LIE SYMMETRY ANALYSIS FOR SOLITON SOLUTIONS OF GENERALISED KADOMTSEV-PETVIASHVILI-BOUSSINESQ EQUATION IN $(3+1)$-DIMENSIONS

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Abstract. The Lie group of infinitesimal transformations technique and similarity reduction is performed for obtaining an exact invariant solution to generalized Kadomstev-Petviashvili-Boussinesq (gKPB) equation in (3+1)-dimensions. We obtain generators of infinitesimal transformations, which provide us a set of Lie algebras. In addition, we get geometric vector fields, a commutator table of Lie algebra, and a group of symmetries. It is observed that the analytic solution (closed-form solutions) to the nonlinear gKPB evolution equations can easily be treated employing the Lie symmetry technique. A detailed geometrical framework related to the nature of the solutions possessing traveling wave, bright and dark soliton, standing wave with multiple breathers, and one-dimensional kink, for the appropriate values of the parameters involved.

1. Introduction

Many nonlinear partial differential equations (PDEs) have been proposed to model various complex physical, chemical, and biological phenomena mathematically throughout the last several decades. For example, diverse theoretical advancements in mathematics such as fluid mechanics (including the interaction of waves, solitary waves, traveling waves, shallow water waves, and rogue waves), theory of turbulence (including analysis of chaos) have been perused in a variety of application to model nonlinear phenomena [1–3]. These nonlinear phenomena are mathematically modeled as a nonlinear system based on the nonlinear system of equations, which is a set of simultaneous differential equations [4–14]. Therefore, the qualitative analysis of various solutions to nonlinear evolution PDEs plays a significant role in such studies. It is known that the nonlinear PDEs are not straightforward to solve. A variety of techniques including the multiple exp-function method [23], generalized symmetry method [15–19], the Bäcklund transformation method, Hirota's bilinear method [20], Pfaffian technique, Darboux transformation [21], the Painlevé analysis, the inverse scattering method [22], Wronskian and Grammian solutions have been widely employed to understand characteristics of nonlinear evolution equations.

In current work, we aim to analyze the following gKPB equation in (3+1)-dimensions [24, 25],

$$\Delta := w_{xxxy} + 3w_x w_{xy} + 3w_y w_{xx} + w_{tt} + w_{xt} + w_{yt} - w_{zz} = 0, \tag{1.1}$$

where $w(x, y, z, t)$ represents the height of the wave at a spatial point (x, y, z) in time t on a three dimensional real space. Removal of the term w_{tt} from the equation (1.1), gives us widely popular Kadomtsev-Petviashvili (KP) equation [26, 27]. It is observed that the KP equation is integrable model and can be represented by a first-order PDE in time, while Kadomstev-Petviashvili-Boussinesq equation can take the form of second-order nonlinear PDE in temporal direction. Moreover, it models both left and right-moving waves.

Wazwaz and El-Tantawy [28] derived the b-type KPB equation and double-soliton solutions by employing the simplified Hirota's method. Yu and Sun [29], constructed a direct bilinear Bäcklund transformation and obtained rational and exponential traveling wave solutions with various wavenumbers. The exact lump solutions through the perturbation expansion technique combined with Hirota's bilinear transformation are described in Kaur and Wazwaz [30]. Lü [31] formulated the

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lump, breather-wave, and interaction solution with few restrictions considering Hirota transformation and symbolic tools of Mathematica. Using Hirota's bilinear transform, Sun and Wazwaz [32] constructed highorder lumps, high-order rogue waves, and hybrid solutions to the KPB equation in (3+1)-dimensions. Recently, Wang *et al.* [33] applied the bifurcation theory of dynamical system to it and presented traveling wave solutions of the KPB equation.
The present work is arranged in the following manner. In Section 2, the Lie symmetry analysis of the gKPB equation in (3+1)-dimensions is obtained. In Section 3, we derive the symmetry groups for the gKPB equation. We shall further perform symmetry reduction and determine the analytic solution in section 4. Section 5 contains a geometrical representation of extracted solutions by providing a 3D solution surface for various values of parameters. This section also includes brief analytical discussions about the nature of obtained solutions. Finally, Section 6 is concluded with remarks and findings.

2. $(3+1)-$dimensions gKPB equation and its Lie symmetry analysis

Using Lie symmetry analysis similarity reductions of the gKPB equation is derived as given in [15, 16]. Let us construct the one parameter Lie group of transformation with $(u_1 = w, x_1 = x, x_2 = y, x_3 = z, x_4 = t)$,

$$\begin{bmatrix} \mathring{\check{\mathrm{x}}} \\ \mathring{\check{\mathrm{y}}} \\ \mathring{\check{\mathrm{z}}} \\ \mathring{\mathrm{t}} \\ \mathring{\check{\mathrm{w}}} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ t \\ w \end{bmatrix} + \varsigma \begin{bmatrix} \xi^1(x,y,z,t,w) \\ \xi^2(x,y,z,t,w) \\ \xi^3(x,y,z,t,w) \\ \tau(x,y,z,t,w) \\ \eta(x,y,z,t,w) \end{bmatrix} + O(\varsigma^2). \tag{2.1}$$

Lie group transformations' generators for the independent and dependent variables are ξ^1, ξ^2, ξ^3, τ, and η, respectively, together with continuous group parameter ς. The aforementioned transformations' vector field is represented as

$$X = \xi^1(x,y,z,t,w)\frac{\partial}{\partial x} + \xi^2(x,y,z,t,w)\frac{\partial}{\partial y} + \xi^3(x,y,z,t,w)\frac{\partial}{\partial z} + \tau(x,y,z,t,w)\frac{\partial}{\partial t} + \eta(x,y,z,t,w)\frac{\partial}{\partial w}.$$

