GLOBAL EXISTENCE AND DECAY ESTIMATE OF SOLUTION TO COMPRESSIBLE QUANTUM NAVIER-STOKES EQUATIONS WITH DAMPING

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Abstract. In this paper, we consider the Cauchy problem of the compressible quantum Navier-Stokes equations with damping in $\mathbb{R}^3$. We first assume that the H^3-norm of the initial data is sufficiently small while the higher derivative can be arbitrarily large, and prove the global existence of smooth solutions. Then the decay estimate of the solution is derived for the initial data in a homogeneous Sobolev space or Besov space with negative exponent. In addition, the usual $L^p - L^2$ $(1 \leq p \leq 2)$ type decay rate is obtained without assuming that the L^p norm of the initial data is sufficiently small.

Keywords Compressible quantum Navier-Stokes equations, Global existence, Decay estimate, Cauchy problem, Damping

1. Introduction

Quantum fluid model is widely used in many fields of natural science, such as weakly interacting Bose gases [12], quantum semiconductors [9], quantum trajectories of Bohm mechanics [32] and so on. In 1927, Madelung [22] discovered the hydrodynamic form of the singlestate Schrödinger equation. In 1993, Ferry-Zhou [9] derived the so-called quantum hydrodynamic equations from the Bloch equation of the density matrix, and in 1994, Gardner [10] derived the same equations from the Wigner equation by the moment method. At the beginning of this century, the dissipative quantum fluid model was proposed. For example, the viscous quantum Euler model is obtained by the moment method applied to the Wigner-Fokker-Planck equation [13]. The Chapman-Enskog expansion in the Wigner equation leads to the quantum Navier-Stokes (QNS) equations under some assumptions [7]. In this paper, we consider the compressible QNS equations with damping:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, & (1.1) \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P - 2\nu \operatorname{div}(\rho \mathbb{D} u) = -r_0 u - r_1 \rho |u|^2 u + 2\kappa^2 \rho \nabla \left(\dfrac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), & (1.2) \end{cases}$$

with initial data

$$(\rho, u)(x,t)|_{t=0} = (\rho_0, u_0) \to (\bar{\rho}, 0), \quad \text{as } |x| \to \infty. \tag{1.3}$$

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Here, $\rho = \rho(x,t)$ is the density, $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity, $u \otimes u$ denotes the matrix made up of the element $u_i u_j$, and $\mathbb{D}u = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$ is the symmetrical part of the velocity gradient. The pressure $P = P(\rho)$ is a smooth function with $P'(\rho) > 0$ for $\rho > 0$, and $\bar{\rho} > 0$ is a constant. The positive constants ν and κ are called viscosity coefficient and dispersion coefficient (or Plank constant), respectively. In particular, the inviscid QNS equation ($\nu = 0$) is a superfluid model (see [20]), and its global weak solution with finite energy is studied in [1, 2]. The expression $\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$ in the last term at the right-hand side of the equation (1.2) satisfies

$$2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \operatorname{div}(\rho\nabla^2\log\rho) = \Delta\nabla\rho + \frac{|\nabla\rho|^2\nabla\rho}{\rho^2} - \frac{\nabla\rho\Delta\rho}{\rho} - \frac{\nabla\rho\cdot\nabla^2\rho}{\rho},$$

we call it the quantum potential or Bohm potential, which comes from the hydrodynamic formula of the singlestate Schrödinger equation, and considered by Wigner [31] for the thermodynamic equilibrium. The damping terms

$$-r_0 u - r_1\rho|u|^2 u,$$

where $r_0 \geq 0,\ r_1 \geq 0$, are proposed by [5].

If $r_0 = r_1 = 0$, i.e., no damping term, the equations (1.1)-(1.2) become the QNS equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, & (1.4)\\ (\rho u)_t + \operatorname{div}(\rho u\otimes u) + \nabla P - 2\nu\operatorname{div}(\rho\mathbb{D}u) = 2\kappa^2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). & (1.5)\end{cases}$$

At present, there are abundant research results on undamped compressible QNS equations. For 1D case, Jüngel [17] proved the global existence of global weak solutions. And for 3D case, Jüngel [18] also proved the global existence of weak solutions on the 3D torus with large initial datas by using a kind of special test functions (i.e. $\rho\varphi$, where φ is smooth) when $\nu < \kappa$. Subsequently, Dong [8] and Jiang [16] extended the results to the cases of $\nu = \kappa$ and $\nu > \kappa$, respectively. In 2015, with the help of adding an extra cold pressure which is introduced in [6], Gisclon et al. [11] gave a different definition of weak solutions and then obtained the global existence. They also studied the limit of the QNS system as $\kappa \to 0$. In 2017, Antonelli-Spirito [3] obtained the global existence of finite energy weak solutions for QNS equations for large initial data in 2D with $\nu > \kappa$ while in 3D coupled with $\frac{9}{8}\kappa^2 > \nu^2 > \kappa^2$. In particular, the concept of weak solutions is standard, which does not contain additional terms like cold pressure or damping. And in 2019, Tang-Zhang [29] obtained the global existence of weak solutions (standard concept) of compressible QNS equations in the critical case of $\nu = \kappa$.

For the case $r_0 > 0,\ r_1 > 0$, Vasseur-Yu [30] have proved that the existence of global weak solutions of the system (1.1)-(1.2) in 3D with large data where they assumed ∇u is well-defined. Later in 2019, Lü et al. [21] obtained some new priori estimates, which help they remove the assumption of ∇u in [30], and they also got the existence of global weak solutions of the system (1.1)-(1.2).

And if $r_0 > 0,\ r_1 = 0$, the damping term is linear. There are some results about the asymptotic limit and the incompressible limit, refer to [19, 33] and the references therein for more detail description.

In this paper, we consider the case of $r_0 > 0,\ r_1 \geq 0$. We prove that if the initial data is small then the Cauchy problem (1.1)-(1.3) has a unique global smooth solution in $\mathbb{R}^3$. Besides, we also derive the time decay rate of the solution. In fact, our results also hold for the case $r_0 = 0$, see remark 2.1.

The rest of this paper is arranged as follows. In section 2, we first present some basic definitions, the main results and a series of auxiliary lemmas. In section 3, we derive the exact energy estimate and prove the local existence of the solution $(\varrho, \nabla\varrho, u)$ of the Cauchy problem (2.1)-(2.3). More specifically, the dissipative estimate of u is derived in Lemma 3.1, and the dissipative estimate of $\varrho, \nabla\varrho$ are derived by an interactive energy estimate in Lemma 3.2, and then the local existence of the solution is obtained by solving the iterative approximation system. In section 4, we derive the evolution of the Sobolev norm and Besov norm with negative exponent of $(\varrho, \nabla\varrho, u)$. In the last section, we complete the proof of the main theorem of this paper.

2. Main results and auxiliary Lemmas

For convenience, we first do some transformation of the system (1.1)-(1.2). We set $\varrho = \rho - \bar{\rho}$, and $P'(1) = 1,\ \bar{\rho} = 1,\ 2\nu = 1,\ \kappa = 1$. In this way, the system (1.1)-(1.2) can be rewritten as

$$\begin{cases} \varrho_t + \operatorname{div} u = S_1, & (2.1) \\ u_t - \operatorname{div} \mathbb{D}u + \nabla\varrho - \Delta\nabla\varrho = S_2, & (2.2) \end{cases}$$

the corresponding initial data is

$$(\varrho, u)(x,t)|_{t=0} = (\varrho_0, u_0) \to (0,0), \quad \text{as } |x| \to \infty, \tag{2.3}$$

where $S_i\ (i = 1,2)$ is defined by

$$\begin{aligned} S_1 =& -\operatorname{div}(\varrho u), \\ S_2 =& -r_0 g(\varrho)u - r_1|u|^2u - u\cdot\nabla u - f(\varrho)\Delta\nabla\varrho - h(\varrho)\nabla\varrho + g(\varrho)\nabla\varrho\cdot\mathbb{D}u \\ &+ \big(g^3(\varrho)|\nabla\varrho|^2\nabla\varrho - g^2(\varrho)\nabla\varrho\Delta\varrho - g^2(\varrho)\nabla\varrho\cdot\nabla^2\varrho\big), \end{aligned}$$

and

$$f(\varrho) = \frac{\varrho}{\varrho+1}, \quad h(\varrho) = \frac{P'(\varrho+1)}{\varrho+1} - \frac{P'(1)}{1}, \quad g(\varrho) = \frac{1}{\varrho+1}.$$

Notation. In this paper, we use the following symbolic conventions. The same letter C may be used in different estimates to represent general constants. Sometimes, in order to show emphasis, $C_1, \overline{C}$ and so on are also used. If $A \leq CB$ holds for a constant $C > 0$, then denote as $A \lesssim B$. We say that A is equivalent to B, if there exist two positive constans $C_1,\ C_2$ such that $C_1B \leq A \leq C_2B$, and denote as $A \sim B$. In particular, C_0 represents a constant that depends on the initial data. The gradient operator $\nabla = \partial_x = \big(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}\big),\ \nabla_i = \partial_{x_i}\ (i = 1,2,3)$. For convenience, we denote

$$\big\|\nabla^l(f,g)\big\|_X^k := \big\|\nabla^l f\big\|_X^k + \big\|\nabla^l g\big\|_X^k,$$

where for integer $l \geq 0$, $\nabla^l = \partial_x^l$ represents the usual l-order spatial derivative, k is a positive integer, and X is a Banach space. For $l < 0$ or l is not a positive integer, the operator Λ^l is defined by

$$\Lambda^l f := \mathscr{F}^{-1}\big(|\xi|^l \mathscr{F} f\big),$$

where $\mathscr{F}$ is the usual Fourier transformation operator and $\mathscr{F}^{-1}$ is its inverse operator.

$L^p(\mathbb{R}^3), 1 \le p \le \infty$ denotes the usual L^p space, with the norm $\|\cdot\|_{L^p}$; $H^N(\mathbb{R}^3)$, $N \in \mathbb{Z}$ denotes the usual Sobolev space, with the norm $\|\cdot\|_{H^N}$; $\dot{H}^{-s}(\mathbb{R}^3), s \in \mathbb{R}$ denotes the homogeneous Sobolev space in $\mathbb{R}^3$, and the norm $\|\cdot\|_{\dot{H}^{-s}}$ is defined by

$$\|f\|_{\dot{H}^{-s}} := \left\|\Lambda^{-s} f\right\|_{L^2};$$

$\dot{B}^{-s}_{2,\infty}(\mathbb{R}^3), s \in \mathbb{R}$ denotes the homogeneous Besov space, and the norm $\|\cdot\|_{\dot{B}^{-s}_{2,\infty}}$ is defined by

$$\|f\|_{\dot{B}^{-s}_{2,\infty}} := \sup_{j\in\mathbb{Z}} 2^{-sj}\left\|\dot{\Delta}_j f\right\|_{L^2(\mathbb{R}^3)},$$

where $\dot{\Delta}_j f$ denotes the homogeneous Littlewood-Paley decomposition for f (see [4]).

For any integer $N \ge 3$, define the energy functional $\mathfrak{E}_N(t)$ and the corresponding dissipation rate $\mathfrak{D}_N(t)$ by

$$\mathfrak{E}_N(t) := \sum_{l=0}^{N} \left\|\nabla^l(\varrho, \nabla\varrho, u)\right\|_{L^2}^2,$$

$$\mathfrak{D}_N(t) := \sum_{l=0}^{N} \left\|\nabla^l(\nabla\varrho, \nabla^2\varrho, u, \nabla u)\right\|_{L^2}^2.$$

The key point to prove the global existence of the solution of the Cauchy problem (2.1)-(2.3) is that, for $k \ge 0$, construct an interactive energy functional (see [14])

$$\begin{aligned}
&\frac{d}{dt}\sum_{l=k}^{k+2}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx + \frac{1}{4}\sum_{l=k}^{k+2}\left\|\nabla^{l+1}(\varrho,\nabla\varrho)\right\|_{L^2}^2 \\
\le& C\|(\varrho,u)\|_{H^3}\sum_{l=k}^{k+2}\left\|\nabla^{l+1}(\varrho,\nabla\varrho,u)\right\|_{L^2}^2 + C\left(\|\varrho\|_{H^3}^2+\|u\|_{H^3}^2\right)\sum_{l=k}^{k+2}\left\|\nabla^{l+1}(\varrho,u)\right\|_{L^2}^2 \\
&+ C\|\varrho\|_{H^3}\|u\|_{H^3}\sum_{l=k}^{k+2}\left\|\nabla^{l+1}\varrho\right\|_{L^2}^2 + 3\sum_{l=k}^{k+2}\left\|\nabla^{l+1}u\right\|_{L^2}^2 + r_0^2\sum_{l=k}^{k+2}\left\|\nabla^l u\right\|_{L^2}^2,
\end{aligned}$$

and then the dissipative estimates of ϱ and $\nabla\varrho$ can be derived (Lemma 3.2). Combined with Lemma 3.1, we can get that for $N \ge 3$, there holds

$$\frac{d}{dt}\mathfrak{E}_N(t) + \mathfrak{D}_N(t) \lesssim \sqrt{\mathfrak{E}_3(t)}\mathfrak{D}_N(t). \tag{2.4}$$

If $\mathfrak{E}_3(0)$ is sufficiently small, then the global existence of the solution in H^N can be derived from (2.4), where $N \ge 3$.

In addition, if the initial data $(\varrho_0, \nabla\varrho_0, u_0)$ belongs to a negative exponential homogeneous Sobolev space or Besov space, then the solution of (2.1)-(2.3) and the further decay rate of its higher order space derivatives can be obtained. In fact, using regularity interpolation method (see Strain-Guo [27], Guo-Wang [15] and Sohinger-Strain [25]), we can develop a general energy method, that is, using a family of scaled energy estimates with minimum derivative counts and interpolation among them to obtain

$$\frac{d}{dt}\mathfrak{E}_k^{k+2} + \mathfrak{D}_k^{k+2} \lesssim \sqrt{\mathfrak{E}_3(t)}\mathfrak{D}_k^{k+2},$$

where $\mathfrak{E}_k^{k+2}$ and $\mathfrak{D}_k^{k+2}$ are defined by (5.1) and (5.2), respectively. Then using the properties of negative homogeneous space, we can use $\mathfrak{E}_k^{k+2}$ to bound $\mathfrak{D}_k^{k+2}$. Thus, we can derive the estimae of time decay by solving an explicit inequality.

2.1. **Main results.** The main results of this paper are described as follows.

Theorem 2.1. *Assume that the initial data*

$$\big(\varrho_0, \nabla\varrho_0, u_0\big) \in H^N \tag{2.5}$$

for any integer $N \geq 3$, and

$$\int_{\mathbb{R}^3} \varrho_0 dx = 0 \quad (\textit{neutral condition}).$$

Then

(I) There is a sufficiently small constant $\delta_0 > 0$ such that, if

$$\mathfrak{E}_3(0) \leq \delta_0, \tag{2.6}$$

then (2.1)-(2.3) admits a unique global solution $(\varrho, \nabla\varrho, u) \in C\big([0,\infty]; H^3\big(\mathbb{R}^3\big)\big)$ satisfying

$$\sup_{0\leq t\leq\infty} \mathfrak{E}_3(t) + \int_0^\infty \mathfrak{D}_3(\tau)d\tau \leq C\mathfrak{E}_3(0). \tag{2.7}$$

Furthermore, if $\mathfrak{E}_N(0) < \infty$ for any $N \geq 4$, then

$$\sup_{0\leq t\leq\infty} \mathfrak{E}_N(t) + \int_0^\infty \mathfrak{D}_N(\tau)d\tau \leq C\mathfrak{E}_N(0). \tag{2.8}$$

(II) Suppose all the assumptions in (I) are true.
If $\big(\varrho_0, \nabla\varrho_0, u_0\big) \in \dot{H}^{-s}$ for some $s \in \big[0, \frac{3}{2}\big)$, then

$$\|(\varrho, \nabla\varrho, u)(t)\|_{\dot{H}^{-s}} \leq C_0, \quad \forall t \geq 0; \tag{2.9}$$

or $\big(\varrho_0, \nabla\varrho_0, u_0\big) \in \dot{B}^{-s}_{2,\infty}$ for some $s \in \big(0, \frac{3}{2}\big]$, then

$$\|(\varrho, \nabla\varrho, u)(t)\|_{\dot{B}^{-s}_{2,\infty}} \leq C_0, \quad \forall t \geq 0. \tag{2.10}$$

Besides, there is decay estimate

$$\big\|\nabla^k(\varrho, \nabla\varrho, u)(t)\big\|_{L^2} \leq C_0(1+t)^{-\frac{k+s}{2}}, \quad k = 0, 1, \cdots, N-2. \tag{2.11}$$

Remark 2.1. *In fact, if $r_0 = 0$, we just need to adjust our proof process a little bit then there is no dissipative term $\big\|\nabla^l u\big\|_{L^2}^2$ meanwhile no $\delta\big\|\nabla^l u\big\|_{L^2}^2$, and we can get that the theorem 2.1 also holds.*

Considering the Hardy-Littlewood-Sobolev theorem (Lemma 2.4), $L^p \subset \dot{H}^{-s}$ holds for $p \in (1, 2]$, where $s = 3\big(\frac{1}{p} - \frac{1}{2}\big)$. However, this embedding result does not hold for $p = 1$. In view of this defect, Sohinger et al. [25] introduced the homogeneous Besov space $\dot{B}^{-s}_{2,\infty}$. Lemma 2.5 shows that, for $p = 1$, $\dot{B}^{-s}_{2,\infty}$ is used to replace $\dot{H}^{-s}$, and the embedded result $L^1 \subset \dot{B}^{-3/2}_{2,\infty}$ holds. In this way, we can derive the optimal $L^p - L^2$ type decay result by Theorem 2.1:

Corollary 2.2. *Under the assumptions of Theorem 2.1, if we replace the hypothesis about* $\dot{H}^{-s}$ *or* $\dot{B}^{-s}_{2,\infty}$ *with* $(\varrho_0, \nabla\varrho_0, u_0) \in L^p$ *for some* $p \in [1,2]$*, then*

$$\left\|\nabla^k(\varrho, \nabla\varrho, u)(t)\right\|_{L^2} \leq C_0(1+t)^{-\frac{k+s_p}{2}}, \quad k = 0, 1, \cdots, N-2, \tag{2.12}$$

where s_p *is defined by*

$$s_p := 3\left(\frac{1}{p} - \frac{1}{2}\right).$$

Remark 2.2. *It should be noted that for the general existence in Theorem 2.1, we only assume that* $\left\|(\varrho_0, \nabla\varrho_0, u_0)\right\|_{H^3}$ *is sufficiently small, while the higher derivative can be arbitrarily large. We restrict* $s < \frac{3}{2}$ *to facilitate the application of Lemma 2.4 to the negative exponential Sobolev estimate of nonlinear terms by* Λ^{-s}*. For* $s \geq \frac{3}{2}$*, this estimate is invalid. In corollary 2.2, in order to make* $p > 1$*, it is also necessary to restrict* $s < \frac{3}{2}$*. In addition, the* L^2 *decay rate of the higher order spatial derivative is obtained by corollary 2.2. By Sobolev interpolation, we can derive the general optimal* L^q $(2 \leq q \leq \infty)$ *time decay estimates of the solution.*

2.2. **Auxiliary lemmas.** In order to prove Theorem 2.1, we give a series of auxiliary lemmas in this section, which will be widely used in the arguments in the next three sections. In particular, Lemma 2.1-Lemma 2.3 plays an important role in the derivation of many prior estimates. We first explain the following Gagliardo-Nirenberg inequality with the form of Sobolev interpolation.

Lemma 2.1. *Let* $0 \leq \gamma,\ \alpha \leq \beta$*. Then*

$$\left\|\nabla^\gamma f\right\|_{L^p} \lesssim \left\|\nabla^\alpha f\right\|_{L^q}^{1-\sigma} \left\|\nabla^\beta f\right\|_{L^r}^{\sigma},$$

where $0 \leq \sigma \leq 1$*, and* γ *satisfies*

$$\frac{\gamma}{3} - \frac{1}{p} = \left(\frac{\alpha}{3} - \frac{1}{q}\right)(1-\sigma) + \left(\frac{\beta}{3} - \frac{1}{r}\right)\sigma.$$

For $p = \infty$*, it is necessary to restrict* $0 < \sigma < 1$*. In particular, the following formulas hold:*

$$for\ 2 \leq p \leq 6,\ \|f\|_{L^p} \lesssim \|f\|_{H^1};\ for\ 6 < p \leq \infty,\ \|f\|_{L^p} \lesssim \|f\|_{H^2}.$$

Proof. see [24, p.125]. □

Lemma 2.2. *Let* $m \geq 1$ *be an integer, then*

$$\left\|\nabla^m(fg)\right\|_{L^p} \lesssim \left\|\nabla^m f\right\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \left\|\nabla^m g\right\|_{L^{p_4}},$$

where $p,\ p_1,\ p_2,\ p_3,\ p_4 \in [1, +\infty]$*, and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Proof. see [23, lemma 3.4, p.129]. □

The following results show that $\left\|\nabla^k f(w)\right\|_{L^p} (p \geq 2)$ can be bounded by $\left\|\nabla^k w\right\|_{L^p}$.

Lemma 2.3. *Let f be a smooth function, and* $w : \mathbb{R}^3 \to \mathbb{R}^n (n \geq 1)$ *satisfies* $\|w\|_{H^3(\mathbb{R}^3)} \leq \delta \leq 1$*. If* $f(w) \sim w$ *or* $f(0) = 0$*, then for any integer* $k \geq 0$*, there is*

$$\left\|\nabla^k f(w)\right\|_{L^p} \leq C(k) \left\|\nabla^k w\right\|_{L^p}.$$

Proof. see [28]. □

If $s \in \left[0, \frac{3}{2}\right)$, according to Hardy-Littlewood-Sobolev theorem, we can use $\|f\|_{L^p}$ to bound $\|f\|_{\dot{H}^{-s}}$.

