Retrieval of the piecewise-constant mass density in the Bergman wave equation

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Abstract

This investigation is concerned with the 2D acoustic scattering problem of a plane wave propagating in a non-lossy, isotropic, homogeneous fluid host and soliciting a linear, isotropic, macroscopically-homogeneous, generally-lossy, flat-plane layer in which the mass density and wavespeed are different from those of the host. The focus is on the inverse problem of the retrieval of the layer mass density. The data is the transmitted pressure field, obtained by simulation (resolution of the forward problem) in exact, explicit form via the domain integral form of the Bergman wave equation. This solution is exact and more explicit in terms of the mass-density contrast (between the host and layer) than the classical solution obtained by separation of variables. A perturbation technique enables the solution (in its form obtained by the domain integral method) to be cast as a series of powers of the mass density contrast, the first three terms of which are employed as the trial models in the treatment of the inverse problem. The aptitude of these models to retrieve the mass density contrast is demonstrated both theoretically and numerically.

Keywords: acoustics, inverse scattering, domain integral equation, small density contrast, retrieval accuracy

Abbreviated title: Retrieval of the mass density in inverse acoustic scattering

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1 Introduction

Fluid acoustics is governed by the partial differential equation [4]

$$\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} p(\mathbf{x},t) - \frac{1}{c(\mathbf{x})^2}\frac{\partial^2 p(\mathbf{x},t)}{\partial t^2} - \frac{\nabla_{\mathbf{x}}\rho(\mathbf{x})}{\rho(\mathbf{x})} \cdot \nabla_{\mathbf{x}} p(\mathbf{x},t) = \rho(\mathbf{x})\nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{x},t) \ ; \ \forall \mathbf{x} \in \mathbb{R}^n \ , \tag{1}$$

wherein $\mathbf{x}$ is the set of cartesian spatial variables for $n = 1, 2$ or 3, t the time variable, p the pressure, ρ the mass density, $\mathbf{f}$ the applied force associated with the source, and

$$\frac{1}{c(\mathbf{x})^2} := \rho(\mathbf{x})\kappa(\mathbf{x}), \tag{2}$$

with $c(\mathbf{x})$ and $\kappa(\mathbf{x})$ the phase velocity (also termed wavespeed) and isentropic compressibility within the fluid medium.

By expanding the time domain pressure and force in the Fourier integrals

$$p(\mathbf{x},t) = \int_{-\infty}^{\infty} p(\mathbf{x},\omega)\exp(-i\omega t)d\omega \ , \tag{3}$$

$$\mathbf{f}(\mathbf{x},t) = \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{x},\omega)\exp(-i\omega t)d\omega \ , \tag{4}$$

wherein $\omega = 2\pi f$ is the angular frequency and f the frequency, one obtains the frequency domain expression of the Bergman partial differential (wave) equation (BPDE) [3]

$$\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} p(\mathbf{x},\omega) + k^2(\mathbf{x},\omega)p(\mathbf{x},\omega) - \frac{\nabla_{\mathbf{x}}\rho(\mathbf{x},\omega)}{\rho(\mathbf{x},\omega)} \cdot \nabla_{\mathbf{x}} p(\mathbf{x},\omega) = \rho(\mathbf{x},\omega)\nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{x},\omega) \ ; \ \forall \mathbf{x} \in \mathbb{R}^n \ , \tag{5}$$

wherein

$$k(\mathbf{x},\omega) := \frac{\omega}{c(\mathbf{x})} \tag{6}$$

is the wavenumber, which is a generally-complex quantity due to the fact that c is assumed to be generally-complex (to account for losses in the configuration).

Radiation is the term which designates the situation in which $\rho(\mathbf{x})$ and $c(\mathbf{x})$ are spatially-constant, i.e $\rho(\mathbf{x}) = \rho^{[0]}$ and $c(\mathbf{x}) = c^{[0]}$ throughout $\mathbb{R}^n$, in which case the pressure field satisfies

$$\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} p^{[0]}(\mathbf{x},\omega) + k^{[0]2}(\mathbf{x},\omega)p^{[0]}(\mathbf{x},\omega) = \rho^{[0]}(\mathbf{x},\omega)\nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{x},\omega) \ ; \ \forall \mathbf{x} \in \mathbb{R}^n \ , \tag{7}$$

wherein $k^{[0]} = \frac{\omega}{c^{[0]}}$. The so-called free-space Green's function (termed $G^{[0]}(\mathbf{x},\omega)$ in a homogeneous medium characterized by the constant parameters $\rho^{[0]}, c^{[0]}$) is the radiation produced by a delta-like singularity source (located at $\mathbf{x}'$) satisfying

$$\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} G^{[0]}(\mathbf{x},\mathbf{x}',\omega) + k^{[0]2}G^{[0]}(\mathbf{x},\omega)(\mathbf{x},\mathbf{x}',\omega) = -\delta(\mathbf{x}-\mathbf{x}') \ ; \ \forall \mathbf{x} \in \mathbb{R}^n \ . \tag{8}$$

Since this source, like all sources, gives rise to waves that are outgoing or incoming at all points infinitely-distant from the source, on physical grounds one chooses only the outgoing wave solution, amounting to what is called the *radiation condition* [4].

Scattering is the term which designates what becomes of the pressure wavefield when $\rho(\mathbf{x})$ and $c(\mathbf{x})$ are *not* spatially-constant. The situation of interest herein is that in which space is occupied by a homogeneous (termed host or outer) medium $M(\rho^{[0]}, c^{[0]})$ containing a generally-inhomogenous inclusion (termed obstacle) occupied by the (inner) medium $M(\rho^{[O]}(\mathbf{x},\omega), c^{[O]}(\mathbf{x},\omega))$. Note that the superscripts $0, O$ designate the outer and obstacle media respectively.

The BPDE is the differential form of the frequency domain governing equation of the scattering problem. As is well-known, it is also possible to cast this problem in integral form. The result is the so-called Bergman domain integral equation (BDIE) [15, 16], valid for the description of both radiation and scattering phenomena in fluids:

$$p(\mathbf{x},\omega) = p^i(\mathbf{x},\omega) + \int_{\mathbb{R}^n} G^{[0]}(\mathbf{x};\mathbf{x}';\omega)\left(\left[\left(k(\mathbf{x}',\omega)\right)^2 - \left(k^{[0]}\right)^2\right] p(\mathbf{x}',\omega) - \frac{\nabla_{\mathbf{x}'}\rho(\mathbf{x}',\omega)}{\rho(\mathbf{x}',\omega)} \cdot \nabla_{\mathbf{x}'} p(\mathbf{x}',\omega)\right) d\Omega(\mathbf{x}') \;\; ; \;\; \forall\, \mathbf{x} \in \mathbb{R}^n \;, \tag{9}$$

wherein $d\Omega$ is the differential volume element in $\mathbb{R}^n$,

$$p^i(\mathbf{x},\omega) = -\int_{\mathbb{R}^n} G^{[0]}(\mathbf{x};\mathbf{x}';\omega)\rho(\mathbf{x}',\omega)\nabla_{\mathbf{x}'} \cdot \mathbf{f}(\mathbf{x}',\omega) d\Omega(\mathbf{x}') \;, \tag{10}$$

which is termed either the solicitation or *incident wavefield*, and use has been made of Green's second identity and the radiation condition (as applied to both $G^{[0]}$ and p) to result in

$$\int_{\mathbb{R}^n} \left[G^{[0]}(\mathbf{x};\mathbf{x}';\omega)\nabla_{\mathbf{x}'} \cdot \nabla_{\mathbf{x}'} p(\mathbf{x}',\omega) - p(\mathbf{x}',\omega)\nabla_{\mathbf{x}'} \cdot \nabla_{\mathbf{x}'} G^{[0]}(\mathbf{x};\mathbf{x}';\omega)\right] d\Omega(\mathbf{x}') = 0 \;\; ; \;\; \forall\, \mathbf{x} \in \mathbb{R}^n \tag{11}$$

Henceforth, we shall: (a) solve the *forward scattering problem* of the determination of $p(\mathbf{x},\omega)$ via the BDIE for a specific (i.e., layer-like) obstacle and solicitation (i.e., plane wave), and (b) from various approximations of the so-obtained rigorous solution for the scattered field solve the *inverse scattering problem* of the retrieval of $\rho(\mathbf{x},\omega)$.

2 Description of the problem of scattering of a plane wave by a layer-like obstacle

The obstacle is assumed to be a flat-faced layer of infinite extent in the x and z directions filled with a fluid, in some ways different from the fluid occupying the homogeneous host.

We restrict our attention to the 2D case (i.e., $n = 2$) which arises when the constitutive parameters, as well as the solicitation, do not depend on $x_3 = z$. Thus, the following discussion takes place in the sagittal $(x - y)$ plane in which the position vector takes the form $\mathbf{x} = (x, y)$.

The traces in the sagittal plane of the upper and lower faces of the layer are the lines $y = 0$ and $y = -h$ respectively, with h the thickness of the layer. The domains above $(y > 0)$, within $(0 > y > -h)$ and below $(y < -h)$ the layer are designated by $\Omega_0, \Omega_1, \Omega_2$, respectively. The media within Ω_0 and Ω_2 are $M(\rho^{[0]}, c^{[0]})$ and the medium occupying Ω_1 is $M(\rho^{[O]}(\mathbf{x}), c^{[O]}(\mathbf{x}))$, the latter being, for the moment, inhomogeneous, but nevertheless depending only on y (later on, particularly in connection with the inverse scattering problem, we shall assume Ω_1 to be occupied by the homogeneous medium $M(\rho^{[1]}, c^{[1]})$).

The solicitation is assumed to take the form of a plane wave $p^i(\mathbf{x},\omega) = a^i(\omega)\exp(i\mathbf{k}^i(\mathbf{x},\omega)\cdot\mathbf{x})$, in which $a^i(\omega)$ is the spectral amplitude, $\mathbf{k}^i = (k_x^{i[0]}, -k_y^{i[0]}) = (k^{[0]}\sin\theta^i, -k^{[0]}\cos\theta^i)$ the incident wavevector and θ^i the angle of incidence (measured counterclockwise from the positive y axis).