By employing the invariance condition $Pr^{(4)}X(\Delta) = 0$, for $\Delta = 0$, the infinitesimal criteria to the invariance of (1.1) is obtained as

$$\eta_{xxxy} + 3\eta_x w_{xy} + 3w_x\eta_{xy} + 3\eta_y w_{xx} + 3\eta_{xx}w_y + \eta_{tt} + \eta_{xt} + d\eta_{yt} - \eta_{zz} = 0. \tag{2.2}$$

Now, implementing the fourth prolongation $Pr^{(4)}X$ of X to (2.2), one can receive the following determining equations

$$\begin{aligned}
&\xi^1_t = \xi^1_x = \frac{\xi^3_z}{3},\ \xi^1_w = 0,\ \xi^1_y = 0,\ \xi^1_z = \frac{\xi^3_t}{2} = 0,\\
&\xi^2_t = \xi^2_z = \xi^2_w = 0,\ \xi^2_y = \xi^3_z,\ \xi^2_z = \frac{\xi^3_t}{2},\\
&\xi^3_w = \xi^3_x = \xi^3_y = \xi^3_{tt} = \xi^3_{tz} = \xi^3_{zz} = 0,\\
&\tau_t = \xi^3_z,\ \tau_x = \tau_w = \tau_y = 0,\ \tau_z = \xi^3_t,\\
&\eta_w = -\frac{\xi^3_z}{3},\ \eta_x = \frac{\xi^3_z}{9} = \eta_y,\ \eta_{tt} = \eta_{zz}.
\end{aligned} \tag{2.3}$$

Upon simplifying the determining equations (2.3), the infinitesimal generators are

$$\begin{aligned}
&\xi^1 = \frac{c_1}{2}z + \frac{c_2}{3}(x+t) + c_6, \quad \xi^2 = \quad 2c_4 y + \frac{a}{c}c_3 t + c_6, \quad \xi^3 = \quad \frac{c_1}{2}z + c_2 y + c3,\\
&\tau = \quad c_1 t + c_2 z + c_3, \quad \eta = \varphi(t-z) + \psi(t+z) + \frac{c_2)}{9}(x+y-3w),
\end{aligned} \tag{2.4}$$

where c_i, $i = 1, \dots, 6$ and $\varphi(t-z)$ & $\psi(t+z)$ are arbitrary constants and arbitrary functions, respectively. Assume $\varphi(t-z) = c_7$, the Lie algebra of infinitesimal symmetries of (1.1) is spanned by following vector fields

$$\begin{aligned} X_1 =\ & \frac{1}{2}(z+2t)\frac{\partial}{\partial x} + \frac{1}{2}z\frac{\partial}{\partial y} + z\frac{\partial}{\partial t} \\ X_2 =\ & \frac{1}{3}(x+t)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{1}{9}(x+y-3w)\frac{\partial}{\partial w} \\ X_3 =& \frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial t}, \quad X_5 = \frac{\partial}{\partial y}, \quad X_6 = \frac{\partial}{\partial x}, \quad X_7 = \frac{\partial}{\partial w}. \end{aligned} \tag{2.5}$$

2.1. **Computation of Lie-Brackets.** The Lie algebra commutation relation, through the Lie bracket table appears to be antisymmetric with $(i,j)^{th}$ entry as $[X_i, X_j] = X_i * X_j = X_i \cdot X_j - X_j \cdot X_i$. Moreover, the diagonal elements of the commutator table are all zero, since $[X_\alpha, X_\beta] = -[X_\beta, X_\alpha]$. Further, the commutator table offer each structure constants in a simplified manner. The following commutation table is constructed using the generators of infinitesimal transformation (2.5),

Table 1. Lie Brackets for the gKPB equation

$[\cdot]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	0	$\frac{X_5-X_6}{2}+X_4$	$-X_3$	0	0	0
X_2	0	0	$-X_3$	$\frac{-X_6}{3}-X_4$	$-X_5-\frac{X_7}{9}$	$-\frac{X_6}{3}-\frac{X_7}{9}$	$\frac{X_7}{3}$
X_3	$\frac{-X_5+X_6}{2}-X_4$	X_3	0	0	0	0	0
X_4	X_3	$\frac{X_6}{3}+X_4$	0	0	0	0	0
X_5	0	$X_5+\frac{X_7}{9}$	0	0	0	0	0
X_6	0	$\frac{X_6}{3}+\frac{X_7}{9}$	0	0	0	0	0
X_7	0	$\frac{-X_7}{3}$	0	0	0	0	0

As depicted in Table 1, a continuous group of transformations for the (3+1)-dimensional gKPB is found by vector fields that span an infinite-dimensional Lie algebra. The linear combinations of generators $X_i, i = 1, 2, \dots, 7$ yield infinite subalgebras for the above Lie algebra.

2.2. **Adjoint Representation.** We shall apply the following formula to evaluate the adjoint representations of vector fields to the Eq. (1.1) [16]

$$Ad(\exp(\varsigma X_i))X_j = \sum_{m=0}^{\infty} \frac{\varsigma^m}{m!}(adX_i)^m(X_j) = X_j - \varsigma[X_i, X_j] + \frac{1}{2}\varsigma^2[X_i,[X_i,X_j]] - \cdots . \tag{2.6}$$

The Table 2.2 enumerates the adjoint representation of Table 1 by applying the formula (2.6)