Lemma 2.4. *Let $0 \le s < \frac{3}{2}$ and $\frac{1}{2} + \frac{s}{3} = \frac{1}{p}$. Then $1 < p \le 2$ and*

$$\|f\|_{\dot{H}^{-s}} \lesssim \|f\|_{L^p}.$$

Proof. see [26, Theroem 1, p.119]. □

Similarly, for homogeneous Besov spaces, if $s \in \left(0, \frac{3}{2}\right]$, then we have the following result.

Lemma 2.5. *Let $0 < s \le \frac{3}{2}$ and $\frac{1}{2} + \frac{s}{3} = \frac{1}{p}$. Then $1 \le p < 2$ and*

$$\|f\|_{\dot{B}^{-s}_{2,\infty}} \lesssim \|f\|_{L^p}.$$

Proof. see [15, Lemma 4.6]. □

Next, we introduce a special Sobolev interpolation inequality.

Lemma 2.6. *Let $s \ge 0$ and integer $l \ge 0$. Then*

$$\left\|\nabla^l f\right\|_{L^2} \le \left\|\nabla^{l+1} f\right\|_{L^2}^{1-\theta} \|f\|_{\dot{H}^{-s}}^{\theta},$$

where $\theta = \frac{1}{l+1+s}$.

Proof. It can be obtained directly from Parseval theorem and Hölder inequality. □

Finally, we will apply the following special Besov interpolation inequality.

Lemma 2.7. *Let $s > 0$ and integer $l \ge 0$. Then*

$$\left\|\nabla^l f\right\|_{L^2} \le \left\|\nabla^{l+1} f\right\|_{L^2}^{1-\theta} \|f\|_{\dot{B}^{-s}_{2,\infty}}^{\theta},$$

where $\theta = \frac{1}{l+1+s}$.

Proof. see [15, Lemma 4.5]. □

3. Energy estimates and local existence

3.1. **Uniform a priori estimate.** We will prove Theorem 2.1 according to the local existence of the solution of (2.1)-(2.3), some uniform a priori estimates and the continuity argument. In this section, starting from the hypothesis of Theorem 2.1, we first analyze the uniform a priori estimate of (2.1)-(2.3) and derive the dissipative estimate of u.

Lemma 3.1. *Suppose all the assumptions in Theorem 2.1 are true, and for $T > 0$,*

$$\sup_{0 \le t \le T} \|(\varrho, \nabla\varrho, u)\|_{H^3} \le \delta << 1.$$

Then, for any integer $k \ge 0$ and $0 \le t \le T$,

$$\begin{aligned} &\frac{d}{dt}\sum_{l=k}^{k+2}\left\|\nabla^l(\varrho, \nabla\varrho, u)\right\|_{L^2}^2 + \sum_{l=k}^{k+2}\left(\left\|\nabla^{l+1}u\right\|_{L^2}^2 + \left\|\nabla^l(\operatorname{div} u)\right\|_{L^2}^2 + \left\|\nabla^l u\right\|_{L^2}^2\right) \\ &\le C\delta\sum_{l=k}^{k+2}\left\|\nabla^l\left(\nabla\varrho, \nabla^2\varrho, u, \nabla u\right)\right\|_{L^2}^2. \end{aligned} \tag{3.1}$$

Proof. For any integer $k \geq 0$, by the ∇^l $(l = k, k+1, k+2)$ energy estimate on ϱ, u for (1.6)-(1.7), respectively, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big(|\nabla^l\varrho|^2+|\nabla^l u|^2\big)dx+\frac{1}{2}\big(\|\nabla^{l+1}u\|_{L^2}^2+\|\nabla^l(\operatorname{div}u)\|_{L^2}^2\big)\\
=&-r_0\int_{\mathbb{R}^3}\nabla^l(g(\varrho)u)\cdot\nabla^l udx-r_1\int_{\mathbb{R}^3}\nabla^l\big(|u|^2u\big)\cdot\nabla^l udx+\int_{\mathbb{R}^3}\nabla^l(\Delta\nabla\varrho)\cdot\nabla^l udx\\
&+\int_{\mathbb{R}^3}\nabla^l(-\varrho\operatorname{div}u)\cdot\nabla^l\varrho dx+\int_{\mathbb{R}^3}\nabla^l\big(-u\cdot\nabla\varrho\big)\cdot\nabla^l\varrho dx-\int_{\mathbb{R}^3}\nabla^l(u\cdot\nabla u)\cdot\nabla^l udx\\
&-\int_{\mathbb{R}^3}\nabla^l(f(\varrho)\Delta\nabla\varrho)\cdot\nabla^l udx-\int_{\mathbb{R}^3}\nabla^l(h(\varrho)\nabla\varrho)\cdot\nabla^l udx\\
&+\int_{\mathbb{R}^3}\nabla^l(g(\varrho)\nabla\varrho\cdot\mathbb{D}u)\cdot\nabla^l udx+\int_{\mathbb{R}^3}\nabla^l(g^3(\varrho)|\nabla\varrho|^2\nabla\varrho)\cdot\nabla^l udx\\
&-\int_{\mathbb{R}^3}\nabla^l\big(g^2(\varrho)\nabla\varrho\Delta\varrho\big)\cdot\nabla^l udx-\int_{\mathbb{R}^3}\nabla^l\big(g^2(\varrho)\nabla\varrho\cdot\nabla^2\varrho\big)\cdot\nabla^l udx\\
:=&\sum_{i=1}^{12}I_i. \qquad (3.2)
\end{aligned}
$$

Next, we will estimate I_i $(i = 1, 2, \cdots, 12)$. Note that $g(\varrho) = 1 - f(\varrho)$ and $f(\varrho) \sim \varrho$, and then applying Hölder inequality, Lemmas 2.1-2.3, we get

$$
\begin{aligned}
I_1=&-r_0\int_{\mathbb{R}^3}\nabla^l(u-f(\varrho)u)\cdot\nabla^l udx\\
\leq&-r_0\|\nabla^l u\|_{L^2}^2+C\|\nabla^l(f(\varrho)u)\|_{L^2}\|\nabla^l u\|_{L^2}\\
\leq&-r_0\|\nabla^l u\|_{L^2}^2+C\big(\|\nabla^l\varrho\|_{L^6}\|u\|_{L^3}+\|\varrho\|_{L^\infty}\|\nabla^l u\|_{L^2}\big)\|\nabla^l u\|_{L^2}\\
\leq&-r_0\|\nabla^l u\|_{L^2}^2+C\big(\|\nabla^{l+1}\varrho\|_{L^2}\|u\|_{H^1}+\|\varrho\|_{H^2}\|\nabla^l u\|_{L^2}\big)\|\nabla^l u\|_{L^2}\\
\leq&-r_0\|\nabla^l u\|_{L^2}^2+C\|(\varrho,u)\|_{H^2}\big(\|\nabla^l u\|_{L^2}^2+\|\nabla^{l+1}\varrho\|_{L^2}^2\big). \qquad (3.3)
\end{aligned}
$$

Using Hölder inequality, Sobolev inequality and Lemma 2.2, lead to

$$
\begin{aligned}
I_2\leq& r_1\|\nabla^l\big(|u|^2u\big)\|_{L^{6/5}}\|\nabla^l u\|_{L^6}\\
\lesssim&\big(\|\nabla^l(u\cdot u)\|_{L^2}\|u\|_{L^3}+\|u\cdot u\|_{L^{3/2}}\|\nabla^l u\|_{L^6}\big)\|\nabla^{l+1}u\|_{L^2}\\
\lesssim&\big(\|\nabla^l u\|_{L^6}\|u\|_{L^3}\|u\|_{L^3}+\|u\|_{L^3}^2\|\nabla^{l+1}u\|_{L^2}\big)\|\nabla^{l+1}u\|_{L^2}\\
\lesssim&\|u\|_{H^1}^2\|\nabla^{l+1}u\|_{L^2}^2. \qquad (3.4)
\end{aligned}
$$

For I_3, by integration by parts, (2.1) and Lemma 2.2, it follows that

$$
\begin{aligned}
I_3=&\int_{\mathbb{R}^3}\nabla^l(\Delta\varrho)\cdot\nabla^l\big(\varrho_t+\varrho\operatorname{div}u+u\cdot\nabla\varrho\big)dx\\
\leq&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^{l+1}\varrho|^2dx+C\big(\|\nabla^l\varrho\|_{L^6}\|\nabla u\|_{L^3}+\|\varrho\|_{L^\infty}\|\nabla^{l+1}u\|_{L^2}\\
&+\|\nabla^l u\|_{L^6}\|\nabla\varrho\|_{L^3}+\|u\|_{L^\infty}\|\nabla^{l+1}\varrho\|_{L^2}\big)\|\nabla^l\Delta\varrho\|_{L^2}\\
\leq&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^{l+1}\varrho|^2dx+C\|(\varrho,u)\|_{H^3}\|\nabla^l(\nabla\varrho,\Delta\varrho,\nabla u)\|_{L^2}^2. \qquad (3.5)
\end{aligned}
$$

For I_4, emploing Hölder inequality and Lemmas 2.1-2.2, we have

$$
I_4\leq\|\nabla^l(-\varrho\operatorname{div}u)\|_{L^{6/5}}\|\nabla^l u\|_{L^6}
$$

$$
\begin{aligned}
&\lesssim\big(\|\nabla^l\varrho\|_{L^6}\|\nabla u\|_{L^{3/2}}+\|\varrho\|_{L^3}\|\nabla^{l+1}u\|_{L^2}\big)\|\nabla^{l+1}\varrho\|_{L^2}\\
&\lesssim\big(\|\nabla^{l+1}\varrho\|_{L^2}\|u\|_{H^2}+\|\varrho\|_{H^1}\|\nabla^{l+1}u\|_{L^2}\big)\|\nabla^{l+1}\varrho\|_{L^2}\\
&\lesssim\|(\varrho,u)\|_{H^2}\|\nabla^l(\nabla\varrho,\nabla u)\|_{L^2}^2, \qquad (3.6)
\end{aligned}
$$

According to the same estimate, we also have

$$
I_5\lesssim\|(\varrho,u)\|_{H^2}\|\nabla^l(\nabla\varrho,\nabla u)\|_{L^2}^2, \qquad (3.7)
$$

$$
I_6\lesssim\|u\|_{H^2}\|\nabla^{l+1}u\|_{L^2}^2. \qquad (3.8)
$$

For I_7, utilize

$$
f(\varrho)\Delta\nabla\varrho=\nabla(f(\varrho)\Delta\varrho)-\nabla f(\varrho)\Delta\varrho,
$$

and then integrating by parts, applying Hölder inequality, Lemma 2.2, Lemma 2.3, Sobolev inequality and Lemma 2.1, we infer that

$$
\begin{aligned}
I_7=&-\int_{\mathbb{R}^3}\nabla^{l+1}(f(\varrho)\Delta\varrho)\cdot\nabla^l u dx+\int_{\mathbb{R}^3}\nabla^l\big(\nabla f(\varrho)\Delta\varrho\big)\cdot\nabla^l u dx\\
=&\int_{\mathbb{R}^3}\nabla^l(f(\varrho)\Delta\varrho)\cdot\big(\nabla\cdot\nabla^l u\big)dx+\int_{\mathbb{R}^3}\nabla^l(\nabla f(\varrho)\Delta\varrho)\cdot\nabla^l u dx\\
\lesssim&\|\nabla^l(f(\varrho)\Delta\varrho)\|_{L^2}\|\nabla^{l+1}u\|_{L^2}+\|\nabla^l(\nabla f(\varrho)\Delta\varrho)\|_{L^{6/5}}\|\nabla^l u\|_{L^6}\\
\lesssim&\big(\|\nabla^l\varrho\|_{L^6}\|\Delta\varrho\|_{L^3}+\|\varrho\|_{L^\infty}\|\nabla^{l+2}\varrho\|_{L^2}\\
&+\|\nabla^{l+1}\varrho\|_{L^2}\|\Delta\varrho\|_{L^3}+\|\nabla\varrho\|_{L^3}\|\nabla^{l+2}\varrho\|_{L^2}\big)\|\nabla^{l+1}u\|_{L^2}\\
\lesssim&\|\varrho\|_{H^3}\|\nabla^l\big(\nabla\varrho,\nabla^2\varrho,\nabla u\big)\|_{L^2}^2. \qquad (3.9)
\end{aligned}
$$

For I_8, note that $h(0)=0$, which satisfies the condition of Lemma 2.3, and then get

$$
I_8\lesssim\|\varrho\|_{H^2}\|\nabla^l(\nabla\varrho,\nabla u)\|_{L^2}^2 \qquad (3.10)
$$

from the estimate of I_4. For I_9, in view of $g(\varrho)=1-f(\varrho)$, give rise to

$$
\begin{aligned}
I_9=&\int_{\mathbb{R}^3}\nabla^l(\nabla\varrho\cdot\mathbb{D}u)\cdot\nabla^l u dx-\int_{\mathbb{R}^3}\nabla^l(f(\varrho)\nabla\varrho\cdot\mathbb{D}u)\cdot\nabla^l u dx\\
:=&I_{9a}+I_{9b}.
\end{aligned}
$$

By Hölder inequality, Lemma 2.1 and Lemma 2.2, get

$$
\begin{aligned}
I_{9a}\lesssim&\|\nabla^l(\nabla\varrho\cdot\nabla u)\|_{L^{6/5}}\|\nabla^l u\|_{L^6}\\
\lesssim&\big(\|\nabla^{l+1}\varrho\|_{L^2}\|\nabla u\|_{L^3}+\|\nabla\varrho\|_{L^3}\|\nabla^{l+1}u\|_{L^2}\big)\|\nabla^{l+1}u\|_{L^2}\\
\lesssim&\|(\varrho,u)\|_{H^2}\|\nabla^l(\nabla\varrho,\nabla u)\|_{L^2}^2.
\end{aligned}
$$

Combined with Lemma 2.3, we have

$$
\begin{aligned}
I_{9b}\lesssim&\|\nabla^l(f(\varrho)\nabla\varrho\cdot\nabla u)\|_{L^{6/5}}\|\nabla^l u\|_{L^6}\\
\lesssim&\big(\|\nabla^{l+1}\varrho\|_{L^2}\|\nabla u\|_{L^3}+\|f(\varrho)\nabla\varrho\|_{L^3}\|\nabla^{l+1}u\|_{L^2}\big)\|\nabla^{l+1}u\|_{L^2}\\
\lesssim&\big(\|\nabla^l\varrho\|_{L^6}\|\nabla\varrho\|_{L^3}+\|\varrho\|_{L^\infty}\|\nabla^{l+1}\varrho\|_{L^2}\big)\|u\|_{H^2}\|\nabla^{l+1}u\|_{L^2}\\
&+\|f(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^3}\|\nabla^{l+1}u\|_{L^2}^2\\
\lesssim&\|\varrho\|_{H^2}\|u\|_{H^2}\|\nabla^l(\nabla\varrho,\nabla u)\|_{L^2}^2+\|\varrho\|_{H^2}\|\nabla^{l+1}u\|_{L^2}^2.
\end{aligned}
$$

Therefore

$$I_9 \lesssim \big(\|(\varrho,u)\|_{H^2}+\|\varrho\|_{H^2}\|u\|_{H^2}\big)\big\|\nabla^l(\nabla\varrho,\nabla u)\big\|_{L^2}^2. \tag{3.11}$$

For I_{10}, denote $F_1(\varrho)=g^3(\varrho)-1$, then $F_1(0)=0$, and

$$\begin{aligned}I_{10}=&\int_{\mathbb{R}^3}\nabla^l\big(|\nabla\varrho|^2\nabla\varrho\big)\cdot\nabla^l u dx+\int_{\mathbb{R}^3}\nabla^l\big(F_1(\varrho)|\nabla\varrho|^2\nabla\varrho\big)\cdot\nabla^l u dx\\ :=&I_{10a}+I_{10b}.\end{aligned}$$

If $l=0$, from Hölder inequality and Lemma 2.1, we deduce

$$\begin{aligned}I_{10a}=&\int_{\mathbb{R}^3}|\nabla\varrho|^2\nabla\varrho\cdot u dx\\ \lesssim&\|\nabla\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^3}\|\nabla\varrho\|_{L^2}\|u\|_{L^6}\\ \lesssim&\|\varrho\|_{H^3}^2\|(\nabla\varrho,\nabla u)\|_{L^2}^2.\end{aligned}$$

If $l\geq 1$, by integration by parts, employ Lemma 2.1 and Lemma 2.2, it follows that

$$\begin{aligned}I_{10a}=&-\int_{\mathbb{R}^3}\nabla^{l-1}\big(\,|\nabla\varrho|^2\,\nabla\varrho\big)\cdot\big(\nabla\cdot\nabla^l u\big)\,dx\\ \lesssim&\big\|\nabla^{l-1}\big(|\nabla\varrho|^2\nabla\varrho\big)\big\|_{L^2}\big\|\nabla^{l+1}u\big\|_{L^2}\\ \lesssim&\big(\big\|\nabla^{l-1}(\nabla\varrho\cdot\nabla\varrho)\big\|_{L^2}\|\nabla\varrho\|_{L^\infty}+\|\nabla\varrho\cdot\nabla\varrho\|_{L^3}\big\|\nabla^l\varrho\big\|_{L^6}\big)\big\|\nabla^{l+1}u\big\|_{L^2}\\ \lesssim&\big(\big\|\nabla^l\varrho\big\|_{L^6}\|\nabla\varrho\|_{L^3}\|\varrho\|_{H^3}+\|\nabla\varrho\|_{L^6}^2\big\|\nabla^{l+1}\varrho\big\|_{L^2}\big)\big\|\nabla^{l+1}u\big\|_{L^2}\\ \lesssim&\|\varrho\|_{H^3}^2\big\|\nabla^l(\nabla\varrho,\nabla u)\big\|_{L^2}^2.\end{aligned}$$

Using the Hölder inequality, Lemmas 2.1-2.3, we further obtain

$$\begin{aligned}I_{10b}\lesssim&\big\|\nabla^l\big(F_1(\varrho)|\nabla\varrho|^2\nabla\varrho\big)\big\|_{L^{6/5}}\|\nabla^l u\|_{L^6}\\ \lesssim&\big(\big\|\nabla^l\varrho\big\|_{L^6}\|(\nabla\varrho\cdot\nabla\varrho)\nabla\varrho\|_{L^{3/2}}+\|\rho\|_{L^\infty}\big\|\nabla^l((\nabla\varrho\cdot\nabla\varrho)\nabla\varrho)\big\|_{L^2}\big)\big\|\nabla^{l+1}u\big\|_{L^2}\\ \lesssim&\|\nabla^{l+1}\varrho\|_{L^2}\|\nabla\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^2}\|\nabla\varrho\|_{L^6}\big\|\nabla^{l+1}u\big\|_{L^2}\\ &+\|\varrho\|_{L^\infty}\big(\big\|\nabla^{l+1}\varrho\big\|_{L^2}\|\nabla\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^\infty}+\|\nabla\varrho\|_{L^\infty}^2\big\|\nabla^{l+1}\varrho\big\|_{L^2}\big)\big\|\nabla^{l+1}u\big\|_{L^2}\\ \lesssim&\|\varrho\|_{H^3}^3\big\|\nabla^l(\nabla\varrho,\nabla u)\big\|_{L^2}^2.\end{aligned}$$

Therefore

$$I_{10}\lesssim\big(\|\varrho\|_{H^3}^2+\|\varrho\|_{H^3}^3\big)\big\|\nabla^l(\nabla\varrho,\nabla u)\big\|_{L^2}^2. \tag{3.12}$$

For I_{11}, denote $F_2(\varrho)=g^2(\varrho)-1$, then $F_2(0)=0$, and

$$\begin{aligned}I_{11}=&-\int_{\mathbb{R}^3}\nabla^l(\nabla\varrho\Delta\varrho)\cdot\nabla^l u dx-\int_{\mathbb{R}^3}\nabla^l\big(F_2(\varrho)\nabla\varrho\Delta\varrho\big)\cdot\nabla^l u dx\\ :=&I_{11a}+I_{11b}.\end{aligned}$$

If $l=0$, directly from the Hölder inequality, we have

$$I_{11a}=-\int_{\mathbb{R}^3}\nabla\varrho\Delta\varrho\cdot u\,dx\lesssim\|\nabla\varrho\|_{L^2}\|\Delta\varrho\|_{L^3}\|u\|_{L^6}\lesssim\|\varrho\|_{H^3}\|(\nabla\varrho,\nabla u)\|_{L^2}^2.$$

If $l\geq 1$, by integration by parts, employ Lemma 2.2 and Sobolev inequality, give rise to

$$I_{11a}=\int_{\mathbb{R}^3}\nabla^{l-1}(\nabla\varrho\Delta\varrho)\cdot\big(\nabla\cdot\nabla^l u\big)dx$$

$$
\begin{aligned}
&\lesssim\big(\|\nabla^l\varrho\|_{L^6}\|\Delta\varrho\|_{L^3}+\|\nabla\varrho\|_{L^\infty}\|\nabla^{l+1}\varrho\|_{L^2}\big)\|\nabla^{l+1}u\|_{L^2}\\
&\lesssim\|\varrho\|_{H^3}\|\nabla^l(\nabla\varrho,\nabla u)\|_{L^2}^2.
\end{aligned}
$$

Applying Hölder inequality, Lemma 2.2, Lemma 2.3, Sobolev inequality and Lemma 2.1, we infer that

$$
\begin{aligned}
I_{11b}\lesssim&\big\|\nabla^l\big(F_2(\varrho)\nabla\varrho\Delta\varrho\big)\big\|_{L^{6/5}}\|\nabla^l u\|_{L^6}\\
\lesssim&\big(\|\nabla^l\varrho\|_{L^6}\|\nabla\varrho\|_{L^3}\|\Delta\varrho\|_{L^3}+\|\varrho\|_{L^3}\times\\
&\big(\|\nabla^{l+1}\varrho\|_{L^6}\|\Delta\varrho\|_{L^3}+\|\nabla\varrho\|_{L^\infty}\|\nabla^{l+2}\varrho\|_{L^2}\big)\big)\|\nabla^{l+1}u\|_{L^2}\\
\lesssim&\|\varrho\|_{H^3}^2\big(\|\nabla^{l+1}\varrho\|_{L^2}+\|\nabla^{l+2}\varrho\|_{L^2}\big)\|\nabla^{l+1}u\|_{L^2}\\
\lesssim&\|\varrho\|_{H^3}^2\big\|\nabla^l\big(\nabla\varrho,\nabla^2\varrho,\nabla u\big)\big\|_{L^2}^2.
\end{aligned}
$$

Therefore

$$
I_{11}\lesssim\big(\|\varrho\|_{H^3}+\|\varrho\|_{H^3}^2\big)\|\nabla^l(\nabla\varrho,\nabla^2\varrho,\nabla u)\|_{L^2}^2. \tag{3.13}
$$

Similarly, we can see that

$$
I_{12}\lesssim\big(\|\varrho\|_{H^3}+\|\varrho\|_{H^3}^2\big)\|\nabla^l(\nabla\varrho,\nabla^2\varrho,\nabla u)\|_{L^2}^2. \tag{3.14}
$$

Putting (3.3)-(3.14) into (3.2), and we deduce

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_{\mathbb{R}^3}\big(|\nabla^l\varrho|^2+|\nabla^{l+1}\varrho|^2+|\nabla^l u|^2\big)dx+\frac12\big(\|\nabla^{l+1}u\|_{L^2}^2+\|\nabla^l(\operatorname{div}u)\|_{L^2}^2\big)+r_0\|\nabla^l u\|_{L^2}^2\\
\lesssim&\|(\varrho,u)\|_{H^3}\|\nabla^l(\varrho,\nabla\varrho,\nabla^2\varrho,u,\nabla u)\|_{L^2}^2+\|\varrho\|_{H^3}\|u\|_{H^3}\|\nabla^l(\nabla\varrho,\nabla^2\varrho,\nabla u)\|_{L^2}^2\\
&+\big(\|\varrho\|_{H^3}^2+\|u\|_{H^1}^2\big)\|\nabla^{l+1}u\|_{L^2}^2.
\end{aligned}
$$

Then Lemma 3.1 is proved. □

Next, we will construct an interactive energy functional to derive the dissipative estimate of ϱ, $\nabla\varrho$. We have the following results.