3 The (forward-scattering) solution of the BDIE

The point of departure is now the Bergman integral equation (6) wherein $G^{[0]}$ is the 2D Helmholtz operator free-space Green's function for medium $M^{[0]}$ given by [8]

$$G^{[0]}(\mathbf{x};\mathbf{x}') = \frac{i}{4}H_0^{(1)}(k^{[0]}(\mathbf{x};\mathbf{x}')) \ , \tag{12}$$

wherein $H_0^{(1)}$ is the zeroth-order Hankel function of the first kind [1]. Note that we have dropped the ω-dependence in this relation and shall do so from now on for all functions that depend on this variable. This Green's function admits the following representation in cartesian coordinates (([8], p. 823):

$$G^{[0]}(\mathbf{x};\mathbf{x}') = \frac{i}{4\pi}\int_{-\infty}^{\infty} \exp[i(k_x(x-x') + k_y^{[0]}|y-y'|)]\frac{dk_x}{k_y^{[0]}} \ , \tag{13}$$

in which

$$k_y^{[j]} = \sqrt{(k^{[j]})^2 - (k_x)^2} \ ; \ \Re k_y^{[j]} \geq 0 \ , \ \Im k_y^{[j]} \geq \ ; \ \omega \geq 0 \ . \tag{14}$$

3.1 Conqsequence in the BDIE of the assumption of invariance with respect to $x_1 = x$

The previously-assumed homogeneity of $M^{[0]}$ and invariance $M^{[O]}$ with respect to x entails

$$k(\mathbf{x}) = k(y) \quad , \quad \rho(\mathbf{x}) = \rho(y) \ . \tag{15}$$

The consequence of this is, on account of the plane wave nature of the incident pressure field:

$$p(\mathbf{x}) = p(y)\exp(ik_x^{i[0]}x) \tag{16}$$

so that, after projection of (6) and employment of the identity

$$\int_{-\infty}^{\infty} \exp[i(k_x^{i[0]} - k_x)x']dx' = 2\pi\delta(k_x^{i[0]} - k_x) \ , \tag{17}$$

(with $\delta(\ )$ the Dirac delta distribution), we are led to the 1D integral equation

$$p(y) = p^i(y) + \\ \frac{i}{2k_y^{i[0]}}\int_{-\infty}^{\infty} dy' \left(\left[\left(k(y')\right)^2 - \left(k^{[0]}\right)^2 \right] p(y') - \frac{\rho_{y'}(y')}{\rho(y')}p_{,y'}(y') \right) \exp(ik_y^{i[0]}|y-y'|) \ ; \ \forall y \in \mathbb{R} \ , \tag{18}$$

in which $\mathfrak{F}(\xi)_{,\xi} := \frac{d\mathfrak{F}}{d\xi}$.

3.2 Consequence in the BDIE of the further assumption of piecewise-constant density with respect to y

We now make the even more radical assumption that the mass density ρ (but not the wavespeed c) is piecewise-constant with respect to y. This translates to the relation

$$\rho(y') = \rho^{[0]} + \left(\rho^{[1]} - \rho^{[0]}\right)[H(y'+h) - H(y')] \ , \tag{19}$$

in which $\rho^{[0]}$ and $\rho^{[1]}$ are constants with respect to the spatial coordinates and $H(\ )$ is the Heaviside distribution. It follows that

$$\rho_{,y'}(y') = \left(\rho^{[1]} - \rho^{[0]}\right)\left[\delta(y'+h) - \delta(y')\right] , \tag{20}$$

so that (18) becomes

$$p(y) = p^i(y) + \frac{i}{2k_y^{i[0]}}\int_{-\infty}^{\infty} dy' \Big(\left[\left(k(y')\right)^2 - \left(k^{[0]}\right)^2 \right] p(y') -$$
$$\frac{\left(\rho^{[1]} - \rho^{[0]}\right)\left[\delta(y'+h) - \delta(y')\right]}{\rho(y')} p_{,y'}(y') \Big) \exp(ik_y^{i[0]}|y-y'|) = p^i(y) + I(y) - K(y) \ ; \ \forall y \in \mathbb{R} \ , \tag{21}$$

wherein

$$I(y) = \frac{i}{2k_y^{i[0]}}\int_{-\infty}^{\infty} dy' \left[\left(k(y')\right)^2 - \left(k^{[0]}\right)^2 \right] p(y') \exp(ik_y^{i[0]}|y-y'|) \ ; \ \forall y \in \mathbb{R} \ , \tag{22}$$

$$K(y) = \frac{i}{2k_y^{i[0]}}\left(\rho^{[1]} - \rho^{[0]}\right)\int_{-\infty}^{\infty} dy')[\delta(y'+h) - \delta(y')]\frac{p_{,y'}(y')}{\rho(y')}) \exp(ik_y^{i[0]}|y-y'|) \ ; \ \forall y \in \mathbb{R} \ , \tag{23}$$

Using the fact that since we assumed at the outset that $c(\mathbf{x}) = c^{[0]}$ in Ω_0 and Ω_2 it follows that

$$I(y) = \frac{i}{2k_y^{i[0]}}\int_{-h}^{0} dy' \left[\left(k(y')\right)^2 - \left(k^{[0]}\right)^2 \right] p(y') \exp(ik_y^{i[0]}|y-y'|) \ ; \ \forall y \in \mathbb{R} \ . \tag{24}$$

Now, let us return to $K(y)$. Using the sifting property of the delta distributions we obtain

$$K(y) = \frac{i}{2k_y^{i[0]}}\left(\rho^{[1]} - \rho^{[0]}\right)\left[\frac{p_{,y'}(-h)}{\rho(-h)} \exp(ik_y^{i[0]}|y+h|) - \frac{p_{,y'}(0)}{\rho(0)} \exp(ik_y^{i[0]}|y|) \right] \ ; \ \forall y \in \mathbb{R} \ , \tag{25}$$

We cannot deal with I and K in analytic manner beyond this point unless further assumptions are made.

3.3 The final assumption of constant wavespeed within the layer

We now make the further assumption that

$$c(\mathbf{x}) = c^{[1]} = \text{const.} \ \Rightarrow k(\mathbf{x}) = k^{[1]} = \text{const.} \ , \tag{26}$$

the principal consequence of which is that it seems legitimate to suppose (this is actually a consequence of separation of variables (SOV) applied to the Helmholtz PDE in Ω_1) that the pressure field within the layer can be represented by the sum of two plane waves:

$$p(y) = a^{[1]} \exp(-ik_y^{i[1]}y) + b^{[1]} \exp(ik_y^{i[1]}y) \ ; \ \forall y \in [-h, 0] \ , \tag{27}$$

($a^{[1]}$ and $b^{[1]}$ are constants) from which it follows (in addition to the continuity property of $\frac{1}{\rho}p_{,y}$ inherent in the BPDE) that

$$\begin{aligned} \tfrac{p_{,y}(-h)}{\rho(-h)} &= \tfrac{ik_y^{i[1]}}{\rho^{[1]}} \left[-a^{[1]} \exp(ik_y^{i[1]}h) + b^{[1]} \exp(-ik_y^{i[1]}h) \right] \\ \tfrac{p_{,y}(0)}{\rho(0)} &= \tfrac{ik_y^{i[1]}}{\rho^{[1]}} \left[-a^{[1]} + b^{[1]} \right] \end{aligned} \ . \tag{28}$$

This suggests that we focus our attention on the integral equation

$$p(y) = I(y) - K(y) \ ; \ \forall y \in]-h, 0[\ . \tag{29}$$

It is easy to find via (27):

$$I(y) = \frac{i}{2k_y^{i[0]}} \left[\left(k^{[1]}\right)^2 - \left(k^{[0]}\right)^2 \right] \times \\ \left(\int_{-h}^{y} dy' p(y') \exp[ik_y^{i[0]}(y-y')] + \int_{y}^{0} dy' p(y') \exp[-ik_y^{i[0]}(y-y')] \right) \ ; \ \forall y \in]-h, 0[\ , \tag{30}$$

or, on account of (27) and the fact that $\left(k_y^{i[1]}\right)^2 - \left(k_y^{i[0]}\right)^2 = \left(k^{[1]}\right)^2 - \left(k^{[0]}\right)^2$:

$$I(y) = a^{[1]} \exp[-ik_y^{i[1]}y] + b^{[1]} \exp[ik_y^{i[1]}y] + \\ \left[-a^{[1]} \left(\frac{k_y^{i[0]} - k_y^{i[1]}}{2k_y^{i[0]}} \right) \exp[i(k_y^{i[1]} + k_y^{i[0]})h] - b^{[1]} \left(\frac{k_y^{i[0]} + k_y^{i[1]}}{2k_y^{i[0]}} \right) \exp[-i(k_y^{i[1]} - k_y^{i[0]})h] \right] \exp[ik_y^{i[0]}y] + \\ \left[-a^{[1]} \left(\frac{k_y^{i[0]} + k_y^{i[1]}}{2k_y^{i[0]}} \right) - b^{[1]} \left(\frac{k_y^{i[0]} - k_y^{i[0]}}{2k_y^{i[0]}} \right) \right] \exp[-ik_y^{i[0]}y] \ ; \ \forall y \in]-h, 0[\ . \tag{31}$$

It is also readily found via (28) that:

$$K(y) = \frac{k_y^{i[1]}}{2k_y^{i[0]}} \left(\frac{\rho^{i[1]} - \rho^{i[0]}}{\rho^{[1]}} \right) \left[a^{[1]} \exp[i(k_y^{i[1]} + k_y^{i[0]})h] - b^{[1]} \exp[-i(k_y^{i[1]} - k_y^{i0})h] \right] \exp[ik_y^{i0}y] - \\ \frac{k_y^{i[1]}}{2k_y^{i[0]}} \left(\frac{\rho^{i[1]} - \rho^{i[0]}}{\rho^{[1]}} \right) \left[a^{[1]} - b^{[1]} \right] \exp[-ik_y^{i[0]}y] \ ; \ \forall y \in]-h, 0[\ . \tag{32}$$

