The Taylor series method of order p and Adams-Bashforth method on time scales

Svetlin G. Georgiev* and İnci M. Erhan†

November 26, 2021

Abstract

A recent study on the Taylor series method of second order and the trapezoidal rule for dynamic equations on time scales has been continued by introducing a derivation of the Taylor series method of arbitrary order p on time scales. The error and convergence analysis of the method is also obtained. The 2 step Adams-Bashforth method for dynamic equations on time scales is concluded and applied to examples of initial value problems for nonlinear dynamic equations. Numerical results are presented and discussed.

1 Introduction

In a recent paper, the Taylor series method of second order and in particular, the trapezoidal rule for dynamic equations on time scales was developed [11]. The authors of the paper presented the derivation of the method and the error analysis for the method. The well known trapezoidal rule which is concluded from the second order Taylor series method was introduced and applied to examples of initial value problems for nonlinear dynamic equations of first order.

The purpose of this paper is to establish a continuation of the study given in [11] by deriving a general Taylor series method of order p for nonlinear dynamic equations on general time scales.

---

*Sorbonne University, Paris, France. E-mail: svetlingeorgiev1@gmail.com

†Atılım University, Department of Mathematics, Ankara, Turkey. E-mail: inci.erhan@atilim.edu.tr, Corresponding Author

[0]MSC 2010:34N05,39A10,65L05

[0]Key words and Phrases: time scale, delta derivative, Taylor series method, Adams-Bashforth method, dynamic equation

The paper is organized as follows. In Section 2, Pötzsche chain rule for functions of arbitrary number of variables is stated and proved. Section 3 contains the derivation of the Taylor series method of order p and is followed by Section 4 in which the error analysis and convergence of the method is presented. The 2-step Adams-Bashforth method is given in Section 5 and applied to two examples of initial value problems associated with nonlinear dynamic equations. Section 6 is devoted to conclusion and remarks.

2 A generalization of Pötzsche's chain rule

Throughout the paper, $\mathbb{T}$ will denote a time scale, σ, ρ, Δ, and μ the forward jump, backward jump, delta derivative operators and graininess function on $\mathbb{T}$, respectively. We refer the reader to [3, 4, 6, 7, 12] for basic concepts on time scales and dynamic equations.

To develop the Taylor series method, we need a chain rule for derivative of a composite function of $n+1$ variables. A chain rule for the case of a two variable functions already exists and is known as the Pötzsche's chain rule. We first recall this rule.

Theorem 2.1. *(Theorem B.3, [8]) For a fixed $a \in \mathbb{T}^\kappa$, let $g : \mathbb{T} \to \mathbb{R}$ and $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be functions such that g, $f(\cdot, g(a))$ are delta differentiable at a and let $U \subset \mathbb{T}$ be a neighborhood of a such that $f(t, \cdot)$ is differentiable for $t \in U \cup \{\sigma(a)\}$, $\frac{\partial}{\partial x} f(\sigma(a), \cdot)$ is continuous on the line segment*

$$\{g(a) + s\mu(a)g^\Delta(a) \in \mathbb{R} \ : \ s \in [0, 1]\}$$

and $\frac{\partial f}{\partial x}$ is continuous at $(a, g(a))$. Then the composition function $F : \mathbb{T} \to \mathbb{R}$ given as $F(t) = f(t, g(t))$ is delta differentiable at a with delta derivative

$$F^\Delta(a) = \Delta_1 f(a, g(a)) + \left(\int_0^1 \frac{\partial}{\partial x} f(\sigma(a), g(a) + s\mu(a)g^\Delta(a))ds \right) g^\Delta(a). \quad (2.1)$$

Here $\Delta_1 f(\cdot, g(a))$ denotes the delta derivative of $f(t, x)$ with respect to its first variable and $\frac{\partial}{\partial x} f(t, \cdot)$ denotes the partial derivative of $f(t, x)$ with respect to its second variable.

The derivation of the method, requires a generalization of this rule. Let $g : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ be a given function. Then for the function $g(t, y_1, \ldots, y_n)$ we denote by $\Delta_1 g(\cdot, y_1, \ldots, y_n)$ its delta derivative.

Theorem 2.2. *For some fixed $t_0 \in \mathbb{T}^\kappa$, let $y_j : \mathbb{T} \to \mathbb{R}$, $j \in \{1, \ldots, n\}$, $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ be continuous functions such that $f(\cdot, y_1(t_0), \ldots, y_n(t_0))$, and y_j, $j \in \{1, \ldots, n\}$, are differentiable at t_0. Let $U \subseteq \mathbb{T}$ be a neighborhood of t_0 such that,*

1. *$f(t,\cdot,\dots,\cdot)$ is continuously-differentiable for $t \in U \cup \{\sigma(t_0)\}$,*

2. *$\Delta_1 f(\cdot, y_1(\cdot), \dots, y_n(\cdot))$ is continuous at t_0,*

3. $$\frac{\partial}{\partial y_j} f(\sigma(t_0), y_1(\sigma(t_0)), \dots, y_{j-1}(\sigma(t_0)), \cdot, y_{j+1}(t), \dots, y_n(t))$$
   is continuous in the line segment
   $$\{y_j(t)+h(y_j(\sigma(t_0))-y_j(t)) \in \mathbb{R} : h \in [0,1]\},\ j \in \{1,\dots,n\},\ \forall t \in U \cup \{t_0\},$$

4. *$\frac{\partial}{\partial y_j} f$ is continuous at $(t_0, y_1(t_0), \dots, y_n(t_0))$.*

Then the composition function $F : \mathbb{T} \to \mathbb{R}$, $F(t) = f(t, y_1(t), y_2(t), \dots, y_n(t))$, is differentiable at t_0 with derivative

$$\begin{aligned}
F^{\Delta}(t_0) &= \Delta_1 f(t_0, y_1(t_0), y_2(t_0), \dots, y_n(t_0)) \\
&+ \left(\int_0^1 \frac{\partial}{\partial y_1} f(\sigma(t_0), y_1(t_0) + h\mu(t_0)y_1^{\Delta}(t_0), y_2(t_0), \dots, y_n(t_0))dh\right) y_1^{\Delta}(t_0) \\
&+ \left(\int_0^1 \frac{\partial}{\partial y_2} f(\sigma(t_0), y_1(\sigma(t_0)), y_2(t_0) + h\mu(t_0)y_2^{\Delta}(t_0), \dots, y_n(t_0))dh\right) y_2^{\Delta}(t_0) \\
&+ \cdots \\
&+ \left(\int_0^1 \frac{\partial}{\partial y_n} f(\sigma(t_0), y_1(\sigma(t_0)), y_2(\sigma(t_0)), \dots, y_{n-1}(\sigma(t_0)), y_n(t_0) + h\mu(t_0)y_n^{\Delta}(t_0))dh\right) y_n^{\Delta}(t_0).
\end{aligned}$$