Table 2. Adjoint table for Lie subalgebras

Ad	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	X_1	X_2	$\frac{X_6-X_5-2X_4}{2}\sin\varsigma + X_3\cos\varsigma$	$X_4+\varsigma X_3$	X_5	X_6	X_7
X_2	X_1	X_2	$X_3 e^{\varsigma}$	$e^{\varsigma}X_4 + e^{\frac{\varsigma}{3}}(X_6+X_7-1)$	$e^{\varsigma}X_5 + e^{\frac{\varsigma}{9}}X_7$	$e^{\frac{\varsigma}{3}}X_6 + e^{\frac{\varsigma}{9}}X_7$	$e^{\frac{-\varsigma}{3}}X_7$
X_3	$X_1+\varsigma(\frac{X_5-X_6}{2}+X_4)$	$X_2-\varsigma X_3$	X_3	X_4	X_5	X_6	X_7
X_4	$X_1-\varsigma X_3$	$X_2-\varsigma(\frac{X_6}{3}+X_4)$	X_3	X_4	X_5	X_6	X_7
X_5	X_1	$X_2-\varsigma(X_5+\frac{X_7}{9})$	X_3	X_4	X_5	X_6	X_7
X_6	X_1	$X_2-\varsigma(\frac{X_6}{3}+\frac{X_7}{9})$	X_3	X_4	X_5	X_6	X_7
X_7	X_1	$X_2+\varsigma\frac{X_7}{3}$	X_3	X_4	X_5	X_6	X_7

If a transformation connects two similar algebras, corresponding invariant solutions are connected with the transformation mentioned earlier. Thus, similar algebras constitute a class. With the aid of Table 1 and Table 2.2 and by prudently applying adjoint maps, the optimal system, with linearly independent vector fields, of one-dimensional subalgebras is spanned by

$$\{X_1, X_3, X_4, X_5, X_6, X_2 + X_7\} \tag{2.7}$$

3. Symmetry groups of $(3+1)-$dimensional gKPB equation

To obtain the group transformations, $\mathbf{G}_i : (x, y, z, t, w) \to (\mathring{\mathrm{x}}, \mathring{\mathrm{y}}, \mathring{\mathrm{z}}, \mathring{\mathrm{t}}, \mathring{\mathrm{w}})$, generated by the vector fields X_i for $i = 1(1)7$, the given system of ordinary differential equations is to be solved

$$\begin{gathered}\frac{d(\mathring{\mathrm{x}}, \mathring{\mathrm{y}}, \mathring{\mathrm{z}}, \mathring{\mathrm{t}}, \mathring{\mathrm{w}})}{d\varsigma} = (\xi^1, \xi^2, \xi^3, \tau, \eta),\\ (\mathring{\mathrm{x}}, \mathring{\mathrm{y}}, \mathring{\mathrm{z}}, \mathring{\mathrm{t}}, \mathring{\mathrm{w}})|_{\varsigma=0} = (\xi^1, \xi^2, \xi^3, \tau, \eta).\end{gathered}$$

The vector fields X_i span following groups $\mathbf{G}_i$ having one continuous parameter ς

$$\begin{aligned}
&\mathbf{G}_1 : (x, y, z, t, w) \to \left(x + \frac{\varsigma}{2}z, y + \frac{\varsigma}{2}z, z + \varsigma t, t + \varsigma z, w\right),\\
&\mathbf{G}_2 : (x, y, z, t, w) \to \left(x + \frac{\varsigma}{3}(x+t), y + \varsigma y, z + \varsigma z, t + \varsigma t, w + \frac{\varsigma}{9}(x + y - 3w)\right),\\
&\mathbf{G}_3 : (x, y, z, t, w) \to (x, y, z + \varsigma, t, w),\\
&\mathbf{G}_4 : (x, y, z, t, w) \to (x, y, z, t + \varsigma, w),\\
&\mathbf{G}_5 : (x, y, z, t, w) \to (x, y + \varsigma, z, t, w),\\
&\mathbf{G}_6 : (x, y, z, t, w) \to (x + \varsigma, y, z, t, w),\\
&\mathbf{G}_7 : (x, y, z, t, w) \to (x, y, z, t, w + \varsigma).
\end{aligned} \tag{3.1}$$

The entry on the right hand side gives the transformed point $\exp(x, y, z, t, w) = (\mathring{\mathrm{x}}, \mathring{\mathrm{y}}, \mathring{\mathrm{z}}, \mathring{\mathrm{t}}, \mathring{\mathrm{w}})$. If $w = f(x, y, z, t)$ is given solution of (1.1), thereby applying the groups $\mathbf{G}_i$, the updated solutions $w_i, i = 1(1)7$, are given as

$$\begin{aligned}
w_1 &= f\left(x - \varsigma\frac{z}{2}, y - \varsigma\frac{z}{2}, z - \varsigma, t - \varsigma\right), \quad w_2 = \frac{1}{3}\left[x + t - f\left(\frac{3x - t}{\varsigma}, \frac{y}{1+\varsigma}, \frac{z}{1+\varsigma}, \frac{t}{1+\varsigma}\right)\right], \qquad (3.2)\\
w_3 &= f(x, y, z - \varsigma, t), \quad w_4 = f(x, y, z, t - \varsigma), \quad w_5 = f(x, y - \varsigma, z, t), \quad w_6 = f(x - \varsigma, y, z, t), \quad w_7 = f(x, y, z, t) - \varsigma.
\end{aligned}$$

4. Symmetry Reduction and Closed-form Solutions

In this section, we seek to derive a group invariant solution for the Eq. (1.1) from the reduced equations. Note that reduced equations are, in turn, derived from invariant functions. It is easy to simplify the characteristic of Lagrange's system to determine invariant functions

$$\frac{dx}{\xi^1} = \frac{dy}{\xi^2} = \frac{dz}{\xi^3} = \frac{dt}{\tau} = \frac{dw}{\eta}. \tag{4.1}$$

Invariance is a remarkable property of the Lie group of transformations method. The solutions obtained under a one-parameter Lie group of transformations are invariant. The Lagrange system of characteristic equations allows group invariant solutions to construct differential equations with the one-less independent variable, resulting in an ordinary differential equation (ODE). The resolution of ODE is back substituted to yield a solution of the primary differential equation.

