal{L}^\infty_T(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\|w\|_{L^1_t(\dot{B}^{\frac{3}{r}}_{r,1})} + \|u\|_{L^1_T(\dot{B}^{1+\frac{3}{p}}_{p,1})}\|w\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{r}}_{r,1})}. \tag{2.13}$$

Lemma 2.13 *Let $1 \le q, r < \infty$ and $\frac{1}{r} - \frac{1}{q} < \frac{1}{3}$. Then we have*

$$\|\Delta_j(v\nabla(-\Delta)^{-1}v)\|_{L^1_t(L^r)} \lesssim d_j 2^{(1-\frac{3}{r})j}\|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})},$$

which implies that

$$\|(v\nabla(-\Delta)^{-1}v)\|_{L^1_t(\dot{B}^{-1+\frac{3}{r}}_{r,1})} \lesssim \|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}. \tag{2.14}$$

3. Estimates of the velocity field u

The purpose of this section is to derive the desired estimates for the horizontal components $u^h = (u^1, u^2)$ and the vertical component u^3 of the velocity field in the framework of weighted Chemin–Lerner type spaces. The main idea is that we introduce some weighted functions and weighted norms to eliminate the difficulties caused by the nonlinear terms involving u^3 and w. Thus we set

$$f_1(t) := \|u^3(\cdot,t)\|_{\dot{B}^{1+\frac{3}{p}}_{p,1}}, \quad f_2(t) := \|u^3(\cdot,t)\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}}, \quad f_3(t) := \|w(\cdot,t)\|_{\dot{B}^{\frac{3}{r}}_{r,1}}. \tag{3.1}$$

For three positive real numbers λ_1, λ_2 and λ_3, we denote $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, and introduce the following three weighted functions:

$$\begin{aligned}
u^h_{\vec{\lambda}} &:= u \exp\Big\{-\lambda_1\int_0^t f_1(\tau)d\tau - \lambda_2\int_0^t f_2(\tau)d\tau - \lambda_3\int_0^t f_3(\tau)d\tau\Big\},\\
\pi_{\vec{\lambda}} &:= \pi \exp\Big\{-\lambda_1\int_0^t f_1(\tau)d\tau - \lambda_2\int_0^t f_2(\tau)d\tau - \lambda_3\int_0^t f_3(\tau)d\tau\Big\},\\
v_{\vec{\lambda}} &:= v \exp\Big\{-\lambda_1\int_0^t f_1(\tau)d\tau - \lambda_2\int_0^t f_2(\tau)d\tau - \lambda_3\int_0^t f_3(\tau)d\tau\Big\}.
\end{aligned}$$

3.1. Estimates of the pressure π

Notice that, using the divergence free condition $\nabla \cdot u = 0$, the term $\nabla \cdot (u \cdot \nabla u)$ has a nice structure:

$$\nabla \cdot (u \cdot \nabla u) = \mathrm{div}_h\,\mathrm{div}_h(u^h \otimes u^h) + 2\partial_3\,\mathrm{div}_h(u^3 u^h) + \partial_3^2(u^3)^2,$$

where for a vector field $u = (u^1, u^2, u^3) = (u^h, u^3)$, we denote $\mathrm{div}_h\, u^h = \partial_1 u^1 + \partial_2 u^2$. Thus, by taking the divergence div to the first equations of (1.3) yields that

$$-\Delta\pi = \mathrm{div}_h\,\mathrm{div}_h(u^h \otimes u^h) + 2\partial_3\,\mathrm{div}_h(u^3 u^h) + \partial_3^2(u^3)^2 + \nabla\cdot(v\nabla(-\Delta)^{-1}v). \tag{3.2}$$

Multiplying (3.2) by the weighted function $\exp\big\{-\sum_{i=1}^3 \lambda_i \int_0^t f_i(\tau)d\tau\big\}$ and applying the divergence free condition $\partial_3 u^3 = -\mathrm{div}_h\, u^h$, we arrive at

$$\nabla\pi_{\vec{\lambda}} = \nabla(-\Delta)^{-1}\Big[\mathrm{div}_h\,\mathrm{div}_h(u^h \otimes u^h_{\vec{\lambda}}) + 2\partial_3\,\mathrm{div}_h(u^3 u^h_{\vec{\lambda}}) - 2\partial_3(u^3\,\mathrm{div}_h\, u^h_{\vec{\lambda}}) + \nabla\cdot(v\nabla(-\Delta)^{-1}v_{\vec{\lambda}})\Big]. \tag{3.3}$$

Applying the dyadic operator Δ_j to (3.3), then taking $L^1_t(L^p)$-norm and using the Hölder inequality and Bernstein inequality (2.4) yield that

$$\begin{aligned}
\|\Delta_j(\nabla\pi_{\vec{\lambda}})\|_{L^1_t(L^p)} \lesssim{}& 2^j\big(\|\Delta_j(u^h \otimes u^h_{\vec{\lambda}})\|_{L^1_t(L^p)} + \|\Delta_j(u^3 u^h_{\vec{\lambda}})\|_{L^1_t(L^p)}\big)\\
&+ \|\Delta_j(u^3\,\mathrm{div}_h\, u^h_{\vec{\lambda}})\|_{L^1_t(L^p)} + \|\Delta_j(v\nabla(-\Delta)^{-1}v_{\vec{\lambda}})\|_{L^1_t(L^p)}.
\end{aligned} \tag{3.4}$$