Lemma 3.2. *Suppose all the assumptions in Lemma 3.1 are true. Then, for any integer $k\ge 0$,*

$$
\begin{aligned}
&\frac{d}{dt}\sum_{l=k}^{k+2}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx+\frac14\sum_{l=k}^{k+2}\|\nabla^{l+1}(\varrho,\nabla\varrho)\|_{L^2}^2\\
\le&C\delta\sum_{l=k}^{k+2}\|\nabla^{l+1}(\varrho,\nabla\varrho,u)\|_{L^2}^2+3\sum_{l=k}^{k+2}\|\nabla^{l+1}u\|_{L^2}^2+r_0^2\sum_{l=k}^{k+2}\|\nabla^l u\|_{L^2}^2.
\end{aligned}\tag{3.15}
$$

Proof. Applying ∇^l $(l=k,\ k+1,\ k+2)$ to (2.2), and taking L^2-inner product with $\nabla^{l+1}\varrho$, we can see that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^l u_t\cdot\nabla^{l+1}\varrho dx+\|\nabla^{l+1}\varrho\|_{L^2}^2+\|\nabla^{l+2}\varrho\|_{L^2}^2\\
=&-r_0\int_{\mathbb{R}^3}\nabla^l(g(\varrho)u)\cdot\nabla^{l+1}\varrho dx-r_1\int_{\mathbb{R}^3}\nabla^l\big(|u|^2u\big)\cdot\nabla^{l+1}\varrho dx\\
&+\int_{\mathbb{R}^3}\nabla^l(\operatorname{div}\mathbb{D}u)\cdot\nabla^{l+1}\varrho dx-\int_{\mathbb{R}^3}\nabla^l(u\cdot\nabla u)\cdot\nabla^{l+1}\varrho dx\\
&-\int_{\mathbb{R}^3}\nabla^l(f(\varrho)\Delta\nabla\varrho)\cdot\nabla^{l+1}\varrho dx-\int_{\mathbb{R}^3}\nabla^l\big(h(\varrho)\nabla\varrho\big)\cdot\nabla^{l+1}\varrho dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3}\nabla^l(g(\varrho)\nabla\varrho\cdot\mathbb{D}u)\cdot\nabla^{l+1}\varrho dx+\int_{\mathbb{R}^3}\nabla^l\big(g^3(\varrho)|\nabla\varrho|^2\nabla\varrho\big)\cdot\nabla^{l+1}\varrho dx\\
&-\int_{\mathbb{R}^3}\nabla^l\big(g^2(\varrho)\nabla\varrho\Delta\varrho\big)\cdot\nabla^{l+1}\varrho dx-\int_{\mathbb{R}^3}\nabla^l\big(g^2(\varrho)\nabla\varrho\cdot\nabla^2\varrho\big)\cdot\nabla^{l+1}\varrho dx\\
:=&\sum_{i=1}^{10}J_i.
\end{aligned}
$$

Substituting (2.1) into the first term at the left of the above formula, we have

$$
\begin{aligned}
\sum_{i=1}^{10}J_i=&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx-\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho_t dx+\big\|\nabla^{l+1}\varrho\big\|_{L^2}^2+\big\|\nabla^{l+2}\varrho\big\|_{L^2}^2\\
=&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx+\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\big(\operatorname{div}u+\operatorname{div}(\varrho u)\big)dx+\big\|\nabla^{l+1}\varrho\big\|_{L^2}^2+\big\|\nabla^{l+2}\varrho\big\|_{L^2}^2\\
=&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx-\big\|\nabla^{l+1}u\big\|_{L^2}^2+\big\|\nabla^{l+1}\varrho\big\|_{L^2}^2+\big\|\nabla^{l+2}\varrho\big\|_{L^2}^2+J_{11},
\end{aligned}\tag{3.16}
$$

where

$$
J_{11}=\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\operatorname{div}(\varrho u)dx.
$$

Now we will estimate J_i $(i=1,2,\cdots,11)$. First, note that $g(\varrho)=1-f(\varrho)$ and $f(\varrho)\sim\varrho$, then by Hölder inequality, Lemmas 2.1-2.3 and Cauchy inequality, lead to

$$
\begin{aligned}
J_1=&-r_0\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx+r_0\int_{\mathbb{R}^3}\nabla^l(f(\varrho)u)\cdot\nabla^{l+1}\varrho dx\\
\le& r_0\big\|\nabla^l u\big\|_{L^2}\big\|\nabla^{l+1}\varrho\big\|_{L^2}+C\big(\big\|\nabla^l f(\varrho)\big\|_{L^6}\|u\|_{L^3}+\|f(\varrho)\|_{L^3}\big\|\nabla^l u\big\|_{L^6}\big)\big\|\nabla^{l+1}\varrho\big\|_{L^2}\\
\le& r_0^2\big\|\nabla^l u\big\|_{L^2}^2+\frac14\big\|\nabla^{l+1}\varrho\big\|_{L^2}^2+C\big(\big\|\nabla^{l+1}\varrho\big\|_{L^2}\|u\|_{H^1}+\|\varrho\|_{H^1}\big\|\nabla^{l+1}u\big\|_{L^2}\big)\big\|\nabla^{l+1}\varrho\big\|_{L^2}\\
\le& r_0^2\big\|\nabla^l u\big\|_{L^2}^2+\frac14\big\|\nabla^{l+1}\varrho\big\|_{L^2}^2+C\|(\varrho,u)\|_{H^1}\big\|\nabla^{l+1}(\varrho,u)\big\|_{L^2}^2.
\end{aligned}\tag{3.17}
$$

According to Hölder inequality, Lemma 2.1 and 2.2, arrive at

$$
\begin{aligned}
J_2\lesssim&\big\|\nabla^l\big(|u|^2u\big)\big\|_{L^2}\big\|\nabla^{l+1}\varrho\big\|_{L^2}\\
\lesssim&\big(\big\|\nabla^l(u\cdot u)\big\|_{L^6}\|u\|_{L^3}+\big\||u|^2\big\|_{L^3}\big\|\nabla^l u\big\|_{L^6}\big)\big\|\nabla^{l+1}\varrho\big\|_{L^2}\\
\lesssim&\big(\big\|\nabla^{l+1}u\big\|_{L^2}\|u\|_{L^\infty}\|u\|_{L^3}+\|u\|_{L^6}^2\big\|\nabla^{l+1}u\big\|_{L^2}\big)\big\|\nabla^{l+1}\varrho\big\|_{L^2}\\
\lesssim&\|u\|_{H^2}^2\big\|\nabla^{l+1}(\varrho,u)\big\|_{L^2}^2.
\end{aligned}\tag{3.18}
$$

For J_3, by integration by parts, then from Cauchy inequality, we get

$$
\begin{aligned}
J_3=&-\int_{\mathbb{R}^3}\nabla^l(\mathbb{D}u)\cdot\nabla^{l+2}\varrho dx\\
\le&\big\|\nabla^{l+1}u\big\|_{L^2}\big\|\nabla^{l+2}\varrho\big\|_{L^2}\\
\le&\big\|\nabla^{l+1}u\big\|_{L^2}^2+\frac14\big\|\nabla^{l+2}\varrho\big\|_{L^2}^2.
\end{aligned}\tag{3.19}
$$

For J_4, from Lemma 2.1 and Lemma 2.2, we have

$$
\begin{aligned}
J_4\lesssim&\big(\big\|\nabla^l u\big\|_{L^6}\|\nabla u\|_{L^3}+\|u\|_{L^\infty}\big\|\nabla^{l+1}u\big\|_{L^2}\big)\big\|\nabla^{l+1}\varrho\big\|_{L^2}\\
\lesssim&\|u\|_{H^2}\big\|\nabla^{l+1}(\varrho,u)\big\|_{L^2}^2.
\end{aligned}\tag{3.20}
$$

For J_5, a process similar to I_7 yields

$$
\begin{aligned}
J_5 =& -\int_{\mathbb{R}^3} \nabla^l(\nabla(f(\varrho)\Delta\varrho) - \nabla f(\varrho)\Delta\varrho) \cdot \nabla^{l+1}\varrho dx \\
=& \int_{\mathbb{R}^3} \nabla^l(f(\varrho)\Delta\varrho) \cdot \big(\nabla \cdot \nabla^{l+1}\varrho\big) dx + \int_{\mathbb{R}^3} \nabla^l(\nabla f(\varrho)\Delta\varrho) \cdot \nabla^{l+1}\varrho dx \\
\lesssim& \big(\|\nabla^l\varrho\|_{L^6}\|\Delta\varrho\|_{L^3} + \|\varrho\|_{L^\infty}\|\nabla^{l+2}\varrho\|_{L^2}\big)\|\nabla^{l+2}\varrho\|_{L^2} \\
&+ \big(\|\nabla^{l+1}\varrho\|_{L^6}\|\Delta\varrho\|_{L^3} + \|\nabla\varrho\|_{L^\infty}\|\nabla^{l+2}\varrho\|_{L^2}\big)\|\nabla^{l+1}\varrho\|_{L^2} \\
\lesssim& \|\varrho\|_{H^3}\|\nabla^{l+1}(\varrho, \nabla\varrho)\|_{L^2}^2.
\end{aligned} \tag{3.21}
$$

For J_6, note that $h(0) = 0$ and apply Lemma 2.3, from the estimate of J_4, we can obtain

$$
J_6 \lesssim \|\varrho\|_{H^2}\|\nabla^{l+1}\varrho\|_{L^2}^2. \tag{3.22}
$$

Now estimate J_7, note that $g(\varrho) = 1 - f(\varrho)$, so J_7 can be written as

$$
\begin{aligned}
J_7 =& \int_{\mathbb{R}^3} \nabla^l(\nabla\varrho \cdot \mathbb{D}u) \cdot \nabla^{l+1}\varrho dx - \int_{\mathbb{R}^3} \nabla^l(f(\varrho) \cdot \nabla\varrho \cdot \mathbb{D}u) \cdot \nabla^{l+1}\varrho dx \\
:=& J_{7a} + J_{7b}.
\end{aligned}
$$

Imitating the estimate of J_4, we obtain

$$
\begin{aligned}
J_{7a} \lesssim& \big(\|\nabla^{l+1}\varrho\|_{L^2}\|\nabla u\|_{L^\infty} + \|\nabla\varrho\|_{L^\infty}\|\nabla^{l+1}u\|_{L^2}\big)\|\nabla^{l+1}\varrho\|_{L^2} \\
\lesssim& \|(\varrho, u)\|_{H^3}\|\nabla^{l+1}(\varrho, u)\|_{L^2}^2.
\end{aligned}
$$

Utilizing Lemmas 2.1-2.3, we also obtain

$$
\begin{aligned}
J_{7b} \lesssim& \big(\|\nabla^l(f(\varrho)\nabla\varrho)\|_{L^2}\|\mathbb{D}u\|_{L^\infty} + \|f(\varrho)\nabla\varrho\|_{L^\infty}\|\nabla^l(\mathbb{D}u)\|_{L^2}\big)\|\nabla^{l+1}\varrho\|_{L^2} \\
\lesssim& \big(\|\nabla^l\varrho\|_{L^6}\|\nabla\varrho\|_{L^3} + \|\varrho\|_{L^\infty}\|\nabla^{l+1}\varrho\|_{L^2}\big)\|\nabla u\|_{L^\infty}\|\nabla^{l+2}\varrho\|_{L^2} \\
&+ \|\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^\infty}\|\nabla^{l+1}u\|_{L^2}\|\nabla^{l+1}\varrho\|_{L^2} \\
\lesssim& \|\varrho\|_{H^2}\|u\|_{H^3}\|\nabla^{l+1}\varrho\|_{L^2}^2 + \|\varrho\|_{H^3}^2\|\nabla^{l+1}(\varrho, u)\|_{L^2}^2.
\end{aligned}
$$

Therefore

$$
J_7 \lesssim \big(\|\varrho\|_{H^3} + \|\varrho\|_{H^3}^2 + \|u\|_{H^3}\big)\|\nabla^{l+1}(\varrho, u)\|_{L^2}^2 + \|\varrho\|_{H^2}\|u\|_{H^3}\|\nabla^{l+1}\varrho\|_{L^2}^2. \tag{3.23}
$$

According to the trick in the estimate of I_{10}, that is, $F_1(\varrho) = g^3(\varrho) - 1$, we have

$$
\begin{aligned}
J_8 =& \int_{\mathbb{R}^3} \nabla^l\big(|\nabla\varrho|^2\nabla\varrho\big) \cdot \nabla^{l+1}\varrho dx + \int_{\mathbb{R}^3} \nabla^l\big(F_1(\varrho)|\nabla\varrho|^2\nabla\varrho\big) \cdot \nabla^{l+1}\varrho dx \\
\lesssim& \big(\|\nabla^l(\nabla\varrho \cdot \nabla\varrho)\|_{L^2}\|\nabla\varrho\|_{L^\infty} + \|\nabla\varrho \cdot \nabla\varrho\|_{L^\infty}\|\nabla^{l+1}\varrho\|_{L^2}\big)\|\nabla^{l+1}\varrho\|_{L^2} \\
&+ \big(\|\nabla^l\varrho\|_{L^6}\||\nabla\varrho|^2\nabla\varrho\|_{L^3} + \|\varrho\|_{L^\infty}\|\nabla^l(|\nabla\varrho|^2\nabla\varrho)\|_{L^2}\big)\|\nabla^{l+1}\varrho\|_{L^2} \\
\lesssim& \big(\|\nabla^{l+1}\varrho\|_{L^2}\|\nabla\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^\infty} + \|\nabla\varrho\|_{L^\infty}^2\|\nabla^{l+1}\varrho\|_{L^2}\big)\|\nabla^{l+1}\varrho\|_{L^2} \\
&+ \big(\|\nabla^{l+1}\varrho\|_{L^2}\|\nabla\varrho\|_{L^\infty}^2\|\nabla\varrho\|_{L^3} + \|\varrho\|_{L^\infty}\|\nabla^l(|\nabla\varrho|^2\nabla\varrho)\|_{L^2}\big)\|\nabla^{l+1}\varrho\|_{L^2} \\
\lesssim& \big(\|\varrho\|_{H^3}^2\|\nabla^{l+1}\varrho\|_{L^2} + \|\varrho\|_{H^3}^3\|\nabla^{l+1}\varrho\|_{L^2} + \|\varrho\|_{H^2}\|\varrho\|_{H^3}^2\|\nabla^{l+1}\varrho\|_{L^2}\big)\|\nabla^{l+1}\varrho\|_{L^2} \\
\lesssim& \big(\|\varrho\|_{H^3}^2 + \|\varrho\|_{H^3}^3\big)\|\nabla^{l+1}\varrho\|_{L^2}^2.
\end{aligned} \tag{3.24}
$$

Similarly, according to the notation in the estimate of I_{11}, that is, $F_2(\varrho)=g^2(\varrho)-1$, we obtain

$$
\begin{aligned}
J_9=&-\int_{\mathbb{R}^3}\nabla^l(\nabla\varrho\Delta\varrho)\cdot\nabla^{l+1}\varrho dx-\int_{\mathbb{R}^3}\nabla^l\big(F_2(\varrho)\nabla\varrho\Delta\varrho\big)\cdot\nabla^{l+1}\varrho dx\\
\lesssim&\big\|\nabla^l(\nabla\varrho\Delta\varrho)\big\|_{L^2}\big\|\nabla^{l+1}\varrho\big\|_{L^2}+\big(\big\|\nabla^l\varrho\big\|_{L^6}\|\nabla\varrho\Delta\varrho\|_{L^3}+\|\varrho\|_{L^\infty}\big\|\nabla^l(\nabla\varrho\Delta\varrho)\big\|_{L^2}\big)\big\|\nabla^{l+1}\varrho\big\|_{L^2}\\
\lesssim&\big(1+\|\varrho\|_{L^\infty}\big)\big(\big\|\nabla^{l+1}\varrho\big\|_{L^6}\|\Delta\varrho\|_{L^3}+\|\nabla\varrho\|_{L^\infty}\big\|\nabla^{l+2}\varrho\big\|_{L^2}\big)\big\|\nabla^{l+1}\varrho\big\|_{L^2}\\
&+\big\|\nabla^{l+1}\varrho\big\|_{L^2}\|\nabla\varrho\|_{L^6}\|\Delta\varrho\|_{L^6}\big\|\nabla^{l+1}\varrho\big\|_{L^2}\\
\lesssim&\big(\|\varrho\|_{H^3}+\|\varrho\|_{H^3}^2\big)\big\|\nabla^{l+1}(\varrho,\nabla\varrho)\big\|_{L^2}^2.
\end{aligned}\tag{3.25}
$$

By the same way, we can see that

$$
J_{10}\lesssim\big(\|\varrho\|_{H^3}+\|\varrho\|_{H^3}^2\big)\big\|\nabla^{l+1}(\varrho,\nabla\varrho)\big\|_{L^2}^2.\tag{3.26}
$$

Finally, from the estimate of J_4, we can also get

$$
\big|J_{11}\big|\lesssim\|(\varrho,u)\|_{H^2}\big\|\nabla^{l+1}(\varrho,u)\big\|_{L^2}^2\tag{3.27}
$$

By substituting (3.17)-(3.27) into (3.16), deduce

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx+\frac14\big\|\nabla^{l+1}(\varrho,\nabla\varrho)\big\|_{L^2}^2\\
\le&C\|(\varrho,u)\|_{H^3}\big\|\nabla^{l+1}(\varrho,\nabla\varrho,u)\big\|_{L^2}^2+C\big(\|\varrho\|_{H^3}^2+\|u\|_{H^2}^2\big)\big\|\nabla^{l+1}(\varrho,u)\big\|_{L^2}^2\\
&+C\|\varrho\|_{H^2}\|u\|_{H^3}\big\|\nabla^{l+1}\varrho\big\|_{L^2}^2+3\big\|\nabla^{l+1}u\big\|_{L^2}^2+r_0^2\big\|\nabla^l u\big\|_{L^2}^2.
\end{aligned}
$$

From this estimate comes (3.15). Lemma 3.2 is proved. □

3.2. **Local existence of solution.** In this section, we purpose to prove the local existence of the solution $(\varrho,\nabla\varrho,u)$ with small H^3-norm. To this end, we will solve the following iterative Cauchy problem for $j\ge0$:

$$
\begin{cases}
\varrho_t^{j+1}+\operatorname{div}u^{j+1}=S_1^{j+1}, & (3.28)\\
u_t^{j+1}-\operatorname{div}\mathbb{D}u^{j+1}+\nabla\varrho^{j+1}-\Delta\nabla\varrho^{j+1}=S_2^{j+1}, & (3.29)
\end{cases}
$$

with intial data

$$
\big(\varrho^{j+1},u^{j+1}\big)(x,t)\big|_{t=0}=\left(\varrho_0,u_0\right)(x)\to(0,0),\ \text{ as }|x|\to\infty,\tag{3.30}
$$

where $S_i^{j+1}(i=1,2)$ is defined by

$$
\begin{aligned}
S_1^{j+1}=&-\varrho^j\operatorname{div}u^{j+1}-u^j\cdot\nabla\varrho^{j+1},\\
S_2^{j+1}=&-r_0g\big(\varrho^j\big)u^{j+1}-r_1|u^j|^2u^{j+1}-u^j\cdot\nabla u^{j+1}\\
&-f\big(\varrho^j\big)\Delta\nabla\varrho^{j+1}-h(\varrho^j)\nabla\varrho^{j+1}+g(\varrho^j)\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\\
&+\big(g^3\big(\varrho^j\big)|\nabla\varrho^j|^2\nabla\varrho^{j+1}-g^2\big(\varrho^j\big)\nabla\varrho^j\Delta\varrho^{j+1}-g^2\big(\varrho^j\big)\nabla\varrho^j\cdot\nabla^2\varrho^{j+1}\big).
\end{aligned}
$$

Here, $\big(\varrho^0,\nabla\varrho^0,u^0\big)=(0,0,0)$ is used as the initial data of the iteration. By solving the above iterative problem, we can construct a solution sequence $(\varrho^j,\nabla\varrho^j,u^j)_{j\ge0}$. For convenience, we denote

$$
\mathscr{G}^j=(\varrho^j,\nabla\varrho^j,u^j)_{j\ge0},\quad \mathscr{G}_0=\big(\varrho_0,\nabla\varrho_0,u_0\big).
$$

In this way, we have the following results.