4.1. **Vector field $\mathbf{X}_1$.**

$$X_1 = \frac{z}{2}\frac{\partial}{\partial x} + \frac{z}{2}\frac{\partial}{\partial y} + t\frac{\partial}{\partial x} + z\frac{\partial}{\partial t}. \tag{4.2}$$

The Eqs. (4.2) and (4.1) are being used to find characteristic equation for Lagrange's system

$$\frac{dx}{\frac{z}{2}} = \frac{dy}{\frac{z}{2}} = \frac{dz}{t} = \frac{dt}{z} = \frac{dw}{0}.$$

The invariant solution for the Eq. (1.1) is

$$w(x,y,z,t) = f(X,Y,Z), \quad \text{with} \quad Y = 2y - t, \quad X = 2x - t, \quad Z = z^2 - t^2. \tag{4.3}$$

From Eqs. (4.3) and (1.1), we get following PDE

$$-f_Z - 4Zf_{ZZ} - f_{YY} - 2f_{XY} - f_{XX} + 24f_X f_{XY} + 24f_Y f_{XX} + 16f_{XXXY} = 0. \tag{4.4}$$

Using similarity transformation method (STM), the infinitesimal generators for Eq. (4.4) are

$$\xi_X = \frac{a_1}{4}X + a_3, \quad \xi_Y = \frac{a_1}{2}Y + a_2, \quad \xi_Z = a_1 Z, \quad \eta_f = a_5 \log Z + \frac{a_1}{48}(X+Y) + a_4, \tag{4.5}$$

where ξ_X, ξ_Y, ξ_Z are the generators of infinitesimal transformations for independent variables $X,\ Y,\ Z$, respectively while η_f for dependent variable f; $a_i,\ i = 1(1)5$, are constants.
A set of vector fields for generators of infinitesimal transformation (4.5) is given by

$$\begin{aligned} \pi_1 &= \frac{X}{4}\frac{\partial}{\partial X} + \frac{Y}{2}\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z} + \frac{X+Y}{48}\frac{\partial}{\partial f}, \\ \pi_2 &= \frac{\partial}{\partial Y}, \quad \pi_3 = \frac{\partial}{\partial X}, \quad \pi_4 = \frac{\partial}{\partial f}, \quad \pi_5 = \log Z\frac{\partial}{\partial f}. \end{aligned} \tag{4.6}$$

4.1.1. *Vector field π_2.*

$$\pi_2 = \frac{\partial}{\partial Y}.$$

The Lagrange's characteristic equations for vector field π_2 are

$$\frac{dX}{0} = \frac{dY}{1} = \frac{dZ}{0} = \frac{df}{0}.$$

In addition, we express the function f in the invariant form shown below.

$$f(X,Y,Z) = H(r,q), \quad r = X, \quad q = Z. \tag{4.7}$$

We reduce Eq. (4.4) as follows

$$H_q + 4H_{qq} + H_{rr} = 0. \tag{4.8}$$

Using back substitution, solution of Eq. (4.8) provides solution of primary equation (1.1) as

$$w(x,y,z,t) = \frac{(z^2-t^2)^{\frac{3}{8}}\left(K_1^2\exp(K_1(2x-t)^2) + K_2\right)\left[\text{BesselJ}(\tfrac{3}{4}, \sqrt{-K_1(z^2-t^2)})K_3 + \text{BesselY}(\tfrac{3}{4}, \sqrt{-K_1(z^2-t^2)})K_4\right]}{\exp(\sqrt{K_1}(2x-t))}, \tag{4.9}$$

where $K_i,\ i = 1, \ldots, 4$ are constants.

4.1.2. *Vector field π_3.*

$$\pi_3 = \frac{\partial}{\partial X}.$$

Lagrange's characteristic equations for vector field π_3 are

$$\frac{dX}{1} = \frac{dY}{0} = \frac{dZ}{0} = \frac{df}{0}.$$

Further, the function f is expressed in the given invariant form

$$f(X,Y,Z) = H(r,q) \quad \text{and} \quad r = Y, \quad q = Z. \tag{4.10}$$

We reduce Eq. (4.4) into following PDE

$$H_q + 4H_{qq} + H_{rr} = 0. \tag{4.11}$$

Using back substitution, the solution of the Eq. (4.11) provides solution of primary equation (1.1)

$$w(x,y,z,t) = \frac{(z^2-t^2)^{\frac{3}{8}}\left(K_1^2\exp(K_1(2y-t)^2)+K_2\right)\left[\mathrm{BesselJ}(\frac{3}{4},\sqrt{-K_1(z^2-t^2)})K_3 + \mathrm{BesselY}(\frac{3}{4},\sqrt{-K_1(z^2-t^2)})K_4\right]}{\exp(\sqrt{K_1}(2y-t))}, \tag{4.12}$$

where K_i, $i = 1, \ldots, 4$ are constants.

4.2. **Vector field $\mathbf{X}_3$.**

$$X_3 = \frac{\partial}{\partial z}.$$

Eq. (4.1) is used to find the related Lagrange equations

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{0} = \frac{dw}{0}. \tag{4.13}$$

The Eq. (1.1) can be converted into the given invariant form

$$w(x,y,z,t) = f(X,Y,T), \quad Y = y \quad X = x, \quad T = t, \tag{4.14}$$

From Eqs. (4.14) and (1.1), we get following PDE

$$f_{XXXY} + 3f_X f_{XY} + 3f_Y f_{XX} + f_{XT} + f_{YT} + f_{TT} = 0. \tag{4.15}$$