The first three terms on the right-hand side of (3.4) have been estimated in [19, 20], and the last term has been bounded in Lemma 2.9, thus we obtain

$$\|\Delta_j(u^h \otimes u^h_{\vec{\lambda}})\|_{L^1_t(L^p)} \lesssim d_j 2^{-\frac{3j}{p}}\|u^h\|_{\mathcal{L}^\infty_t(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\|u^h_{\vec{\lambda}}\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})}, \tag{3.5}$$

$$\|\Delta_j(u^3 u^h_{\bar\lambda})\|_{L^1_t(L^p)} \lesssim d_j 2^{-\frac{3j}{p}}\big(\|u^h_{\bar\lambda}\|^{\frac12}_{L^1_t(\dot B^{1+\frac3p}_{p,1})}\|u^h_{\bar\lambda}\|^{\frac12}_{\mathcal{L}^1_{t,f_2}(\dot B^{-1+\frac3p}_{p,1})} + \|u^h_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-1+\frac3p}_{p,1})}\big), \tag{3.6}$$

$$\|\Delta_j(u^3 \operatorname{div}_h u^h_{\bar\lambda})\|_{L^1_t(L^p)} \lesssim d_j 2^{j(1-\frac3p)}\big(\|u^h_{\bar\lambda}\|^{\frac12}_{L^1_t(\dot B^{1+\frac3p}_{p,1})}\|u^h_{\bar\lambda}\|^{\frac12}_{\mathcal{L}^1_{t,f_2}(\dot B^{-1+\frac3p}_{p,1})} + \|u^h_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-1+\frac3p}_{p,1})}\big), \tag{3.7}$$

$$\|\Delta_j(v\nabla(-\Delta)^{-1}v_{\bar\lambda})\|_{L^1_t(L^p)} \lesssim d_j 2^{(1-\frac3p)j}\|v\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac3q}_{q,1})}\|v_{\bar\lambda}\|_{L^1_t(\dot B^{\frac3q}_{q,1})}. \tag{3.8}$$

Taking the above estimates (3.5)-(3.8) into (3.4), we get

$$\begin{aligned}\|\Delta_j(\nabla\pi_{\bar\lambda})\|_{L^1_t(L^p)} \lesssim d_j 2^{j(1-\frac3p)}\Big[&\|u^h_{\bar\lambda}\|^{\frac12}_{L^1_t(\dot B^{1+\frac3p}_{p,1})}\|u^h_{\bar\lambda}\|^{\frac12}_{\mathcal{L}^1_{t,f_2}(\dot B^{-1+\frac3p}_{p,1})} + \|u^h_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-1+\frac3p}_{p,1})}\\ &+ \|u^h\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})}\|u^h_{\bar\lambda}\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})} + \|v\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac3q}_{q,1})}\|v_{\bar\lambda}\|_{L^1_t(\dot B^{\frac3q}_{q,1})}\Big],\end{aligned} \tag{3.9}$$

which clearly implies that

$$\begin{aligned}\|\nabla\pi_{\bar\lambda}\|_{L^1_t(\dot B^{-1+\frac3p}_{p,1})} \lesssim\ & \|u^h_{\bar\lambda}\|^{\frac12}_{L^1_t(\dot B^{1+\frac3p}_{p,1})}\|u^h_{\bar\lambda}\|^{\frac12}_{\mathcal{L}^1_{t,f_2}(\dot B^{-1+\frac3p}_{p,1})} + \|u^h_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-1+\frac3p}_{p,1})}\\ &+ \|u^h\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})}\|u^h_{\bar\lambda}\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})} + \|v\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac3q}_{q,1})}\|v_{\bar\lambda}\|_{L^1_t(\dot B^{\frac3q}_{q,1})}.\end{aligned} \tag{3.10}$$

On the other hand, in order to deal with the vertical component u^3, we also need the following two estimates from [20]:

$$\|\Delta_j(u^3 u^h)\|_{L^1_t(L^p)} \lesssim d_j 2^{-\frac{3j}{p}}\big(\|u^h\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})}\|u^3\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})} + \|u^h\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})}\|u^3\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})}\big), \tag{3.11}$$

$$\|\Delta_j(u^3 \operatorname{div}_h u^h)\|_{L^1_t(L^p)} \lesssim d_j 2^{j(1-\frac3p)}\big(\|u^h\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})}\|u^3\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})} + \|u^h\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})}\|u^3\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})}\big). \tag{3.12}$$

Based on the above two estimates, we can exactly follow the same lines as derivation of (3.10) by taking $\lambda_1=\lambda_2=\lambda_3=0$ to obtain that the pressure π satisfies the following estimate:

$$\begin{aligned}\|\nabla\pi\|_{L^1_t(\dot B^{-1+\frac3p}_{p,1})} \le C\Big(&\|u^h\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})}\big(\|u^h\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})} + \|u^3\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})}\big)\\ &+ \|u^h\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})}\|u^3\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})} + \|v\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac3q}_{q,1})}\|v\|_{L^1_t(\dot B^{\frac3q}_{q,1})}\Big).\end{aligned} \tag{3.13}$$

3.2. Estimate of the horizontal components u^h

Considering the first equations of (1.3), it is clear that the horizontal components u^h satisfies the following equations:

$$\partial_t u^h_{\bar\lambda} + (\sum_{i=1}^3 \lambda_i f_i(t))u^h_{\bar\lambda} - \Delta u^h_{\bar\lambda} = -u\cdot\nabla u^h_{\bar\lambda} - \nabla_h\pi_{\bar\lambda} - v\nabla_h(-\Delta)^{-1}v_{\bar\lambda}. \tag{3.14}$$

Applying the operator Δ_j to (3.14), then taking L^2 inner product of the resulting equations with $|\Delta_j u^h_{\bar\lambda}|^{p-2}\Delta_j u^h_{\bar\lambda}$, we obtain

$$\begin{aligned}\frac1p\frac{d}{dt}\|\Delta_j u^h_{\bar\lambda}\|^p_{L^p} + (\sum_{i=1}^3\lambda_i f_i(t))\|\Delta_j u^h_{\bar\lambda}\|^p_{L^p} - \int_{\mathbb{R}^3}\Delta\Delta_j u^h_{\bar\lambda}\cdot|\Delta_j u^h_{\bar\lambda}|^{p-2}\Delta_j u^h_{\bar\lambda}dx\\ = -\int_{\mathbb{R}^3}\Delta_j\big(u\cdot\nabla u^h_{\bar\lambda} + \nabla_h\pi_{\bar\lambda} + v\nabla_h(-\Delta)^{-1}v_{\bar\lambda}\big)|\Delta_j u^h_{\bar\lambda}|^{p-2}\Delta_j u^h_{\bar\lambda}dx.\end{aligned} \tag{3.15}$$