Proof. Let $s \in (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$, $s \neq \sigma(t_0)$, for $\delta > 0$ small enough, and $s < \sigma(t_0)$ if $\sigma(t_0) > t_0$. Then

$$\begin{aligned}
&F(\sigma(t_0)) - F(s) \\
&= f(\sigma(t_0), y_1(\sigma(t_0)), y_2(\sigma(t_0)), \dots, y_n(\sigma(t_0))) - f(s, y_1(s), y_2(s), \dots, y_n(s)) \\
&= f(\sigma(t_0), y_1(s), y_2(s), \dots, y_n(s)) - f(s, y_1(s), y_2(s), \dots, y_n(s)) \\
&+ f(\sigma(t_0), y_1(\sigma(t_0)), y_2(s), \dots, y_n(s)) - f(\sigma(t_0), y_1(s), y_2(s), \dots, y_n(s)) \\
&+ f(\sigma(t_0), y_1(\sigma(t_0)), y_2(\sigma(t_0)), \dots, y_n(s)) - f(\sigma(t_0), y_1(\sigma(t_0)), y_2(s), \dots, y_n(s)) \\
&+ \cdots \\
&+ f(\sigma(t_0), y_1(\sigma(t_0), y_2(\sigma(t_0)), \dots, y_n(\sigma(t_0))) - f(\sigma(t_0), y_1(\sigma(t_0)), y_2(\sigma(t_0)), \dots, y_n(s))
\end{aligned}$$

Then, we have

$$F(\sigma(t_0)) - F(s)$$

$$= f(\sigma(t_0), y_1(s), y_2(s), \ldots, y_n(s)) - f(s, y_1(s), y_2(s), \ldots, y_n(s))$$

$$+ \left(\int_0^1 \frac{\partial}{\partial y_1} f(\sigma(t_0), y_1(s) + h(y_1(\sigma(t_0)) - y_1(s)), y_2(s), \ldots, y_n(s)) dh \right) (y_1(\sigma(t_0)) - y_1(s))$$

$$+ \left(\int_0^1 \frac{\partial}{\partial y_2} f(\sigma(t_0), y_1(\sigma(t_0)), y_2(s) + h(y_2(\sigma(t_0)) - y_2(s)), \ldots, y_n(s)) dh \right) (y_2(\sigma(t_0)) - y_2(s))$$

$$+ \cdots$$

$$+ \left(\int_0^1 \frac{\partial}{\partial y_n} f(\sigma(t_0), y_1(\sigma(t_0)), y_2(\sigma(t_0)), \ldots, y_n(s) + h(y_n(\sigma(t_0)) - y_n(s))) dh \right)$$

$$\times (y_n(\sigma(t_0)) - y_n(s)).$$

If $\sigma(t_0) > t_0$, by the Mean value theorem there exist $\xi_1, \xi_2 \in [s, \sigma(t_0)) = [s, t_0]$ so that

$$\Delta_1 f(\xi_1, y_1(s), y_2(s), \ldots, y_n(s))(\sigma(t_0) - s) \leq f(\sigma(t_0), y_1(s), y_2(s), \ldots, y_n(s))$$

$$-f(s, y_1(s), y_2(s), \ldots, y_n(s)) \leq \Delta_1 f(\xi_2, y_1(s), y_2(s), \ldots, y_n(s))(\sigma(t_0) - s)$$

and

$$\Delta_1 f(t_0, y_1(t_0), y_2(t_0), \ldots, y_n(t_0)) = \lim_{s \to t_0} \Delta_1 f(\xi_1, y_1(s), y_2(s), \ldots, y_n(s))$$

$$\leq \lim_{s \to t_0} \frac{1}{\sigma(t_0) - s} \left(f(\sigma(t_0), y_1(s), y_2(s), \ldots, y_n(s)) - f(s, y_1(s), y_2(s), \ldots, y_n(s)) \right)$$

$$\leq \lim_{s \to t_0} \Delta_1 f(\xi_2, y_1(s), y_2(s), \ldots, y_n(s))$$

$$= \Delta_1 f(t_0, y_1(t_0), y_2(t_0), \ldots, y_n(t_0)).$$

If $\sigma(t_0) = t_0$, by the Mean value theorem, there exist ξ_1, ξ_2 between s and t_0 so that

$$\Delta_1 f(\xi_1, y_1(s), y_2(s), \ldots, y_n(s))(t_0 - s) \leq f(t_0, y_1(s), y_2(s), \ldots, y_n(s))$$

$$-f(s, y_1(s), y_2(s), \ldots, y_n(s)) \leq \Delta_1 f(\xi_2, y_1(s), y_2(s), \ldots, y_n(s))(t_0 - s).$$

In this case, if $s < t_0$ we have

$$
\begin{aligned}
\Delta_1 f(t_0, y_1(t_0), y_2(t_0), \ldots, y_n(t_0)) &= \lim_{s\to t_0-} \Delta_1 f(\xi_1, y_1(s), y_2(s), \ldots, y_n(s)) \\
&\le \lim_{s\to t_0-} \frac{1}{t_0 - s}\left(f(\sigma(t_0), y_1(s), y_2(s), \ldots, y_n(s)) - f(s, y_1(s), y_2(s), \ldots, y_n(s))\right) \\
&\le \lim_{s\to t_0-} \Delta_1 f(\xi_2, y_1(s), y_2(s), \ldots, y_n(s)) \\
&= \Delta_1 f(t_0, y_1(t_0), y_2(t_0), \ldots, y_n(t_0)),
\end{aligned}
$$

and if $s > t_0$ we have

$$
\begin{aligned}
\Delta_1 f(t_0, y_1(t_0), y_2(t_0), \ldots, y_n(t_0)) &= \lim_{s\to t_0+} \Delta_1 f(\xi_1, y_1(s), y_2(s), \ldots, y_n(s)) \\
&\ge \lim_{s\to t_0+} \frac{1}{t_0 - s}\left(f(\sigma(t_0), y_1(s), y_2(s), \ldots, y_n(s)) - f(s, y_1(s), y_2(s), \ldots, y_n(s))\right) \\
&\ge \lim_{s\to t_0+} \Delta_1 f(\xi_2, y_1(s), y_2(s), \ldots, y_n(s)) \\
&= \Delta_1 f(t_0, y_1(t_0), y_2(t_0), \ldots, y_n(t_0)).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\lim_{s\to t_0}\Bigg(\Bigg(\int_0^1 \frac{\partial}{\partial y_j} f(\sigma(t_0), y_1(\sigma(t_0)), \ldots, y_{j-1}(\sigma(t_0)), y_j(s) + h(y_j(\sigma(t_0)) - y_j(s)), \\
&\qquad y_{j+1}(s), \ldots, y_n(s))dh\Bigg)\frac{y_j(\sigma(t_0)) - y_j(t_0)}{\sigma(t_0) - s}\Bigg) \\
&= \lim_{s\to t_0}\Bigg(\int_0^1 \frac{\partial}{\partial y_j} f(\sigma(t_0), y_1(\sigma(t_0)), \ldots, y_{j-1}(\sigma(t_0)), y_j(s) + h(y_j(\sigma(t_0)) - y_j(s)), \\
&\qquad y_{j+1}(s), \ldots, y_n(s))dh\Bigg)\lim_{s\to t_0}\frac{y_j(\sigma(t_0)) - y_j(t_0)}{\sigma(t_0) - s} \\
&= \Bigg(\int_0^1 \frac{\partial}{\partial y_j} f(\sigma(t_0), y_1(\sigma(t_0)), \ldots, y_{j-1}(\sigma(t_0)), y_j(t_0) + h(y_j(\sigma(t_0)) - y_j(t_0)), \\
&\qquad y_{j+1}(t_0), \ldots, y_n(t_0))dh\Bigg)y_j^{\Delta}(t_0) \\
&= \Bigg(\int_0^1 \frac{\partial}{\partial y_j} f(\sigma(t_0), y_1(\sigma(t_0)), \ldots, y_{j-1}(\sigma(t_0)), y_j(t_0) + h\mu(t_0)y_j^{\Delta}(t_0), y_{j+1}(t_0), \ldots, y_n(t_0))dh\Bigg) \\
&\qquad \times y_j^{\Delta}(t_0), \quad j \in \{1, \ldots, n\}.
\end{aligned}
$$