We find traveling wave solutions of Eq. (4.15) is

$$f(X,Y,T) = 2K_2\tanh\left(K_2X + K_3Y + \left(-\frac{K_2+K_3}{2} - \frac{1}{2}\sqrt{-16K_2^3K_3 + K_2^2 + 2K_2K_3 + K_3^2}\right)T + K_1\right) + K_5, \tag{4.16}$$

$$f(X,Y,T) = 2K_2\tanh\left(K_2X + K_3Y + \left(-\frac{K_2+K_3}{2} + \frac{1}{2}\sqrt{-16K_2^3K_3 + K_2^2 + 2K_2K_3 + K_3^2}\right)T + K_1\right) + K_5. \tag{4.17}$$

We obtain following traveling wave solutions of Eq. (1.1)

$$w(x,y,z,t) = 2K_2\tanh\left(K_2x + K_3y + \left(-\frac{K_2+K_3}{2} - \frac{1}{2}\sqrt{-16K_2^3K_3 + K_2^2 + 2K_2K_3 + K_3^2}\right)t + K_1\right) + K_5, \tag{4.18}$$

$$w(x,y,z,t) = 2K_2\tanh\left(K_2x + K_3y + \left(-\frac{K_2+K_3}{2} + \frac{1}{2}\sqrt{-16K_2^3K_3 + K_2^2 + 2K_2K_3 + K_3^2}\right)t + K_1\right) + K_5, \tag{4.19}$$

where K_i, $i = 1, \ldots, 5$ are constants.

Apply the similarity transformation approach to Eq.(4.15) to produce the infinitesimal generators shown below

$$\xi_X = \frac{a_1}{3}(X+T) + a_4, \quad \xi_Y = a_1Y + a_3, \quad \tau_T = a_1T + a_2, \quad \eta_f = \frac{a_1}{9}(X+Y-3f) + a_5T + a_6,$$

where a_i, $i = 1, \ldots, 6$ are arbitrary constants. Vector fields associated to aforementioned infinitesimal generators are given by

$$\begin{aligned} \pi_1 = &\ \frac{1}{3}(T+X)\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + T\frac{\partial}{\partial T} + \frac{1}{9}(X - 3f + Y)\frac{\partial}{\partial f}, \\ \pi_2 = &\ \frac{\partial}{\partial T}, \quad \pi_3 = \frac{\partial}{\partial Y}, \quad \pi_4 = \frac{\partial}{\partial X}, \quad \pi_5 = T\frac{\partial}{\partial f}, \quad \pi_6 = \frac{\partial}{\partial f}. \end{aligned} \tag{4.20}$$

4.2.1. *Vector field π_2.*

$$\pi_2 = \frac{\partial}{\partial T}.$$

Invariant function $H(r,q)$ and invariant variable $r,\ q$ are produced by the characteristic equations of Lagrange's system for Eq. (4.15)

$$f(X,Y,T) = H(r,q), \quad r = X, \quad q = Y. \tag{4.21}$$

Further, the reduction of Eq. (4.15) is

$$3H_r H_{rq} + 3H_q H_{rr} + H_{rrrq} = 0. \tag{4.22}$$

Apply Lie group of transformations method on Eq. (4.22) to obtain infinitesimal generators

$$\xi_r = b_1 r + b_2, \quad \xi_q = P(q), \quad \eta_H = -b_1 H + b_3, \quad b_1, b_2, b_3 \text{ are constants.}$$

Assuming $P(q) = 0$, the invariant solution $H(r,q)$ is written as

$$H(r,q) = \frac{G(\zeta)}{r + b_2}, \quad \text{with similarity variable} \quad \zeta = q. \tag{4.23}$$

Using above invariant function, Eq. (4.22) is reduced into following ODE

$$+ 3G'^2(\zeta) + 6G'(\zeta)G(\zeta) - 6G'(\zeta) = 0. \tag{4.24}$$

We back substitute the solutions of Eq. (4.24) to obtain solutions of Eq. (1.1)

$$w(x,y,z,t) = K_3 + \frac{1 + K_1 \exp(-2y)}{x + K_2}, \tag{4.25}$$

$$w(x,y,z,t) = K_3 + \frac{K_1}{x + K_2}, \qquad K_1, K_2, K_3 \quad \text{are constants.} \tag{4.26}$$

4.2.2. *Vector field π_3.*

$$\pi_3 = \frac{\partial}{\partial Y}.$$

Lagrange's characteristic equations for vector field π_3 are

$$\frac{dX}{0} = \frac{dY}{1} = \frac{dT}{0} = \frac{df}{0}.$$

By solving these equations, one can express f in terms of $H(r,q)$ as

$$f(X,Y,T) = H(r,q), \qquad r = X, \quad q = T. \tag{4.27}$$

Further, the Eq. (4.15) simplifies to

$$H_{qq} + H_{rq} = 0. \tag{4.28}$$

By back substituting the solutions of Eq. (4.28), we find solutions of the Eq. (1.1) as

$$w(x,y,z,t) = K_7 \tanh^3(K_3(t-x) - K_1) + K_5 \tanh(K_3(t-x) - K_1) + K_4, \tag{4.29}$$

$$w(x,y,z,t) = K_7 \tanh^3(K_3(t-x)-K_1) + K_6 \tanh^2(K_3(t-x)-K_1) + K_5 \tanh(K_3(t-x)-K_1) + K_4, \tag{4.30}$$

where K_i, $i = 1(1)7$, are constants.