Thanks to Lemma 2.5, there exists a positive constant κ such that

$$-\int_{\mathbb{R}^3}\Delta\Delta_j u^h_{\bar\lambda}\cdot|\Delta_j u^h_{\bar\lambda}|^{p-2}\Delta_j u^h_{\bar\lambda}dx \ge \kappa 2^{2j}\|\Delta_j u^h_{\bar\lambda}\|^p_{L^p},$$

whence a similar argument as that in [6] gives rise to

$$\begin{aligned}\frac{d}{dt}\|\Delta_j u^h_{\bar\lambda}\|_{L^p} + (\sum_{i=1}^3\lambda_i f_i(t))\|\Delta_j u^h_{\bar\lambda}\|_{L^p} + \kappa 2^{2j}\|\Delta_j u^h_{\bar\lambda}\|_{L^p}\\ \le \|\Delta_j(u\cdot\nabla u^h_{\bar\lambda})\|_{L^p} + \|\Delta_j\nabla_h\pi_{\bar\lambda}\|_{L^p} + \|\Delta_j(v\nabla_h(-\Delta)^{-1}v_{\bar\lambda})\|_{L^p}.\end{aligned}$$

Integrating the above inequality on $[0, t]$ yields that

$$\begin{aligned}\|\Delta_j u^h_{\bar\lambda}\|_{L^\infty_t(L^p)} &+ (\sum_{i=1}^3 \lambda_i f_i(t))\|\Delta_j u^h_{\bar\lambda}\|_{L^1_t(L^p)} + \kappa 2^{2j}\|\Delta_j u^h_{\bar\lambda}\|_{L^1_t(L^p)} \le 2^j\big(\|\Delta_j(u^h\otimes u^h_{\bar\lambda})\|_{L^1_t(L^p)} \\ &+ \|\Delta_j(u^3 u^h_{\bar\lambda})\|_{L^1_t(L^p)}\big) + \|\Delta_j\nabla_h \pi_{\bar\lambda}\|_{L^1_t(L^p)} + \|\Delta_j(v\nabla_h(-\Delta)^{-1}v_{\bar\lambda})\|_{L^1_t(L^p)}. \end{aligned} \tag{3.16}$$

The right-hand side of (3.16) has been estimated in (3.5), (3.6), (3.8) and (3.9), thus there exists a positive constant C_1 such that

$$\begin{aligned}\|u^h_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})} &+ \sum_{i=1}^3 \lambda_i \|u^h_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_i}(\dot B^{-1+\frac3p}_{p,1})} + \kappa\|u^h_{\bar\lambda}\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})} \\ &\le \|u^h_0\|_{\dot B^{-1+\frac3p}_{p,1}} + \frac{\kappa}{4}\|u^h_{\bar\lambda}\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})} + C_1\big(\|u^h\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})}\|u^h_{\bar\lambda}\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})} \\ &+ \|u^h_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-1+\frac3p}_{p,1})} + \|u^h_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_2}(\dot B^{-1+\frac3p}_{p,1})} + \|v\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac3q}_{q,1})}\|v_{\bar\lambda}\|_{L^1_t(\dot B^{\frac3q}_{q,1})}\big).\end{aligned} \tag{3.17}$$

3.3. Estimate of the vertical component u^3

Observing that the vertical component u^3 satisfies the following equation:

$$\partial_t u^3 - \Delta u^3 = -u\cdot\nabla u^3 - \partial_3\pi - v\partial_3(-\Delta)^{-1}v. \tag{3.18}$$

As the derivation of (3.16), and using the following identity:

$$u\cdot\nabla u^3 = \mathrm{div}(uu^3) = \mathrm{div}_h(u^h u^3) - 2u^3\,\mathrm{div}_h\, u^h,$$

one has

$$\begin{aligned}\|\Delta_j u^3\|_{L^\infty_t(L^p)} + \kappa 2^{2j}\|\Delta_j u^3\|_{L^1_t(L^p)} &\le \|u^3_0\|_{L^p} + C\big(\|\Delta_j(u\cdot\nabla u^3)\|_{L^1_t(L^p)} \\ &+ \|\Delta_j\partial_3\pi\|_{L^1_t(L^p)} + \|\Delta_j(v\partial_3(-\Delta)^{-1}v)\|_{L^1_t(L^p)}\big) \\ &\le \|u^3_0\|_{L^p} + C\big(2^j\|\Delta_j(u^h u^3)\|_{L^1_t(L^p)} + \|\Delta_j(u^3\,\mathrm{div}_h\,u^h)\|_{L^1_t(L^p)} \\ &+ \|\Delta_j\partial_3\pi\|_{L^1_t(L^p)} + \|\Delta_j(v\partial_3(-\Delta)^{-1}v)\|_{L^1_t(L^p)}\big).\end{aligned} \tag{3.19}$$

The right-hand side of (3.19) has been estimated in (3.11), (3.12), (3.13) and (2.10), and substituting these estimates into (3.19), we obtain that there exists a positive constant C_2 such that

$$\begin{aligned}\|u^3\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})} + \kappa\|u^3\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})} &\le \|u^3_0\|_{\dot B^{-1+\frac3p}_{p,1}} + C_2\big(\|u^h\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})}(\|u^h\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})} + \|u^3\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})}) \\ &+ \|u^h\|_{L^1_t(\dot B^{1+\frac3p}_{p,1})}\|u^3\|_{\mathcal{L}^\infty_t(\dot B^{-1+\frac3p}_{p,1})} + \|v\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac3q}_{q,1})}\|v\|_{L^1_t(\dot B^{\frac3q}_{q,1})}\big).\end{aligned} \tag{3.20}$$

4. Estimates of the densities v and w

In this section, we intend to derive the estimates for the charged densities v and w. As we pointed out before, the crucial ingredient is to introduce the proper weighted functions to eliminate the difficulties caused by the nonlinear terms of the third and fourth equations of system (1.3), and we shall use different weighted functions to tackle with v and w.