It follows that (29) takes the form:

$$a^{[1]} \exp[-ik_y^{i[1]}y] + b^{[1]} \exp[ik_y^{i[1]}y] = a^{[0]} \exp[-ik_y^{i[0]}y] + a^{[1]} \exp[-ik_y^{i[1]}y] + b^{[1]} \exp[ik_y^{i[1]}y] + \\ \left[-a^{[1]} \left(\frac{k_y^{i[0]} - k_y^{i[1]}}{2k_y^{i[0]}} \right) \exp[i(k_y^{i[1]} + k_y^{i[1]})h] - b^{[1]} \left(\frac{k_y^{i[0]} + k_y^{i[1]}}{2k_y^{i[0]}} \right) \exp[-i(k_y^{i[1]} - k_y^{i[0]})h] \right] \exp[ik_y^{i[0]}y] + \\ \left[-a^{[1]} \left(\frac{k_y^{i[0]} + k_y^{i[1]}}{2k_y^{i[0]}} \right) - b^{[1]} \left(\frac{k_y^{i[0]} - k_y^{i[1]}}{2k_y^{i[0]}} \right) \right] \exp[-ik_y^{i[0]}y] - \\ \frac{k_y^{i[1]}}{2k_y^{i[0]}} \left(\frac{\rho^{i[1]} - \rho^{i[0]}}{\rho^{[1]}} \right) \left[a^{[1]} \exp[i(k_y^{i[1]} + k_y^{i0})h] - b^{[1]} \exp[-i(k_y^{i[1]} - k_y^{i0})h] \right] \exp[ik_y^{i[0]}y] - \\ \frac{k_y^{i[1]}}{2k_y^{i[0]}} \left(\frac{\rho^{i[1]} - \rho^{i[0]}}{\rho^{[1]}} \right) \left[a^{[1]} - b^{[1]} \right] \exp[-ik_y^{i[0]}y] \ ; \ \forall y \in]-h, 0[\ , \tag{33}$$

from which, after cancelations of like terms, and the fact that the term multiplying $\exp\left[-ik_y^{i[0]}\right]$ must be equal to $a^{[0]}$ and the terms multiplying $\exp\left[ik_y^{i[0]}\right]$ must vanish if the relation (33) is to hold for every y in $]-h,0[$, that we are left with the two results:

$$a^{[0]}=\left[a^{[1]}\left(\frac{k_y^{i[0]}+k_y^{i[1]}}{2k_y^{i[0]}}\right)+b^{[1]}\left(\frac{k_y^{i[0]}-k_y^{i[1]}}{2k_y^{i[0]}}\right)\right]-\left[a^{[1]}-b^{[1]}\right]\frac{k_y^{i[1]}}{2k_y^{i[0]}}\left(\frac{\rho^{i[1]}-\rho^{i[0]}}{\rho^{[1]}}\right)\ , \tag{34}$$

$$0=\left[a^{[1]}\left(\frac{k_y^{i[0]}-k_y^{i[1]}}{2k_y^{i[0]}}\right)\exp[i(k_y^{i[1]}+k_y^{i0})h]+b^{[1]}\left(\frac{k_y^{i[0]}+k_y^{i[1]}}{2k_y^{i[0]}}\right)\exp[-i(k_y^{i[1]}-k_y^{i0})h]\right]+$$
$$\frac{k_y^{i[1]}}{2k_y^{i[0]}}\left(\frac{\rho^{i[1]}-\rho^{i[0]}}{\rho^{[1]}}\right)\left[a^{[1]}\exp[i(k_y^{i[1]}+k_y^{i0})h]-b^{[1]}\exp[-i(k_y^{i[1]}-k_y^{i0})h]\right]\ . \tag{35}$$

These two equations can be put in the form:

$$a^{[0]}2\frac{k_y^{i[0]}}{\rho^{[0]}}=a^{[1]}\left[\left(\frac{k_y^{i[0]}}{\rho^{[0]}}+\frac{k_y^{i[1]}}{\rho^{[0]}}\right)-\frac{k_y^{i[1]}}{\rho^{[0]}}\epsilon\right]+b^{[1]}\left[\left(\frac{k_y^{i[0]}}{\rho^{[0]}}-\frac{k_y^{i[1]}}{\rho^{[0]}}\right)+\frac{k_y^{i[1]}}{\rho^{[0]}}\epsilon\right]\ , \tag{36}$$

$$0=a^{[1]}e^{ik_y^{i[1]}h}\left[\left(\frac{k_y^{i[0]}}{\rho^{[0]}}-\frac{k_y^{i[1]}}{\rho^{[0]}}\right)+\frac{k_y^{i[1]}}{\rho^{[0]}}\epsilon\right]+b^{[1]}e^{-ik_y^{i[1]}h}\left[\left(\frac{k_y^{i[0]}}{\rho^{[0]}}+\frac{k_y^{i[1]}}{\rho^{[0]}}\right)-\frac{k_y^{i[1]}}{\rho^{[0]}}\epsilon\right]\ , \tag{37}$$

wherein

$$\epsilon=\frac{\rho^{[1]}-\rho^{[0]}}{\rho^{[1]}}\ , \tag{38}$$

is the *mass density contrast*. The solution of the two linear equations (36)-(37) in the two unknowns $a^{[1]}$, $b^{[1]}$ is then readily found to be:

$$a^{[1]}=a^{[0]}\left(\frac{k_y^{i[0]}\left[k_y^{i[0]}+k_y^{i[1]}(1-\epsilon)\right]e^{-ik_y^{i[1]}h}}{2k_y^{i[1]}k_y^{i[0]}(1-\epsilon)\cos(k_y^{i[1]}h)-\left[\left(k_y^{i[0]}\right)^2+\left(k_y^{i[1]}\right)^2(1-\epsilon)^2\right]i\sin(k_y^{i[1]}h)}\right)\ , \tag{39}$$

$$b^{[1]}=a^{[0]}\left(\frac{k_y^{i[0]}\left[k_y^{i[1]}(1-\epsilon)-k_y^{i[0]}\right]e^{ik_y^{i[1]}h}}{2k_y^{i[1]}k_y^{i[0]}(1-\epsilon)\cos(k_y^{i[1]}h)-\left[\left(k_y^{i[0]}\right)^2+\left(k_y^{i[1]}\right)^2(1-\epsilon)^2\right]i\sin(k_y^{i[1]}h)}\right)\ . \tag{40}$$

Since $1-\epsilon=\rho^{[0]}/\rho^{[1]}$ it is easy to see that (39)-(40) are identical to the well-known [15] SOV counterparts.

3.4 BDIE solutions for the pressure fields in Ω_0 and Ω_2

We now turn to what is essential for undertaking the resolution of the inverse problem, which is that of establishing suitable expressions of the pressure fields in Ω_0 and Ω_2 which is where these fields can be easily-measured. These fields are necessarily an outcome of the solution obtained for the field within the layer, as is expressed by the Bergman integral representations of these fields:

$$p(y)=p^i(y)+I(y)-K(y)\ ;\ \forall y\in]-\infty,-h[\ ,\ \forall y\in]0,\infty[\ . \tag{41}$$

wherein $I(y)$ and $K(y)$ (=0 for $\epsilon=0$) are functions of the previously-found $p(y)$; $y\in]-h,0[$.

Let us first consider the field in Ω_0. Proceeding as previously, we find

$$I(y)=\frac{1}{2}e^{ik_y^{i[0]}y}\times \left[a^{[1]}\left(\frac{k_y^{i[0]}-k_y^{i[1]}}{k_y^{i[0]}}\right)+b^{[1]}\left(\frac{k_y^{i[0]}+k_y^{i[1]}}{k_y^{i[0]}}\right)-a^{[1]}\left(\frac{k_y^{i[0]}-k_y^{i[1]}}{k_y^{i[0]}}\right)e^{i(k_y^{i[0]}+k_y^{i[1]})h} -b^{[1]}\left(\frac{k_y^{i[0]}+k_y^{i[1]}}{k_y^{i[0]}}\right)e^{i(k_y^{i[0]}-k_y^{i[1]})h}\right]\ ;\ \forall y\in]0,\infty[\ , \quad (42)$$

$$K(y)\ =\ -\frac{\epsilon}{2}\frac{k_y^{i[1]}}{k_y^{i[0]}}e^{ik_y^{i[0]}y}\Big[-a^{[1]}e^{i(k_y^{i[0]}+k_y^{i[1]})h}+b^{[1]}e^{i(k_y^{i[0]}-k_y^{i[1]})h}+a^{[1]}-b^{[1]}\Big]\ ;\ \forall y\ \in]0,\infty[\ , \quad (43)$$

so that

$$p(y)=p^i(y)-\frac{1}{2k_y^{i[0]}}e^{ik_y^{i[0]}y}\times \Big(a^{[1]}\Big[k_y^{i[0]}-k_y^{i[1]}(1-\epsilon)\Big]+b^{[1]}\Big[k_y^{i[0]}+k_y^{i[1]}(1-\epsilon)\Big]+a^{[1]}\Big[-k_y^{i[0]}+k_y^{i[1]}(1-\epsilon)\Big]e^{i(k_y^{i[0]}+k_y^{i[1]})h}+ b^{[1]}\Big[-k_y^{i[0]}-k_y^{i[1]}(1-\epsilon)\Big]e^{i(k_y^{i[0]}-k_y^{i[1]})h}\Big)\ ;\ \forall y\in]0,\infty[\ , \quad (44)$$

which, upon the introduction of (39)-(40), yields

$$p(y)=b^{[0]}\exp(ik_y^{i[0]}y)\ ;\ \forall y\in]0,\infty[\ , \quad (45)$$

wherein

$$b^{[0]}=a^{[0]}\left(\frac{\left[\left(k_y^{i[1]}\right)^2(1-\epsilon)^2-\left(k_y^{i[0]}\right)^2\right]i\sin(k_y^{i[1]}h)}{2k_y^{i[1]}k_y^{i[0]}(1-\epsilon)\cos(k_y^{i[1]}h)-\left[\left(k_y^{i[0]}\right)^2+\left(k_y^{i[1]}\right)^2(1-\epsilon)^2\right]i\sin(k_y^{i[1]}h)}\right)\ . \quad (46)$$

Since $1-\epsilon=\rho^{[0]}/\rho^{[1]}$ it is easy to see that (46) is identical to its SOV counterpart ([16]).