Lemma 3.3. *Suppose the assumptions of Theorem 2.1 hold for $N=3$. Then*

$$\begin{aligned}
&\frac{d}{dt}\|\mathscr{G}^{j+1}\|_{H^3}^2+\|\nabla u^{j+1}\|_{H^3}^2+\|\operatorname{div} u^{j+1}\|_{H^3}^2+\|u^{j+1}\|_{H^3}^2\\
\le&C\|(\varrho^j,u^j)\|_{H^3}\|(\varrho^{j+1},\nabla\varrho^{j+1},\nabla^2\varrho^{j+1},u^{j+1},\nabla u^{j+1})\|_{H^3}^2\\
&+C(\|\varrho^j\|_{H^3}+\|\varrho^j\|_{H^3}^2+\|\varrho^j\|_{H^3}^3+\|u^j\|_{H^3}^2)\|\nabla(\varrho^{j+1},u^{j+1})\|_{H^3}^2,
\end{aligned}\tag{3.31}$$

where C is a positive constant independent of j.

Proof. Taking the ∇^l $(0\le l\le 3)$ energy estimates for (3.28)-(3.29) on ϱ^{j+1},u^{j+1}, respectively, we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^l(\varrho^{j+1},u^{j+1})\|_{L^2}^2\\
=&-r_0\int_{\mathbb{R}^3}\nabla^l(g(\varrho^j)u^{j+1})\cdot\nabla^l u^{j+1}dx-r_1\int_{\mathbb{R}^3}\nabla^l(|u^j|^2u^{j+1})\cdot\nabla^l u^{j+1}dx\\
&-\int_{\mathbb{R}^3}\nabla^l(g(\varrho^j)\operatorname{div} u^{j+1})\cdot\nabla^l\varrho^{j+1}dx-\int_{\mathbb{R}^3}\nabla^l(u^j\cdot\nabla\varrho^{j+1})\cdot\nabla^l\varrho^{j+1}dx\\
&+\int_{\mathbb{R}^3}\nabla^l(\operatorname{div}\mathbb{D}u^{j+1})\cdot\nabla^l u^{j+1}dx+\int_{\mathbb{R}^3}\nabla^l(\Delta\nabla\varrho^{j+1})\cdot\nabla^l u^{j+1}dx\\
&-\int_{\mathbb{R}^3}\nabla^l(u^j\cdot\nabla u^{j+1})\cdot\nabla^l u^{j+1}dx-\int_{\mathbb{R}^3}\nabla^l(f(\varrho^j)\Delta\nabla\varrho^{j+1})\cdot\nabla^l u^{j+1}dx\\
&-\int_{\mathbb{R}^3}\nabla^l(h(\varrho^j)\nabla\varrho^{j+1})\cdot\nabla^l u^{j+1}dx+\int_{\mathbb{R}^3}\nabla^l(g(\varrho^j)\nabla\varrho^j\cdot\mathbb{D}u^{j+1})\cdot\nabla^l u^{j+1}dx\\
&+\int_{\mathbb{R}^3}\nabla^l(g^3(\varrho^j)|\nabla\varrho^j|^2\nabla\varrho^{j+1})\cdot\nabla^l u^{j+1}dx-\int_{\mathbb{R}^3}\nabla^l(g^2(\varrho^j)\nabla\varrho^j\Delta\varrho^{j+1})\cdot\nabla^l u^{j+1}dx\\
&-\int_{\mathbb{R}^3}\nabla^l(g^2(\varrho^j)\nabla\varrho^j\cdot\nabla^2\varrho^{j+1})\cdot\nabla^l u^{j+1}dx\\
:=&\sum_{i=1}^{13}K_i.
\end{aligned}\tag{3.32}$$

Similar to the proof of Lemma 3.1, we now deal with K_i $(i=1,2,\cdots,13)$. First, applying Hölder inequality, Lemmas 2.1-2.3 and Cauchy inequality, we have

$$\begin{aligned}
K_1=&-r_0\int_{\mathbb{R}^3}\nabla^l u^{j+1}\cdot\nabla^l u^{j+1}dx+r_0\int_{\mathbb{R}^3}\nabla^l(f(\varrho^j)u^{j+1})\cdot\nabla^l u^{j+1}dx\\
\le&-r_0\|\nabla^l u^{j+1}\|_{L^2}^2+C(\|\nabla^l\varrho^j\|_{L^2}\|u^{j+1}\|_{L^3}+\|\varrho^j\|_{L^3}\|\nabla^l u^{j+1}\|_{L^2})\|\nabla^l u^{j+1}\|_{L^6}\\
\le&-r_0\|\nabla^l u^{j+1}\|_{L^2}^2+C(\|\varrho^j\|_{H^3}\|u^{j+1}\|_{H^1}+\|\varrho^j\|_{H^1}\|u^{j+1}\|_{H^3})\|\nabla u^{j+1}\|_{H^3}\\
\le&-r_0\|\nabla^l u^{j+1}\|_{L^2}^2+C\|\varrho^j\|_{H^3}(\|u^{j+1}\|_{H^3}^2+\|\nabla u^{j+1}\|_{H^3}^2).
\end{aligned}\tag{3.33}$$

$$\begin{aligned}
K_2\lesssim&\|\nabla^l(|u^j|^2u^{j+1})\|_{L^{6/5}}\|\nabla^l u^{j+1}\|_{L^6}\\
\lesssim&(\|\nabla^l(u^j\cdot u^j)\|_{L^{3/2}}\|u^{j+1}\|_{L^6}+\|u^j\cdot u^j\|_{L^{3/2}}\|\nabla^l u^{j+1}\|_{L^6})\|\nabla u^{j+1}\|_{H^3}\\
\lesssim&(\|\nabla^l u^j\|_{L^2}\|u^j\|_{L^6}+\|u^j\|_{L^3}^2)\|\nabla u^{j+1}\|_{H^3}^2\\
\lesssim&\|u^j\|_{H^3}^2\|\nabla u^{j+1}\|_{H^3}^2.
\end{aligned}\tag{3.34}$$

It is obviously that

$$K_3\lesssim\|\varrho^j\|_{H^3}\|(\varrho^{j+1},\nabla u^{j+1})\|_{H^3}^2,\tag{3.35}$$

$$K_4 \lesssim \|u^j\|_{H^3}\|\nabla\varrho^{j+1}\|_{H^3}^2, \tag{3.36}$$

$$K_7 \lesssim \|u^j\|_{H^3}\|\nabla u^{j+1}\|_{H^3}^2, \tag{3.37}$$

$$K_9 \lesssim \|\varrho^j\|_{H^3}\|\nabla(\varrho^{j+1}, u^{j+1})\|_{H^3}^2. \tag{3.38}$$

Now, by integration by parts, arrive at

$$\begin{aligned}
K_5 =& -\int_{\mathbb{R}^3}\nabla^l(\mathbb{D}u^{j+1})\cdot\nabla^{l+1}u^{j+1}dx\\
=& -\frac{1}{2}\int_{\mathbb{R}^3}\nabla^{l+1}u^{j+1}\cdot\nabla^{l+1}u^{j+1}dx-\frac{1}{2}\int_{\mathbb{R}^3}\nabla^l\big((\nabla u^{j+1})^{\mathrm{T}}\big)\cdot\nabla^{l+1}u^{j+1}dx\\
=& -\frac{1}{2}\big(\|\nabla^{l+1}u^{j+1}\|_{L^2}^2+\|\nabla^l(\operatorname{div}u^{j+1})\|_{L^2}^2\big).
\end{aligned}\tag{3.39}$$

For K_6, by integration by parts, combined with (3.28), we infer that

$$\begin{aligned}
K_6 =& -\int_{\mathbb{R}^3}\nabla^l(\Delta\varrho^{j+1})\cdot(\nabla\cdot\nabla^l u^{j+1})dx\\
=& \int_{\mathbb{R}^3}\nabla^l(\Delta\varrho^{j+1})\cdot\nabla^l\big(\varrho_t^{j+1}+\varrho^j\operatorname{div}u^{j+1}+u^j\cdot\nabla\varrho^{j+1}\big)dx\\
=& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^{l+1}\varrho^{j+1}|^2dx+\int_{\mathbb{R}^3}\nabla^l(\Delta\varrho^{j+1})\cdot\nabla^l\big(\varrho^j\operatorname{div}u^{j+1}+u^j\cdot\nabla\varrho^{j+1}\big)dx\\
\le& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^{l+1}\varrho^{j+1}|^2dx+C\|\nabla^{l+2}\varrho^{j+1}\|_{L^2}\big(\|\nabla^l\varrho^j\|_{L^2}\|\nabla u^{j+1}\|_{L^\infty}\\
&+\|\nabla^l u^j\|_{L^2}\|\nabla\varrho^{j+1}\|_{L^\infty}+\|u^j\|_{L^\infty}\|\nabla^{l+1}\varrho^{j+1}\|_{L^2}\big)\\
\le& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^{l+1}\varrho^{j+1}|^2dx+C\|(\varrho^j,u^j)\|_{H^3}\|(\nabla\varrho^{j+1},\nabla^2\varrho^{j+1},\nabla u^{j+1})\|_{H^3}^2.
\end{aligned}\tag{3.40}$$

For K_8, imitating the estimate of J_7, give

$$\begin{aligned}
K_8 =& -\int_{\mathbb{R}^3}\nabla^l\big(\nabla(f(\varrho^j)\Delta\varrho^{j+1})-\nabla f(\varrho^j)\Delta\varrho^{j+1}\big)\cdot\nabla^l u^{j+1}dx\\
=& -\int_{\mathbb{R}^3}\nabla^{l+1}\big(f(\varrho^j)\Delta\varrho^{j+1}\big)\cdot\nabla^l u^{j+1}dx+\int_{\mathbb{R}^3}\nabla^l\big(\nabla f(\varrho^j)\Delta\varrho^{j+1}\big)\cdot\nabla^l u^{j+1}dx\\
:=& K_{8a}+K_{8b}.
\end{aligned}$$

By integration by parts, and then applying Lemmas 2.1-2.3, derive

$$K_{8a}\lesssim\|\varrho^j\|_{H^3}\|(\nabla^2\varrho^{j+1},\nabla u^{j+1})\|_{H^3}^2.$$

If $l=0$, it is obviously that

$$K_{8b}\lesssim\|\varrho^j\|_{H^3}\|(\nabla^2\varrho^{j+1},\nabla u^{j+1})\|_{H^3}^2.$$

If $1\le l\le 3$, by integration by parts, we have

$$\begin{aligned}
K_{8b}\lesssim&\big(\|\nabla^l\varrho^j\|_{L^2}\|\Delta\varrho^{j+1}\|_{L^\infty}+\|\varrho^j\|_{L^3}\|\nabla^{l+1}\varrho^{j+1}\|_{L^6}\big)\|\nabla^{l+1}u^{j+1}\|_{L^2}\\
\lesssim&\|\varrho^j\|_{H^3}\big(\|\nabla^2\varrho^{j+1}\|_{H^3}^2+\|\nabla u^{j+1}\|_{H^3}^2\big).
\end{aligned}$$

Therefore

$$K_8\lesssim\|\varrho^j\|_{H^3}\big(\|\nabla^2\varrho^{j+1}\|_{H^3}^2+\|\nabla u^{j+1}\|_{H^3}^2\big). \tag{3.41}$$

For K_{10}, in virtue of $g\big(\varrho^j\big)=1-f\big(\varrho^j\big)$, then

$$K_{10}=\int_{\mathbb{R}^3}\nabla^l\big(\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\big)\cdot\nabla^l u^{j+1}dx-\int_{\mathbb{R}^3}\nabla^l\big(f\big(\varrho^j\big)\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\big)\cdot\nabla^l u^{j+1}dx.$$

If $l=0$, directly from Hölder inequality,

$$\begin{aligned}K_{10}\lesssim&\big\|\nabla\varrho^j\big\|_{L^3}\big\|\mathbb{D}u^{j+1}\big\|_{L^2}\big\|u^{j+1}\big\|_{L^6}+\big\|f\big(\varrho^j\big)\big\|_{L^6}\big\|\nabla\varrho^j\big\|_{L^6}\big\|\mathbb{D}u^{j+1}\big\|_{L^2}\big\|u^{j+1}\big\|_{L^6}\\\lesssim&\big(\big\|\varrho^j\big\|_{H^3}+\big\|\varrho^j\big\|_{H^3}^2\big)\big\|\nabla u^{j+1}\big\|_{H^3}^2.\end{aligned}$$

If $1\le l\le 3$, then

$$\begin{aligned}K_{10}=&-\int_{\mathbb{R}^3}\nabla^{l-1}\big(\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\big)\cdot\big(\nabla\cdot\nabla^l u^{j+1}\big)dx+\int_{\mathbb{R}^3}\nabla^{l-1}\big(f\big(\varrho^j\big)\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\big)\cdot\big(\nabla\cdot\nabla^l u^{j+1}\big)dx\\\lesssim&\big(\big\|\nabla^l\varrho^j\big\|_{L^2}\big\|\mathbb{D}u^{j+1}\big\|_{L^\infty}+\big\|\nabla\varrho^j\big\|_{L^3}\big\|\nabla^{l-1}\big(\mathbb{D}u^{j+1}\big)\big\|_{L^6}\big)\big\|\nabla^{l+1}u^{j+1}\big\|_{L^2}+\\&\big(\big\|f\big(\varrho^j\big)\nabla\varrho^j\big\|_{L^3}\big\|\nabla^{l-1}\big(\mathbb{D}u^{j+1}\big)\big\|_{L^6}+\big\|\nabla^{l-1}\big(f\big(\varrho^j\big)\nabla\varrho^j\big)\big\|_{L^2}\big\|\mathbb{D}u^{j+1}\big\|_{L^\infty}\big)\big\|\nabla^{l+1}u^{j+1}\big\|_{L^2}\\\lesssim&\big\|\varrho^j\big\|_{H^3}\big\|\nabla u^{j+1}\big\|_{H^3}^2+\big(\big\|\varrho^j\big\|_{H^3}^2\big\|\nabla^{l+1}u^{j+1}\big\|_{L^2}+\big\|\varrho^j\big\|_{H^3}\big\|\nabla u^{j+1}\big\|_{H^2}\big)\big\|\nabla^{l+1}u^{j+1}\big\|_{L^2}\\\lesssim&\big(\big\|\varrho^j\big\|_{H^3}+\big\|\varrho^j\big\|_{H^3}^2\big)\big\|\nabla u^{j+1}\big\|_{H^3}^2.\end{aligned}$$

Therefore

$$K_{10}\lesssim\big(\big\|\varrho^j\big\|_{H^3}+\big\|\varrho^j\big\|_{H^3}^2\big)\big\|\nabla u^{j+1}\big\|_{H^3}^2.\tag{3.42}$$

Further, for K_{11}, we have

$$\begin{aligned}K_{11}=&\int_{\mathbb{R}^3}\nabla^l\big(\big|\nabla\varrho^j\big|^2\nabla\varrho^{j+1}\big)\cdot\nabla^l u^{j+1}dx+\int_{\mathbb{R}^3}\nabla^l\big(F_1\big(\varrho^j\big)\big|\nabla\varrho^j\big|^2\nabla\varrho^{j+1}\big)\cdot\nabla^l u^{j+1}dx\\:=&K_{11a}+K_{11b}.\end{aligned}$$

If $l=0$, directly from Hölder inequality,

$$\begin{aligned}K_{11a}\lesssim&\big\|\nabla\varrho^j\big\|_{L^6}\big\|\nabla\varrho^j\big\|_{L^6}\big\|\nabla\varrho^{j+1}\big\|_{L^2}\big\|u^{j+1}\big\|_{L^6}\\\lesssim&\big\|\varrho^j\big\|_{H^3}^2\big(\big\|\nabla\varrho^{j+1}\big\|_{H^3}^2+\big\|\nabla u^{j+1}\big\|_{H^3}^2\big),\end{aligned}$$

combined with Lemma 2.3, give

$$\begin{aligned}K_{11b}\lesssim&\big\|\varrho^j\big\|_{L^3}\big\|\nabla\varrho^j\big\|_{L^\infty}^2\big\|\nabla\varrho^{j+1}\big\|_{L^2}\big\|u^{j+1}\big\|_{L^6}\\\lesssim&\big\|\varrho^j\big\|_{H^3}^3\big(\big\|\nabla\varrho^{j+1}\big\|_{H^3}^2+\big\|\nabla u^{j+1}\big\|_{H^3}^2\big),\end{aligned}$$

If $1\le l\le 3$, by integration by parts, then applying Lemma 2.2, Sobolev inequality and Lemma 2.1, derive

$$\begin{aligned}K_{11a}=&-\int_{\mathbb{R}^3}\nabla^{l-1}\big(|\nabla\varrho^j|^2\nabla\varrho^{j+1}\big)\cdot\big(\nabla\cdot\nabla^l u^{j+1}\big)dx\\\lesssim&\big(\big\|\nabla^{l-1}\big(\nabla\varrho^j\cdot\nabla\varrho^j\big)\big\|_{L^2}\big\|\nabla\varrho^{j+1}\big\|_{L^\infty}+\big\|\big(\nabla\varrho^j\cdot\nabla\varrho^j\big)\big\|_{L^3}\big\|\nabla^l u^{j+1}\big\|_{L^6}\big)\big\|\nabla^{l+1}u^{j+1}\big\|_{L^2}\\\lesssim&\big(\big\|\nabla^l\varrho^j\big\|_{L^2}\big\|\nabla\varrho^j\big\|_{L^\infty}\big\|\nabla\varrho^{j+1}\big\|_{L^\infty}+\big\|\nabla\varrho^j\big\|_{L^6}^2\big\|\nabla^{l+1}\varrho^{j+1}\big\|_{L^2}\big)\big\|\nabla^{l+1}u^{j+1}\big\|_{L^2}\\\lesssim&\big\|\varrho^j\big\|_{H^3}^2\big(\big\|\nabla\varrho^{j+1}\big\|_{H^3}^2+\big\|\nabla u^{j+1}\big\|_{H^3}^2\big),\end{aligned}$$

combined with Lemma 2.3, we obtain

$$\begin{aligned}K_{11b}=&-\int_{\mathbb{R}^3}\nabla^{l-1}\big(F_1\big(\varrho^j\big)\big|\nabla\varrho^j\big|^2\nabla\varrho^{j+1}\big)\cdot\big(\nabla\cdot\nabla^l u^{j+1}\big)dx\\\lesssim&\big(\big\|\nabla^{l-1}\varrho^j\big\|_{L^2}\big\|\nabla\varrho^j\big\|_{L^\infty}^2\big\|\nabla\varrho^{j+1}\big\|_{L^\infty}+\big\|\varrho^j\big\|_{L^\infty}\big\|\nabla^{l-1}\big(\big|\nabla\varrho^j\big|^2\nabla\varrho^{j+1}\big)\big\|_{L^2}\big)\big\|\nabla^{l+1}u^{j+1}\big\|_{L^2}\end{aligned}$$

$$\lesssim\big(\|\varrho^j\|_{H^3}^3\|\nabla\varrho^{j+1}\|_{H^2}+\|\varrho^j\|_{H^2}\|\nabla^{l-1}\big(|\nabla\varrho^j|^2\nabla\varrho^{j+1}\big)\|_{L^2}\big)\|\nabla u^{j+1}\|_{H^3}.$$

In view of

$$\begin{aligned}\|\nabla^{l-1}\big(|\nabla\varrho^j|^2\nabla\varrho^{j+1}\big)\|_{L^2}\lesssim&\|\nabla^{l-1}\big(\nabla\varrho^j\cdot\nabla\varrho^j\big)\|_{L^2}\|\nabla\varrho^{j+1}\|_{L^\infty}+\|\nabla\varrho^j\cdot\nabla\varrho^j\|_{L^3}\|\varrho^{j+1}\|_{L^6}\\ \lesssim&\|\nabla^l\varrho^j\|_{L^2}\|\nabla\varrho^j\|_{L^\infty}\|\nabla\varrho^{j+1}\|_{H^2}+\|\nabla\varrho^j\|_{L^6}^2\|\nabla^{l+1}\varrho^{j+1}\|_{L^2}\\ \lesssim&\|\varrho^j\|_{H^3}^2\|\nabla\varrho^{j+1}\|_{H^3},\end{aligned}$$

therefore, if $1\le l\le 3$, then

$$K_{11b}\lesssim\|\varrho^j\|_{H^3}^3\big(\|\nabla\varrho^{j+1}\|_{H^3}^2+\|\nabla u^{j+1}\|_{H^3}^2\big).$$

Thus, we deduce

$$K_{11}\lesssim\big(\|\varrho^j\|_{H^3}^2+\|\varrho^j\|_{H^3}^3\big)\big(\|\nabla\varrho^{j+1}\|_{H^3}^2+\|\nabla u^{j+1}\|_{H^3}^2\big).\tag{3.43}$$