4.1. Estimate of density v

To deal with v, we mainly use $f_1(t)$ to eliminate the difficulties caused the term $u\cdot\nabla v$, and the weighted function $f_3(t)$ to eliminate the difficulties caused the term $\nabla\cdot(w\nabla(-\Delta)^{-1}v)$. It follows the third equation of (1.3) that

$$\partial_t v_{\bar\lambda} + (\sum_{i=1}^3 \lambda_i f_i(t))v_{\bar\lambda} - \Delta v_{\bar\lambda} = -u\cdot\nabla v_{\bar\lambda} + \nabla\cdot(w\nabla(-\Delta)^{-1}v_{\bar\lambda}). \tag{4.1}$$

Applying the dyadic operator Δ_j to (4.1), then taking L^2 inner product of the resulting equation with $|\Delta_j v_{\bar\lambda}|^{q-2}\Delta_j v_{\bar\lambda}$ and applying Lemma 2.5, we see that

$$\frac1q\frac{d}{dt}\|\Delta_j v_{\bar\lambda}\|^q_{L^q} + (\sum_{i=1}^3\lambda_i f_i(t))\|\Delta_j v_{\bar\lambda}\|^q_{L^q} + \kappa 2^{2j}\|\Delta_j v_{\bar\lambda}\|^q_{L^q}$$

$$\leq -\int_{\mathbb{R}^3}(\Delta_j(u\cdot\nabla v_{\bar\lambda})+\Delta_j\nabla\cdot(w\nabla(-\Delta)^{-1}v_{\bar\lambda})|\Delta_j v_{\bar\lambda}|^{q-2}\Delta_j v_{\bar\lambda}dx. \tag{4.2}$$

Moreover, applying Bony's paraproduct decomposition (2.3), one has

$$u\cdot\nabla v_{\bar\lambda}=T_u\nabla v_{\bar\lambda}+T_{\nabla v_{\bar\lambda}}u+R(u,\nabla v_{\bar\lambda}),$$

which combining the standard commutator's argument gives us to

$$\begin{aligned}\int_{\mathbb{R}^3}\Delta_j(T_u\nabla v_{\bar\lambda})|\Delta_j v_{\bar\lambda}|^{q-2}\Delta_j v_{\bar\lambda}dx=&\sum_{|j'-j|\le 5}\int_{\mathbb{R}^3}[\Delta_j;S_{j'-1}u]\Delta_{j'}\nabla v_{\bar\lambda}|\Delta_j v_{\bar\lambda}|^{q-2}\Delta_j v_{\bar\lambda}dx\\&+\sum_{|j'-j|\le 5}\int_{\mathbb{R}^3}(S_{j'-1}u-S_{j-1}u)\Delta_j\Delta_{j'}\nabla v_{\bar\lambda}|\Delta_j v_{\bar\lambda}|^{q-2}\Delta_j v_{\bar\lambda}dx\\&-\frac{1}{q}\int_{\mathbb{R}^3}S_{j-1}(\mathrm{div}\,u)\Delta_j\Delta_{j'}v_{\bar\lambda}|\Delta_j v_{\bar\lambda}|^{q-2}\Delta_j v_{\bar\lambda}dx.\end{aligned}\tag{4.3}$$

Hence, taking the above estimate (4.3) into (4.2), and using the divergence free condition $\nabla\cdot u=0$ and the argument for the L^q energy estimate in [6], we obtain

$$\begin{aligned}\|\Delta_j v_{\bar\lambda}\|_{L^q}&+\int_0^t(\sum_{i=1}^3\lambda_i f_i(\tau))\|\Delta_j v_{\bar\lambda}(\tau)\|_{L^q}d\tau+\kappa 2^{2j}\int_0^t\|\Delta_j v_{\bar\lambda}(\tau)\|_{L^q}d\tau\le\|\Delta_j v_0\|_{L^q}\\&+C\Big(\sum_{|j'-j|\le5}(\|[\Delta_j;S_{j'-1}u]\Delta_{j'}\nabla v_{\bar\lambda}\|_{L^1_t(L^q)}+\|(S_{j'-1}u-S_{j-1}u)\Delta_j\Delta_{j'}\nabla v_{\bar\lambda}\|_{L^1_t(L^q)})\\&+\|\Delta_j(T_{\nabla v_{\bar\lambda}}u)\|_{L^1_t(L^q)}+\|\Delta_j(R(u,\nabla v_{\bar\lambda}))\|_{L^1_t(L^q)}+\|\Delta_j\nabla\cdot(w\nabla(-\Delta)^{-1}v_{\bar\lambda})\|_{L^1_t(L^q)}\Big).\end{aligned}\tag{4.4}$$

In the following we estimate the terms on the right-hand side of (4.4) one by one. Applying Lemma 2.4 and Definition 2.1, the first two terms can be estimated as

$$\begin{aligned}&\sum_{|j'-j|\le5}\|[\Delta_j;S_{j'-1}u]\Delta_{j'}\nabla v_{\bar\lambda}\|_{L^1_t(L^q)}\\&\lesssim\sum_{|j'-j|\le5}(\|S_{j'-1}\nabla u^h\|_{L^1_t(L^\infty)}\|\Delta_{j'}v_{\bar\lambda}\|_{L^\infty_t(L^q)}+\int_0^t\|S_{j'-1}\nabla u^3(\tau)\|_{L^\infty}\|\Delta_{j'}v_{\bar\lambda}(\tau)\|_{L^q}d\tau)\\&\lesssim\sum_{|j'-j|\le5}(d_{j'}2^{j'(2-\frac{3}{q})}\|u^h\|_{L^1_t(\dot B^{1+\frac{3}{p}}_{p,1})}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}+\int_0^t\|u^3(\tau)\|_{\dot B^{1+\frac{3}{p}}_{p,1}}\|\Delta_{j'}v_{\bar\lambda}(\tau)\|_{L^q}d\tau)\\&\lesssim d_j2^{j(2-\frac{3}{q})}(\|u^h\|_{L^1_t(\dot B^{1+\frac{3}{p}}_{p,1})}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}+\|v_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-2+\frac{3}{q}}_{q,1})}),\end{aligned}\tag{4.5}$$

$$\begin{aligned}&\sum_{|j'-j|\le5}\|(S_{j'-1}u-S_{j-1}u)\Delta_j\Delta_{j'}\nabla v_{\bar\lambda}\|_{L^1_t(L^q)}\\&\lesssim\sum_{|j'-j|\le5}(\|(S_{j'-1}\nabla u^h-S_{j-1}\nabla u^h)\|_{L^1_t(L^\infty)}\|\Delta_j v_{\bar\lambda}\|_{L^\infty_t(L^q)}\\&\quad+\int_0^t\|(S_{j'-1}\nabla u^3-S_{j-1}\nabla u^3)(\tau)\|_{L^\infty}\|\Delta_j v_{\bar\lambda}(\tau)\|_{L^q}d\tau)\\&\lesssim d_j2^{j(2-\frac{3}{q})}\|u^h\|_{L^1_t(\dot B^{1+\frac{3}{p}}_{p,1})}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}+\int_0^t\|u^3(\tau)\|_{\dot B^{1+\frac{3}{p}}_{p,1}}\|\Delta_j v_{\bar\lambda}(\tau)\|_{L^q}d\tau\\&\lesssim d_j2^{j(2-\frac{3}{q})}(\|u^h\|_{L^1_t(\dot B^{1+\frac{3}{p}}_{p,1})}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}+\|v_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-2+\frac{3}{q}}_{q,1})}).\end{aligned}\tag{4.6}$$