Let us next consider the field in Ω_2. Proceeding as previously, we find

$$I(y)=-\frac{1}{2}e^{-ik_y^{i[0]}y}\times \left[a^{[1]}\left(\frac{k_y^{i[0]}+k_y^{i[1]}}{k_y^{i[0]}}\right)\left[1-e^{-i(k_y^{i[0]}-k_y^{i[1]})h}\right]+b^{[1]}\left(\frac{k_y^{i[0]}-k_y^{i[1]}}{k_y^{i[0]}}\right)\left[1-e^{-i(k_y^{i[0]}+k_y^{i[1]})h}\right]\right]\ ;\ \forall y\in]-\infty,-h[\ , \quad (47)$$

$$K(y)=-\frac{\epsilon}{2}\frac{k_y^{i[1]}}{k_y^{i[0]}}e^{-ik_y^{i[0]}y}\Big[-a^{[1]}e^{-i(k_y^{i[0]}-k_y^{i[1]})h}+b^{[1]}e^{-i(k_y^{i[0]}+k_y^{i[1]})h}+a^{[1]}-b^{[1]}\Big]\ ;\ \forall y\in]-\infty,-h[\ , \quad (48)$$

so that

$$p(y) = p^i(y) - \frac{1}{2k_y^{i[0]}} e^{-ik_y^{i[0]}y} \times$$

$$\Big(a^{[1]} \left[k_y^{i[0]} + k_y^{i[1]}(1-\epsilon) \right] + b^{[1]} \left[k_y^{i[0]} - k_y^{i[1]}(1-\epsilon) \right] + a^{[1]} \left[-k_y^{i[0]} - k_y^{i[1]}(1-\epsilon) \right] e^{-i(k_y^{i[0]} - k_y^{i[1]})h} +$$
$$b^{[1]} \left[-k_y^{i[0]} + k_y^{i[1]}(1-\epsilon) \right] e^{-i(k_y^{i[0]} + k_y^{i[1]})h} \Big) \; ; \; \forall y \in]-\infty, -h[\; , \quad (49)$$

which, upon the introduction of (39)-(40), yields

$$p(y) = a^{[2]} \exp(-ik_y^{i[0]}y) \; ; \; \forall y \in]-\infty, -h[\; , \quad (50)$$

wherein

$$a^{[2]} = a^{[0]} \left(\frac{2k_y^{i[1]}k_y^{i[0]}(1-\epsilon)\exp(-ik_y^{i[0]}h)}{2k_y^{i[1]}k_y^{i[0]}(1-\epsilon)\cos(k_y^{i[1]}h) - \left[\left(k_y^{i[0]}\right)^2 + \left(k_y^{i[1]}\right)^2 (1-\epsilon)^2 \right] i\sin(k_y^{i[1]}h)} \right) . \quad (51)$$

Since $1 - \epsilon = \rho^{[0]}/\rho^{[1]}$ it is easy to see that (51) is identical to its SOV counterpart [16].

The important point to note here is that the BDIE expressions for $b^{[0]}, a^{[1]}, b^{[1]}, a^{[2]}$ enable, contrary to their SOV counterparts, to account, in very explicit manner, for the mass density contrast (expressed by ϵ) of the layer obstacle configuration. We shall see further on that this has important consequences, especially in the inverse scattering context.

3.5 Approximate models, obtained by a perturbation method, for the transmission coefficient for later use in the inverse problem

What will be offered here for the transmission coefficient $a^{[2]}$ is applicable, as well to the other coefficients $b^{[0]}, a^{[1]}, b^{[1]}$.

Eq. (51) is of the form

$$a^{[2]} = \frac{a + b\epsilon}{c + d\epsilon + f\epsilon^2} \; , \quad (52)$$

in which:

$$\begin{aligned} a &= a^{[0]} 2k_y^{i[1]}k_y^{i[0]} \exp(-ik_y^{i[0]}h) \\ b &= -a \\ c &= 2k_y^{i[1]}k_y^{i[0]}\cos(k_y^{i[1]}h) - \left[\left(k_y^{i[0]}\right)^2 + \left(k_y^{i[1]}\right)^2 \right] i\sin(k_y^{i[1]}h) \\ d &= -2k_y^{i[1]}k_y^{i[0]}\cos(k_y^{i[1]}h) + 2\left(k_y^{i[1]}\right)^2 i\sin(k_y^{i[1]}h) \\ f &= -\left(k_y^{i[1]}\right)^2 i\sin(k_y^{i[1]}h) \end{aligned} \quad . \quad (53)$$

so that

$$a^{[2]} = \left(\frac{a}{c} + \frac{b}{c}\epsilon\right) \sum_{l=0}^{\infty} \left[-\epsilon\left(\frac{d}{c} + \frac{f}{c}\epsilon\right)\right]^l \ . \tag{54}$$

This relation suggests that the transmission coefficient can be expressed in iterative manner as follows:

$$\begin{array}{rcl} a^{[2](0)} = & & \frac{a}{c} \\ a^{[2](1)} = & & a^{[2](0)} + \left(\frac{b}{c} - \frac{ad}{c^2}\right)\epsilon \\ a^{[2](2)} = & & a^{[2](1)} + \left(-\frac{af}{c^2} - \frac{bd}{c^2} + \frac{ad^2}{c^3}\right)\epsilon^2 \\ \ldots . & & \ldots . \end{array} \ . \tag{55}$$

It is readily-verified that the zeroth-order approximation $a^{[2](0)}$ of $a^{[2]}$ is the constant-density (i.e., the $\epsilon = 0$) solution for this coefficient. Similarly, the first-order approximation $a^{[2](1)}$ of $a^{[2]}$ constitutes the first correction, presumably-valid for very small ϵ, of the constant-density solution. This approximation is further corrected via the second-order approximation $a^{[2](2)}$, presumably-valid for small ϵ, and so on.

Numerical comparisons are provided between the zeroth-, first-, and second-order approximations of $a^{[2]}$ and its exact counterpart in [16].

4 The inverse-scattering problem

4.1 Preliminaries

The following material concerns the retrieval of ϵ from data relating to the measured field p in Ω_2.

Actually, we shall not measure this field, but rather obtain it by simulation, the latter appealing to the exact solution of the forward problem.

Furthermore, we shall assume that the configurations of the forward and inverse problems are exactly the same, this meaning that (a) the obstacle is the same layer (i.e., same: thickness, macroscopic homogeneity invariant with respect to the three space coordinates, location in space, but differing a priori only by the mass density contrast and/or the wavespeed), (b) the obstacle is submitted to the same radiation (i.e., same plane wave), and (c) the measurement location and device are the same (actually a device capable of detecting with perfect accuracy the amplitude and phase of the pressure on the lower face of the layer).

Inversion of this sort normally is the outcome of the comparison of two pressure fields (at the above-mentioned location): 1) the one predicted by a trial model of the layer involving a trial value of the to-be-retrieved parameter (which may initially be far-removed from the actual value of this parameter as it intervenes in the data) and 2) the simulated pressure field. If the difference (the meaning of this word will be detailed further on) of the trial field from the measured (simulated) data field is larger than some prescribed value, then the trial parameter in the trial model is changed in some hopefully-rational manner and the new trial model is again compared to the data. This comparison procedure is repeated as many times as necessary to obtain a difference equal to, or smaller than, a prescribed value. This is the procedure for retrieving $c^{[1]}$ [16]. However, inversion is simpler when the difference between the predictions of (the pressure fields) trial model and the simulated data are imposed at the outset to vanish and the equation translating this fact can be solved in explicit, mathematical manner. This is the procedure [12] we adopt for retrieving the density contrast ϵ.

In all the following material, upper-case letters will designate trial parameters (and the corresponding trial fields) and lower-case letters will designate the actual parameters (and data fields) involved in the simulation model employed to generate the data. When a parameter in the trial model is not varied (i.e., not the object of the retrieval), it is termed 'prior', and due to the aforementioned assumptions, all the priors are considered to be equal to the corresponding parameters in the simulation model. This means, in particular, that when trying to retrieve the mass density contrast, we assume the layer wavespeed to be equal to its true value in the simulation model.

The simulation model employed to generate the data appeals to the exact BDIE (identical to the SOV) solution for the transmitted pressure. We shall employ three trial models to retrieve the density contrast: these are based on the zeroth, first and second-order ϵ approximations of the pressure field, and we shall show that these three inversions can be carried out in explicit, mathematical manner.

This will enable us to show that the employment of approximate trial models, such as the one based on constant-density assumption, lead to impossible or inaccurate retrievals, much as do the employment of exact trial models with one or several trial priors being different from the corresponding parameters employed in the simulation model [7].

4.2 Explicit retrievals of the mass density contrast

The simulated (via the DD-SOV model) field in the transmission half-space is

$$p(x,z) = a^{[2]} \exp[i(k_y^{i[0]}x - k_y^{i[0]}y)] \ . \tag{56}$$

We assume the data is collected at $x=0,\ y=-h$, so that the simulated pressure data is

$$p(0,-h) = a^{[2]}(\epsilon) \exp[ik_y^{i[0]}h)] \ . \tag{57}$$

The trial fields, at the same location, for the three trial models, are of the general form

$$P(0,-h) = A^{[2](l)}(E) \exp[iK_y^{i[0]}H] \ ; \ l=0,1,2 \ , \tag{58}$$

wherein E is the trial mass density contrast, and due to the assumption of the identity of the priors, $K_y^{i[0]} = k_y^{i[0]}$, $H=h$, so that

$$P^{(l)}(0,-h) = A^{[2](l)} \exp[ik_y^{i[0]}h] \ ; \ l=0,1,2 \ , \tag{59}$$

wherein, it must be recalled:

$$A^{[2](l)} = \sum_{j=0}^{l} A_j^{[2]} E^j \ ; \ l=0,1,2 \ , \tag{60}$$

with the coefficients $A_j^{[2]}$ not depending on E.

The so-called *cost function* $\mathcal{K}$ is a function of the difference of $P^{(l)}(0,-h)$ from $p(0,-h)$

$$\mathcal{K}^{(l)}(E^{(l)}) = \mathcal{F}\left(P^{(l)}(0,-h) - p(0,-h)\right) \ ; \ l=0,1,2 \ , \tag{61}$$

the object being to find the trial density contrast $E^{(l)}$ that minimizes $\mathcal{K}^{(l)}$ for each l. For reasons that will become obvious, we shall choose $\mathcal{F}\left((P^{(l)}(0,-h)-p(0,-h)\right)=P^{(l)}(0,-h)-p(0,-h))$ and the minimum to be zero, so that the inverse problems reduce to finding $E^{(l)}$ from

$$\mathcal{K}^{(l)}(E^{(l)})=P^{(l)}(0,-h)-p(0,-h)=0\ , \tag{62}$$

or, more explicitly,

$$\left(A^{[2](l)}-a^{[2]}\right)\exp[ik_y^{i[0]}h]=0 \tag{63}$$

which implies

$$A^{[2](l)}-a^{[2]}=0\ , \tag{64}$$

First consider the case $l=0$. On account of (60) the previous equation reduces to

$$A_0^{[2]}-a^{[2]}=0\ , \tag{65}$$

which possesses *no solution* for the mass density contrast since $A_0^{[2]}$ does not depend on this parameter. Thus, the somewhat-trivial conclusion is that it is not possible to retrieve the density contrast when the constant-density assumption is made.