4.2.3. *Vector field π_4.*

$$\pi_4 = \frac{\partial}{\partial X}.$$

Lagrange's characteristic equations are calculated as follows

$$\frac{dX}{1} = \frac{dY}{0} = \frac{dT}{0} = \frac{df}{0}.$$

Solution of the characteristic equations provide the function f in the terms of invariant function $H(r,q)$

$$f(X,Y,T) = H(r,q), \quad r = Y, \quad q = T. \tag{4.31}$$

Using above invariant function, the reduced PDE from the Eq. (4.15) is

$$H_{ss} + H_{rs} = 0. \tag{4.32}$$

The solutions of the Eq. (1.1) are

$$w(x,y,z,t) = K_7 \tanh^3(K_3(t-y)-K_1) + K_5 \tanh(K_3(t-y)-K_1) + K_4, \tag{4.33}$$

$$w(x,y,z,t) = K_7 \tanh^3(K_3(t-y)-K_1) + K_6 \tanh^2(K_3(t-y)-K_1) + K_5 \tanh(K_3(t-y)-K_1) + K_4, \tag{4.34}$$

where K_i, $i = 1(1)7$, are constants.

4.3. **Vector field $\mathbf{X}_4$.**

$$X_4 = \frac{\partial}{\partial t}.$$

The characteristic equations of Lagrange are as follows

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{1} = \frac{dw}{0}.$$

The Eq. (1.1) is written in the form of invariant function $f(X,Y,Z)$

$$w(x,y,z,t) = f(X,Y,Z), \qquad Y = y, \quad X = x, \quad Z = z. \tag{4.35}$$

Plugging Eq.(4.35) into Eq. (1.1) yields the following PDE

$$3f_X f_{XY} + 3f_Y f_{XX} + f_{XXXY} - f_{ZZ} = 0. \tag{4.36}$$

Equation (4.36) has following traveling wave type solutions

$$f(X,Y,Z) = 2K_2 \tanh(K_2 X + K_3 Y - 2\sqrt{K_2^3 K_3}\ Z + K_1) + K_5, \tag{4.37}$$

$$f(X,Y,Z) = 2K_2 \tanh(K_2 X + K_3 Y + 2\sqrt{K_2^3 K_3}\ Z + K_1) + K_5. \tag{4.38}$$

Therefore, solutions of main Eq. (1.1) can be written as

$$w(x,y,z,t) = 2K_2 \tanh(K_2 x + K_3 y - 2\sqrt{K_2^3 K_3}\ z + K_1) + K_5, \tag{4.39}$$

$$w(x,y,z,t) = 2K_2 \tanh(K_2 x + K_3 y + 2\sqrt{K_2^3 K_3}\ z + K_1) + K_5, \tag{4.40}$$

where K_i, $i = 1(1)5$, are constants.
Apply the similarity transformation approach to Eq.(4.36) to produce the infinitesimal generators shown below

$$\xi_X = \frac{1}{3}(2a_1 - a_3)X + a_5, \quad \xi_Y = a_3 Y + a_4, \quad \xi_Z = a_1 Z + a_2, \quad \eta_f = -\frac{1}{3}(2a_1 - a_3)f + a_6 Z + a_7,$$

where a_i, $i = 1(1)7$, are arbitrary constants. The vector fields associated to these infinitesimal generators are

$$\begin{aligned} \pi_1 = & \frac{2X}{3}\frac{\partial}{\partial X} + Z\frac{\partial}{\partial Z} - \frac{2f}{3}\frac{\partial}{\partial f}, \quad \pi_2 = \frac{\partial}{\partial Z}, \\ \pi_3 = & -\frac{1}{3}X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + \frac{1}{3}f\frac{\partial}{\partial f}, \quad \pi_4 = \frac{\partial}{\partial Y}, \\ \pi_5 = & \frac{\partial}{\partial X}, \quad \pi_6 = Z\frac{\partial}{\partial f}, \quad \pi_7 = \frac{\partial}{\partial f}. \end{aligned} \tag{4.41}$$

4.3.1. *Vector field π_1.*

$$\pi_1 = \frac{2X}{3}\frac{\partial}{\partial X} + Z\frac{\partial}{\partial Z} - \frac{2f}{3}\frac{\partial}{\partial f}. \tag{4.42}$$

Equation (4.42) and (4.1) are used to find the related Lagrange system

$$\frac{dX}{\frac{2X}{3}} = \frac{dY}{0} = \frac{dZ}{Z} = \frac{df}{\frac{-2f}{3}}.$$

Invariant functions are formed by reducing Eq. (4.36) to its similarity

$$f(X,Y,Z) = \frac{H(r,q)}{Z^{\frac{2}{3}}}, \quad r = \frac{X^3}{Z^2}, \quad q = Y. \tag{4.43}$$

We produce following PDE with one less independent variables using Eqs. (4.43) and (1.1)

$$\begin{aligned} - & \tfrac{10}{9}H - \tfrac{26}{3}rH_r + 6H_{rq} + 27r^{\frac{4}{3}}H_rH_{rq} - 4r^2H_{rr} + 18r^{\frac{1}{3}}H_rH_q \\ & +27r^{\frac{4}{3}}H_qH_{rr} + 54rH_{rrq} + 27r^2H_{rrrq} = 0. \end{aligned} \tag{4.44}$$

Apply the similarity transformation approach to Eq.(4.44) to produce the infinitesimal generators shown below

$$\xi_r = -b_1 r, \quad \xi_s = b_1 q + b_2, \quad \eta_f = \frac{b_1}{3}H, \quad b_1, b_2 \text{ are constants.}$$

Invariant function and invariant variable are produced by the characteristic equations of Lagrange's system for Eq. (4.44)

$$H(r,q) = (q+b_2)^{\frac{1}{3}}G(\zeta), \quad \zeta = r(q+b_2). \tag{4.45}$$

The Eq. (4.45) reduces Eq. (4.44) into a fourth order nonlinear ODE

$$\begin{aligned} & -\tfrac{10}{9}G(\zeta) + (8 - \tfrac{26}{3}\zeta)G'(\zeta) + 6\zeta^{\frac{1}{3}}G(\zeta)G\prime(\zeta) + 54\zeta^{\frac{4}{3}}G'^2(132\zeta - 4\zeta^2)G''(\zeta) + 9\zeta^{\frac{4}{3}}G(\zeta)G''(\zeta) \\ & +54\zeta^{\frac{7}{3}}G'(\zeta)G''(\zeta) + 63\zeta^2G'''(\zeta) + 2\zeta^3G''''(\zeta) = 0. \end{aligned} \tag{4.46}$$

In this case, we did not find an exact solution, though one can easily treat them numerically.