Therefore

$$\begin{aligned}
&\lim_{s\to t_0}\frac{F(\sigma(t_0))-F(s)}{\sigma(t_0)-s}\\
&\quad=\quad \lim_{s\to t_0}\frac{f(\sigma(t_0),y_1(s),y_2(s),\ldots,y_n(s))-f(s,y_1(s),y_2(s),\ldots,y_n(s))}{\sigma(t_0)-s}\\
&+\lim_{s\to t_0}\left(\left(\int_0^1\frac{\partial}{\partial y_1}f(\sigma(t_0),y_1(s)+h(y_1(\sigma(t_0))-y_1(s)),y_2(s),\ldots,y_n(s))dh\right)\right.\\
&\left.\times\frac{y_1(\sigma(t_0))-y_1(s)}{\sigma(t_0)-s}\right)\\
&+\lim_{s\to t_0}\left(\left(\int_0^1\frac{\partial}{\partial y_2}f(\sigma(t_0),y_1(\sigma(t_0)),y_2(s)+h(y_2(\sigma(t_0))-y_2(s)),\ldots,y_n(s))dh\right)\right.\\
&\left.\times\frac{y_2(\sigma(t_0))-y_2(s)}{\sigma(t_0)-s}\right)\\
&+\cdots\\
&+\lim_{s\to t_0}\left(\left(\int_0^1\frac{\partial}{\partial y_n}f(\sigma(t_0),y_1(\sigma(t_0)),y_2(\sigma(t_0)),\ldots,y_n(s)+h(y_n(\sigma(t_0))-y_n(s)))dh\right)\right.\\
&\left.\times\frac{y_n(\sigma(t_0))-y_n(s)}{\sigma(t_0)-s}\right)\\
=\quad&\Delta_1 f(t_0,y_1(t_0),y_2(t_0),\ldots,y_n(t_0))\\
&+\left(\int_0^1\frac{\partial}{\partial y_1}f(\sigma(t_0),y_1(t_0)+h\mu(t_0)y_1^\Delta(t_0),y_2(t_0),\ldots,y_n(t_0))dh\right)y_1^\Delta(t_0)\\
&+\left(\int_0^1\frac{\partial}{\partial y_2}f(\sigma(t_0),y_1(\sigma(t_0)),y_2(t_0)+h\mu(t_0)y_2^\Delta(t_0),\ldots,y_n(t_0))dh\right)y_2^\Delta(t_0)\\
&+\cdots\\
&+\left(\int_0^1\frac{\partial}{\partial y_n}f(\sigma(t_0),y_1(\sigma(t_0)),y_2(\sigma(t_0)),\ldots,y_{n-1}(\sigma(t_0)),y_n(t_0)+h\mu(t_0)y_n^\Delta(t_0))dh\right)y_n^\Delta(t_0).
\end{aligned}$$

This completes the proof. □

3 The Taylor series method of order p

In this section we derive the Taylor series method of order $p\geq 2$. As a matter of fact, we generalize the derivation of Taylor series method of order 2 given in [11].

Let $\mathbb{T}$ be a time scale and Δ denotes the differentiation operator in $\mathbb{T}$ as usual. Suppose that $p \in \mathbb{N}$, $p \geq 2$, $t_0, t_f \in \mathbb{T}$, $t_0 < t_f < \infty$, $r > 0$ be such that $t, t+r \in [t_0, t_f]$. Consider the initial value problem (IVP)

$$\begin{cases} x^{\Delta}(t) &= f(t, x(t)), \quad t \in [t_0, t_f], \\ \\ x(t_0) &= x_0, \end{cases} \tag{3.1}$$

where $x_0 \in \mathbb{R}$ is a given constant, the function f satisfies the following conditions

$$(\mathbf{H1}) \begin{cases} |f(t,x)| \leq A, \quad t \in \mathbb{T}, \quad x \in \mathbb{R}, \\ \text{there exists } g_k(t, x(t), \ldots, x^{\Delta^k}(t)) = (f(t,x(t)))^{\Delta^k}, \quad k \in \{1, \ldots, p-1\}, \\ \text{such that } \left|\dfrac{\partial f}{\partial y}(t,z)\right| \leq A, \quad |\Delta_1 g_k(t, y_1, \ldots, y_{k+1})| \leq A, \\ \text{and } \left|\dfrac{\partial}{\partial y_j} g_k(t, y_1, \ldots, y_{k+1})\right| \leq A, \quad j \in \{1, \ldots, k+1\}, \\ \text{for any } t \in \mathbb{T} \text{ and for } z, y_j \in \mathbb{R}, j \in \{1, \ldots, p-1\}, \\ \text{where } pe^{t_f - t_0} A < 1 \text{ and } A > 0. \end{cases}$$

By the Taylor formula on time scales (see Theorem 2.1, [11]), we get

$$\begin{aligned} x(t+r) &= x(t) + h_1(t+r,t)x^{\Delta}(t) + h_2(t+r,t)x^{\Delta^2}(t) + \cdots + h_p(t+r,t)x^{\Delta^p}(t) \\ &\quad + \int_t^{\rho^p(t+r)} h_p(t+r, \sigma(u)) x^{\Delta^{p+1}}(u)\Delta u. \end{aligned}$$

Let

$$R_p(t) = \int_t^{\rho^p(t+r)} h_p(t+r, \sigma(u)) x^{\Delta^{p+1}}(u)\Delta u,$$

be the remainder term. Let also, $t_0 < t_1 < \ldots < t_{m+1} = t_f$ be a partition of the interval $[t_0, t_f]$ such that $t_{n+1} = t_n + r_{n+1} \in \mathbb{T}$, $r_{n+1} > 0$, $n \in \{0, \ldots, m\}$. Then

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + h_1(t_{n+1}, t_n)x^{\Delta}(t_n) + h_2(t_{n+1}, t_n)x^{\Delta^2}(t_n) \\ &\quad + \cdots + h_p(t_{n+1}, t_n)x^{\Delta^p}(t_n) + R_p(t_{n+1}). \end{aligned}$$

Neglecting the remainder term $R_p(t)$, we obtain

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + h_1(t_{n+1}, t_n)x^{\Delta}(t_n) + h_2(t_{n+1}, t_n)x^{\Delta^2}(t_n) \\ &\quad + \cdots + h_p(t_{n+1}, t_n)x^{\Delta^p}(t_n). \end{aligned}$$

Set

$$x_n^{\Delta^k} = x^{\Delta^k}(t_n), \quad x_n^{\Delta^k \sigma} = x^{\Delta^k}(\sigma(t_n)), \quad k \in \{0, \ldots, p\}.$$

Thus, we get

$$x_{n+1} = x_n + r_{n+1} x_n^{\Delta} + h_2(t_{n+1}, t_n) x_n^{\Delta^2} + \cdots + h_p(t_{n+1}, t_n) x_n^{\Delta^p}. \tag{3.2}$$