Next consider the case $l=1$. On account of (60) the comparison equation (64) reduces to the linear equation (in terms of the mass density contrast)

$$A_0^{[2]}+A_1^{[2]}E^{(1)}-a^{[2]}=0\ , \tag{66}$$

whose *single solution* is simply

$$E^{(1)}=-\left(\frac{A_0^{[2]}-a^{[2]}}{A_1^{[2]}}\right)\ . \tag{67}$$

This shows that the employment of the linearized approximation $A^{[2](1)}$ of $A^{[2]}$ as the trial model enables: (1) not only to solve the inverse problem in explicit, mathematical form, (2) but also to obtain a *unique* solution of this problem. These are the reasons why linearization is often employed in dealing with inverse problems. A last property of this solution: it may be complex even when ϵ is real, this being due to the fact that the different terms in (67) are generally-complex.

Finally, consider the $l=2$ case. On account of (60) the comparison equation (64) reduces to the quadratic equation (in terms of the mass density contrast)

$$A_0^{[2]}+A_1^{[2]}E^{(2)}+A_2^{[2]}\left(E^{(2)}\right)^2-a^{[2]}=0\ , \tag{68}$$

whose *two solutions* are simply

$$E^{(2)\pm}=-\frac{A_1^{[2]}}{2A_2^{[2]}}\pm\frac{\sqrt{\left(A_1^{[2]}\right)^2-4A_2^{[2]}\left(A_0^{[2]}-a^{[2]}\right)}}{2A_2^{[2]}}\ . \tag{69}$$

This shows that the employment of the second-order approximation $A^{[2](2)}$ of $A^{[2]}$ as the trial model enables: (1) not only to solve the inverse problem in explicit, mathematical form, (2) but also to obtain a *non-unique* solution of this problem. In fact, we find two solutions to this problem, and there does not appear to exist any easily-discernible relation between these two solutions on the one hand and the sole solution of the the $l = 1$ problem on the other hand. A last property of the second-order solution: it may be complex even when ϵ is real, this being due to the fact that the different terms in (69) are generally-complex.

It has been suggested that to alleviate the non-uniqueness problem, and/or improve the accuracy of the retrievals, one should take advantage of the fact that a retrieved constitutive parameter should not depend on the characteristics of the solicitation (because of the implicit assumption of isotropy of the media in presence). We translate this remark here to: $E^{[2]}$ should not depend on the angle of incidence θ^i, so that to possibly alleviate the non-uniqueness problem and/or increase the accuracy of the retrieval, we generalize the comparison equation (65) to N realizations which differ from each other only by the choice of incident angle.

The chosen incident angles form the set

$$\boldsymbol{\Theta}^i = \{\theta_1^i, \theta_2^i, ..., \theta_N^i\} \ , \tag{70}$$

and for the n-th realization, the simulated transmission coefficient is $a^{[2]}(\theta_n^i)$ while its trial counterpart is $A^{[2]}(\theta_n^i)$ so that the comparison equation now takes the form

$$\sum_{n=1}^{N} \left(A^{[2]}(\theta_n^i) - a^{[2]}(\theta_n^i) \right) = 0 \ . \tag{71}$$

which is the same as (65) when $N = 1$.

Proceeding as previously, we make the expansion (for each incident angle realization)

$$A^{[2](l)}(\theta_n^i) = \sum_{m=1}^{l} A_m^{[2]}(\theta_n^i) \left(E^{(l)} \right)^m \ , \tag{72}$$

whence (71) becomes

$$\sum_{n=1}^{N} \left(\sum_{m=1}^{l} A_m^{[2]}(\theta_n^i) \left(E^{(l)} \right)^m - a^{[2]}(\theta_n^i) \right) = 0 \ . \tag{73}$$

For $l = 1$ this equation yields the unique solution

$$E^{(1)} = -\left(\frac{\mathcal{A}_0^{[2]} - \alpha^{[2]}}{\mathcal{A}_1^{[2]}} \right) \ , \tag{74}$$

whereas for $l = 2$ it gives rise to the two solutions

$$E^{(2)\pm} = -\frac{\mathcal{A}_1^{[2]}}{2\mathcal{A}_2^{[2]}} \pm \frac{\sqrt{\left(\mathcal{A}_1^{[2]}\right)^2 - 4\mathcal{A}_2^{[2]} \left(\mathcal{A}_0^{[2]} - \alpha^{[2]} \right)}}{2\mathcal{A}_2^{[2]}} \ . \tag{75}$$

wherein:

$$\mathcal{A}_m^{[2]} = \sum_{n=1}^{N} A_m^{[2]}(\theta_n^i) \ \ , \ \ \alpha^{[2]} = \sum_{n=1}^{N} a^{[2]}(\theta_n^i) \ . \tag{76}$$

As previously, the $l = 1$ and $l = 2$ retrievals for the mass density contrast may turn out to be complex even though ϵ is real, this being due to the fact that the different terms in (74) and (75) are generally-complex.

4.3 Numerical results for the retrieval of the mass density contrast

The aim of the computations was to compare numerically the actual density contrast ϵ to the retrievals thereof: (a) $E^{(1)}$) obtained via the first-order ϵ trial model, and (b) $E^{(2)}$ obtained via the second-order ϵ trial model, as a function of the various parameters $\Re\left(c^{[1]}\right)$, $\Im\left(c^{[1]}\right)$, ϵ, f, h and θ^i, the other parameters being fixed at the following values: $\rho^{[0]} = 1000\ Kgm^{-3}$, $c^{[0]} = 1500\ ms^{-1}$, $a^{[0]} = 1$. As concerns the set of incident angle(s), we shall designate it in condensed form by $\mathbf{\Theta}^i = (\theta^i_b, \theta^i_e, N)$, which means that N equally-spaced values of θ^i are chosen for the retrieval ranging from θ^i_b to θ^i_e.

4.3.1 First and second-order retrievals as a function of the frequency f

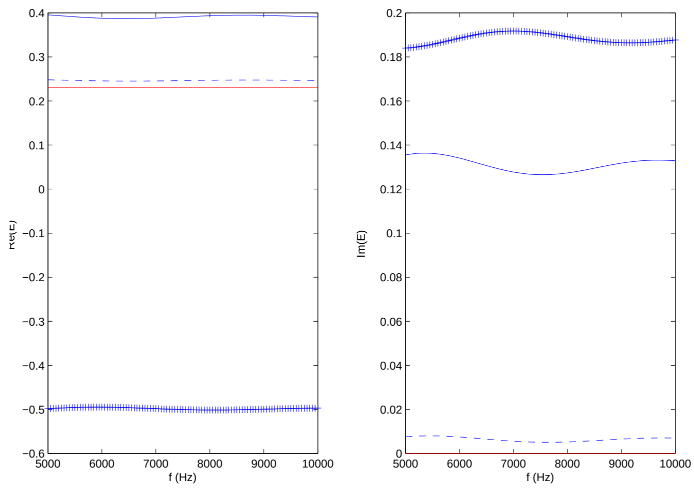

Figure 1: Retrieved (E) mass density contrast compared to the actual (ϵ) mass density contrast as a function of the frequency f. The left(right)-hand panels depict the real(imaginary) parts of ϵ (red——), $E^{(1)}$ (blue ——), $E^{(2)-}$ (blue - - - -), $E^{(2)+}$ (blue + + + +). Case $\rho^{[1]} = 1300\ Kgm^{-3}$, $c^{[1]} = 1700 - 210i\ ms^{-1}$, $h = 0.2\ m$, $\mathbf{\Theta}^i = (0°, 0°, 1)$.

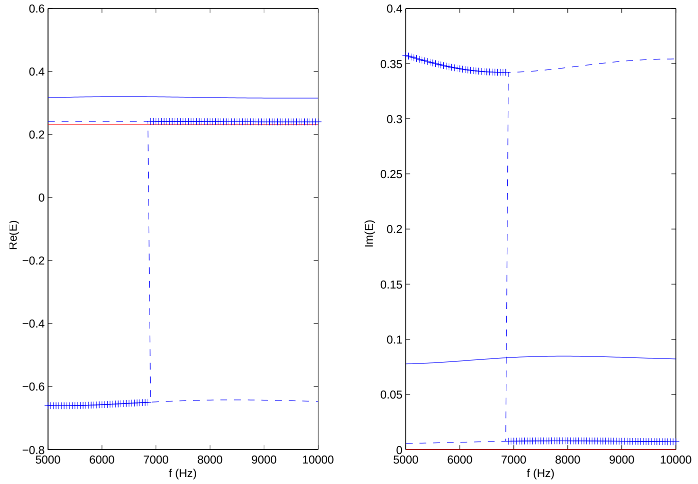

Figure 2: Same as fig. 1 except that $\mathbf{\Theta}^i = (40°, 40°, 1)$.

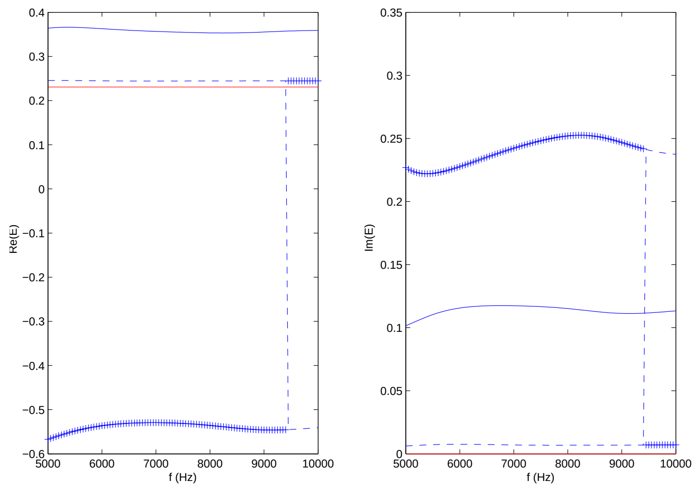

Figure 3: Same as fig. 1 except that $\mathbf{\Theta}^i = (0°, 40°, 5)$.