For the term involving $T_{\nabla v_{\bar\lambda}}u$, we consider the following two cases: in the case $1<q\le p<6$, one can choose $\widetilde{q}$ $(1<\widetilde{q}\le\infty)$ such that $\frac{1}{q}=\frac{1}{p}+\frac{1}{\widetilde{q}}$, then applying Lemma 2.4 yields that

$$\begin{aligned}\|\Delta_j(T_{\nabla v_{\bar\lambda}}u)\|_{L^1_t(L^q)}\lesssim&\sum_{|j'-j|\le5}(\|S_{j'-1}\nabla_h v_{\bar\lambda}\|_{L^\infty_t(L^{\widetilde{q}})}\|\Delta_{j'}u^h\|_{L^1_t(L^p)}\\&+\int_0^t\|S_{j'-1}\partial_3 v_{\bar\lambda}(\tau)\|_{L^{\widetilde{q}}}\|\Delta_{j'}u^3(\tau)\|_{L^p}d\tau)\\\lesssim&\sum_{k\le j'-2}2^{k(1+\frac{3}{q}-\frac{3}{\widetilde{q}})}\|\Delta_k v_{\bar\lambda}\|_{L^\infty_t(L^q)}\|\Delta_{j'}u^h\|_{L^1_t(L^p)}\end{aligned}$$

$$
\begin{aligned}
&+2^{-j(1+\frac{3}{p})}\sum_{|j'-j|\le 5}\sum_{k\le j'-2}2^{k(1+\frac{3}{p})}\int_0^t\|u^3(\tau)\|_{\dot B^{1+\frac{3}{p}}_{p,1}}\|\Delta_k v_{\bar\lambda}(\tau)\|_{L^q}\,d\tau\\
\lesssim&\sum_{k\le j'-2}d_k 2^{k(3+\frac{3}{p}-\frac{3}{q})}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}\|\Delta_{j'}u^h\|_{L^1_t(L^p)}\\
&+2^{-j(1+\frac{3}{p})}\sum_{|j'-j|\le 5}\sum_{k\le j'-2}2^{k(3+\frac{3}{p}-\frac{3}{q})}d_k\|v_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-2+\frac{3}{q}}_{q,1})}\\
\lesssim& d_j 2^{j(2-\frac{3}{q})}\big(\|u^h\|_{L^1_t(\dot B^{1+\frac{3}{p}}_{p,1})}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}+\|v_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-2+\frac{3}{q}}_{q,1})}\big),
\end{aligned}\tag{4.7}
$$

while in the case $1<p<q$, one can directly estimate that

$$
\begin{aligned}
\|\Delta_j(T_{\nabla v_{\bar\lambda}}u)\|_{L^1_t(L^q)}\lesssim& 2^{(\frac{3}{p}-\frac{3}{q})j}\sum_{|j'-j|\le 5}\Big(\|S_{j'-1}\nabla_h v_{\bar\lambda}\|_{L^\infty_t(L^\infty)}\|\Delta_{j'}u^h\|_{L^1_t(L^p)}\\
&+\int_0^t\|S_{j'-1}\partial_3 v_{\bar\lambda}(\tau)\|_{L^\infty}\|\Delta_{j'}u^3(\tau)\|_{L^p}\,d\tau\Big)\\
\lesssim& 2^{(\frac{3}{p}-\frac{3}{q})j}\sum_{k\le j'-2}2^{(1+\frac{3}{q})k}\|\Delta_k v_{\bar\lambda}\|_{L^\infty_t(L^q)}\|\Delta_{j'}u^h\|_{L^1_t(L^p)}\\
&+d_j 2^{-j(1+\frac{3}{q})}\sum_{|j'-j|\le 5}\sum_{k\le j'-2}2^{(1+\frac{3}{q})k}\int_0^t\|u^3(\tau)\|_{\dot B^{1+\frac{3}{p}}_{p,1}}\|\Delta_k v_{\bar\lambda}(\tau)\|_{L^q}\,d\tau\\
\lesssim& 2^{(\frac{3}{p}-\frac{3}{q})j}\sum_{k\le j'-2}d_k 2^{3k}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}\|\Delta_{j'}u^h\|_{L^1_t(L^p)}\\
&+2^{-j(1+\frac{3}{q})}\sum_{|j'-j|\le 5}\sum_{k\le j'-2}2^{3k}d_k\|v_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-2+\frac{3}{q}}_{q,1})}\\
\lesssim& d_j 2^{j(2-\frac{3}{q})}\big(\|u^h\|_{L^1_t(\dot B^{1+\frac{3}{p}}_{p,1})}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}+\|v_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-2+\frac{3}{q}}_{q,1})}\big).
\end{aligned}\tag{4.8}
$$