To compute x_{n+1} we need to determine $x_n^{\Delta^q}$ for $q \in \{1, \ldots, p\}$. From the dynamic equation in the IVP (3.1) we determine x_n^{Δ} as

$$x_n^{\Delta} = f(t_n, x_n).$$

Now, we will determine $x_n^{\Delta^2}$, ..., $x_n^{\Delta^p}$. By the generalized Pötzsche chain rule, we have

$$\begin{aligned}
(f(t, x(t)))^{\Delta} &= \Delta_1 f(t, x(t)) + \left(\int_0^1 \frac{\partial}{\partial y_1} f(\sigma(t), x(t) + h\mu(t) x^{\Delta}(t)) dh \right) x^{\Delta}(t) \\
&= g_1(t, x(t), x^{\Delta}(t))
\end{aligned}$$

and for $q \in \{2, \ldots, p\}$ we compute

$$\begin{aligned}
&(f(t, x(t)))^{\Delta^q} \\
&= \left(g_{q-1}(t, x(t), x^{\Delta}(t), \ldots, x^{\Delta^{q-1}}(t)) \right)^{\Delta} \\
&= \Delta_1 g_{q-1}(t, x(t), x^{\Delta}(t), \ldots, x^{\Delta^{q-1}}(t)) \\
&+ \left(\int_0^1 \frac{\partial}{\partial y_1} g_{q-1}(\sigma(t), x(t) + h\mu(t) x^{\Delta}(t), x^{\Delta}(t), \ldots, x^{\Delta^{q-1}}(t)) dh \right) x^{\Delta}(t) \\
&+ \left(\int_0^1 \frac{\partial}{\partial y_2} g_{q-1}(\sigma(t), x(\sigma(t)), x^{\Delta}(t) + h\mu(t) x^{\Delta^2}(t), \ldots, x^{\Delta^{q-1}}(t)) dh \right) x^{\Delta^2}(t) \\
&+ \cdots \\
&+ \left(\int_0^1 \frac{\partial}{\partial y_q} g_{q-1}(\sigma(t), x(\sigma(t)), x^{\Delta}(\sigma(t)), \ldots, x^{\Delta^{q-1}}(t) + h\mu(t) x^{\Delta^q}(t)) dh \right) x^{\Delta^q}(t),
\end{aligned}$$

for $t \in \mathbb{T}^\kappa$. Therefore, we have

$$
\begin{aligned}
x_n^{\Delta^2} &= \Delta_1 f(t_n, x_n) + \left(\int_0^1 \frac{\partial}{\partial y_1} f(\sigma(t_n), x_n + h\mu(t_n)x_n^{\Delta})dh\right) x_n^{\Delta},\\
x_n^{\Delta^3} &= \Delta_1 g_1(t_n, x_n, x_n^{\Delta})\\
&\quad + \left(\int_0^1 \frac{\partial}{\partial y_1} g_1(\sigma(t_n), x_n + h\mu(t_n)x_n^{\Delta}, x_n^{\Delta})dh\right) x_n^{\Delta}\\
&\quad + \left(\int_0^1 \frac{\partial}{\partial y_2} g_1(\sigma(t_n), x_n^{\sigma}, x_n^{\Delta} + h\mu(t_n)x_n^{\Delta^2})dh\right) x_n^{\Delta^2}\\
&\vdots\\
x_n^{\Delta^{p+1}} &= \Delta_1 g_{p-1}(t_n, x_n, x_n^{\Delta}, \ldots, x_n^{\Delta^{p-1}})\\
&\quad + \left(\int_0^1 \frac{\partial}{\partial y_1} g_{p-1}(\sigma(t_n), x_n + h\mu(t_n)x_n^{\Delta}, x_n^{\Delta}, \ldots, x_n^{\Delta^{p-1}})dh\right) x_n^{\Delta}\\
&\quad + \left(\int_0^1 \frac{\partial}{\partial y_2} g_{p-1}(\sigma(t_n), x_n^{\sigma}, x_n^{\Delta} + h\mu(t_n)x_n^{\Delta^2}, \ldots, x_n^{\Delta^{p-1}})dh\right) x_n^{\Delta^2}\\
&\quad + \cdots\\
&\quad + \left(\int_0^1 \frac{\partial}{\partial y_p} g_{p-1}(\sigma(t_n), x_n^{\sigma}, x_n^{\Delta\sigma}, \ldots, x_n^{\Delta^{p-1}} + h\mu(t_n)x_n^{\Delta^p})dh\right) x_n^{\Delta^p},
\end{aligned}
$$

from where we can find $x_n^{\Delta^2}$, …, $x_n^{\Delta^{p+1}}$.

4 Convergence and error analysis of the Taylor series method

Now, we will investigate the convergence of the Taylor series method of order p. We will use the following property of the monomials $h_q(t,s)$, $q \in \mathbb{N}_0$, which is proved in [9].

Theorem 4.1. *(Theorem 1.61, [9]). The following estimate*

$$0 \le h_q(t,s) \le \frac{(t-s)^q}{q!}, \quad t \ge s, \tag{4.1}$$

holds for all $q \in \mathbb{N}$.

By the condition $(H1)$ and the dynamic equation in (3.1), we find

$$|x^{\Delta}(t)| \leq A, \quad t \in [t_0, t_f].$$

Next, we estimate

$$\begin{aligned}
\left|x^{\Delta^2}(t)\right| &\leq |\Delta_1 f(t, x(t))| + \left(\int_0^1 \left|\frac{\partial}{\partial y_1} f(\sigma(t), x(t) + h\mu(t)x^{\Delta}(t))\right| dh\right) |x^{\Delta}(t)| \\
&\leq A(1+A), \quad t \in [t_0, t_f],
\end{aligned}$$

and

$$\begin{aligned}
&\left|x^{\Delta^3}(t)\right| \leq \left|\Delta_1 g_1(t, x(t), x^{\Delta}(t))\right| \\
&+ \left(\int_0^1 \left|\frac{\partial}{\partial y_1} g_1(\sigma(t), x(t) + h\mu(t)x^{\Delta}(t), x^{\Delta}(t))\right| dh\right) |x^{\Delta}(t)| \\
&+ \left(\int_0^1 \left|\frac{\partial}{\partial y_2} g_1(\sigma(t), x(\sigma(t)), x^{\Delta}(t) + h\mu(t)x^{\Delta^2}(t))\right| dh\right) \left|x^{\Delta^2}(t)\right| \\
&\leq A + A^2 + A(A + A^2) \\
&= A(1+A)^2,
\end{aligned}$$

$$\begin{aligned}
&\left|x^{\Delta^4}(t)\right| \leq \left|\Delta_1 g_2(t, x(t), x^{\Delta}(t), x^{\Delta^2}(t))\right| \\
&+ \left(\int_0^1 \left|\frac{\partial}{\partial y_1} g_2(\sigma(t), x(t) + h\mu(t)x^{\Delta}(t), x^{\Delta}(t), x^{\Delta^2}(t))\right| dh\right) |x^{\Delta}(t)| \\
&+ \left(\int_0^1 \left|\frac{\partial}{\partial y_2} g_2(\sigma(t), x(\sigma(t)), x^{\Delta}(t) + h\mu(t)x^{\Delta^2}(t), x^{\Delta^2}(t))\right| dh\right) \left|x^{\Delta^2}(t)\right| \\
&+ \left(\int_0^1 \left|\frac{\partial}{\partial y_3} g_2(\sigma(t), x(\sigma(t)), x^{\Delta}(\sigma(t)), x^{\Delta^2}(t) + h\mu(t)x^{\Delta^3}(t))\right| dh\right) |x^{\Delta^3}(t)| \\
&\leq A + A^2 + A(A + A^2) + A\left((A + A^2) + A(A + A^2)\right) \\
&= A + A^2 + 2A(A + A^2) + A^2(A + A^2) \\
&= A(1+A)^3,
\end{aligned}$$