Now start to estimate K_{12}. Write K_{12} as

$$K_{12}=-\int_{\mathbb{R}^3}\nabla^l\big(\nabla\varrho^j\Delta\varrho^{j+1}\big)\cdot\nabla^l u^{j+1}dx-\int_{\mathbb{R}^3}\nabla^l\big(F_2(\varrho^j)\nabla\varrho^j\Delta\varrho^{j+1}\big)\cdot\nabla^l u^{j+1}dx.$$

If $l=0$, K_{12} can be estimated as

$$\begin{aligned}K_{12}=&\int_{\mathbb{R}^3}\nabla\big(\nabla\varrho^j u^{j+1}\big)\cdot\nabla\varrho^{j+1}dx+\int_{\mathbb{R}^3}\nabla\big(F_2(\varrho^j)\nabla\varrho^j u^{j+1}\big)\cdot\nabla\varrho^{j+1}dx\\ \lesssim&\big(\|\nabla^2\varrho^j\|_{L^3}\|u^{j+1}\|_{L^6}+\|\nabla\varrho^j\|_{L^\infty}\|\nabla u^{j+1}\|_{L^2}\big)\|\nabla\varrho^{j+1}\|_{L^2}\\ &+\big(\|\nabla\varrho^j\|_{L^3}\|\nabla\varrho^j\|_{L^\infty}\|u^{j+1}\|_{L^6}+\|\varrho^j\|_{L^\infty}\times\\ &\big(\|\nabla^2\varrho^j\|_{L^3}\|u^{j+1}\|_{L^6}+\|\nabla\varrho^j\|_{L^\infty}\|\nabla u^{j+1}\|_{L^2}\big)\big)\|\nabla\varrho^{j+1}\|_{L^2}\\ \lesssim&\big(\|\varrho^j\|_{H^3}+\|\varrho^j\|_{H^3}^2\big)\big(\|\nabla\varrho^{j+1}\|_{H^3}^2+\|\nabla u^{j+1}\|_{H^3}^2\big).\end{aligned}$$

If $1\le l\le 3$, employ Lemma 2.3, K_{12} can be similarly estimated as

$$\begin{aligned}K_{12}=&\int_{\mathbb{R}^3}\nabla^{l-1}\big(\nabla\varrho^j\Delta\varrho^{j+1}\big)\cdot\big(\nabla\cdot\nabla^l u^{j+1}\big)dx+\int_{\mathbb{R}^3}\nabla^{l-1}\big(F_2(\varrho^j)\nabla\varrho^j\Delta\varrho^{j+1}\big)\cdot\big(\nabla\cdot\nabla^l u^{j+1}\big)dx\\ \lesssim&\big(\|\nabla^l\varrho^j\|_{L^2}\|\Delta\varrho^{j+1}\|_{L^\infty}+\|\nabla\varrho^j\|_{L^\infty}\|\nabla^{l+1}\varrho^{j+1}\|_{L^2}\big)\|\nabla^{l+1}u^{j+1}\|_{L^2}\\ &+\big(\|\nabla^{l-1}\varrho^j\|_{L^2}\|\nabla\varrho^j\|_{L^\infty}\|\Delta\varrho^{j+1}\|_{L^\infty}+\|\varrho^j\|_{L^\infty}\times\\ &\big(\|\nabla^l\varrho^j\|_{L^2}\|\Delta\varrho^{j+1}\|_{L^\infty}+\|\nabla\varrho^j\|_{L^\infty}\|\nabla^{l+1}\varrho^{j+1}\|_{L^2}\big)\big)\|\nabla^{l+1}u^{j+1}\|_{L^2}\\ \lesssim&\big(\|\varrho^j\|_{H^3}+\|\varrho^j\|_{H^3}^2\big)\big(\|\nabla\varrho^{j+1}\|_{H^3}^2+\|\nabla u^{j+1}\|_{H^3}^2\big).\end{aligned}$$

Thus, we can see that

$$K_{12}\lesssim\big(\|\varrho^j\|_{H^3}+\|\varrho^j\|_{H^3}^2\big)\big(\|\nabla\varrho^{j+1}\|_{H^3}^2+\|\nabla u^{j+1}\|_{H^3}^2\big).\tag{3.44}$$

By the same way, we can also get

$$K_{13}\lesssim\big(\|\varrho^j\|_{H^3}+\|\varrho^j\|_{H^3}^2\big)\big(\|\nabla\varrho^{j+1}\|_{H^3}^2+\|\nabla u^{j+1}\|_{H^3}^2\big).\tag{3.45}$$

Putting (3.33)-(3.45) into (3.32) and we deduce (3.31). Lemma 3.4 is proved. ☐

Similar to Lemma 3.2, we will then derive the dissipative estimates of $\varrho^{j+1},\nabla\varrho^{j+1}$. We have

Lemma 3.4. *Suppose all the assumptions in Lemma 3.3 are true. Then*

$$\begin{aligned}
&\frac{d}{dt}\sum_{l=0}^{3}\int_{\mathbb{R}^3}\nabla^l u^{j+1}\cdot\nabla^{l+1}\varrho^{j+1}dx+\frac{1}{2}\big(\|\nabla\varrho^{j+1}\|_{H^3}^2+\|\nabla^2\varrho^{j+1}\|_{H^3}^2\big)\\
\le&3\|\nabla u^{j+1}\|_{H^3}^2+r_0^2\|u^{j+1}\|_{H^3}^2+C\|u^j\|_{H^3}^2\|(\nabla^2\varrho^{j+1},\nabla u^{j+1})\|_{H^3}^2\\
&+C\|(\varrho^j,u^j)\|_{H^3}\|(\nabla\varrho^{j+1},\nabla^2\varrho^{j+1},u^{j+1},\nabla u^{j+1})\|_{H^3}^2\\
&+C\big(\|\varrho^j\|_{H^3}+\|\varrho^j\|_{H^3}^2+\|\varrho^j\|_{H^3}^3\big)\|(\nabla\varrho^{j+1},\nabla^2\varrho^{j+1},\nabla u^{j+1})\|_{H^3}^2,
\end{aligned}\tag{3.46}$$

where C is a positive constant independent of j.

Proof. By ∇^l $(0\le l\le 3)$ acting on both ends of (3.29), and then taking the inner product with $\nabla^{l+1}\varrho^{j+1}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^l u_t^{j+1}\cdot\nabla^{l+1}\varrho^{j+1}dx+\|\nabla^{l+1}\varrho^{j+1}\|_{L^2}^2\\
=&-r_0\int_{\mathbb{R}^3}\nabla^l\big(g(\varrho^j)u^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx-r_1\int_{\mathbb{R}^3}\nabla^l\big(|u^j|^2u^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\
&+\int_{\mathbb{R}^3}\nabla^l\big(\operatorname{div}\mathbb{D}u^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx+\int_{\mathbb{R}^3}\nabla^l\big(\Delta\nabla\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\
&-\int_{\mathbb{R}^3}\nabla^l\big(u^j\cdot\nabla u^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx-\int_{\mathbb{R}^3}\nabla^l\big(f(\varrho^j)\Delta\nabla\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\
&-\int_{\mathbb{R}^3}\nabla^l\big(h(\varrho^j)\nabla\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx+\int_{\mathbb{R}^3}\nabla^l\big(g(\varrho^j)\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\
&+\int_{\mathbb{R}^3}\nabla^l\big(g^3(\varrho^j)|\nabla\varrho^j|^2\nabla\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx-\int_{\mathbb{R}^3}\nabla^l\big(g^2(\varrho^j)\nabla\varrho^j\Delta\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\
&-\int_{\mathbb{R}^3}\nabla^l\big(g^2(\varrho^j)\nabla\varrho^j\cdot\nabla^2\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\
:=&\sum_{i=1}^{11}M_i.
\end{aligned}\tag{3.47}$$

Substituting (3.28) into (3.47), give

$$\begin{aligned}
\sum_{i=1}^{11}M_i=&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^l u^{j+1}\cdot\nabla^{l+1}\varrho^{j+1}dx+\int_{\mathbb{R}^3}\nabla^l\operatorname{div}u^{j+1}\cdot\nabla^l\varrho_t^{j+1}dx+\|\nabla^{l+1}\varrho^{j+1}\|_{L^2}^2\\
=&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^l u^{j+1}\cdot\nabla^{l+1}\varrho^{j+1}dx+\|\nabla^{l+1}\varrho^{j+1}\|_{L^2}^2-\|\nabla^{l+1}u^{j+1}\|_{L^2}^2+M_{12},
\end{aligned}\tag{3.48}$$

where

$$M_{12}=\int_{\mathbb{R}^3}\nabla^l u^{j+1}\cdot\nabla^{l+1}\big(\varrho^j\operatorname{div}u^{j+1}+u^j\cdot\nabla\varrho^{j+1}\big)dx.$$

As before, we are now dealing with M_i $(i=1,2,\cdots,12)$. First of all, we applying Hölder inequality, Lemmas 2.1-2.3 and Cauchy inequality, give rise to

$$\begin{aligned}
M_1=&-r_0\int_{\mathbb{R}^3}\nabla^l u^{j+1}\cdot\nabla^{l+1}\varrho^{j+1}dx+r_0\int_{\mathbb{R}^3}\nabla^l\big(f(\varrho^j)u^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\
\le&r_0\|\nabla^l u^{j+1}\|_{L^2}\|\nabla^{l+1}\varrho^{j+1}\|_{L^2}+C\big(\|\nabla^l\varrho^j\|_{L^2}\|u^{j+1}\|_{L^\infty}+\|\varrho^j\|_{L^3}\|\nabla^l u^{j+1}\|_{L^6}\big)\|\nabla^{l+1}\varrho^{j+1}\|_{L^2}\\
\le&r_0^2\|\nabla^l u^{j+1}\|_{L^2}^2+\frac{1}{2}\|\nabla^{l+1}\varrho^{j+1}\|_{L^2}^2+C\|\varrho^j\|_{H^3}\|(\nabla\varrho^{j+1},u^{j+1},\nabla u^{j+1})\|_{H^3}^2.
\end{aligned}\tag{3.49}$$

For M_2, we have

$$\begin{aligned}
M_2 \lesssim & \big\|\nabla^l\big(\big|u^j\big|^2 u^{j+1}\big)\big\|_{L^{6/5}}\big\|\nabla^{l+1}\varrho^{j+1}\big\|_{L^6}\\
\lesssim & \big(\big\|\nabla^l\big(u^j\cdot u^j\big)\big\|_{L^{3/2}}\big\|u^{j+1}\big\|_{L^6}+\big\|\big|u^j\big|^2\big\|_{L^{3/2}}\big\|\nabla^l u^{j+1}\big\|_{L^6}\big)\big\|\nabla^{l+2}\varrho^{j+1}\big\|_{L^2}\\
\lesssim & \big(\big\|\nabla^l u^j\big\|_{L^6}\big\|u^j\big\|_{L^6}\big\|\nabla u^{j+1}\big\|_{L^2}+\big\|u^j\big\|_{L^3}^2\big\|\nabla^{l+1}u^{j+1}\big\|_{L^2}\big)\big\|\nabla^{l+2}\varrho^{j+1}\big\|_{L^2}\\
\lesssim & \big\|u^j\big\|_{H^3}^2\big\|\nabla\big(\nabla\varrho^{j+1},u^{j+1}\big)\big\|_{L^2}^2.
\end{aligned} \tag{3.50}$$

From the estimates in Lemma 3.2, it is easy to get

$$M_3 \le \big\|\nabla^{l+1}u^{j+1}\big\|_{L^2}^2+\frac{1}{4}\big\|\nabla^{l+2}\varrho^{j+1}\big\|_{L^2}^2, \tag{3.51}$$

$$M_4 = -\big\|\nabla^{l+2}\varrho^{j+1}\big\|_{L^2}^2, \tag{3.52}$$

$$M_5 \lesssim \big\|u^j\big\|_{H^3}\big\|\nabla\big(\varrho^{j+1},u^{j+1}\big)\big\|_{H^3}^2, \tag{3.53}$$

$$M_7 \lesssim \big\|\varrho^j\big\|_{H^3}\big\|\nabla\varrho^{j+1}\big\|_{H^3}^2, \tag{3.54}$$

$$\big|M_{12}\big| \lesssim \big\|\big(\varrho^j,u^j\big)\big\|_{H^3}\big\|\nabla\big(\varrho^{j+1},u^{j+1}\big)\big\|_{H^3}^2. \tag{3.55}$$

For M_6, we can estimate it as

$$\begin{aligned}
M_6 = & -\int_{\mathbb{R}^3}\nabla^l\big(\nabla f(\varrho^j)\Delta\varrho^{j+1}-\nabla f(\varrho^j)\Delta\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\
= & \int_{\mathbb{R}^3}\nabla^l\big(f(\varrho^j)\Delta\varrho^{j+1}\big)\cdot\big(\nabla\cdot\nabla^{l+1}\varrho^{j+1}\big)dx+\int_{\mathbb{R}^3}\nabla^l\big(\nabla f\big(\varrho^j\big)\Delta\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\
\lesssim & \big\|\varrho^j\big\|_{H^3}\big\|\nabla^2\varrho^{j+1}\big\|_{H^3}^2.
\end{aligned} \tag{3.56}$$

Now let us estimate M_8. Pay attention to

$$\begin{aligned}
M_8 = & \int_{\mathbb{R}^3}\nabla^l\big(g\big(\varrho^j\big)\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\
= & \int_{\mathbb{R}^3}\nabla^l\big(\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx-\int_{\mathbb{R}^3}\nabla^l\big(f\big(\varrho^j\big)\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx.
\end{aligned}$$

If $l=0$, in view of $g\big(\varrho^j\big)\in(0,1)$, and applying Hölder inequality, give

$$M_8 \lesssim \big\|\nabla\varrho^j\big\|_{L^\infty}\big\|\mathbb{D}u^{j+1}\big\|_{L^2}\big\|\nabla\varrho^{j+1}\big\|_{L^2} \lesssim \big\|\varrho^j\big\|_{H^3}\big\|\nabla\big(\varrho^{j+1},u^{j+1}\big)\big\|_{L^2}^2.$$

If $1\le l\le 3$, M_8 can be estimated as

$$\begin{aligned}
M_8 = & -\int_{\mathbb{R}^3}\nabla^{l-1}\big(\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\big)\cdot\big(\nabla\cdot\nabla^{l+1}\varrho^{j+1}\big)dx\\
& +\int_{\mathbb{R}^3}\nabla^{l-1}\big(f\big(\varrho^j\big)\nabla\varrho^j\cdot\mathbb{D}u^{j+1}\big)\cdot\big(\nabla\cdot\nabla^{l+1}\varrho^{j+1}\big)dx\\
\lesssim & \big(\big\|\nabla^l\varrho^j\big\|_{L^2}\big\|\mathbb{D}u^{j+1}\big\|_{L^\infty}+\big\|\nabla\varrho^j\big\|_{L^\infty}\big\|\nabla^{l-1}\big(\mathbb{D}u^{j+1}\big)\big\|_{L^2}\big)\big\|\nabla^{l+2}\varrho^{j+1}\big\|_{L^2}\\
& +\big(\big(\big\|\nabla^{l-1}\varrho^j\big\|_{L^2}\big\|\nabla\varrho^j\big\|_{L^\infty}+\big\|\varrho^j\big\|_{L^\infty}\big\|\nabla^l\varrho^j\big\|_{L^2}\big)\big\|\nabla u^{j+1}\big\|_{L^2}\\
& +\big\|\varrho^j\big\|_{L^\infty}\big\|\nabla\varrho^j\big\|_{L^\infty}\big\|\nabla^{l-1}\big(\mathbb{D}u^{j+1}\big)\big\|_{L^2}\big)\big\|\nabla^{l+2}\varrho^{j+1}\big\|_{L^2}\\
\lesssim & \big(\big\|\varrho^j\big\|_{H^3}+\big\|\varrho^j\big\|_{H^3}^2\big)\big\|\nabla\big(\nabla\varrho^{j+1},u^{j+1}\big)\big\|_{H^3}^2.
\end{aligned}$$

Synthesize the above two expressions to get

$$M_8 \lesssim \big(\big\|\varrho^j\big\|_{H^3}+\big\|\varrho^j\big\|_{H^3}^2\big)\big\|\nabla\big(\varrho^{j+1},\nabla\varrho^{j+1},u^{j+1}\big)\big\|_{H^3}^2. \tag{3.57}$$

In addition, we can also obtain

$$\begin{aligned}M_9=&\int_{\mathbb{R}^3}\nabla^l\big(\big|\nabla\varrho^j\big|^2\nabla\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx+\int_{\mathbb{R}^3}\nabla^l\big(F_1\big(\varrho^j\big)\big|\nabla\varrho^j\big|^2\nabla\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\ \lesssim&\big(\big\|\varrho^j\big\|_{H^3}^2+\big\|\varrho^j\big\|_{H^3}^3\big)\big(\big\|\nabla\varrho^{j+1}\big\|_{H^3}^2+\big\|\nabla^2\varrho^{j+1}\big\|_{H^3}^2\big). \qquad (3.58)\\ M_{11}=&-\int_{\mathbb{R}^3}\nabla^l\big(\nabla\varrho^j\cdot\nabla^2\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx-\int_{\mathbb{R}^3}\nabla^l\big(F_2\big(\varrho^j\big)\nabla\varrho^j\cdot\nabla^2\varrho^{j+1}\big)\cdot\nabla^{l+1}\varrho^{j+1}dx\\ \lesssim&\big(\big\|\varrho^j\big\|_{H^3}+\big\|\varrho^j\big\|_{H^3}^2\big)\big(\big\|\nabla\varrho^{j+1}\big\|_{H^3}^2+\big\|\nabla^2\varrho^{j+1}\big\|_{H^3}^2\big). \qquad (3.59)\end{aligned}$$

The estimate from M_{11} directly implies

$$M_{10}\lesssim\big(\big\|\varrho^j\big\|_{H^3}+\big\|\varrho^j\big\|_{H^3}^2\big)\big(\big\|\nabla\varrho^{j+1}\big\|_{H^3}^2+\big\|\nabla^2\varrho^{j+1}\big\|_{H^3}^2\big). \qquad (3.60)$$

Putting (3.49)-(3.60) into (3.48) and summing up with $l=0,1,2,3$, we deduce (3.46). Lemma 3.4 is proved. □

According to Lemma 3.3 and Lemma 3.4, we can now prove the local existence of solutions to the Cauchy problem (2.1)-(2.3).

Theorem 3.1. *Suppose the assumptions of Theorem 2.1 hold for $N=3$. Then there exists constants $\varepsilon_1>0,\ T_1>0,\ M_0>0$ such that, if $\|\mathscr{G}_0\|_{H^3}\le\varepsilon_1$, then for each $j\ge0$, $\mathscr{G}^j\in C\big([0,T_1,];H^3\big)$ is well-defined and satisfies*

$$\sup_{0\le t\le T_1}\big\|\mathscr{G}^j(t)\big\|_{H^3}\le M_0,\quad\forall j\ge0. \qquad (3.61)$$

In addition, $\big(\mathscr{G}^j\big)_{j\ge0}$ is a Cauchy sequence in the Banach space $C\big(\big[0,T_1\big];H^3\big)$, whose limit function $\mathscr{G}\in C\big(\big[0,T_1\big];H^3\big)$ satisfies

$$\sup_{0\le t\le T_1}\|\mathscr{G}(t)\|_{H^3}\le M_0. \qquad (3.62)$$

Here, $\mathscr{G}=(\varrho,\nabla\varrho,u)$ is the solution of (2.1)-(2.3) on $\big[0,T_1\big]$, and there is at most one solution satisfying (3.62) on $\big[0,T_1\big]$.

Proof. We first prove (3.61) by mathematical induction. For $j=0$, (3.61) is obviously true, because $\mathscr{G}^0=0$ is assumed. Now suppose (3.61) is true for $j\ge0$ and sufficiently small $M_0>0$, where the size of M_0 will be defined later. We need to prove that (3.61) is also true for $j+1$. Combined with Lemma 3.3 and Lemma 3.4, according to the inductive hypothesis, we have

$$\begin{aligned}&\frac{d}{dt}\big\|\mathscr{G}^{j+1}\big\|_{H^3}^2+\big\|\nabla u^{j+1}\big\|_{H^3}^2+\big\|\operatorname{div}u^{j+1}\big\|_{H^3}^2+\big\|u^{j+1}\big\|_{H^3}^2\\ \le&CM_0\big\|\big(\varrho^{j+1},\nabla\varrho^{j+1},\nabla^2\varrho^{j+1},u^{j+1},\nabla u^{j+1}\big)\big\|_{H^3}^2\\ &+C\big(M_0+M_0^2+M_0^3\big)\big\|\nabla\big(\varrho^{j+1},u^{j+1}\big)\big\|_{H^3}^2, \qquad (3.63)\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dt}\sum_{l=0}^3\int_{\mathbb{R}^3}\nabla^lu^{j+1}\cdot\nabla^{l+1}\varrho^{j+1}dx+\frac12\big(\big\|\nabla\varrho^{j+1}\big\|_{H^3}^2+\big\|\nabla^2\varrho^{j+1}\big\|_{H^3}^2\big)\\ \le&3\big\|\nabla u^{j+1}\big\|_{H^3}^2+r_0^2\big\|u^{j+1}\big\|_{H^3}^2+CM_0^2\big\|\big(\nabla^2\varrho^{j+1},\nabla u^{j+1}\big)\big\|_{H^3}^2\\ &+CM_0\big\|\big(\nabla\varrho^{j+1},\nabla^2\varrho^{j+1},u^{j+1},\nabla u^{j+1}\big)\big\|_{H^3}^2\end{aligned}$$

$$+ C\big(M_0 + M_0^2 + M_0^3\big)\big\|\big(\nabla \varrho^{j+1}, \nabla^2 \varrho^{j+1}, \nabla u^{j+1}\big)\big\|_{H^3}^2. \tag{3.64}$$

Now define

$$\big\|\tilde{\mathscr{G}}^{j+1}\big\|_{H^3}^2 = \big\|\mathscr{G}^{j+1}\big\|_{H^3}^2 + \tilde{\delta} \sum_{l=0}^{3} \int_{\mathbb{R}^3} \nabla^l u^{j+1} \cdot \nabla^{l+1} \varrho^{j+1} dx,$$

where $\tilde{\delta} > 0$ is sufficiently small. Then $\big\|\tilde{\mathscr{G}}^{j+1}\big\|_{H^3}^2$ is equivalent to $\big\|\mathscr{G}^{j+1}\big\|_{H^3}^2$.