For the remaining term $\|\Delta_j(R(u,\nabla v_{\bar\lambda}))\|_{L^1_t(L^q)}$, we split the estimate into the following two parts: In the case $\frac{1}{3}<\frac{1}{p}+\frac{1}{q}\le 1$, we get

$$
\begin{aligned}
\|\Delta_j(R(u,\nabla v_{\bar\lambda}))\|_{L^1_t(L^q)}\lesssim& 2^{(1+\frac{3}{p})j}\sum_{j'\ge j-N_0}\|\Delta_{j'}u^h\|_{L^1_t(L^p)}\|\widetilde\Delta_{j'}v_{\bar\lambda}\|_{L^\infty_t(L^q)}\\
&+2^{(1+\frac{3}{p})j}\sum_{j'\ge j-N_0}\int_0^t\|\widetilde\Delta_{j'}u^3(\tau)\|_{L^p}\|\Delta_{j'}v_{\bar\lambda}(\tau)\|_{L^q}\,d\tau\\
\lesssim& 2^{(1+\frac{3}{p})j}\sum_{j'\ge j-N_0}d_{j'}2^{j'(1-\frac{3}{p}-\frac{3}{q})}\|u^h\|_{L^1_t(\dot B^{1+\frac{3}{p}}_{p,1})}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}\\
&+2^{(1+\frac{3}{p})j}\sum_{j'\ge j-N_0}2^{-(1+\frac{3}{p})j'}\int_0^t\|u^3(\tau)\|_{\dot B^{1+\frac{3}{p}}_{p,1}}\|\Delta_{j'}v_{\bar\lambda}(\tau)\|_{L^q}\,d\tau\\
\lesssim& d_j 2^{j(2-\frac{3}{q})}\|u^h\|_{L^1_t(\dot B^{1+\frac{3}{p}}_{p,1})}\big(\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}+\|v_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-2+\frac{3}{q}}_{q,1})}\big);
\end{aligned}\tag{4.9}
$$

while in the case $\frac{1}{p}+\frac{1}{q}>1$, we choose q' $(1<q'<\infty)$ such that $\frac{1}{q}+\frac{1}{q'}=1$, then applying Lemma 2.4 again yields that

$$
\begin{aligned}
\|\Delta_j(R(u,\nabla v_{\bar\lambda}))\|_{L^1_t(L^q)}\lesssim& 2^{j(3-\frac{3}{q})}\sum_{j'\ge j-N_0}\|\Delta_{j'}u^h\|_{L^1_t(L^{q'})}\|\widetilde\Delta_{j'}\nabla_h v_{\bar\lambda}\|_{L^\infty_t(L^q)}\\
&+2^{j(3-\frac{3}{q})}\sum_{j'\ge j-N_0}\int_0^t\|\widetilde\Delta_{j'}u^3(\tau)\|_{L^{q'}}\|\Delta_{j'}\partial_3 v_{\bar\lambda}(\tau)\|_{L^q}\,d\tau\\
\lesssim& 2^{j(3-\frac{3}{q})}\sum_{j'\ge j-N_0}2^{3j'(\frac{1}{p}-\frac{1}{q'})}\|\Delta_{j'}u^h\|_{L^1_t(L^p)}\|\widetilde\Delta_{j'}\nabla_h v_{\bar\lambda}\|_{L^\infty_t(L^q)}\\
&+2^{j(3-\frac{3}{q})}\sum_{j'\ge j-N_0}2^{3j'(\frac{1}{p}-\frac{1}{q'})}\int_0^t\|\widetilde\Delta_{j'}u^3(\tau)\|_{L^p}\|\Delta_{j'}\partial_3 v_{\bar\lambda}(\tau)\|_{L^q}\,d\tau\\
\lesssim& 2^{j(3-\frac{3}{q})}\sum_{j'\ge j-N_0}d_{j'}2^{-j'}\big(\|u^h\|_{L^1_t(\dot B^{1+\frac{3}{p}}_{p,1})}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}+\|v_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-2+\frac{3}{q}}_{q,1})}\big)\\
\lesssim& d_j 2^{j(2-\frac{3}{q})}\big(\|u^h\|_{L^1_t(\dot B^{1+\frac{3}{p}}_{p,1})}\|v_{\bar\lambda}\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{3}{q}}_{q,1})}+\|v_{\bar\lambda}\|_{\mathcal{L}^1_{t,f_1}(\dot B^{-2+\frac{3}{q}}_{q,1})}\big).
\end{aligned}\tag{4.10}
$$

Putting all above estimates together, we conclude that

$$
\begin{aligned}
&\|\Delta_j v_{\vec{\lambda}}\|_{L^q}+\sum_{i=1}^{3}\lambda_i\int_0^t f_i(\tau)\|\Delta_j v_{\vec{\lambda}}(\tau)\|_{L^q}d\tau+\kappa 2^{2j}\int_0^t\|\Delta_j v_{\vec{\lambda}}(\tau)\|_{L^q}d\tau\le\|\Delta_j v_0\|_{L^q}\\
&+Cd_j 2^{j(2-\frac{3}{q})}\big(\|u^h\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})}\|v_{\vec{\lambda}}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}+\|v_{\vec{\lambda}}\|_{\mathcal{L}^1_{t,f_1}(\dot{B}^{-2+\frac{3}{q}}_{q,1})}\big)+C\|\Delta_j\nabla\cdot(w\nabla(-\Delta)^{-1}v_{\vec{\lambda}})\|_{L^1_t(L^q)},
\end{aligned}
$$

which directly implies that

$$
\begin{aligned}
&\|v_{\vec{\lambda}}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}+\sum_{i=1}^{3}\lambda_i\|v_{\vec{\lambda}}\|_{\mathcal{L}^1_{t,f_i}(\dot{B}^{-2+\frac{3}{q}}_{q,1})}+\kappa\|v_{\vec{\lambda}}\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}\le\|v_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}\\
&+C\big(\|u^h\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})}\|v_{\vec{\lambda}}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}+\|v_{\vec{\lambda}}\|_{\mathcal{L}^1_{t,f_1}(\dot{B}^{-2+\frac{3}{q}}_{q,1})}+\|w\nabla(-\Delta)^{-1}v_{\vec{\lambda}}\|_{\mathcal{L}^1_t(\dot{B}^{-1+\frac{3}{q}}_{q,1})}\big).
\end{aligned}
\tag{4.11}
$$

According to the Minkowski's inequality, it is readily to see that

$$
\|w\nabla(-\Delta)^{-1}v_{\vec{\lambda}}\|_{\mathcal{L}^1_t(\dot{B}^{-1+\frac{3}{q}}_{q,1})}\approx\int_0^t\|w(\tau)\nabla(-\Delta)^{-1}v_{\vec{\lambda}}(\tau)\|_{\dot{B}^{-1+\frac{3}{q}}_{q,1}}d\tau.
$$