$$
\begin{aligned}
&\left|x^{\Delta^5}(t)\right| \le \left|\Delta_1 g_3(t, x(t), x^{\Delta}(t), x^{\Delta^2}(t), x^{\Delta^3}(t))\right| \\
&+ \left(\int_0^1 \left|\frac{\partial}{\partial y_1} g_3(\sigma(t), x(t) + h\mu(t)x^{\Delta}(t), x^{\Delta}(t), x^{\Delta^2}(t), x^{\Delta^3}(t))\right| dh\right) |x^{\Delta}(t)| \\
&+ \left(\int_0^1 \left|\frac{\partial}{\partial y_2} g_3(\sigma(t), x(\sigma(t)), x^{\Delta}(t) + h\mu(t)x^{\Delta^2}(t), x^{\Delta^2}(t), x^{\Delta^3}(t))\right| dh\right) \left|x^{\Delta^2}(t)\right| \\
&+ \left(\int_0^1 \left|\frac{\partial}{\partial y_3} g_3(\sigma(t), x(\sigma(t)), x^{\Delta}(\sigma(t)), x^{\Delta^2}(t) + h\mu(t)x^{\Delta^3}(t), x^{\Delta^3}(t))\right| dh\right) \left|x^{\Delta^3}(t)\right| \\
&+ \left(\int_0^1 \left|\frac{\partial}{\partial y_4} g_3(\sigma(t), x(\sigma(t)), x^{\Delta}(\sigma(t)), x^{\Delta^2}(\sigma(t)), x^{\Delta^3}(t) + h\mu(t)x^{\Delta^4}(t))\right| dh\right) \left|x^{\Delta^4}(t)\right| \\
&\le A + A^2 + A(A + A^2) + A(1 + A)(A + A^2) + A(A + A^2)(1 + A)^2 \\
&= (A + A^2)(1 + A + A + A^2 + A + 2A^2 + A^3) \\
&= A(1 + A)^4, \quad t \in [t_0, t_f],
\end{aligned}
$$

so that we deduce

$$
\begin{aligned}
\left|x^{\Delta^p}(t)\right| &\le A(1 + A)^{p-1}, \\
\left|x^{\Delta^{p+1}}(t)\right| &\le A(1 + A)^p, \quad t \in [t_0, t_f].
\end{aligned}
$$

Moreover, for the remaider terms

$$
R_q(r) = \int_t^{\rho^q(t+r)} h_q(t + r, \sigma(u)) x^{\Delta^{q+1}}(u)\Delta u, \quad q \in \{1, \ldots, p\}, \quad t \in [t_0, t_f],
$$

employing the estimate (4.1) and the fact that $\rho^q(t + r) - t \le t + r - t = r$, we get

$$
\begin{aligned}
|R_q(r)| &\le \int_t^{\rho^q(t+r)} h_q(t + r, \sigma(u)) \left|x^{\Delta^{q+1}}(u)\right| \Delta u \\
&\le \frac{r^q}{q!} A(1 + A)^q \left(\rho^q(t + r) - t\right) \qquad (4.2) \\
&\le \frac{r^{q+1}}{q!} A(1 + A)^q, \quad q \in \{1, \ldots, p\}.
\end{aligned}
$$

Therefore

$$
R_q(r) = O\left(r^{q+1}\right), \quad q \in \{1, \ldots, p\}.
$$

Denote

$$
e_n^{\Delta^k} = x^{\Delta^k}(t_n) - x_n^{\Delta^k}, \quad k \in \{0, \ldots, p - 1\}.
$$

Taking into account the fact that

$$\begin{aligned}
x_n^{\Delta} &= f(t_n, x_n), \quad x^{\Delta}(t_n) = f(t_n, x(t_n)),\\
x_n^{\Delta^q} &= g_{q-1}(t_n, x_n, x_n^{\Delta}, \ldots, x_n^{\Delta^{q-1}}),\\
x^{\Delta^q}(t_n) &= g_{q-1}(t_n, x(t_n), x^{\Delta}(t_n), \ldots, x^{\Delta^{q-1}}(t_n)), \quad q \in \{2, \ldots, p\}
\end{aligned}$$

we have

$$\begin{aligned}
x(t_{n+1}) &= x(t_n) + r_{n+1} f(t_n, x(t_n)) + h_2(t_{n+1}, t_n) g_1(t_n, x(t_n), x^{\Delta}(t_n))\\
&+ \cdots + h_p(t_{n+1}, t_n) g_{p-1}(t_n, x(t_n), \ldots, x^{\Delta^{p-1}}(t_n)) + R_p(r_{n+1})
\end{aligned}$$

and

$$\begin{aligned}
x_{n+1} &= x_n + r_{n+1} f(t_n, x_n) + h_2(t_{n+1}, t_n) g_1(t_n, x_n, x_n^{\Delta})\\
&+ \cdots + h_p(t_{n+1}, t_n) g_{p-1}(t_n, x_n, \ldots, x_n^{\Delta^{p-1}}).
\end{aligned}$$

Then

$$\begin{aligned}
&x(t_{n+1}) - x_{n+1} = (x(t_n) - x_n) + r_{n+1}(f(t_n, x(t_n)) - f(t_n, x_n))\\
&+ \ h_2(t_{n+1}, t_n) \left(g_1(t_n, x(t_n), x^{\Delta}(t_n)) - g_1(t_n, x_n, x_n^{\Delta})\right)\\
&+ \ \cdots\\
&+ \ h_p(t_{n+1}, t_n) \left(g_{p-1}(t_n, x(t_n), \ldots, x^{\Delta^{p-1}}(t_n)) - g_{p-1}(t_n, x_n, \ldots, x_n^{\Delta^{p-1}})\right)\\
&+ \ R_p(r_{n+1}).
\end{aligned}$$

Note that by the condition $(H1)$, the Mean value theorem for f and g_k implies that

$$f(t_n, x(t_n)) - f(t_n, x_n) = \frac{\partial f}{\partial y}(t_n, \xi_1^0)(x(t_n) - x_n) = \frac{\partial f}{\partial y}(t_n, \xi_1^0) e_n,$$

where ξ_1^0 is between $x(t_n)$ and x_n, and $\frac{\partial f}{\partial y}$ stands for the partial derivative with respect to the second variable. Also,

$$\begin{aligned}
&g_k(t_n, x(t_n), x^{\Delta}(t_n), \ldots, x^{\Delta^k}(t_n)) - g_k(t_n, x_n, x_n^{\Delta}, \ldots, x_n^{\Delta^k})\\
&= \ g_k(t_n, x(t_n), x^{\Delta}(t_n), \ldots, x^{\Delta^k}(t_n)) - g_k(t_n, x_n, x^{\Delta}(t_n), \ldots, x^{\Delta^k}(t_n))\\
&+ \ g_k(t_n, x_n, x^{\Delta}(t_n), \ldots, x^{\Delta^k}(t_n)) - g_k(t_n, x_n, x_n^{\Delta}, \ldots, x^{\Delta^k}(t_n))\\
&+ \ \cdots\\
&+ \ g_k(t_n, x_n, x_n^{\Delta}, \ldots, x_n^{\Delta^{k-1}}, x^{\Delta^k}(t_n)) - g_k(t_n, x_n, x_n^{\Delta}, \ldots, x_n^{\Delta^{k-1}}, x_n^{\Delta^k})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial y_1} g_k(t_n, \xi_1^k, x^{\Delta}(t_n), \ldots, x^{\Delta^k}(t_n)) e_n \\
&+ \frac{\partial}{\partial y_2} g_k(t_n, x_n, \xi_2^k, \ldots, x^{\Delta^k}(t_n)) e_n^{\Delta} \\
&+ \cdots \\
&+ \frac{\partial}{\partial y_{k+1}} g_k(t_n, x_n, \ldots, x_n^{\Delta^{k-1}}, \xi_{k+1}^k) e_n^{\Delta^k}, , \quad k \in \{1, \ldots, p-1\},
\end{aligned}
$$