By the linear combination of (3.63)+(3.64)$\times\tilde{\delta}$ and the smallness of $\tilde{\delta}$ (provided $3\tilde{\delta} < 1,\ r_0^2\,\tilde{\delta} < 1$), we can see that

$$\begin{aligned}
&\frac{d}{dt}\big\|\mathscr{G}^{j+1}\big\|_{H^3}^2 + \lambda\big\|\big(\nabla \varrho^{j+1}, \nabla^2 \varrho^{j+1}, \nabla u^{j+1}\big)\big\|_{H^3}^2 \\
\leq& C\big(M_0 + M_0^2 + M_0^3\big)\big\|\nabla\big(\nabla \varrho^{j+1}, u^{j+1}\big)\big\|_{H^3}^2 + C\big(M_0 + M_0^2 + M_0^3\big)\big\|\mathscr{G}^{j+1}\big\|_{H^3}^2,
\end{aligned} \tag{3.65}$$

where $\lambda \in (0,1)$ is a constant. Applying Gronwall inequality to (3.65), we obtain

$$\begin{aligned}
&\big\|\mathscr{G}^{j+1}(t)\big\|_{H^3}^2 + \lambda \int_0^t \big\|\big(\nabla \varrho^{j+1}, \nabla^2 \varrho^{j+1}, \nabla u^{j+1}\big)(s)\big\|_{H^3}^2 ds \\
\leq\ & \mathrm{e}^{C\left(M_0+M_0^2+M_0^3\right)t}\left[\varepsilon_1^2 + C\big(M_0 + M_0^2 + M_0^3\big)\int_0^t \big\|\nabla\big(\nabla \varrho^{j+1}, u^{j+1}\big)(s)\big\|_{H^3}^2 ds\right]
\end{aligned}$$

for any $0 \leq t \leq T_1$. Therefore, if we choose $\varepsilon_1 > 0,\ T_1 > 0$ and $M_0 > 0$ to be sufficiently small, we can deduce

$$\big\|\mathscr{G}^{j+1}(t)\big\|_{H^3}^2 + \lambda' \int_0^t \big\|\big(\nabla \varrho^{j+1}, \nabla^2 \varrho^{j+1}, \nabla u^{j+1}\big)(s)\big\|_{H^3}^2 ds \leq M_0^2 \tag{3.66}$$

for any $0 \leq t \leq T_1$, where $\lambda' \in (0,1)$ is a constant. This implies that if (3.61) is true for $j \geq 0$, then it is also true for $j+1$.

Furthermore, we assert that

$$\mathscr{G}^{j+1}(t) \in C\big(\big[0, T_1\big]; H^3\big), \quad \forall j \geq 0. \tag{3.67}$$

In fact, we set $0 \leq s < t \leq T_1$, then from (3.65) and (3.66) we can derive

$$\begin{aligned}
&\big|\big\|\mathscr{G}^{j+1}(t)\big\|_{H^3}^2 - \big\|\mathscr{G}^{j+1}(s)\big\|_{H^3}^2\big| = \left|\int_s^t \frac{d}{d\tau}\big\|\mathscr{G}^{j+1}(\tau)\big\|_{H^3}^2 d\tau\right| \\
\leq& C\big(M_0 + M_0^2 + M_0^3\big)\int_s^t \big\|\nabla\big(\nabla \varrho^{j+1}, u^{j+1}\big)(\tau)\big\|_{H^3}^2 d\tau + C\big(M_0 + M_0^2 + M_0^3\big)\int_s^t \big\|\mathscr{G}^{j+1}(\tau)\big\|_{H^3}^2 d\tau \\
\leq& C\big(M_0 + M_0^2 + M_0^3\big)\int_s^t \big\|\nabla\big(\nabla \varrho^{j+1}, u^{j+1}\big)(\tau)\big\|_{H^3}^2 d\tau + C\big(M_0 + M_0^2 + M_0^3\big)\int_s^t M_0^2 d\tau,
\end{aligned}$$

this implies (3.67).

Next, we consider the convergence of the sequence $\big(\mathscr{G}^j\big)_{j\geq 0}$. According to the investigation of (3.28)-(3.29), for problem

$$\begin{cases}
\big(\varrho^{j+1} - \varrho^j\big)_t + \operatorname{div}\big(u^{j+1} - u^j\big) = S_1^{j+1} - S_1^j, \\
\big(u^{j+1} - u^j\big)_t - \operatorname{div}\mathbb{D}\big(u^{j+1} - u^j\big) + \nabla\big(\varrho^{j+1} - \varrho^j\big) - \Delta\nabla\big(\varrho^{j+1} - \varrho^j\big) = S_2^{j+1} - S_2^j,
\end{cases}$$

where $S_i^{j+1} - S_i^j$ $(i = 1, 2)$ is defined by

$$\begin{aligned}
S_1^{j+1} - S_1^j =& -\big[\varrho^j \operatorname{div}\big(u^{j+1} - u^j\big) + \big(\varrho^j - \varrho^{j-1}\big)\operatorname{div} u^j + u^j \cdot \nabla\big(\varrho^{j+1} - \varrho^j\big) + \big(u^j - u^{j-1}\big)\cdot\nabla\varrho^j\big], \\
S_2^{j+1} - S_2^j =& -r_0\big(g\big(\varrho^j\big) - g\big(\varrho^{j-1}\big)\big)u^j - r_0 g\big(\varrho^j\big)\big(u^{j+1} - u^j\big)
\end{aligned}$$

$$
\begin{aligned}
&- r_1\big(|u^j|^2 - |u^{j-1}|^2\big)u^j - r_1|u^j|^2\big(u^{j+1} - u^j\big)\\
&- \big(u^j - u^{j-1}\big)\cdot\nabla u^j - u^j\cdot\nabla\big(u^{i+1} - u^j\big)\\
&- \big(f(\varrho^j) - f(\varrho^{j-1})\big)\Delta\nabla\varrho^j - f(\varrho^j)\Delta\nabla\big(\varrho^{j+1} - \varrho^j\big)\\
&- \big(h(\varrho^j) - h(\varrho^{j-1})\big)\nabla\varrho^j - h(\varrho^j)\nabla\big(\varrho^{j+1} - \varrho^j\big)\\
&+ \big(g(\varrho^j)\nabla\varrho^j - g(\varrho^{i-1})\nabla\varrho^{j-1}\big)\cdot\mathbb{D}u^j + g(\varrho^j)\nabla\varrho^j\cdot\mathbb{D}\big(u^{j+1} - u^j\big)\\
&+ \big(g^3(\varrho^j)|\nabla\varrho^j|^2 - g^3(\varrho^{j-1})|\nabla\varrho^{j-1}|^2\big)\nabla\varrho^j + g^3(\varrho^j)|\nabla\varrho^j|^2\nabla\big(\varrho^{j+1} - \varrho^j\big)\\
&- \big(g^2(\varrho^j)\nabla\varrho^j - g^2(\varrho^{j-1})\nabla\varrho^{j-1}\big)\Delta\varrho^j - g^2(\varrho^j)\nabla\varrho^j\Delta\big(\varrho^{j+1} - \varrho^j\big)\\
&- \big(g^2(\varrho^j)\nabla\varrho^j - g^2(\varrho^{j-1})\nabla\varrho^{j-1}\big)\cdot\nabla^2\varrho^j - g^2(\varrho^j)\nabla\varrho^j\cdot\nabla^2\big(\varrho^{j+1} - \varrho^j\big),
\end{aligned}
$$

similar to the derivation of (3.65), we can get

$$
\begin{aligned}
&\frac{d}{dt}\big\|\mathscr{G}^{j+1} - \mathscr{G}^j\big\|_{H^3}^2 + \lambda''\big\|\big(\nabla\big(\varrho^{j+1} - \varrho^j\big), \nabla^2\big(\varrho^{j+1} - \varrho^j\big), \nabla\big(u^{j+1} - u^j\big)\big)\big\|_{H^3}^2\\
\le& C\big(M_0 + M_0^2 + M_0^3\big)\big\|\nabla\big(\nabla\big(\varrho^{j+1} - \varrho^j\big), u^{j+1} - u^j\big)\big\|_{H^3}^2 + C\big(M_0 + M_0^2 + M_0^3\big)\big\|\mathscr{G}^{j+1} - \mathscr{G}^j\big\|_{H^3}^2\\
&+ C\big(M_0 + M_0^2 + M_0^3\big)\big\|\mathscr{G}^j - \mathscr{G}^{j-1}\big\|_{H^3}^2 + C\big\|\mathscr{G}^j - \mathscr{G}^{j-1}\big\|_{H^3}^2\big\|\nabla\big(\nabla\varrho^j, u^j\big)\big\|_{H^3}^2,
\end{aligned}
\tag{3.68}
$$

where $\lambda'' \in (0,1)$ is a constant. By applying Gronwall inequality to (3.68) and combining with (3.66), it follows that

$$
\begin{aligned}
&\big\|\big(\mathscr{G}^{j+1} - \mathscr{G}^j\big)(t)\big\|_{H^3}^2 + \lambda''\int_0^t \big\|\big(\nabla\big(\varrho^{j+1} - \varrho^j\big), \nabla^2\big(\varrho^{j+1} - \varrho^j\big), \nabla\big(u^{j+1} - u^j\big)\big)(s)\big\|_{H^3}^2 ds\\
\le& \mathrm{e}^{C(M_0 + M_0^2 + M_0^3)t}\Big[C\big(M_0 + M_0^2 + M_0^3\big)\int_0^t \big\|\nabla\big(\nabla\big(\varrho^{j+1} - \varrho^j\big), u^{j+1} - u^j\big)(s)\big\|_{H^3}^2 ds\\
&+ C\big(M_0 + M_0^2 + M_0^3\big)T_1\sup_{0\le s\le T_1}\big\|\big(\mathscr{G}^j - \mathscr{G}^{j-1}\big)(s)\big\|_{H^3}^2 + CM_0^2\sup_{0\le s\le T_1}\big\|\big(\mathscr{G}^j - \mathscr{G}^{j-1}\big)(s)\big\|_{H^3}^2\Big]
\end{aligned}
$$

for any $0 \le t \le T_1$. In virtue of M_0 and T_1 are sufficiently small, there exists a constant $\beta \in (0,1)$ such that

$$
\sup_{0\le t\le T_1}\big\|\big(\mathscr{G}^{j+1} - \mathscr{G}^j\big)(t)\big\|_{H^3}^2 \le \beta\sup_{0\le t\le T_1}\big\|\big(\mathscr{G}^j - \mathscr{G}^{j-1}\big)(t)\big\|_{H^3}^2
\tag{3.69}
$$

for any $j \ge 1$. It can be seen that $\big(\mathscr{G}^j\big)_{j\ge 0}$ is a Cauchy sequence in the Banach space $C\big([0,T_1];H^3\big)$ from (3.69), and the limit function

$$
\mathscr{G} = \mathscr{G}^0 + \lim_{n\to\infty}\sum_{j=0}^{n}\big(\mathscr{G}^{j+1} - \mathscr{G}^j\big)
$$

exists, and $\mathscr{G} \in C\big([0,T_1];H^3\big)$. In addition,

$$
\sup_{0\le t\le T_1}\|\mathscr{G}(t)\|_{H^3} \le \sup_{0\le t\le T_1}\varliminf_{j\longrightarrow\infty}\big\|\mathscr{G}^j(t)\big\|_{H^3} \le M_0,
$$

which means that (3.62) is true.

Finally, the local solution of the Cauchy problem (2.1)-(2.3) is unique in $C\big([0,T_1];H^3\big)$. In fact, let $\mathscr{G}(t)$ and $\tilde{\mathscr{G}}(t)$ be two local solutions satisfying (3.62), then it can be concluded from the derivation of (3.69) that

$$
\sup_{0\le t\le T_1}\big\|\mathscr{G}(t) - \tilde{\mathscr{G}}(t)\big\|_{H^3}^2 \le \beta'\sup_{0\le t\le T_1}\big\|\mathscr{G}(t) - \tilde{\mathscr{G}}(t)\big\|_{H^3}^2,
$$

where $\beta' \in (0,1)$ is a constant. This implies that $\mathscr{G}(t)=\tilde{\mathscr{G}}(t)$. So far, we have completed the proof of Theorem 3.1. □

4. Sobolev and Besov estimates with negative exponent

In this section, we will derive the evolution of the negative Sobolev and Besov norm of the solution $(\varrho,\nabla\varrho,u)$ to Cauchy problem (2.1)-(2.3). In order to deal with nonlinear terms conveniently, we only consider the case of $s\in\big[0,\frac{3}{2}\big]$. First of all, we have the following results for $s\in\big[0,\frac{3}{2}\big)$.

Lemma 4.1. *Suppose all the assumptions in Theorem 2.1 are true. Then, for $s\in\big(0,\frac{1}{2}\big]$, there holds*

$$\begin{aligned}&\frac{d}{dt}\|(\varrho,\nabla\varrho,u)\|^2_{\dot H^{-s}}+\|\nabla u\|^2_{\dot H^{-s}}+\|\operatorname{div}u\|^2_{\dot H^{-s}}\\ \lesssim&\big(\|\nabla\varrho\|^2_{H^2}+\|\nabla\varrho\|^3_{H^2}+\|\nabla u\|^2_{H^1}+\|\nabla\varrho\|_{L^2}\big\|\nabla^2u\big\|_{H^1}\\ &+\|\nabla\varrho\|_{H^1}\|u\|_{L^2}+\|u\|_{H^1}\|\nabla u\|^2_{H^1}\big)\|(\varrho,\nabla\varrho,u)\|_{\dot H^{-s}};\end{aligned}\tag{4.1}$$

and for $s\in\big(\frac{1}{2},\frac{3}{2}\big)$, there holds

$$\begin{aligned}&\frac{d}{dt}\|(\varrho,\nabla\varrho,u)\|^2_{\dot H^{-s}}+\|\nabla u\|^2_{\dot H^{-s}}+\|\operatorname{div}u\|^2_{\dot H^{-s}}\\ \lesssim&\left(\|\varrho\|_{H^1}+\|u\|_{L^2}\right)^{s-1/2}\left(\|\nabla\varrho\|_{H^1}+\|\nabla u\|_{L^2}\right)^{3/2-s}\left(\|\nabla\varrho\|_{H^2}+\|\nabla u\|_{H^1}\right.\\ &\left.+\|\nabla\varrho\|_{L^2}\|\nabla\varrho\|_{H^2}+\|u\|_{L^2}+\|\nabla u\|_{L^2}\|u\|_{H^1}\right)\|(\varrho,\nabla\varrho,u)\|_{\dot H^{-s}}.\end{aligned}\tag{4.2}$$

Remark 4.1. *If $s=0$, then the operator Λ^{-s} is an identical operator. By direct energy estimates, we can conclude that (4.1) also holds.*

Proof. Taking the Λ^{-s} $(s>0)$ energy estimates for (2.1)-(2.2) on ϱ, u and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big(\big|\Lambda^{-s}\varrho\big|^2+|\Lambda^{-s}u|^2\big)dx+\frac{1}{2}\int_{\mathbb{R}^3}\big(\big|\Lambda^{-s}(\nabla u)\big|^2+\big|\Lambda^{-s}(\operatorname{div}u)\big|^2\big)dx\\ =&-r_0\int_{\mathbb{R}^3}\Lambda^{-s}(g(\varrho)u)\cdot\Lambda^{-s}udx-r_1\int_{\mathbb{R}^3}\Lambda^{-s}\big(|u|^2u\big)\cdot\Lambda^{-s}udx\\ &+\int_{\mathbb{R}^3}\Lambda^{-s}(-\varrho\operatorname{div}u)\cdot\Lambda^{-s}\varrho dx+\int_{\mathbb{R}^3}\Lambda^{-s}(-u\cdot\nabla\varrho)\cdot\Lambda^{-s}\varrho dx\\ &+\int_{\mathbb{R}^3}\Lambda^{-s}(\Delta\nabla\varrho)\cdot\Lambda^{-s}udx-\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla u)\cdot\Lambda^{-s}udx\\ &-\int_{\mathbb{R}^3}\Lambda^{-s}\big(f(\varrho)\Delta\nabla\varrho\big)\cdot\Lambda^{-s}udx-\int_{\mathbb{R}^3}\Lambda^{-s}\big(h(\varrho)\nabla\varrho\big)\cdot\Lambda^{-s}udx\\ &+\int_{\mathbb{R}^3}\Lambda^{-s}\big(g(\varrho)\nabla\varrho\cdot\mathbb{D}u\big)\cdot\Lambda^{-s}udx\\ &+\int_{\mathbb{R}^3}\Lambda^{-s}\big(g^3(\varrho)|\nabla\varrho|^2\nabla\varrho-g^2(\varrho)\nabla\varrho\Delta\varrho-g^2(\varrho)\nabla\varrho\cdot\nabla^2\varrho\big)\cdot\Lambda^{-s}udx\\ :=&\sum_{i=1}^{10}Q_i.\end{aligned}\tag{4.3}$$

First, if $s \in \left(0, \frac{1}{2}\right]$, then $\frac{1}{2} + \frac{s}{3} < 1$ and $\frac{3}{s} \geq 6$. Note that $g(\varrho) = 1 - f(\varrho)$, and then apply Hölder inequality, Lemma 2.4, Lemma 2.3, Lemma 2.1, Sobolev inequality and Young inequality, we have

$$\begin{aligned}
Q_1 =& -r_0 \int_{\mathbb{R}^3} \left|\Lambda^{-s} u\right|^2 dx + r_0 \int_{\mathbb{R}^3} \Lambda^{-s}(f(\varrho)u) \cdot \Lambda^{-s} u dx \\
\lesssim& \left\|\Lambda^{-s}(f(\varrho)u)\right\|_{L^2} \left\|\Lambda^{-s} u\right\|_{L^2} \lesssim \left\|f(\varrho)u\right\|_{L^{\frac{1}{1/2+s/3}}} \left\|\Lambda^{-s} u\right\|_{L^2} \\
\lesssim& \|\varrho\|_{L^{3/s}} \|u\|_{L^2} \left\|\Lambda^{-s} u\right\|_{L^2} \lesssim \|\nabla \varrho\|_{L^2}^{1/2+s} \left\|\nabla^2 \varrho\right\|_{L^2}^{1/2-s} \|u\|_{L^2} \left\|\Lambda^{-s} u\right\|_{L^2} \\
\lesssim& \left\|\nabla \varrho\right\|_{H^1} \|u\|_{L^2} \left\|\Lambda^{-s} u\right\|_{L^2}. \qquad (4.4)\\
Q_2 \lesssim& \left\|\Lambda^{-s}\left(|u|^2 u\right)\right\|_{L^2} \left\|\Lambda^{-s} u\right\|_{L^2} \lesssim \|(u \cdot u)u\|_{L^{\frac{1}{1/2+3/s}}} \left\|\Lambda^{-s} u\right\|_{L^2} \\
\lesssim& \|u \cdot u\|_{L^2} \|u\|_{L^{3/s}} \left\|\Lambda^{-s} u\right\|_{L^2} \lesssim \|u\|_{L^6} \|u\|_{L^3} \left\|\nabla u\right\|_{L^2}^{1/2+s} \left\|\nabla^2 u\right\|_{L^2}^{1/2-s} \left\|\Lambda^{-s} u\right\|_{L^2} \\
\lesssim& \|u\|_{H^1} \|\nabla u\|_{H^1}^2 \left\|\Lambda^{-s} u\right\|_{L^2}. \qquad (4.5)
\end{aligned}$$

For Q_5, by integration by parts, then substitute (2.1), and apply the Young inequality, we can estimate as

$$\begin{aligned}
Q_5 =& \int_{\mathbb{R}^3} \Lambda^{-s}(\Delta \varrho) \cdot \Lambda^{-s}\big(\varrho_t + \varrho \operatorname{div} u + u \cdot \nabla \varrho\big) dx \\
=& -\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} \left|\Lambda^{-s} \nabla \varrho\right|^2 dx + \int_{\mathbb{R}^3} \Lambda^{-s}(\nabla \varrho) \cdot \Lambda^{-s}(-\nabla(\varrho \operatorname{div} u) - \nabla(u \cdot \nabla \varrho)) dx \\
\leq& -\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} \left|\Lambda^{-s} \nabla \varrho\right|^2 dx + C\left\|\Lambda^{-s}(-\nabla(\varrho \operatorname{div} u) - \nabla(u \cdot \nabla \varrho))\right\|_{L^2} \left\|\Lambda^{-s}(\nabla \varrho)\right\|_{L^2} \\
\leq& -\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} \left|\Lambda^{-s} \nabla \varrho\right|^2 dx + C\|(-\nabla(\varrho \operatorname{div} u) - \nabla(u \cdot \nabla \varrho))\|_{L^{\frac{1}{1/2+s/3}}} \left\|\Lambda^{-s}(\nabla \varrho)\right\|_{L^2} \\
\leq& -\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} \left|\Lambda^{-s} \nabla \varrho\right|^2 dx + C\big(\|\nabla \varrho\|_{L^{3/s}} \|\nabla u\|_{L^2} + \|\varrho\|_{L^{3/s}} \left\|\nabla^2 u\right\|_{L^2} \\
&+ \|u\|_{L^{3/s}} \left\|\nabla^2 \varrho\right\|_{L^2}\big) \left\|\Lambda^{-s}(\nabla \varrho)\right\|_{L^2} \\
\leq& -\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} \left|\Lambda^{-s} \nabla \varrho\right|^2 dx + C\big(\left\|\nabla^2 \varrho\right\|_{L^2}^{1/2+s} \left\|\nabla^3 \varrho\right\|_{L^2}^{1/2-s} \|\nabla u\|_{L^2} \\
&+ \|\nabla \varrho\|_{L^2}^{1/2+s} \left\|\nabla^2 \varrho\right\|_{L^2}^{1/2-s} \left\|\nabla^2 u\right\|_{L^2} + \|\nabla u\|_{L^2}^{1/2+s} \left\|\nabla^2 u\right\|_{L^2}^{1/2-s} \left\|\nabla^2 \varrho\right\|_{L^2}\big) \left\|\Lambda^{-s}(\nabla \varrho)\right\|_{L^2} \\
\leq& -\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} \left|\Lambda^{-s} \nabla \varrho\right|^2 dx + C\big(\|\nabla \varrho\|_{H^2}^2 + \|\nabla u\|_{H^1}^2\big) \left\|\Lambda^{-s}(\nabla \varrho)\right\|_{L^2}. \qquad (4.6)
\end{aligned}$$