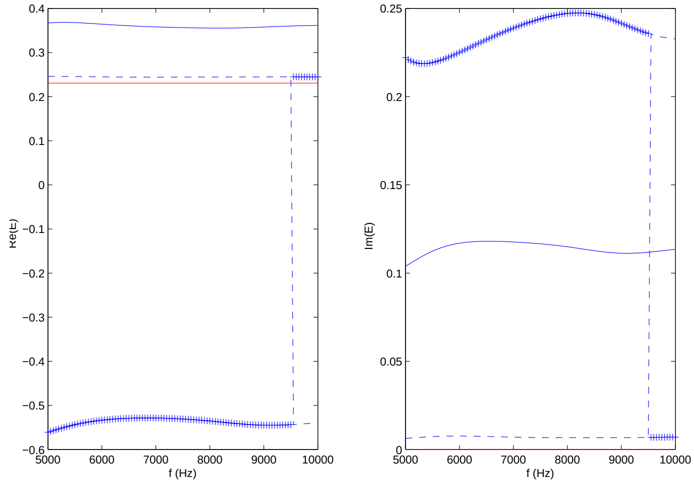

Figure 4: Same as fig. 1 except that $\mathbf{\Theta}^i = (0°, 40°, 9)$.

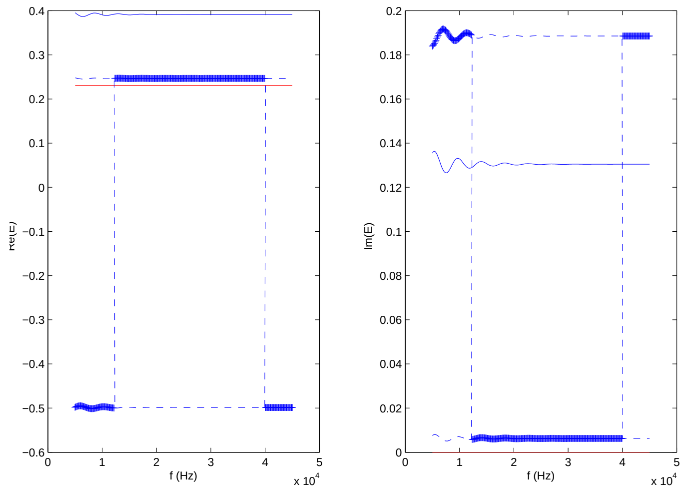

Figure 5: Retrieved (E) mass density contrast compared to the actual (ϵ) mass density contrast as a function of, and for a different range of, the frequency f . The left(right)-hand panels depict the real(imaginary) parts of ϵ (red——), $E^{(1)}$ (blue ——), $E^{(2)-}$ (blue - - - -), $E^{(2)+}$ (blue + + + +). Case $\rho^{[1]} = 1300\ Kgm^{-3}$, $c^{[1]} = 1700 - 210i\ ms^{-1}$, $h = 0.2\ m$, $\mathbf{\Theta}^i = (0°, 0°, 1)$.

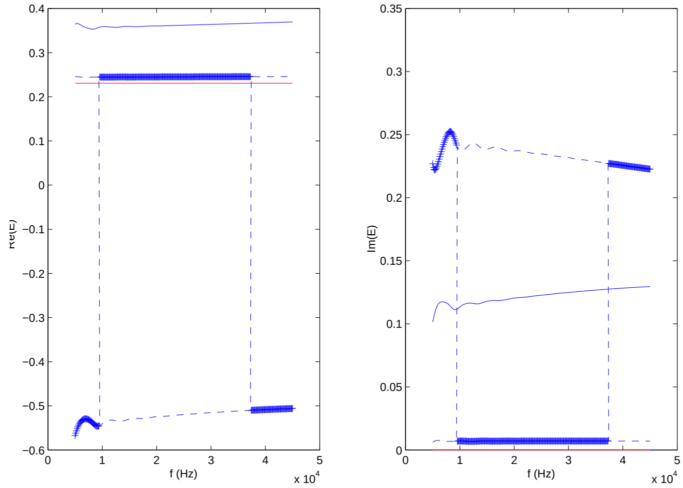

Figure 6: Same as fig. 5 except that $\mathbf{\Theta}^i = (0°, 40°, 5)$.

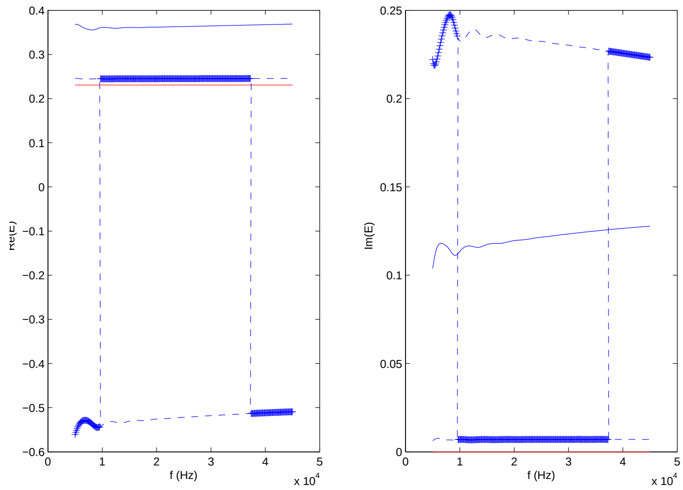

Figure 7: Same as fig. 5 except that $\mathbf{\Theta}^i = (0°, 40°, 9)$.

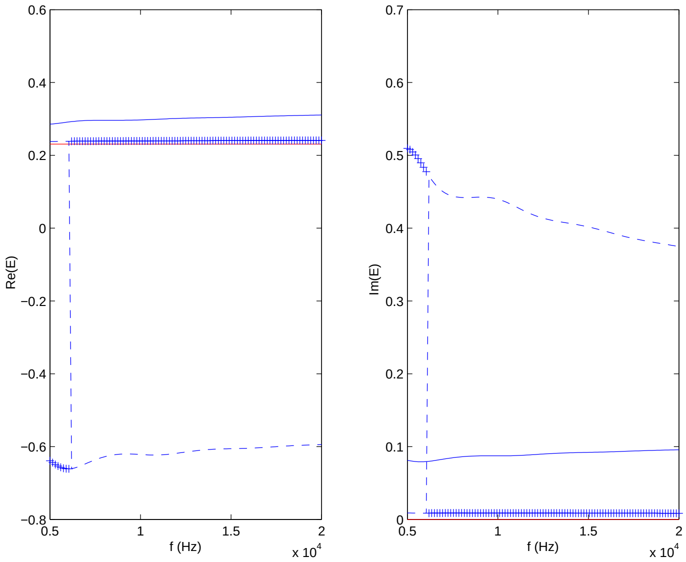

Figure 8: Retrieved (E) mass density contrast compared to the actual (ϵ) mass density contrast as a function of, and for a different range of, the frequency f . The left(right)-hand panels depict the real(imaginary) parts of ϵ (red——), $E^{(1)}$ (blue ——), $E^{(2)-}$ (blue - - - -), $E^{(2)+}$ (blue $+ + + +$). Case $\rho^{[1]} = 1300\ Kgm^{-3}$, $c^{[1]} = 1700 - 210i\ ms^{-1}$, $h = 0.2\ m$, $\mathbf{\Theta}^i = (30^\circ, 60^\circ, 5)$.

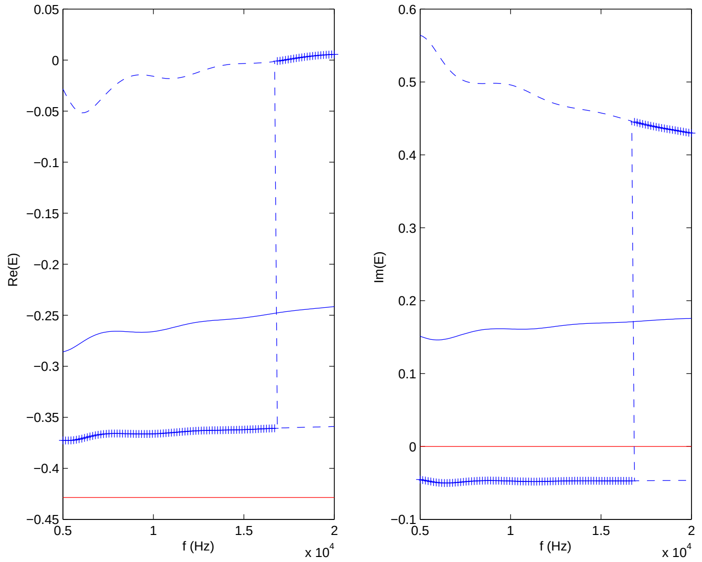

Figure 9: Same as fig. 8 except that $\rho^{[1]} = 700\ Kgm^{-3}$.

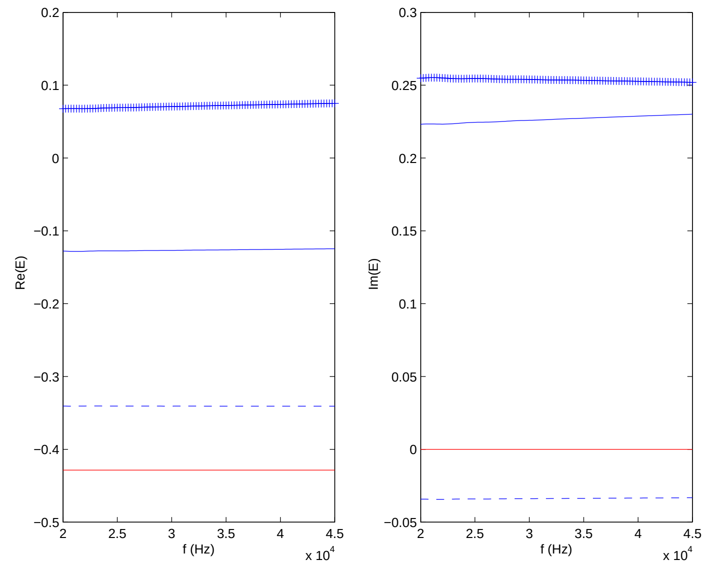

Figure 10: Same as fig. 9 except that $\mathbf{\Theta}^i = (0^\circ, 30^\circ, 5)$.