4.3.2. *Vector field π_2.*

$$4\pi_2 = \frac{\partial}{\partial Z},$$

Invariant functions are formed by reducing Eq. (4.36) to its similarity

$$f(X,Y,Z) = H(r,q), \quad r = X, \quad q = Y. \tag{4.47}$$

The Eq. (4.36) is reduced in a PDE with two independent variables

$$3H_rH_{rq} + 3H_qH_{rr} + H_{rrrq} = 0. \tag{4.48}$$

Eq. (4.48), by applying Lie group analysis, yields the infinitesimal generators

$$\xi_r = b_1 r + b_2, \quad \xi_q = P(q), \quad \eta_H = -b_1 H + b_3, \quad b_1, b_2, b_3 \text{ are constants.}$$

Assuming $P(q)=0$, the function $H(r,q)$ is written in the form of invariant function $G(\zeta)$

$$H(r,s)=\frac{G(\zeta)}{r+b_2},\quad \text{where}\quad \zeta=s. \tag{4.49}$$

Upon simplifying above invariant function, the reduction of Eq. (4.48) into a first order ODE is

$$6G(\zeta)G'(\zeta)-6G'(\zeta)+3G'^{2}(\zeta)=0 \tag{4.50}$$

To find solutions of the Eq. (1.1), we back substitute the solutions of Eq. (4.50)

$$w(x,y,z,t)=K_3+\frac{1+K_1\exp(-2y)}{x+K_2}, \tag{4.51}$$

$$w(x,y,z,t)=K_3+\frac{K_1}{x+K_2}, \tag{4.52}$$

where K_i, $i=1,2,3$ are constants.

4.3.3. *Vector field π_3.*

$$\pi_3=-\frac{X}{3}\frac{\partial}{\partial X}+Y\frac{\partial}{\partial Y}+\frac{f}{3}\frac{\partial}{\partial f}.$$

Lagrange Characteristic equations for vector field π_3 are

$$\frac{dX}{\frac{-X}{3}}=\frac{dY}{Y}=\frac{dZ}{0}=\frac{df}{\frac{f}{3}}.$$

The Eq. (4.36) is converted into following invariant form

$$f(X,Y,Z)=Y^{\frac{1}{3}}H(r,q),\quad r=X^3Y,\quad q=Z. \tag{4.53}$$

Using above invariant functions, reduction of Eq. (4.36) is

$$-H_{qq}+8H_r+132rH_{rr}+54r^{\frac{4}{3}}H_r^2+54r^{\frac{7}{3}}H_rH_{rr}+6r^{\frac{1}{3}}HH_r+9r^{\frac{4}{3}}HH_{rr}+144r^2H_{rrr}+27r^3H_{rrrr}=0. \tag{4.54}$$

When applied to Eq. (4.54), the Lie group analysis method produces the infinitesimal generators below

$$\xi_r=2b_1r,\quad \xi_q=b_1q+b_2,\quad \eta_H=-\frac{2}{3}b_1H,\quad b_1,b_2 \text{ are constants.}$$

The characteristic equations are

$$\frac{dr}{2r}=\frac{dq}{q+\overset{*}{b}_2}=\frac{dH}{\frac{-2H}{3}},\quad \text{where}\quad \overset{*}{b}_2=\frac{b_2}{b_1}.$$

Reduction of Eq. (4.54) to its similarity is

$$\zeta=\frac{r}{q+\overset{*}{b}_2},\quad H(r,q)=\frac{G(\zeta)}{r^{\frac{1}{3}}}, \tag{4.55}$$

The Eq. (4.55) reduces equation (4.54) into a fourth order nonlinear ODE

$$\begin{aligned}&-\tfrac{10}{9}G(\zeta)-26\zeta G'(\zeta)+8G\prime(\zeta)+6\zeta^{\frac{1}{3}}G'G(\zeta)+54\zeta^{\frac{4}{3}}G'(\zeta)-4\zeta^2G''(\zeta)G(\zeta)\\&+132\zeta G''(\zeta)+9\zeta^{\frac{4}{3}}G(\zeta)G''(\zeta)+54\zeta^{\frac{7}{3}}G''(\zeta)G(\zeta)+144\zeta^2G'''(\zeta)+27\zeta^3G''''(\zeta)=0.\end{aligned} \tag{4.56}$$

In this case, we did not find an exact solution, though it is easy to solve them by available discretization techniques.

4.3.4. *Vector field π_4.*

$$\pi_4 = \frac{\partial}{\partial Y}.$$

The vector field π_4 provide similarity reduction of Eq. (4.36) as follows

$$f(X,Y,Z) = H(r,q), \quad \text{where,} \quad r = X, \; q = Z. \tag{4.57}$$

By using Eq. (4.57), reduction of Eq. (4.36) is

$$H_{ss} = 0. \tag{4.58}$$

Hence, Eq. (1.1) has following solution for this case

$$w(x,y,z,t) = \alpha(x)z + \beta(x). \tag{4.59}$$

4.3.5. *Vector field π_5.*

$$\pi_4 = \frac{\partial}{\partial X}.$$

The vector field π_4 provide similarity reduction of the Eq. (4.36) as follows

$$f(X,Y,Z) = H(r,q), \quad r = Y, \; q = Z. \tag{4.60}$$

Equation (4.36) can be simplified as follows by using Eq. (4.60)

$$H_{qq} = 0. \tag{4.61}$$

Hence, we obtain

$$w(x,y,z,t) = \alpha(y)z + \beta(y) \tag{4.62}$$

as one of a solution for Eq. (1.1).