Then we can apply Lemma 2.7 by setting $s_1=\frac{3}{r}$, $s_2=-1+\frac{3}{q}$, $p_1=r$ and $p_2=q$ to obtain that

$$
\|w\nabla(-\Delta)^{-1}v_{\vec{\lambda}}\|_{\dot{B}^{-1+\frac{3}{q}}_{q,1}}\lesssim\|w\|_{\dot{B}^{\frac{3}{r}}_{r,1}}\|\nabla(-\Delta)^{-1}v_{\vec{\lambda}}\|_{\dot{B}^{-1+\frac{3}{q}}_{q,1}}\lesssim\|w\|_{\dot{B}^{\frac{3}{r}}_{r,1}}\|v_{\vec{\lambda}}\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}},
$$

which yields that

$$
\|w\nabla(-\Delta)^{-1}v_{\vec{\lambda}}\|_{\mathcal{L}^1_t(\dot{B}^{-1+\frac{3}{q}}_{q,1})}\lesssim\int_0^t\|w(\tau)\|_{\dot{B}^{\frac{3}{r}}_{r,1}}\|v_{\vec{\lambda}}(\tau)\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}d\tau\lesssim\|v_{\vec{\lambda}}\|_{\mathcal{L}^1_{t,f_3}(\dot{B}^{-2+\frac{3}{q}}_{q,1})}.
\tag{4.12}
$$

Taking (4.12) into (4.11), we obtain that there exists a positive constant C_3 such that

$$
\begin{aligned}
&\|v_{\vec{\lambda}}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}+\sum_{i=1}^{3}\lambda_i\|v_{\vec{\lambda}}\|_{\mathcal{L}^1_{t,f_i}(\dot{B}^{-2+\frac{3}{q}}_{q,1})}+\kappa\|v_{\vec{\lambda}}\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}\le\|v_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}\\
&\qquad+C_3\big(\|u^h\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})}\|v_{\vec{\lambda}}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}+\|v_{\vec{\lambda}}\|_{\mathcal{L}^1_{t,f_1}(\dot{B}^{-2+\frac{3}{q}}_{q,1})}+\|v_{\vec{\lambda}}\|_{\mathcal{L}^1_{t,f_3}(\dot{B}^{-2+\frac{3}{q}}_{q,1})}\big).
\end{aligned}
\tag{4.13}
$$

4.2. *Estimate of density w*

For any positive real number λ_1, recall that $f_1(t)=\|u^3(t)\|_{\dot{B}^{1+\frac{3}{p}}_{p,1}}$, and we denote

$$
w_{\lambda_1}:=w\exp(-\lambda_1\int_0^t f_1(\tau)d\tau),\quad v_{\lambda_1}:=v\exp(-\lambda_1\int_0^t f_1(\tau)d\tau).
$$

It follows the fourth equation of (1.1) that

$$
\partial_t w_{\lambda_1}+\lambda_1 f_1(t)w_{\lambda_1}-\Delta w_{\lambda_1}=-u\cdot\nabla w_{\lambda_1}+\nabla\cdot(v\nabla(-\Delta)^{-1}v_{\lambda_1}).
\tag{4.14}
$$

Arguing like the derivation of (4.1) yields that

$$
\begin{aligned}
&\|w_{\lambda_1}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{r}}_{r,1})}+\lambda_1\|w_{\lambda_1}\|_{\mathcal{L}^1_{t,f_1}(\dot{B}^{-2+\frac{3}{r}}_{r,1})}+\kappa\|w_{\lambda_1}\|_{L^1_t(\dot{B}^{\frac{3}{r}}_{r,1})}\le\|w_0\|_{\dot{B}^{-2+\frac{3}{r}}_{r,1}}\\
&+C\big(\|u^h\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})}\|w_{\lambda_1}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{r}}_{r,1})}+\|w_{\lambda_1}\|_{\mathcal{L}^1_{t,f_1}(\dot{B}^{-2+\frac{3}{r}}_{r,1})}+\|v\nabla(-\Delta)^{-1}v_{\lambda_1}\|_{L^1_t(\dot{B}^{-1+\frac{3}{r}}_{r,1})}\big).
\end{aligned}
\tag{4.15}
$$

Applying Lemma 2.13, one has

$$
\|v\nabla(-\Delta)^{-1}v_{\lambda_1}\|_{L^1_t(\dot{B}^{-1+\frac{3}{r}}_{r,1})}\lesssim\|v_{\lambda_1}\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})},
$$

which back to (4.15), we conclude that there exists a positive constant C_4 such that

$$
\begin{aligned}
&\|w_{\lambda_1}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{r}}_{r,1})}+\lambda_1\|w_{\lambda_1}\|_{\mathcal{L}^1_{t,f_1}(\dot{B}^{-2+\frac{3}{r}}_{r,1})}+\kappa\|w_{\lambda_1}\|_{L^1_t(\dot{B}^{\frac{3}{r}}_{r,1})}\le\|w_0\|_{\dot{B}^{-2+\frac{3}{r}}_{r,1}}\\
&+C_4\big(\|u^h\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})}\|w_{\lambda_1}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{r}}_{r,1})}+\|w_{\lambda_1}\|_{\mathcal{L}^1_{t,f_1}(\dot{B}^{-2+\frac{3}{r}}_{r,1})}+\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})}\|v_{\lambda_1}\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}\big).
\end{aligned}
\tag{4.16}
$$

5. Proofs of Theorems 1.1 and 1.2

The proof of Theorem 1.1 is simple. Once one gets the above desired bilinear estimates (2.9)–(2.14), one can follow exactly the same procedure as [32] to prove that there exists $T > 0$ such that the system (1.3) admits a unique solution (u, v, w) on $[0, T]$ satisfying (1.8). Moreover, if the initial data is sufficiently small, then the above local solution is actually a global one, for details, please see [32].