where ξ_j^k is between $x^{\Delta^{j-1}}(t_n)$ and $x_n^{\Delta^{j-1}}$, $j \in \{1, \ldots, k+1\}$ and the $\frac{\partial}{\partial y_j}$ denotes the partial derivative with respect to the $(j+1)$-st variable. Consequently,

$$
\begin{aligned}
e_{n+1} &= e_n + r_{n+1} \frac{\partial f}{\partial y_1}(t_n, \xi_1^0) e_n \\
&+ h_2(t_{n+1}, t_n) \left(\frac{\partial g_1}{\partial y_1}(t_n, \xi_1^1, x^{\Delta}(t_n)) e_n + \frac{\partial g_1}{\partial y_2}(t_n, x_n, \xi_2^1) e_n^{\Delta} \right) \\
&+ \cdots \\
&+ h_p(t_{n+1}, t_n) \left(\frac{\partial g_{p-1}}{\partial y_1}(t_n, \xi_1^{p-1}, x^{\Delta}(t_n), \ldots, x^{\Delta^{p-1}}(t_n)) e_n \right. \\
&+ \frac{\partial g_{p-1}}{\partial y_2}(t_n, x_n, \xi_2^{p-1}, \ldots, x^{\Delta^{p-1}}(t_n)) e_n^{\Delta} \\
&+ \cdots \\
&\left. + \frac{\partial g_{p-1}}{\partial y_p}(t_n, x_n, x_n^{\Delta}, \ldots, \xi_p^{p-1}) e_n^{\Delta^{p-1}} \right) \\
&+ R_p(r_{n+1}).
\end{aligned}
$$

Let $r_{max} = \max\{r_1, \ldots, r_{m+1}\}$. Since $t_f < \infty$, there is a constant $B > 0$ such that

$$
\frac{1}{p!} r_{max} A (1 + A)^p \left(e^{r_{max}} A + 1 \right) \leq B.
$$

Then

$$
\begin{aligned}
|e_{n+1}| &\leq (1 + h_1(t_{n+1}, t_n) + \cdots + h_p(t_{n+1}, t_n)) \, A \left(|e_n| + |e_n^{\Delta}| + \cdots + \left| e_n^{\Delta^{p-1}} \right| \right) + |R_p(r_{n+1})| \\
&\leq \left(1 + r_{max} + \frac{r_{max}^2}{2!} + \cdots + \frac{r_{max}^p}{p!} \right) A \left(|e_n| + |e_n^{\Delta}| + \cdots + |e_n^{\Delta^{p-1}}| \right) + |R_p(r_{n+1})| \\
&\leq e^{r_{max}} A \left(|e_n| + |e_n^{\Delta}| + \cdots + |e_n^{\Delta^{p-1}}| \right) + |R_p(r_{n+1})|
\end{aligned}
$$

In a similar way, we make the following estimates.

$$\begin{aligned}
\left|e_{n+1}^{\Delta}\right| &\leq e^{r_{max}} A\left(\left|e_n^{\Delta}\right| + \cdots + \left|e_n^{\Delta^{p-1}}\right|\right) + |R_{p-1}(r_{n+1})|, \\
\left|e_{n+1}^{\Delta^2}\right| &\leq e^{r_{max}} A\left(\left|e_n^{\Delta^2}\right| + \cdots + \left|e_n^{\Delta^{p-1}}\right|\right) + |R_{p-2}(r_{n+1})|, \\
&\cdots \\
\left|e_n^{\Delta^{p-2}}\right| &\leq e^{r_{max}} A\left|e_n^{\Delta^{p-1}}\right| + |R_2(r_{n+1})|, \\
\left|e_{n+1}^{\Delta^{p-1}}\right| &\leq |R_1(r_{n+1})|.
\end{aligned}$$

Let

$$B_n = |e_n| + \left|e_n^{\Delta}\right| + \cdots + \left|e_n^{\Delta^{p-1}}\right|.$$

Then

$$B_{n+1} \leq p e^{r_{max}} A B_n + |R_1(r_{n+1})| + \cdots + |R_p(r_{n+1})|.$$

Observe that from (4.2) we get

$$\begin{aligned}
|R_1(r_{n+1})| + \cdots + |R_p(r_{n+1})| &\leq r_{max}^2 A(1+A) + \frac{r_{max}^3}{2!} A(1+A)^2 \\
&+ \cdots + \frac{r_{max}^{p+1}}{p!} A(1+A)^p \\
&\leq r_{max}^2 A(1+A)^p \left(1 + \frac{r_{max}}{2!} + \cdots + \frac{r_{max}^{p-1}}{p!}\right) \\
&\leq r_{max}^2 A(1+A)^p \left(1 + r_{max} + \frac{r_{max}^2}{2!} + \cdots + \frac{r_{max}^{p-1}}{(p-1)!}\right) \\
&\leq r_{max}^2 A(1+A)^p e^{r_{max}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
B_{n+1} &\le pe^{r_{max}}AB_n + r_{max}^2 A(1+A)^p e^{r_{max}} \\
&\le pe^{r_{max}}A\left(pe^{r_{max}}AB_{n-1} + r_{max}^2 A(1+A)^p e^{r_{max}}\right) + r_{max}^2 A(1+A)^p e^{r_{max}} \\
&= \left(pe^{r_{max}}A\right)^2 B_{n-1} + \left(pe^{r_{max}}A + 1\right) r_{max}^2 A(1+A)^p e^{r_{max}} \\
&\le \cdots \\
&\le \left(pe^{r_{max}}A\right)^{n+1} B_0 + \left(\left(pe^{r_{max}}A\right)^n + \cdots + pe^{r_{max}}A + 1\right) r_{max}^2 A(1+A)^p e^{r_{max}} \\
&\le r_{max}^2 A(1+A)^p e^{r_{max}} \sum_{j=0}^{\infty} \left(pe^{r_{max}}A\right)^j \\
&\le r_{max}^2 A(1+A)^p e^{t_f - t_0} \sum_{j=0}^{\infty} \left(pe^{t_f - t_0}A\right)^j \\
&= \frac{1}{1 - pe^{t_f - t_0}A} r_{max}^2 A(1+A)^p e^{t_f - t_0}.
\end{aligned}$$

In the last inequality we have used the fact that $B_0 = 0$ and $r_{max} \le t_f - t_0$. Consequently

$$|e_n| + \left|e_n^{\Delta}\right| + \cdots + \left|e_n^{\Delta^{p-1}}\right| = O\left(r_{max}^2\right).$$

5 The 2-step Adams-Bashforth method: AB(2)

In this section we consider the special case of the Taylor series method of order p, which in the case of $\mathbb{T} = \mathbb{R}$ reduces to the numerical method known as the 2-step Adams-Bashforth method [13]. We shall call this method the 2-step Adams-Bashforth method on time scales.