What changes between the figures 1-4 (relative to the narrow low-frequency range $f = [5\ KHz, 10\ KHz]$) and the figures 5-7 (relative to the wider, including high-frequencies, range $f = [0\ KHz, 50\ KHz]$) is $\mathbf{\Theta}^i$. The increase of the number of incident-angle realizations is observed to have the unexpected effect of decreasing the quality of the first-order mass-density retrievals. It is also observed that the second-order retrievals are divided into two widely-separated branches, one of which is rather close to the exact mass density ϵ, with the $+$ and $-$ solutions suddenly shifting between these two branches as the frequency increases. These sudden shifts have been observed in other inverse problems and can be qualified as 'instability' which is known to be one of the characteristics (although usually associated with data, rather than trial model, error [10, 5]) of the 'solutions' of ill-posed mathematical problems. The increase of the number N of realizations seems not to affect the height of the branch closest to ϵ, which suggests that this might be a useful indicator for distinguishing the 'good branch' from the 'bad' branch. Also the increase of N seems, for small N, to shift to higher frequencies the moment of shift from one branch to the other, but for larger N, ceases to produce this possibly-useful effect. Besides this, the 'good branches' of the second-order retrievals are clearly-closer than the corresponding first-order retrievals to the actual mass-density ϵ, as one would expect, and clearly, the first-order retrievals are way off mark. Other than these remarks , the noticeable feature of figs. 5-7, also discernible to some extent in figs. 1-4, is the *dispersive nature* of the retrieved mass density contrast even though the actual mass density contrast was assumed to be non-dispersive (i.e., to not depend on f). This dispersive nature is seen to particularly affect: 1) the first-order retrieval and 2) the particular second-order retrieval (of the two possible second-order retrievals) which is the farthest from the actual mass density contrast. This suggests that the dispersion of the retrieval is induced by trial model error and is all the larger the larger is the trial model error.

What changes between the two figures 8-9 is $\rho^{[1]}$. Other than the previous remarks which apply as well here as for a different layer mass density (and therefore different ϵ), one gets the impression that the effects of changing $\mathbf{\Theta}^i$ are not obviously-beneficial. These remarks also hold for figs. 9-10 which differ from each other by the choice (but not the number) of incident angles.

4.3.2 First and second-order retrievals as a function of the actual density contrast ϵ

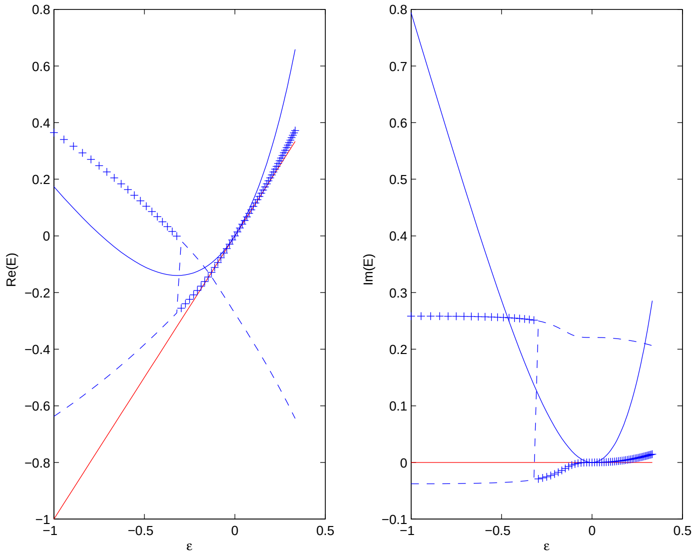

Figure 11: Retrieved (E) mass density contrast compared to the actual (ϵ) mass density contrast as a function of the actual density contrast ϵ. The left(right)-hand panels depict the real(imaginary) parts of ϵ (red——), $E^{(1)}$ (blue ——), $E^{(2)-}$ (blue - - - -), $E^{(2)+}$ (blue + + + +). Case $f = 20000\ Hz$, $c^{[1]} = 1700 - 210i\ ms^{-1}$, $h = 0.2\ m$, $\mathbf{\Theta}^i = (0°, 30°, 5)$.

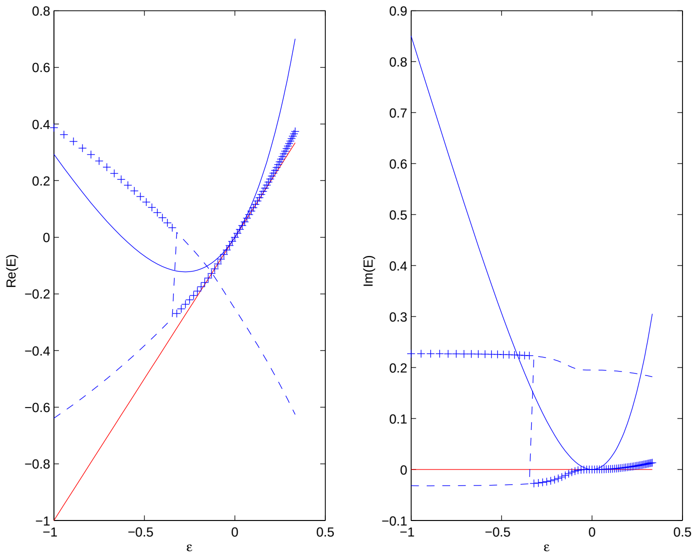

Figure 12: Same as fig. 11 except that $\mathbf{\Theta}^i = (0°, 0°, 1)$.

What changes between the two figures 11 and 12 is again $\mathbf{\Theta}^i$ which is seen to have a small effect on the first-order, but no visible effect on the second-order retrievals. Otherwise, the patterns are the same as previously: first-order retrieval rather far from the actual ϵ except for very small ϵ, and double-branch second-order retrievals, the 'bad' one of which is really way off from ϵ except for very small ϵ, and the 'good' one of which is quite close to ϵ, especially in the interval $\epsilon \approx [-0.2, 0.2]$, with the + and − solutions suddenly shifting between these two branches as ϵ increases. Due to this latter behavior there does not appear to exist a reliable way to spot the 'good' retrieval for a given ϵ. This is, in fact, only possible when one disposes of rather narrow-range a priori information (such as: E should lie in the interval $[E_{min}, E_{max}]$, with the difference of the two values being as small as possible) as to where the retrieved parameter should lie. However, if the trial model is rather crude, as in the case of the first-order model, the single retrieved solution might lie outside the range $[E_{min}, E_{max}]$ in which case one would be led to the conclusion that the retrieval (with this crude model) is impossible, this being an incorrect conclusion for a wider range $[E_{min}, E_{max}]$, so that one should be cautious about the use of a priori information for the retrieval.

4.3.3 First and second-order retrievals as a function of the layer thickness h

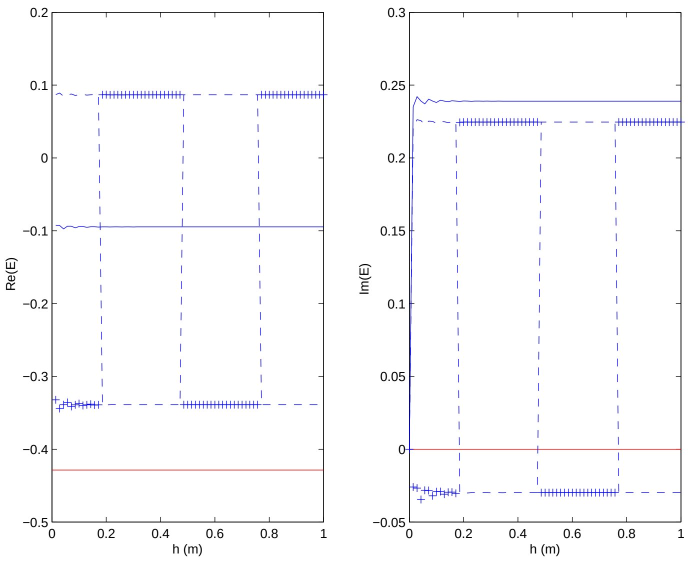

Figure 13: Retrieved (E) mass density contrast compared to the actual (ϵ) mass density contrast as a function of the layer thickness h. The left(right)-hand panels depict the real(imaginary) parts of ϵ (red——), $E^{(1)}$ (blue ——), $E^{(2)-}$ (blue - - - -), $E^{(2)+}$ (blue + + + +). Case $f = 20000\ Hz$, $\rho^{[1]} = 700\ Kgm^{-3}$, $c^{[1]} = 1700 - 210i\ ms^{-1}$, $\mathbf{\Theta}^i = (0°, 0°, 1)$.

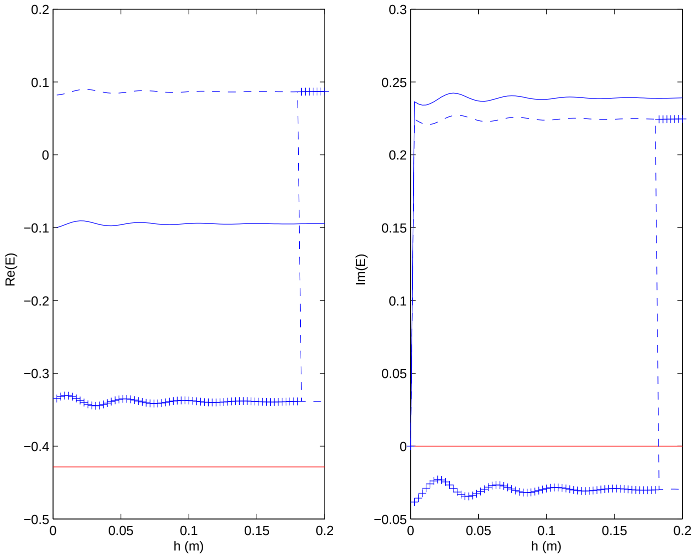

Figure 14: Same as fig. 13 for a different range of h.

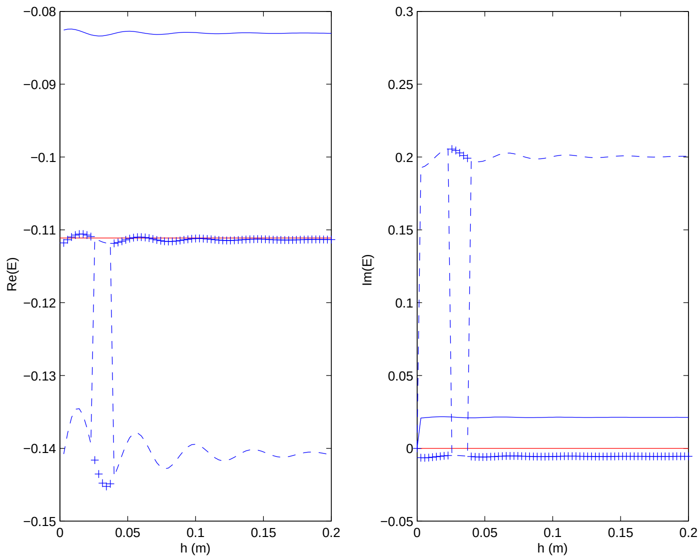

Figure 15: Same as fig. 14 except that $\rho^{[1]} = 900\ Kgm^{-3}$.