Now we present the proof of Theorem 1.2. Let us denote by T_* the maximal existence time of local solution (u, v, w) satisfying (1.8). Then to prove Theorem 1.2, it suffices to prove $T_* = \infty$ under the initial condition (1.9). We prove it by contradiction. Assume that $T_* < \infty$, based on the estimates (3.17), (3.20), (4.13) and (4.16), let η be a small enough positive constant which the exact value will be determined later, we define T_η by

$$T_\eta := \max\Big\{t \in [0, T_*):\ \|u^h\|_{\mathcal{L}^\infty_t(\dot{B}^{-1+\frac{3}{p}}_{p,1})} + \kappa\|u^h\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})} + \|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})} + \kappa\|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})} \le \eta\Big\}. \tag{5.1}$$

Taking $\lambda_1 \ge 2C_1$, $\lambda_2 \ge 2C_1$ and $\eta \le \frac{\kappa}{4C_1}$, we can derive from (3.17) to get that

$$\|u^h_\lambda\|_{\mathcal{L}^\infty_t(\dot{B}^{-1+\frac{3}{p}}_{p,1})} + \frac{\kappa}{2}\|u^h_\lambda\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})} \le \|u^h_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + \frac{\kappa}{4}\|v_\lambda\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}. \tag{5.2}$$

On the other hand, taking $\lambda_1 \ge 2C_3$, $\lambda_3 \ge 2C_3$ in (4.13), and $\eta \le \frac{\kappa}{2C_3}$ one obtains that

$$\|v_\lambda\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})} + 2\kappa\|v_\lambda\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})} \le 2\|v_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}. \tag{5.3}$$

As a consequence, we obtain from (5.2)–(5.3) that for all $t \le T_\eta$, it holds that

$$\begin{aligned}
&\|u^h\|_{\mathcal{L}^\infty_t(\dot{B}^{-1+\frac{3}{p}}_{p,1})} + \kappa\|u^h\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})} + \|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})} + \kappa\|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}\\
&\le 2\big(\|u^h_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + \|v_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}\big) \times \exp\Big\{\int_0^t (\lambda_1 f_1(\tau) + \lambda_2 f_2(\tau) + \lambda_3 f_3(\tau))d\tau\Big\}\\
&= 2\big(\|u^h_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + \|v_0\|_{\dot{B}^{-2+\frac{3}{q}}_{q,1}}\big) \times \exp\Big\{\int_0^t (\lambda_1\|u^3(\tau)\|_{\dot{B}^{1+\frac{3}{p}}_{p,1}} + \lambda_2\|u^3(\tau)\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}} + \lambda_3\|w(\tau)\|_{\dot{B}^{\frac{3}{r}}_{r,1}})d\tau\Big\}.
\end{aligned} \tag{5.4}$$

Thanks to (3.20), by choosing $\eta \le \frac{\kappa}{2C_2}$, it holds that for all $t \le T_\eta$,

$$\|u^3\|_{\mathcal{L}^\infty_t(\dot{B}^{-1+\frac{3}{p}}_{p,1})} + \kappa\|u^3\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})} \le 2\|u^3_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + 2\eta. \tag{5.5}$$

Back to (4.16), by taking $\lambda_1 > 2C_4$ and $\eta \le \frac{\kappa}{2C_4}$, one gets

$$\|w_{\lambda_1}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{r}}_{r,1})} + \kappa\|w_{\lambda_1}\|_{L^1_t(\dot{B}^{\frac{3}{r}}_{r,1})} \le 2\|w_0\|_{\dot{B}^{-2+\frac{3}{r}}_{r,1}} + \eta,$$

which using (5.5) yields that

$$\begin{aligned}
\|w\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{r}}_{r,1})} + \kappa\|w\|_{L^1_t(\dot{B}^{\frac{3}{r}}_{r,1})} &\le \big(2\|w_0\|_{\dot{B}^{-2+\frac{3}{r}}_{r,1}} + \eta\big) \times \exp\Big\{\int_0^t \lambda_1\|u^3(\tau)\|_{\dot{B}^{1+\frac{3}{p}}_{p,1}}d\tau\Big\}\\
&\le \big(2\|w_0\|_{\dot{B}^{-2+\frac{3}{r}}_{r,1}} + \eta\big) \times \exp\Big\{\frac{2\lambda_1}{\kappa}(\|u^3_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + \eta)\Big\}.
\end{aligned} \tag{5.6}$$

Besides, it follows the interpolation inequality in Lemma 2.6 that

$$\begin{aligned}
\|u^3\|^2_{L^2_t(\dot{B}^{\frac{3}{p}}_{p,1})} &\le C\|u^3\|_{L^\infty_t(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\|u^3\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})}\\
&\le C\|u^3\|_{\mathcal{L}^\infty_t(\dot{B}^{-1+\frac{3}{p}}_{p,1})}\|u^3\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})}\\
&\le \frac{C}{\kappa}(\|u^3_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + \eta)^2.
\end{aligned} \tag{5.7}$$

Taking above estimates (5.5)–(5.7) into (5.4), we obtain that there exists a positive constant C_5 which depends on κ and η such that for all $t \le T_\eta$, it holds that

$$\|u^h\|_{\mathcal{L}^\infty_t(\dot{B}^{-1+\frac{3}{p}}_{p,1})} + \kappa\|u^h\|_{L^1_t(\dot{B}^{1+\frac{3}{p}}_{p,1})} + \|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{3}{q}}_{q,1})} + \kappa\|v\|_{L^1_t(\dot{B}^{\frac{3}{q}}_{q,1})}$$

$$
\begin{aligned}
&\leq 2\big(\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|v_0\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}\big) \times \exp\Big\{ \int_0^t (\lambda_1 \|u^3(\tau)\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} + \lambda_2 \|u^3(\tau)\|^2_{\dot{B}_{p,1}^{\frac{3}{p}}} + \lambda_3 \|w(\tau)\|_{\dot{B}_{r,1}^{\frac{3}{r}}}) d\tau \Big\} \\
&\leq 2\big(\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|v_0\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}\big) \times \exp\Big\{ C_5 \big(\|u_0^3\|^2_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + (\|w_0\|_{\dot{B}_{r,1}^{-2+\frac{3}{r}}} + 1) \exp\big\{ C_5 \|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} \big\} + 1 \big) \Big\}. \qquad (5.8)
\end{aligned}
$$

Finally we conclude that if we take C_0 large enough and c_0 small enough in (1.9), then it follows from (5.8) that

$$
\|u^h\|_{\mathcal{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \kappa \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} + \|v\|_{\mathcal{L}_t^\infty(\dot{B}_{q,1}^{-2+\frac{3}{q}})} + \kappa \|v\|_{L_t^1(\dot{B}_{q,1}^{\frac{3}{q}})} \leq \frac{\eta}{2}
$$

for all $t < T_\eta$, which contradicts with the maximality of T_η, thus $T^* = \infty$. We complete the proof of Theorem 1.2.

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