Consider again the IVP (3.1). Suppose that $r, l > 0$, $t, t+r, t-l \in [t_0, t_f]$, $\rho^2(t+r), \rho(t-l) \in [t_0, t_f]$. Applying the Taylor formula of the second order, we compute

$$x(t+r) = x(t) + h_1(t+r,t)x^{\Delta}(t) + h_2(t+r,t)x^{\Delta^2}(t) + R_2(r) \tag{5.1}$$

and applying the Taylor formula of the first order, we get

$$\begin{aligned}
x^{\Delta}(t-l) &= x^{\Delta}(t) + h_1(t-l,t)x^{\Delta^2}(t) + R_1(l) \\
&= x^{\Delta}(t) - lx^{\Delta^2}(t) + R_1(l),
\end{aligned}$$

whereupon

$$x^{\Delta^2}(t) = \frac{1}{l}\left(x^{\Delta}(t) - x^{\Delta}(t-l)\right) + \frac{1}{l}R_1(l).$$

We put this expression in (5.1) and we find

$$\begin{aligned} x(t+r) &= x(t) + rx^{\Delta}(t) + \frac{h_2(t+r,t)}{l}\left(x^{\Delta}(t) - x^{\Delta}(t-l) + R_1(l)\right) + R_2(r) \\ &= x(t) + rf(t,x(t)) + \frac{h_2(t+r,t)}{l}\left(f(t,x(t)) - f(t-l,x(t-l))\right) \\ &+ \frac{h_2(t+r,t)}{l}R_1(l) + R_2(r). \end{aligned} \tag{5.2}$$

Assume that $\{t_0 < t_1 < \ldots < t_{m+1} = t_f\}$ is a partition of the interval $[t_0, t_f]$ such that $t_{n+1} = t_n + r_{n+1} \in \mathbb{T}$, $r_{n+1} > 0$, $n \in \{0, \ldots, m\}$. Take $t = t_n$, $r = r_{n+1}$, $l = r_n$ in (5.2) and we obtain

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + r_{n+1}f(t_n, x(t_n)) + \frac{h_2(t_{n+1},t_n)}{r_n}\left(f(t_n,x(t_n)) - f(t_{n-1},x(t_{n-1}))\right) \\ &+ \frac{h_2(t_{n+1},t_n)}{r_n}R_1(r_n) + R_2(r_{n+1}). \end{aligned}$$

Let $x_n = x(t_n)$, $f_n = f(t_n, x(t_n))$. Then, neglecting the remainder terms, we arrive at the 2-step Adams-Bashforth method (AB(2) method).

$$x_{n+1} = x_n + r_{n+1}f_n + \frac{h_2(t_{n+1},t_n)}{r_n}(f_n - f_{n-1})$$

or

$$x_{n+1} = x_n + \left(r_{n+1} + \frac{h_2(t_{n+1},t_n)}{r_n}\right)f_n - \frac{h_2(t_{n+1},t_n)}{r_n}f_{n-1}. \tag{5.3}$$

Remark 5.1. *1. Note that the 2-step Adams-Bashforth method* (5.3) *is of order* $(1+O(r_n))O(r_{n+1}^2)$.

2. If $\mathbb{T} = \mathbb{R}$ *and* $r_n = h$ *is constant, then we have* $h_2(t_{n+1},t_n) = \frac{(t_{n+1}-t_n)^2}{2} = \frac{h^2}{2}$ *and hence,* (5.3) *takes the form*

$$x_{n+1} = x_n + \left(h + \frac{h}{2}\right)f_n - \frac{h}{2}f_{n-1} = x_n + \frac{3h}{2}f_n - \frac{h}{2}f_{n-1},$$

which is the classical 2-step Adams-Bashforth method.

3. *The initial condition $x(t_0) = x_0$ provides the first term of the sequence $\{x_n\}$, but one needs the second term x_1 in order to compute the following terms of the sequence. For the computation of x_1, one can use the Euler method on time scales given in [5] or the trapezoidal rule on time scales [11].*

Below, we apply the method to specific examples of initial value problems associated with nonlinear dynamic equations.

Example 5.1. As a first example we consider the initial value problem for the Beverton-Holt model given as follows.

$$x^{\Delta}(t) = \frac{\alpha x(t)}{1+\beta x(t)}, \quad x(0) = x_0, \tag{5.4}$$

where α, β are real numbers. This model has various applications in population dynamics [1, 2]. Take $\mathbb{T} = \mathbb{N}_0$ and $[t_0, t_f] = [0, 20]$. The monomial h_2 on this time scale is in the form $h_2(t,s) = \dfrac{(t-s)(t-s-1)}{2}$. If we take constant step size $r_n = h$, then $m = \frac{20}{h}$ and $t_n = nh$ for $n \in \{0, \ldots, m\}$. In this case the AB (2) formula (5.3) takes the form

$$\begin{aligned} x_{n+1} &= x_n + \left(h + \frac{h(h-1)}{2h}\right)\frac{\alpha x_n}{1+\beta x_n} - \frac{h(h-1)}{2h}\frac{\alpha x_{n-1}}{1+\beta x_{n-1}} \\ &= x_n + \frac{3h-1}{2}\frac{\alpha x_n}{1+\beta x_n} - \frac{h-1}{2}\frac{\alpha x_{n-1}}{1+\beta x_{n-1}}. \end{aligned}$$

Starting with $x_0 = x(0)$ we use the Euler method introduced in [5] to compute x_1 which gives

$$x_1 = x_0 + h\frac{\alpha x_0}{1+\beta x_0},$$

and then compute the sequence x_n, $n \in \{2, \ldots, m\}$ by using the AB (2) method.

On the other hand, it is easy to see that the exact solution of the problem can be obtained by writing the dynamic equation in (5.4) as a difference equation, that is,

$$\begin{aligned} x_0 &= x(0), \\ x_{n+1} &= x_n + \frac{\alpha x_n}{1+\beta x_n}, \quad n \in \{0, \ldots, 19\}. \end{aligned}$$

The solutions computed with the AB (2) method and the exact solutions for different choices of x_0, α and β and h are given in Figures 1-4. When $h = 1$, the approximate solution is the same as the exact solution. However, for $h = 2$ an error is observed.

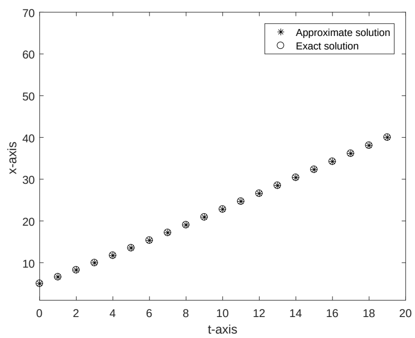

Figure 1: Computed and exact values of the solution with $\alpha = 1.5, \beta = 0.75, x_0 = 1$ and $h = 1$.

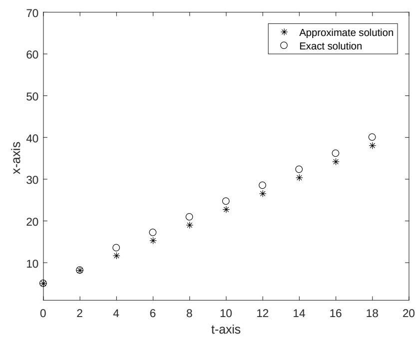

Figure 2: Computed and exact values of the solution with $\alpha = 1.5, \beta = 0.75, x_0 = 1$ and $h = 2$.

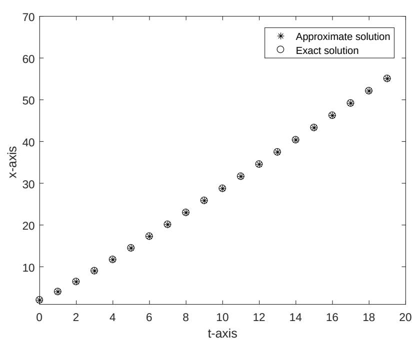

Figure 3: Computed and exact values of the solution with $\alpha = 3, \beta = 1, x_0 = 2$ and $h = 1$.