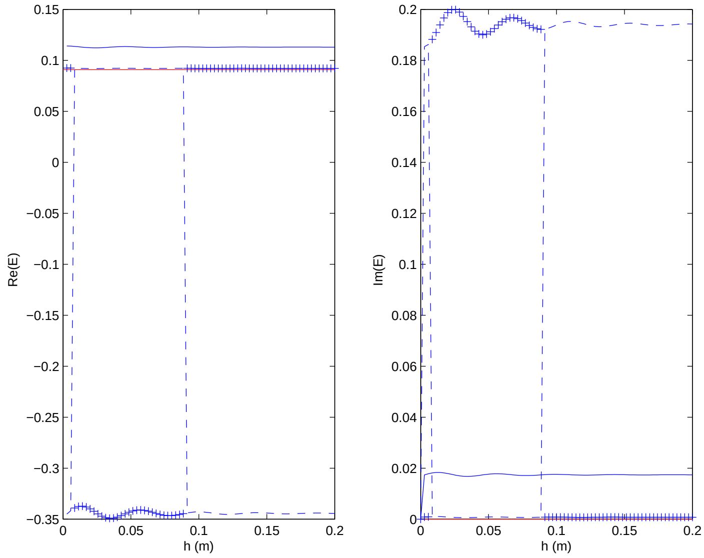

Figure 16: Same as fig. 14 except that $\rho^{[1]} = 1100\ Kgm^{-3}$.

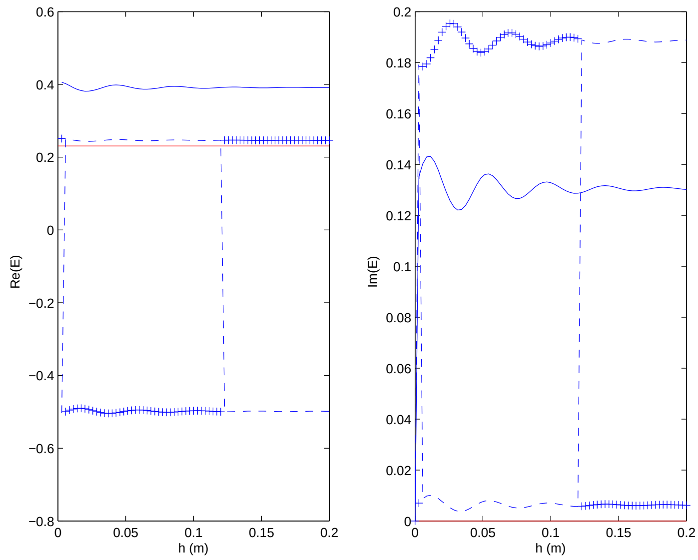

Figure 17: Same as fig. 14 except that $\rho^{[1]} = 1300\ Kgm^{-3}$.

What changes between fig. 13 and fig. 14 is the range of h, the latter figure constituting a sort of zoom to highlight the dispersive (with respect to h) behavior of the retrievals.

What changes between fig. 14 and fig. 17 is $\rho^{[1]}$, but this does not change the general pattern of behavior. Again, the dispersive (with respect to h) effects are quite apparent, more so for the first-order and second-order 'bad' branch retrievals than for the second-order 'good' branch retrievals, thus providing a possible manner of spotting the retrieval branch that is closest to the actual mass density contrast ϵ.

In the retrievals of figs. 13-17 it should be noticed that we have renounced at considering more than one incident angle realization because the figures in the previous subsections showed that taking more than one incident angle into account does not consistently improve the accuracy nor the uniqueness of the retrievals.

5 Conclusions

This investigation dealt in fine with the question of how small must the mass density contrast ϵ (which is unknown when it is the sought-for-parameter) be for the retrieval of ϵ to be reliable if at the outset the small (or zero)-ϵ assumption is incorporated in the trial model?

The trial model is nothing but a (method of) solution of a forward problem. Since our inquiry had to do with a small-ϵ assumption, we first searched for a small-ϵ, preferably mathematical, rather than numerical, solution of the forward-scattering problem. This cannot be easily-done for an obstacle of arbitrary shape, so that the first idea was to treat the obstacle as being a flat-faced layer submitted to a plane wave. The mathematically-explicit solution to this forward-scattering problem is easily-obtained by separation of variables applied to the BPDE, but the way ϵ intervenes in this solution is not easy to discern. For this reason, we chose to search for the solution via the domain integral formulation, whereby we found the same solution as previously, but in a form in which ϵ clearly emerges. Moreover, this new form of the solution lends itself to a perturbation analysis by which the solution, and thus the sought-for trial model (of similar form to what is found in [12, 11, 2]) is expressed as a series of powers of ϵ, the coefficients of which were obtained in explicit, algebraic form.

What we called the zeroth-order trial model results from neglecting all except the first terms in the series, and this model (which does not involve ϵ and therefore is as if ϵ were equal to zero) is nothing but the one resulting from the constant-density assumption. We showed that the inverse problem of the retrieval of ϵ is trivially-impossible using this zeroth-order trial model.

Next, we called the first-order trial model the one that results from neglecting all except the first two terms in the series. The employment of this linearized (in terms of ϵ) model enabled, via comparison with simulated data relative to the transmitted field, the retrieval problem to be cast as a linear algebraic equation which was easily-solved for the mass density contrast. The solution was thus found to exist and be unique, these being features of inverse problems that are often strived-for.

Finally, we called the second-order trial model the one that results from neglecting all except the first three terms in the series. The employment of this model enabled, via comparison with simulated data relative to the transmitted field, the retrieval problem to be cast as a quadratic equation which was again easily and explicitly-solved for the mass density contrast. The solution was found to exist, but not to be unique since the quadratic equation possesses two solutions, both of which are equally-plausible.

Since there is no obvious relation between the solutions for the mass density contrast obtained via these three trial models, one can say that either we are not obtaining a solution, obtaining a single solution, or obtaining two solutions that all seem to differ from each other solely on account of the fact of employing different trial models to solve the inverse problem. This result is similar to what happens when one or more of the priors (i.e., the parameters that are not retrieved, but assumed to be known) in the trial model are, in fact, different from the corresponding parameters in the model employed to simulate the scattered field data, or different from what is guessed to be these parameters in actually-measured data [7, 9].

We showed that the use of approximate trial models (i.e., those corresponding to the various orders of ϵ) lead (this was shown numerically) to several remarkable features concerning the retrieved mass density contrast:
1) complex retrievals of what is a real quantity,
2) frequency dispersion of this parameter although the latter is assumed at the outset to be non-dispersive,
3) branch-hopping for the second-order trial model solutions which means, for example, that the solution skips from one branch to another in apparently-haphazard fashion as the frequency (or another parameter) increases,
4) first-order (i.e., based on a very small density contrast assumption) retrievals that are way off mark,
5) one of the second-order solutions is more affected by frequency dispersion, and is farther from the actual mass density contrast, than the other solution, which fact furnishes a possible means of spotting the 'right' retrieval, provided, of course, the to-be-retrieved parameter is known a priori to actually be non- or feebly-dispersive in the frequency range of interest.

Not unsurprisingly, we found that the 'right' second-order retrievals were generally-closer than the first-order retrievals to the actual ϵ for a diversity of priors and frequencies. In fact, the 'right' second-order retrieval was found to be practically-coincident with the actual ϵ for $\epsilon \in [-0.2, 0.2]$. Finally, we found that increasing the amount of angle of incidence realizations does not have a systematically-beneficial effect on the quality of the retrievals.

It follows from the preceding remarks and [16] that the mass density contrast (ϵ, which equals zero when the constant density assumption is made) is a constitutive parameter that plays an important role in the forward problem solution for the scattered field transmitted by the fluid-like layer solicited by a plane acoustic wave, and consequently this role is as, if not more, important in the inverse problem context in which one strives to retrieve the mass density contrast ϵ from transmitted field data (the same is probably true for the other constitutive parameters and perhaps for the characteristics of the solicitation).

Another, more general, conclusion of this study (underlined in our previous investigation [12] for a much simpler scattering configuration), has to do with another aspect of the non-uniqueness of inverse problem 'solutions': there exist as many of such 'solutions' as the number of mathematical translations (here related to the number of terms in the ϵ series) of the trial models one might want to employ.

Choosing the 'most accurate' trial model does not eliminate this paradox when real, measured, data is compared to the trial model, because it is not certain a priori that this data is as 'exact' (i.e., devoid of systematic and random errors) as the the trial model and this discrepancy can lead to a difficulty whose origin is described in the next paragraph. Moreover, the adopted trial model necessarily involves a number (which usually is larger the more exact is the trial model) of priors (i.e., the parameters which are not retrieved, but whose values are assumed to be known) and the uncertain knowledge one may have concerning these priors can induce the same type of non-uniqueness of the retrieval [9] as the variety of mathematical expressions for the trial-model. As far as we know, no method, including FWI (full-wave inversion [6, 11, 14]), are immune to these pathological aspects of inverse problem 'solving'.

Finally, a nagging issue in the inverse problem context concerns how to obtain what one can qualify as the 'best' retrieval. If, for example, the sought-for parameter is the mass density contrast ϵ, then the 'best' retrieval is obviously attained when the trial parameter E is equal to ϵ, in which case, whatever the definition of the cost function, the latter attains the value zero. In our study, we employed exact simulated data and compared it to approximate trial models with the objective of attaining a zero cost, the result of which generally led to a solution $E \neq \epsilon$ which cannot be qualified as 'best' even though it is associated with zero cost. The question is then: how to obtain $E = \epsilon$ and zero cost at the same time? The answer is that the simulated pressure field data must be identical to the trial model prediction of the pressure field, this meaning, that we should have simulated the data with the same ϵ series as the one employed for the trial model (with, of course, the mass-density contrast being denoted by ϵ in the data series of $l+1$ terms, and by E in the trial model series of $L+1$ terms), which would have led, after imposing the zero- cost condition, to $E = \epsilon$, whatever the number $(l+1 = L+1 > 1)$ of terms in both these series. From this one must conclude that the 'best' retrieval is obtained by tailoring the trial model to the data (this is called committing the 'inverse crime' [13]), which means that if the data is 'bad', the trial model should be 'bad' and if the data is 'good', the trial model should be 'good'. Of course, tailoring of this sort is only possible when the data is simulated, which means, that when the data is real (as opposed to simulated), it is: 1) impossible to commit the 'inverse crime' and 2) impossible to obtain the 'best' retrieval of the sought-for parameter(s).

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