Similarly, we can see that

$$\begin{aligned}
Q_3 \lesssim& \big(\|\nabla \varrho\|_{H^1}^2 + \|\nabla u\|_{L^2}^2\big) \left\|\Lambda^{-s} \varrho\right\|_{L^2}, \qquad (4.7)\\
Q_4 \lesssim& \big(\|\nabla \varrho\|_{L^2}^2 + \|\nabla u\|_{H^1}^2\big) \left\|\Lambda^{-s} \varrho\right\|_{L^2}, \qquad (4.8)\\
Q_6 \lesssim& \big(\|\nabla u\|_{H^1}^2 + \|\nabla u\|_{L^2}^2\big) \left\|\Lambda^{-s} u\right\|_{L^2}, \qquad (4.9)\\
Q_7 \lesssim& \big(\|\nabla \varrho\|_{H^1}^2 + \|\nabla \Delta \varrho\|_{L^2}^2\big) \left\|\Lambda^{-s} u\right\|_{L^2}, \qquad (4.10)\\
Q_8 \lesssim& \big(\|\nabla \varrho\|_{H^1}^2 + \|\nabla \varrho\|_{L^2}^2\big) \left\|\Lambda^{-s} u\right\|_{L^2}. \qquad (4.11)
\end{aligned}$$

For Q_9, note that $g(\varrho) = 1 - f(\varrho)$ and $f(\varrho) \in (0, 1)$, give rise to

$$Q_9 = \int_{\mathbb{R}^3} \Lambda^{-s}(\nabla \varrho \cdot \mathbb{D} u) \cdot \Lambda^{-s} u dx - \int_{\mathbb{R}^3} \Lambda^{-s}(f(\varrho) \nabla \varrho \cdot \mathbb{D} u) \cdot \Lambda^{-s} u dx$$

$$
\begin{aligned}
&\lesssim\big(\|\nabla\varrho\cdot\mathbb{D}u\|_{L^{\frac{1}{1/2+s/3}}}+\|f(\varrho)\nabla\varrho\cdot\mathbb{D}u\|_{L^{\frac{1}{1/2+s/3}}}\big)\big\|\Lambda^{-s}u\big\|_{L^2}\\
&\lesssim\big(\|\nabla\varrho\|_{L^2}\|\mathbb{D}u\|_{L^{3/s}}+\|f(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^2}\|\mathbb{D}u\|_{L^{3/s}}\big)\big\|\Lambda^{-s}u\big\|_{L^2}\\
&\lesssim\|\nabla\varrho\|_{L^2}\big\|\nabla^2u\big\|_{L^2}^{1/2+s}\big\|\nabla^3u\big\|_{L^2}^{1/2-s}\big\|\Lambda^{-s}u\big\|_{L^2}\\
&\lesssim\|\nabla\varrho\|_{L^2}\big\|\nabla^2u\big\|_{H^1}\big\|\Lambda^{-s}u\big\|_{L^2}.
\end{aligned}
\tag{4.12}
$$

Finally, we infer that

$$
\begin{aligned}
Q_{10}\leq&\big(\big\|\Lambda^{-s}\big(g^3(\varrho)|\nabla\varrho|^2\nabla\varrho\big)\big\|_{L^2}+\big\|\Lambda^{-s}\big(g^2(\varrho)\nabla\varrho\Delta\varrho\big)\big\|_{L^2}+\big\|\Lambda^{-s}\big(g^2(\varrho)\nabla\varrho\cdot\nabla^2\varrho\big)\big\|_{L^2}\big)\big\|\Lambda^{-s}u\big\|_{L^2}\\
\lesssim&\big(\big\|g^3(\varrho)|\nabla\varrho|^2\nabla\varrho\big\|_{L^{\frac{1}{1/2+s/3}}}+\big\|g^2(\varrho)\nabla\varrho\Delta\varrho\big\|_{L^{\frac{1}{1/2+s/3}}}+\big\|g^2(\varrho)\nabla\varrho\cdot\nabla^2\varrho\big\|_{L^{\frac{1}{1/2+s/3}}}\big)\big\|\Lambda^{-s}u\big\|_{L^2}\\
\lesssim&\big(\|g(\varrho)\|_{L^\infty}^3\|\nabla\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^{3/s}}\|\nabla\varrho\|_{L^2}+\|g(\varrho)\|_{L^\infty}^2\|\nabla\varrho\|_{L^{3/s}}\|\nabla^2\varrho\|_{L^2}\big)\big\|\Lambda^{-s}u\big\|_{L^2}\\
\lesssim&\big\|\nabla^2\varrho\big\|_{L^2}^{1/2+s}\big\|\nabla^3\varrho\big\|_{L^2}^{1/2-s}\big(\|\nabla\varrho\|_{L^2}\|\nabla\varrho\|_{H^2}+\big\|\nabla^2\varrho\big\|_{L^2}\big)\big\|\Lambda^{-s}u\big\|_{L^2}\\
\lesssim&\big(\|\nabla\varrho\|_{H^2}^2+\|\nabla\varrho\|_{H^2}^3\big)\big\|\Lambda^{-s}u\big\|_{L^2}.
\end{aligned}
\tag{4.13}
$$

Putting (4.4)-(4.13) into (4.3) and we get (4.1).

Next, we consider $s\in\big(\frac{1}{2},\frac{3}{2}\big)$, in which $\frac{1}{2}+\frac{s}{3}<1$ and $2<\frac{3}{s}<6$. Similar to the estimation process of (4.4), we have

$$
\begin{aligned}
Q_1\lesssim&\|f(\varrho)u\|_{L^{\frac{1}{1/2+s/3}}}\big\|\Lambda^{-s}u\big\|_{L^2}\\
\lesssim&\|\varrho\|_{L^{3/s}}\|u\|_{L^2}\big\|\Lambda^{-s}u\big\|_{L^2}\\
\lesssim&\|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|u\|_{L^2}\big\|\Lambda^{-s}u\big\|_{L^2}.
\end{aligned}
\tag{4.14}
$$

From the estimation process of (4.5), give

$$
\begin{aligned}
Q_2\lesssim&\|(u\cdot u)u\|_{L^{\frac{1}{1/2+s/3}}}\big\|\Lambda^{-s}u\big\|_{L^2}\\
\lesssim&\|u\|_{L^6}\|u\|_{L^3}\|u\|_{L^{3/s}}\big\|\Lambda^{-s}u\big\|_{L^2}\\
\lesssim&\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}\|u\|_{H^1}\big\|\Lambda^{-s}u\big\|_{L^2}.
\end{aligned}
\tag{4.15}
$$

For Q_5, we have

$$
\begin{aligned}
Q_5=&\int_{\mathbb{R}^3}\Lambda^{-s}(\Delta\varrho)\cdot\Lambda^{-s}\big(\varrho_t+\varrho\operatorname{div}u+u\cdot\nabla\varrho\big)dx\\
=&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big|\Lambda^{-s}\nabla\varrho\big|^2dx+\int_{\mathbb{R}^3}\Lambda^{-s}(\nabla\varrho)\cdot\Lambda^{-s}(-\nabla(u\cdot\nabla\varrho)-\nabla(\varrho\operatorname{div}u))dx\\
\leq&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big|\Lambda^{-s}\nabla\varrho\big|^2dx+C\big\|\Lambda^{-s}(-\nabla(u\cdot\nabla\varrho)-\nabla(\varrho\operatorname{div}u))\big\|_{L^2}\big\|\Lambda^{-s}(\nabla\varrho)\big\|_{L^2}\\
\leq&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big|\Lambda^{-s}\nabla\varrho\big|^2dx+C\big\|(-\nabla(u\cdot\nabla\varrho)-\nabla(\varrho\operatorname{div}u))\big\|_{L^{\frac{1}{1/2+s/3}}}\big\|\Lambda^{-s}(\nabla\varrho)\big\|_{L^2}\\
\leq&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big|\Lambda^{-s}\nabla\varrho\big|^2dx+C\big(\|u\|_{L^{3/s}}\big\|\nabla^2\varrho\big\|_{L^2}\\
&+\|\nabla\varrho\|_{L^{3/s}}\|\nabla u\|_{L^2}+\|\varrho\|_{L^{3/s}}\big\|\nabla^2u\big\|_{L^2}\big)\big\|\Lambda^{-s}(\nabla\varrho)\big\|_{L^2}\\
\leq&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big|\Lambda^{-s}\nabla\varrho\big|^2dx+C\big(\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\big\|\nabla^2\varrho\big\|_{L^2}\\
&+\|\nabla\varrho\|_{L^2}^{s-1/2}\big\|\nabla^2\varrho\big\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}+\|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\big\|\nabla^2u\big\|_{L^2}\big)\big\|\Lambda^{-s}(\nabla\varrho)\big\|_{L^2}.
\end{aligned}
\tag{4.16}
$$

Similarly it is easy to get the following:

$$Q_3 \lesssim \|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}\left\|\Lambda^{-s}\varrho\right\|_{L^2}, \tag{4.17}$$

$$Q_4 \lesssim \|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla\varrho\|_{L^2}\left\|\Lambda^{-s}\varrho\right\|_{L^2}, \tag{4.18}$$

$$Q_6 \lesssim \|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}\left\|\Lambda^{-s}u\right\|_{L^2}, \tag{4.19}$$

$$Q_7 \lesssim \|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla\Delta\varrho\|_{L^2}\left\|\Lambda^{-s}u\right\|_{L^2}, \tag{4.20}$$

$$Q_8 \lesssim \|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla\varrho\|_{L^2}\left\|\Lambda^{-s}u\right\|_{L^2}. \tag{4.21}$$

Like the estimation process of (4.12), we have

$$\begin{aligned}
Q_9 \lesssim & \|g(\varrho)\nabla\varrho\cdot\mathbb{D}u\|_{L^{\frac{1}{1/2+s/3}}}\left\|\Lambda^{-s}u\right\|_{L^2}\\
\lesssim & \|g(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^{3/s}}\|\nabla u\|_{L^2}\left\|\Lambda^{-s}u\right\|_{L^2}\\
\lesssim & \|\nabla\varrho\|_{L^2}^{s-1/2}\|\nabla^2\varrho\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}\left\|\Lambda^{-s}\varrho\right\|_{L^2}.
\end{aligned} \tag{4.22}$$

Finally, for Q_{10}, we can now estimate it as

$$\begin{aligned}
Q_{10} \lesssim & \Big(\left\|g^3(\varrho)|\nabla\varrho|^2\nabla\varrho\right\|_{L^{\frac{1}{1/2+s/3}}}+\left\|g^2(\varrho)\nabla\varrho\Delta\varrho\right\|_{L^{\frac{1}{1/2+s/3}}}+\left\|g^2(\varrho)\nabla\varrho\cdot\nabla^2\varrho\right\|_{L^{\frac{1}{1/2+s/3}}}\Big)\left\|\Lambda^{-s}u\right\|_{L^2}\\
\lesssim & \Big(\|g(\varrho)\|_{L^\infty}^3\|\nabla\varrho\|_{L^\infty}\|\nabla\varrho\|_{L^{3/s}}\|\nabla\varrho\|_{L^2}+\|g(\varrho)\|_{L^\infty}^2\|\nabla\varrho\|_{L^{3/s}}\left\|\nabla^2\varrho\right\|_{L^2}\Big)\left\|\Lambda^{-s}u\right\|_{L^2}\\
\lesssim & \|\nabla\varrho\|_{L^2}^{s-1/2}\left\|\nabla^2\varrho\right\|_{L^2}^{3/2-s}\Big(\|\nabla\varrho\|_{L^2}\|\nabla\varrho\|_{H^2}+\left\|\nabla^2\varrho\right\|_{L^2}\Big)\left\|\Lambda^{-s}u\right\|_{L^2}.
\end{aligned} \tag{4.23}$$

Putting (4.14)-(4.23) into (4.3) and we get (4.2). Lemma 4.1 is proved. □

Similar to the case of homogeneous Sobolev estimation, we derive the evolution of the negative Besov norm of the solution $(\varrho,\nabla\varrho,u)$. Now consider $s\in\left(0,\frac{3}{2}\right]$, we have the following results.

Lemma 4.2. *Suppose all the assumptions in Theorem 2.1 are true, $j\in\mathbb{Z}$. Then, for $s\in\left(0,\frac{1}{2}\right]$, there holds*

$$\begin{aligned}
&\frac{d}{dt}\left(2^{-2sj}\left\|\dot\Delta_j\varrho\right\|_{L^2}^2+2^{-2sj}\left\|\dot\Delta_j(\nabla\varrho)\right\|_{L^2}^2+2^{-2sj}\left\|\dot\Delta_j u\right\|_{L^2}^2\right)\\
&\quad+2^{-2sj}\left\|\dot\Delta_j(\nabla u)\right\|_{L^2}^2+2^{-2sj}\left\|\dot\Delta_j(\operatorname{div}u)\right\|_{L^2}^2\\
&\lesssim\left(\|\nabla\varrho\|_{H^2}^2+\|\nabla\varrho\|_{H^2}^3+\|\nabla\varrho\|_{L^2}\left\|\nabla^2u\right\|_{H^1}\right.\\
&\quad\left.+\|\nabla\varrho\|_{H^1}\|u\|_{L^2}+\|\nabla u\|_{H^1}^2+\|u\|_{H^1}\|\nabla u\|_{H^1}^2\right)\\
&\quad\times\left(\|\varrho\|_{\dot B^{-s}_{2,\infty}}+\|\nabla\varrho\|_{\dot B^{-s}_{2,\infty}}+\|u\|_{\dot B^{-s}_{2,\infty}}\right);
\end{aligned} \tag{4.24}$$

and for $s\in\left(\frac{1}{2},\frac{3}{2}\right]$, there holds

$$\begin{aligned}
&\frac{d}{dt}\left(2^{-2sj}\left\|\dot\Delta_j\varrho\right\|_{L^2}^2+2^{-2sj}\left\|\dot\Delta_j(\nabla\varrho)\right\|_{L^2}^2+2^{-2sj}\left\|\dot\Delta_j u\right\|_{L^2}^2\right)\\
&\quad+2^{-2sj}\left\|\dot\Delta_j(\nabla u)\right\|_{L^2}^2+2^{-2sj}\left\|\dot\Delta_j(\operatorname{div}u)\right\|_{L^2}^2\\
&\lesssim\left(\|\varrho\|_{H^1}+\|u\|_{L^2}\right)^{s-1/2}\left(\|\nabla\varrho\|_{H^1}+\|\nabla u\|_{L^2}\right)^{3/2-s}\\
&\quad\times\left(\|\nabla\varrho\|_{H^2}+\|\nabla\varrho\|_{L^2}\|\nabla\varrho\|_{H^2}+\|\nabla u\|_{H^1}+\|u\|_{L^2}+\|\nabla u\|_{L^2}\|u\|_{H^1}\right)\\
&\quad\times\left(\|\varrho\|_{\dot B^{-s}_{2,\infty}}+\|\nabla\varrho\|_{\dot B^{-s}_{2,\infty}}+\|u\|_{\dot B^{-s}_{2,\infty}}\right).
\end{aligned} \tag{4.25}$$

Proof. By the $\dot{\Delta}_j$ energy estimates for (2.1)-(2.2) on $\varrho,\ u$, and multiplication with 2^{-2sj}, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(2^{-2sj}\big\|\dot{\Delta}_j\varrho\big\|_{L^2}^2+2^{-2sj}\big\|\dot{\Delta}_j u\big\|_{L^2}^2\big)+\frac{1}{2}\big(2^{-2sj}\big\|\dot{\Delta}_j(\nabla u)\big\|_{L^2}^2+2^{-2sj}\big\|\dot{\Delta}_j(\operatorname{div} u)\big\|_{L^2}^2\big)\\
=&-r_0\ 2^{-2sj}\int_{\mathbb{R}^3}\dot{\Delta}_j\big(g(\varrho)u\big)\cdot\dot{\Delta}_j u dx-r_1\ 2^{-2sj}\int_{\mathbb{R}^3}\dot{\Delta}_j\big(|u|^2u\big)\cdot\dot{\Delta}_j u dx\\
&+2^{-2sj}\int_{\mathbb{R}^3}\dot{\Delta}_j\big(-\varrho\operatorname{div} u\big)\cdot\dot{\Delta}_j\varrho dx+2^{-2sj}\int_{\mathbb{R}^3}\dot{\Delta}_j\big(-u\cdot\nabla\varrho\big)\cdot\dot{\Delta}_j\varrho dx\\
&-2^{-2sj}\int_{\mathbb{R}^3}\dot{\Delta}_j\big(\Delta\nabla\varrho\big)\cdot\dot{\Delta}_j u dx-2^{-2sj}\int_{\mathbb{R}^3}\dot{\Delta}_j\big(u\cdot\nabla u\big)\cdot\dot{\Delta}_j u dx\\
&-2^{-2sj}\int_{\mathbb{R}^3}\dot{\Delta}_j\big(f(\varrho)\Delta\nabla\varrho\big)\cdot\dot{\Delta}_j u dx-2^{-2sj}\int_{\mathbb{R}^3}\dot{\Delta}_j\big(h(\varrho)\nabla\varrho\big)\cdot\dot{\Delta}_j u dx\\
&+2^{-2sj}\int_{\mathbb{R}^3}\dot{\Delta}_j\big(g(\varrho)\nabla\varrho\cdot\mathbb{D}u\big)\cdot\dot{\Delta}_j u dx\\
&+2^{-2sj}\int_{\mathbb{R}^3}\dot{\Delta}_j\big(g^3(\varrho)|\nabla\varrho|^2\nabla\varrho-g^2(\varrho)\nabla\varrho\Delta\varrho-g^2(\varrho)\nabla\varrho\cdot\nabla^2\varrho\big)\cdot\dot{\Delta}_j u dx.
\end{aligned}$$

Then combined with Lemma 2.5, we can prove Lemma 4.2. When applying Lemma 2.5, we only need to pay attention to

$$2^{-sj}\|\dot{\Delta}_j f\|_{L^2}\le\sup_{j\in\mathbb{Z}}2^{-sj}\big\|\dot{\Delta}_j f\big\|_{L^2}=\|f\|_{\dot{B}^{-s}_{2,\infty}}\lesssim\|f\|_{L^{\frac{1}{1/2+s/3}}}.$$

In addition, except replacing the $\dot{H}^{-s}$ norm by the $\dot{B}^{-s}_{2,\infty}$ norm and allowing $s=\frac{3}{2}$, the proof process is exactly the same as that of Lemma 4.1. Here we omit the later proof. □

5. The proof of the main theorem

Based on the prior estimates derived in the previous two sections and the local existence of (2.1)-(2.3) given by theorem 3.1. If the H^3-norm of the initial data is sufficiently small, then we can prove the global existence of the solution. To do this, define

$$\mathfrak{E}_k^{k+2}:=\sum_{l=k}^{k+2}\big\|\nabla^l(\varrho,\nabla\varrho,u)\big\|_{L^2}^2,\tag{5.1}$$

$$\mathfrak{D}_k^{k+2}:=\sum_{l=k}^{k+2}\big\|\nabla^l\big(\nabla\varrho,\nabla^2\varrho,u,\nabla u\big)\big\|_{L^2}^2.\tag{5.2}$$

Let us prove the main results of this paper.