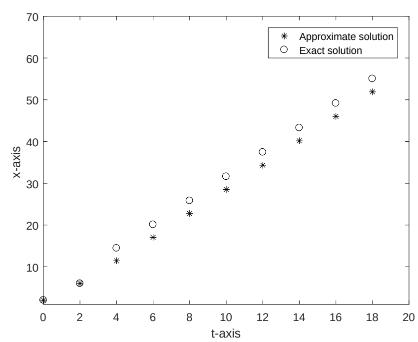

Figure 4: Computed and exact values of the solution with $\alpha = 3, \beta = 1, x_0 = 2$ and $h = 2$.

Example 5.2. Consider the initial value problem

$$x^{\Delta}(t) = \frac{1}{1+t^2} + \frac{t}{1+(x(t))^2}, \quad x(0) = x_0, \tag{5.5}$$

which is solved by the trapezoidal rule in [11]. As in [11], we take $\mathbb{T} = a\mathbb{N}_0$ for some $a > 0$ and $[t_0, t_f] = [0, 20]$. The monomial h_2 on this time scale is in the form $h_2(t,s) = \dfrac{(t-s)(t-s-a)}{2}$. If we take constant step size $r_n = h$, then $m = \frac{20}{h}$ and $t_n = nh$ for $n \in \{0, \dots, m\}$. In this case the AB(2) formula (5.3) takes the form

$$\begin{aligned} x_{n+1} &= x_n + \frac{3h-a}{2}\left(\frac{1}{1+(t_n)^2} + \frac{t_n}{1+(x_n)^2}\right) \\ &- \frac{h-a}{2}\left(\frac{1}{1+(t_{n-1})^2} + \frac{t_{n-1}}{1+(x_{n-1})^2}\right). \end{aligned}$$

Starting with $x_0 = x(0)$ we use the Euler method introduced in [5] to compute x_1 which gives

$$x_1 = x_0 + h\left(\frac{1}{1+(t_0)^2} + \frac{t_0}{1+(x_0)^2}\right),$$

and then compute the sequence x_n, $n \in \{2,\ldots,m\}$ by using the AB (2) method.

From the discrete structure of the time scale $a\mathbb{N}_0$, the dynamic equation in (5.5) can be written as a difference equation, that is,

$$\begin{aligned} x_0 &= x(0), \\ x_{n+1} &= x_n + a\left(\frac{1}{1+(an)^2} + \frac{an}{1+(x_n)^2}\right), \quad n \in \{0,\ldots,19\}, \end{aligned}$$

and hence, can be solved analytically on the inteval $[0, 20]$. The solutions computed with the AB (2) method and the exact solutions for different choices of x_0, h are given in Figures 5-6. The errors in the computation with the 2-step Adams Bashforth method and the trapezoidal rule are compared in Figures 7-8. The figures show that there is no significant difference between

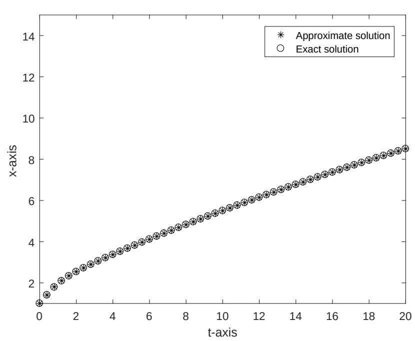

Figure 5: Computed and exact values of the solution with $x_0 = 1, a = 0.2$ and $h = 0.4$.

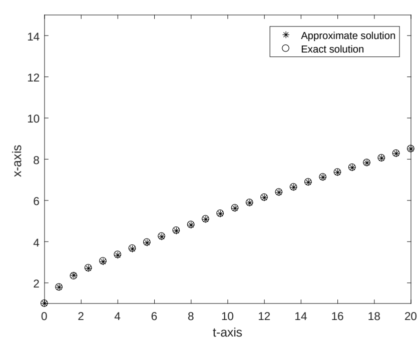

Figure 6: Computed and exact values of the solution with $x_0 = 1, a = 0.2$ and $h = 0.8$.

the exact and computed solutions because h is small. The comparison of the errors in trapezoidal rule and AB (2) method show that both errors have small magnitude since the chosen values of h are small.

6 Conclusion

This paper can be regarded as a continuation of [11] and presents a general Taylor series method of order p. As is known, in the continuous case, that is, if $\mathbb{T} = \mathbb{R}$, different numerical methods can be obtained from the Taylor series

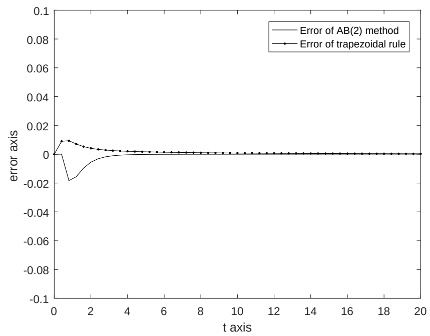

Figure 7: Errors in the computed solutions with AB(2) method and trapezoidal rule for $x_0 = 1, a = 0.2$ and $h = 0.4$.

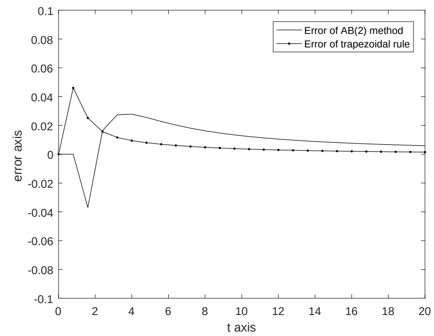

Figure 8: Errors in the computed solutions with AB(2) method and trapezoidal rule for $x_0 = 1, a = 0.2$ and $h = 0.8$.

method of order p. Here we derived the 2 step Adams-Bashforth method, however, it is possible to obtain other numerical schemes. In this sense, this paper provides different perspectives for those who study numerical methods for dynamic equations on time scales.

References

[1] R.J.H. Beverton and S. J. Holt, On the Dynamics of Exploited Fish Populations, Volume 19 of Fishery investigations (Great Britain, Ministry of Agriculture, Fisheries, and Food), (London: H. M. Stationery off.).

[2] M. Bohner and H. Warth, The Beverton- Holt dynamic equation, Applicable Analysis, 86 (2007) 1007-1015.

[3] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2003.

[4] M. Bohner and S. Georgiev, Multivariable Dynamic Calculus on Time Scales, Springer, 2016.

[5] M. Bohner, S. Georgiev and İ. M. Erhan, The Euler method for dynamic equations on general time scales, Nonlinear Studies, submitted (2020).

[6] M. Bohner and R. P. Agarwal, Basic Calculus on Time Scales and some of its Applications, Result.Math. 35 (1999) 3-22.

[7] S. Georgiev, Integral Equations on Time Scales, Atlantis Press, 2016.

[8] S. Georgiev, Functional Dynamic Equations on Time Scales, Springer, 2019.

[9] S. Georgiev, Fractional Dynamic Calculus and Fractional Dynamic Equations on Time Scales, Springer, 2018.

[10] S. Georgiev and I. Erhan, Nonlinear Integral Equations on Time Scales, Nova Science Publishers, 2019.

[11] S. Georgiev and I. Erhan, The Taylor series method and trapezoidal rule on time scales, Applied Mathematics and Computation, 378 (2020) .

[12] S. Hilger, Analysis on measure chains - a unified approach to continuous and discrete calculus, Results Math., 18 (1990) 18-56.

[13] A. Iserles, A First Course in the Numerical Analysis of Differential Equations, Cambridge University Press, 1996.