Proof of Theorem 2.1. Let $k\ge 0$ and $N\ge\max\{k+2,\ 3\}$. From (3.1) in Lemma 3.1 and (3.15) in Lemma 3.2, we have

$$\begin{aligned}
&\frac{d}{dt}\sum_{l=k}^{k+2}\big\|\nabla^l(\varrho,\nabla\varrho,u)\big\|_{L^2}^2+\sum_{l=k}^{k+2}\big(\big\|\nabla^{l+1}u\big\|_{L^2}^2+\big\|\nabla^l(\operatorname{div} u)\big\|_{L^2}^2+\big\|\nabla^l u\big\|_{L^2}^2\big)\\
&\le C_1\delta\sum_{l=k}^{k+2}\big\|\nabla^l\big(\nabla\varrho,\nabla^2\varrho,u,\nabla u\big)\big\|_{L^2}^2,
\end{aligned}\tag{5.3}$$

and

$$\frac{d}{dt}\sum_{l=k}^{k+2}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx+\frac{1}{4}\sum_{l=k}^{k+2}\big\|\nabla^{l+1}(\varrho,\nabla\varrho)\big\|_{L^2}^2$$

$$\leq C_2\delta\sum_{l=k}^{k+2}\big\|\nabla^{l+1}(\varrho,\nabla\varrho,u)\big\|_{L^2}^2+3\sum_{l=k}^{k+2}\big\|\nabla^{l+1}u\big\|_{L^2}^2+r_0^2\sum_{l=k}^{k+2}\big\|\nabla^{l}u\big\|_{L^2}^2. \tag{5.4}$$

Linear combination (5.3)+(5.4)$\times 4(C_1+C_2)\delta$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\sum_{l=k}^{k+2}\big\|\nabla^{l}(\varrho,\nabla\varrho,u)\big\|_{L^2}^2+4\big(C_1+C_2\big)\delta\sum_{l=k}^{k+2}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx\right\}\\
\leq&\big(4C_2\big(C_1+C_2\big)\delta^2+C_1\delta-\big(C_1+C_2\big)\delta\big)\sum_{l=k}^{k+2}\big\|\nabla^{l+1}\varrho\big\|_{L^2}^2\\
&+\big(4C_2\big(C_1+C_2\big)\delta^2+C_1\delta-\big(C_1+C_2\big)\delta\big)\sum_{l=k}^{k+2}\big\|\nabla^{l+2}\varrho\big\|_{L^2}^2\\
&+\big(4\big(C_1+C_2\big)r_0^2\delta+C_1\delta-1\big)\big)\sum_{l=k}^{k+2}\big\|\nabla^{l}u\big\|_{L^2}^2\\
&+\big(4C_2\big(C_1+C_2\big)\delta^2+12\big(C_1+C_2\big)\delta+C_1\delta-1\big)\sum_{l=k}^{k+2}\big\|\nabla^{l+1}u\big\|_{L^2}^2.
\end{aligned}$$

According to the smallness of $\delta\in(0,1)$, there exists a constant $C_3>0$, such that

$$\begin{aligned}
&\frac{d}{dt}\left\{\sum_{l=k}^{k+2}\big\|\nabla^{l}(\varrho,\nabla\varrho,u)\big\|_{L^2}^2+4\big(C_1+C_2\big)\delta\sum_{l=k}^{k+2}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx\right\}\\
&+C_3\sum_{l=k}^{k+2}\big\|\nabla^{l}\big(\nabla\varrho,\nabla^2\varrho,u,\nabla u\big)\big\|_{L^2}^2\leq 0.
\end{aligned}\tag{5.5}$$

Now define

$$\tilde{\mathfrak{E}}_k^{k+2}(t)=\frac{1}{C_3}\left\{\sum_{l=k}^{k+2}\big\|\nabla^{l}(\varrho,\nabla\varrho,u)\big\|_{L^2}^2+4\big(C_1+C_2\big)\delta\sum_{l=k}^{k+2}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho dx\right\},$$

In view of δ sufficiently small, lead to $\tilde{\mathfrak{E}}_k^{k+2}(t)$ is equivalent to $\mathfrak{E}_k^{k+2}(t)$.

Thus, we can see that

$$\frac{d}{dt}\mathfrak{E}_k^{k+2}(t)+C\mathfrak{D}_k^{k+2}(t)\leq 0. \tag{5.6}$$

Below, we prove the global existence and decay estimates of solutions of (2.1)-(2.3), respectively.

(I) Global existence.

Step 1. Global small solution in H^3-norm.

Let $k=0,1$ in (5.6), respectively, and then add the two formulas, then for any $T>0$, we have

$$\mathfrak{E}_3(t)+\int_0^t\mathfrak{D}_3(s)ds\leq\overline{C}\mathfrak{E}_3(0),\quad\forall t\in[0,T]. \tag{5.7}$$

Now choose the positive constant

$$\varepsilon_0:=\min\big\{\delta,\varepsilon_1\big\},$$

where δ and ε_1 are given in Lemma 3.1 and Theorem 3.1, respectively. Then take the small constant δ_0 to satisfy

$$\sqrt{\mathfrak{E}_3(0)} \leq \sqrt{\delta_0} := \frac{\varepsilon_0}{2\sqrt{1+\overline{C}}}. \tag{5.8}$$

For the Cauchy problem (2.1)-(2.3), set

$$T := \sup\big\{t : \sup_{0\leq s\leq t} \sqrt{\mathfrak{E}_3(s)} \leq \varepsilon_0\big\}.$$

Note that

$$\sqrt{\mathfrak{E}_3(0)} \leq \frac{\varepsilon_0}{2\sqrt{1+\overline{C}}} < \varepsilon_0 \leq \varepsilon_1,$$

and $\mathscr{G} \in C\big([0,T_1];H^3\big)$ given in Theorem 3.1, we can show $T > 0$. If T is a finite value, then from the definition of T, we get

$$\sup_{0\leq s\leq T} \sqrt{\mathfrak{E}_3(s)} = \varepsilon_0.$$

On the other hand, according to (5.7) and (5.8), give rise to

$$\sup_{0\leq s\leq T} \sqrt{\mathfrak{E}_3(s)} \leq \sqrt{\overline{C}}\sqrt{\mathfrak{E}_3(0)} \leq \frac{\varepsilon_0\sqrt{\overline{C}}}{2\sqrt{1+\overline{C}}} \leq \frac{\varepsilon_0}{2},$$

which contradicts the previous formula. So it can only be $T = \infty$. This implies that the local solution obtained in Theorem 3.1 can extend time to infinity. In this way, there is a unique global solution in $C\big([0,\infty];H^3\big)$ for the Cauchy problem (2.1)-(2.3), and (2.7) holds.

Step 2. Global solution in H^N-norm.

Considering Lemma 3.3 and Lemma 3.4, through a similar argumentation, we can infer the local existence of $\mathscr{G}(t)$ in $C\big([0,T_1];H^N\big)$ $(N \geq 3)$. Combined with the existence of the global solution with small H^3-norm obtained in step 1, the existence of the global solution in H^N can be obtained.

In fact, using (2.5)-(2.6), re-estimating Lemma 3.3 and Lemma 3.4, we infer that

$$\begin{aligned}
&\frac{d}{dt}\|\mathscr{G}^{j+1}\|_{H^N}^2 + \|\nabla u^{j+1}\|_{H^N}^2 + \|\operatorname{div} u^{j+1}\|_{H^N}^2 + \|u^{j+1}\|_{H^N}^2\\
&\leq C\|\left(\varrho^j,u^j\right)\|_{H^3}^{\alpha_1}\|\left(\varrho^j,u^j\right)\|_{H^N}^{1-\alpha_1}\|\left(\varrho^{j+1},\nabla\varrho^{j+1},\nabla^2\varrho^{j+1},u^{j+1},\nabla u^{j+1}\right)\|_{H^N}^2\\
&\quad + C\Big(\|\varrho^j\|_{H^3}^{\alpha_2}\|\varrho^j\|_{H^N}^{1-\alpha_2} + \|\varrho^j\|_{H^3}^{2\alpha_3}\|\varrho^j\|_{H^N}^{2(1-\alpha_3)} + \|\varrho^j\|_{H^3}^{3\alpha_4}\|\varrho^j\|_{H^N}^{3(1-\alpha_4)}\\
&\quad + \|u^j\|_{H^3}^{2\alpha_5}\|u^j\|_{H^N}^{2(1-\alpha_5)}\Big)\|\nabla\left(\varrho^{j+1},u^{j+1}\right)\|_{H^N}^2,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}\sum_{l=0}^{N}\int_{\mathbb{R}^3}\nabla^l u^{j+1}\cdot\nabla^{l+1}\varrho^{j+1}dx + \frac{1}{2}\big(\|\nabla\varrho^{j+1}\|_{H^N}^2 + \|\nabla^2\varrho^{j+1}\|_{H^N}^2\big)\\
&\leq 3\|\nabla u^{j+1}\|_{H^N}^2 + r_0^2\|u^{j+1}\|_{H^N}^2 + C\|u^j\|_{H^3}^{2\alpha_6}\|u^j\|_{H^N}^{2(1-\alpha_6)}\|\big(\nabla^2\varrho^{j+1},\nabla u^{j+1}\big)\|_{H^N}^2\\
&\quad + C\|\big(\varrho^j,u^j\big)\|_{H^3}^{\alpha_7}\|\big(\varrho^j,u^j\big)\|_{H^N}^{1-\alpha_7}\|\big(\nabla\varrho^{j+1},\nabla^2\varrho^{j+1},u^{j+1},\nabla u^{j+1}\big)\|_{H^N}^2\\
&\quad + C\big(\|\varrho^j\|_{H^3}^{\alpha_8}\|\varrho^j\|_{H^N}^{1-\alpha_8} + \|\varrho^j\|_{H^3}^{2\alpha_9}\|\varrho^j\|_{H^N}^{2(1-\alpha_9)} + \|\varrho^j\|_{H^3}^{3\alpha_{10}}\|\varrho^j\|_{H^N}^{3(1-\alpha_{10})}\big)\\
&\quad \times\|\big(\nabla\varrho^{j+1},\nabla^2\varrho^{j+1},\nabla u^{j+1}\big)\|_{H^N}^2.
\end{aligned}$$

Here, $\alpha_i \in (0,1)$, $i = 1, 2, \cdots, 10$ are all constants. In this way, if we assume that the H^3-norm of the initial data is sufficiently small, we can derive the global existence of the solution $(\varrho, \nabla\varrho, u)$ of the Cauchy problem (2.1)-(2.3) in H^N. In fact, for any fixed constant $M > 0$, assuming that $\|\mathscr{G}_0\|_{H^N} \le M$, then according to the proof of Theorem 3.1 and the smallness of M_0 in (3.61), we can obtain $\left\|\mathscr{G}^j(t)\right\|_{H^N} \le M$, $\forall t \in [0, T_1]$. Here T_1 is given in Theorem 3.1. In addition, we can verify that the limit function of $\mathscr{G}^j(t)$ is indeed the solution of (2.1)-(2.3) on $[0, T_1]$. The results here, combined with the previous a priori estimates, imply the global solution $(\varrho, \nabla\varrho, u) \in C\big([0,\infty]; H^N\big)$. Finally, taking $k = 0, 1, \cdots, N-2$ in (5.6) and take the sum of k, then we get (2.8) by integrating the time on $[0, \infty]$.

(II) Decay estimate.

From (5.6), it can be seen that $\mathfrak{D}_k^{k+2}$ is weaker than $\mathfrak{E}_k^{k+2}$, which indicates that there is no exponential decay. We need to bound the missing terms in the energy, that is to say, we use $\mathfrak{E}_k^{k+2}$ to bound $\left\|\nabla^k(\varrho, \nabla\varrho, u)\right\|_{L^2}^2$. For this reason, we need to apply the interpolation inequality between the Sobolev norm of positive and negative exponents.

For now, let us assume that (2.9) and (2.10) have been established. So, for $s > 0$ and $k \ge 0$, apply Lemma 2.6 or Lemma 2.7, we have

$$
\begin{aligned}
\left\|\nabla^k(\varrho, \nabla\varrho, u)\right\|_{L^2} \le & \left\|\nabla(\varrho, \nabla\varrho, u)\right\|_{\dot{H}^{-s}}^{\frac{1}{k+1+s}} \left\|\nabla^{k+1}(\varrho, \nabla\varrho, u)\right\|_{L^2}^{\frac{k+s}{k+1+s}} \\
\le & C\left\|\nabla^k(\nabla\varrho, \nabla^2\varrho, u, \nabla u)\right\|_{L^2}^{\frac{k+s}{k+1+s}},
\end{aligned} \tag{5.9}
$$

or

$$
\begin{aligned}
\left\|\nabla^k(\varrho, \nabla\varrho, u)\right\|_{L^2} \le & \left\|\nabla(\varrho, \nabla\varrho, u)\right\|_{\dot{B}_{2,\infty}^{-s}}^{\frac{1}{k+1+s}} \left\|\nabla^{k+1}(\varrho, \nabla\varrho, u)\right\|_{L^2}^{\frac{k+s}{k+1+s}} \\
\le & C\left\|\nabla^k(\nabla\varrho, \nabla^2\varrho, u, \nabla u)\right\|_{L^2}^{\frac{k+s}{k+1+s}}.
\end{aligned} \tag{5.10}
$$

By substituting (5.9) or (5.10) into (5.6), it can be obtain

$$
\frac{d}{dt}\mathfrak{E}_k^{k+2}(t) + C\big[\mathfrak{E}_k^{k+2}(t)\big]^{1+\alpha} \le 0, \tag{5.11}
$$

where $\alpha = \frac{1}{k+s}$, $k = 0, 1, \cdots, N-2$. By directly solving the inequality (5.11), we can get

$$
\mathfrak{E}_k^{k+2}(t) \le \big\{\big[\mathfrak{E}_k^{k+2}(0)\big]^{-\alpha} + \alpha t\big\}^{-\frac{1}{\alpha}} \le C_0(1+t)^{-(k+s)}. \tag{5.12}
$$

This implies (2.11) to be established.

The remaining part is to prove the estimates (2.9) and (2.10). First, starting from Lemma 4.1, we can consider proving (2.11). However, it is not possible to prove all $s \in \big[0, \frac{1}{2}\big]$ at this time, but the different values of s must be examined. For $s = 0$, the conclusion is trivial. For $s \in \big(0, \frac{1}{2}\big]$, integrating (4.1) with respect to time, combined with the definition of $\mathfrak{E}_3(t)$, $\mathfrak{D}_3(t)$, deduce

$$
\begin{aligned}
\|(\varrho, \nabla\varrho, u)(t)\|_{\dot{H}^{-s}}^2 \lesssim & \left\|(\varrho_0, \nabla\varrho_0, u_0)\right\|_{\dot{H}^{-s}}^2 + \int_0^t \big(\sqrt{\mathfrak{E}_3(\tau)} + 1\big)\mathfrak{D}_3(\tau)\|(\varrho, \nabla\varrho, u)(\tau)\|_{\dot{H}^{-s}} d\tau \\
\le & C_0\Big(1 + \sup_{0\le\tau\le t}\|(\varrho, \nabla\varrho, u)(\tau)\|_{\dot{H}^{-s}}\Big),
\end{aligned} \tag{5.13}
$$

this implies that (2.9) holds for $s \in \big(0, \frac{1}{2}\big]$, and then (2.11) holds for $s \in \big(0, \frac{1}{2}\big]$. For $s \in \big(\frac{1}{2}, \frac{3}{2}\big)$, we find that it can not be demonstrated by the above method for the case of $s \in \big(0, \frac{1}{2}\big]$. However, for $s \in \left(\frac{1}{2}, \frac{3}{4}\right]$, there is $\left(\varrho_0, \nabla \varrho_0, u_0\right) \in \dot{H}^{-\frac{1}{2}}$, due to $\dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'}$, $\forall s' \in [0, s]$. In this way, from (2.11) we have the decay estimate

$$\mathfrak{E}_k^{k+2}(t) \leq C_0(1+t)^{-\left(k+\frac{1}{2}\right)}, \quad k=0,1,\cdots,N-2. \tag{5.14}$$

Now define

$$\overline{\mathfrak{D}}_3(t)=\sum_{l=0}^{3}\left\|\nabla^l\left(\nabla \varrho, \nabla^2 \varrho, \nabla u\right)\right\|_{L^2}^2.$$

We examine the definitions of $\mathfrak{E}_3(t)$, $\overline{\mathfrak{D}}_3(t)$ and $\mathfrak{E}_k^{k+2}(t)$, and then take $k=0,1$ in (5.14), respectively, give rise to

$$\sqrt{\mathfrak{E}_3(t)} \leq C_0(1+t)^{-\frac{1}{4}}, \quad \sqrt{\overline{\mathfrak{D}}_3(t)} \leq C_0(1+t)^{-\frac{3}{4}}. \tag{5.15}$$

Now, integrating (4.2) on time, substitute (5.15), and then apply Hölder inequality to know that, for $s \in \left(\frac{1}{2}, \frac{3}{4}\right]$, there holds

$$\begin{aligned}
&\left\|(\varrho, \nabla \varrho, u)(t)\right\|_{\dot{H}^{-s}}^2 \\
\lesssim &\left\|\left(\varrho_0, \nabla \varrho_0, u_0\right)\right\|_{\dot{H}^{-s}}^2+\int_0^t\left(\|\varrho\|_{H^1}+\|u\|_{L^2}\right)^{s-1/2}\left(\|\nabla \varrho\|_{H^1}+\|\nabla u\|_{L^2}\right)^{3/2-s} \\
&\times\left(\left(\sqrt{\mathfrak{E}_3(\tau)}+1\right) \sqrt{\overline{\mathfrak{D}}_3(\tau)}+\|u\|_{L^2}\right)\|(\varrho, \nabla \varrho, u)(\tau)\|_{\dot{H}^{-s}} d\tau \\
\lesssim &\left\|\left(\varrho_0, \nabla \varrho_0, u_0\right)\right\|_{\dot{H}^{-s}}^2+\int_0^t\left(\left(\sqrt{\mathfrak{E}_3(\tau)}\right)^{s-1/2}\left(\sqrt{\overline{\mathfrak{D}}_3(\tau)}\right)^{3/2-s}\left(\sqrt{\mathfrak{E}_3(\tau)}+1\right) \sqrt{\overline{\mathfrak{D}}_3(\tau)}\right. \\
&\left.+\left(\|\varrho\|_{H^1}+\|u\|_{L^2}\right)^{s-1/2}\left(\sqrt{\overline{\mathfrak{D}}_3(\tau)}\right)^{3/2-s}\|u\|_{L^2}\right)\|(\varrho, \nabla \varrho, u)(\tau)\|_{\dot{H}^{-s}} d\tau \\
\lesssim &\left\|\left(\varrho_0, \nabla \varrho_0, u_0\right)\right\|_{\dot{H}^{-s}}^2+\sup_{0 \leq \tau \leq t}\|(\varrho, \nabla \varrho, u)(\tau)\|_{\dot{H}^{-s}} \\
&\times \int_0^t\left((1+\tau)^{\frac{s}{2}-\frac{7}{4}}+(1+\tau)^{\frac{s}{2}-1}\|u\|_{L^2}+(1+\tau)^{\frac{3}{4}s-\frac{9}{8}}\|u\|_{L^2}^{s+\frac{1}{2}}\right) d\tau \\
\leq &C_0\left(1+\sup_{0 \leq \tau \leq t}\|(\varrho, \nabla \varrho, u)(\tau)\|_{\dot{H}^{-s}}\left(\int_0^t(1+\tau)^{\frac{s}{2}-\frac{7}{4}} d\tau+\right.\right. \\
&\left.\left.\left(\int_0^t(1+\tau)^{s-2} d\tau\right)^{1/2}\left(\int_0^t\|u\|_{L^2}^2 d\tau\right)^{1/2}+\left(\int_0^t(1+\tau)^{-\frac{3}{2}} d\tau\right)^{1/2}\left(\int_0^t\|u\|_{L^2}^2 d\tau\right)^{1/2}\right)\right) \\
\leq &C_0\left(1+\sup_{0 \leq \tau \leq t}\|(\varrho, \nabla \varrho, u)(\tau)\|_{\dot{H}^{-s}}\right),
\end{aligned} \tag{5.16}$$

The last inequality is that $\frac{s}{2}-\frac{7}{4}<-1$ and $2-s<-1$ for $s \in \left(\frac{1}{2}, \frac{3}{4}\right]$, combined with (5.7), it can be seen that the integral with respect to time is finite. From (5.16), it is known that (2.9) holds for $s \in \left(\frac{1}{2}, \frac{3}{4}\right]$, and then (2.11) holds for $s \in \left(\frac{1}{2}, \frac{3}{4}\right]$. For $s \in \left(\frac{3}{4}, 1\right]$, take $s=\frac{3}{4}$ in (2.11), there is decay estimate

$$\mathfrak{E}_k^{k+2}(t) \leq C_0(1+t)^{-\left(k+\frac{3}{4}\right)}, \quad k=0,1,\cdots,N-2.$$

Then take $k=0,1$, respectively, we have

$$\sqrt{\mathfrak{E}_3(t)} \leq C_0(1+t)^{-\frac{3}{8}}, \quad \sqrt{\overline{\mathfrak{D}}_3(t)} \leq C_0(1+t)^{-\frac{7}{8}}.$$

From the derivation process of (5.16), we can get that (2.9) holds for $s \in \left(\frac{3}{4}, 1\right]$, and then (2.11) holds for $s \in \left(\frac{3}{4}, 1\right]$. For $s \in \left(1, \frac{3}{2}\right)$, take $s=1$ in (2.11), by the same way, we can get (2.9) holds for $s \in \left(1, \frac{3}{2}\right)$. Thus, we obtain that (2.9) holds for $s \in \left[0, \frac{3}{2}\right)$. Finally, for (2.10), this is the result of the corresponding Besov norm case, and the proof is completely similar to the above argument. We only need to use Lemma 4.2 instead of Lemma 4.1, and examine the different values of s. It should be noted that the result of Lemma 4.2 is not expressed in the form of $\dot{B}^{-s}_{2,\infty}$ norm, but after integrating (4.24) and (4.25) with respect to time, the results similar to (5.13) and (5.16) can be obtained, then the corresponding results in the form of $\dot{B}^{-s}_{2,\infty}$ norm can be obtained by taking the supremum of $j \in \mathbb{Z}$ at both ends of the inequality. On the other hand, if $\dot{B}^{-s}_{2,\infty}$ norm is used instead of $\dot{H}^{-s}$ norm for $s = \frac{3}{2}$, the right-hand side of (5.16) will blow up. Therefore, for the argument of (2.10), we can discuss it in five cases: $s \in \left(0, \frac{1}{2}\right), s \in \left(\frac{1}{2}, \frac{3}{4}\right], \quad s \in \left(\frac{3}{4}, 1\right], \quad s \in \left(1, \frac{5}{4}\right]$ and $s \in \left(\frac{5}{4}, \frac{3}{2}\right]$. Thus, we can infer that (2.10) holds for $s \in \left(0, \frac{3}{2}\right]$. So far, we have completed the proof of Theorem 2.1. $\square$

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