4.4. **Vector field $\mathbf{X}_5$.**

$$X_5 = \frac{\partial}{\partial y}. \tag{4.63}$$

With the aid of Eqs. (4.63) and (4.1), we find following characteristic equations

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{0} = \frac{dt}{0} = \frac{dw}{0}.$$

Invariant function $f(X,Z,T)$ formed by reducing Eq. (1.1)

$$w(x,y,z,t) = f(X,Z,T), \quad X = x, \quad T = t, \quad Z = z. \tag{4.64}$$

By putting Eq.(4.64) into Eq. (1.1), the following PDE is generated

$$f_{XT} + f_{TT} - f_{ZZ} = 0. \tag{4.65}$$

By solving Eq. (4.65), we back substitute its solutions to obtain following travelling wave type solutions

$$w(x,y,z,t) = K_8 \tanh^3\left(K_3 t - \tfrac{K_3^2 - K_4^2}{K_3}x + K_4 z + K_1\right) + K_6 \tanh\left(K_3 t - \tfrac{K_3^2 - K_4^2}{K_3}x + K_4 z + K_1\right) + K_5, \tag{4.66}$$

$$\begin{aligned} w(x,y,z,t) &= K_8 \tanh^3\left(K_3 t - \tfrac{K_3^2 - K_4^2}{K_3}x + K_4 z + K_1\right) + K_7 \tanh^2\left(K_3 t - \tfrac{K_3^2 - K_4^2}{K_3}x + K_4 z + K_1\right) \\ &\quad + K_6 \tanh\left(K_3 t - \tfrac{K_3^2 - K_4^2}{K_3}x + K_4 z + K_1\right) + K_4, \end{aligned} \tag{4.67}$$

where K_i, $i = 1(1)8$, are constants.

4.5. Vector field $\mathbf{X}_6$.

$$X_6 = \frac{\partial}{\partial x}. \tag{4.68}$$

The following characteristic equations of Lagrange are found using Eqs. (4.68) and (4.1)

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{0} = \frac{dw}{0}.$$

The Eq. (1.1) is converted to the following invariant form when new similarity variables are introduced

$$w(x, y, z, t) = f(Y, Z, T), \quad Y = y, \quad Z = z, \quad T = t. \tag{4.69}$$

From Eqs. (4.69) and (1.1), following PDE is obtained

$$f_{YT} + f_{TT} - f_{ZZ} = 0. \tag{4.70}$$

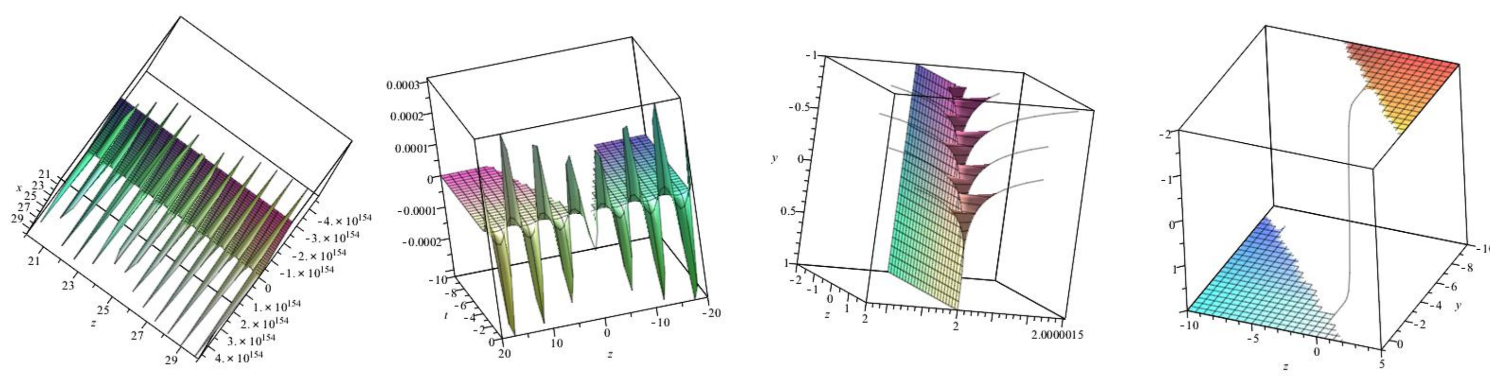
Upon solving Eq. (4.70), we back substitute its solutions to obtain following travelling wave type solutions

$$w(x,y,z,t) = K_8 \tanh^3\left(K_3 t - \tfrac{K_3^2 - K_4^2}{K_3} y + K_4 z + K_1\right) + K_6 \tanh\left(K_3 t - \tfrac{K_3^2 - K_4^2}{K_3} y + K_4 z + K_1\right) + K_5, \tag{4.71}$$

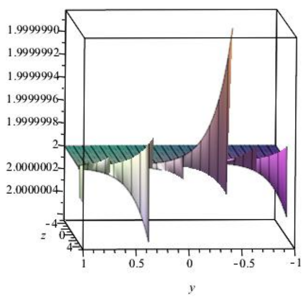
$$\begin{aligned} w(x,y,z,t) &= K_8 \tanh^3\left(K_3 t - \tfrac{K_3^2 - K_4^2}{K_3} y + K_4 z + K_1\right) + K_7 \tanh^2\left(K_3 t - \tfrac{K_3^2 - K_4^2}{K_3} y + K_4 z + K_1\right) \\ &\quad + K_6 \tanh\left(K_3 t - \tfrac{K_3^2 - K_4^2}{K_3} y + K_4 z + K_1\right) + K_4, \end{aligned} \tag{4.72}$$

where K_i, $i = 1(1)8$, are arbitrary constant.

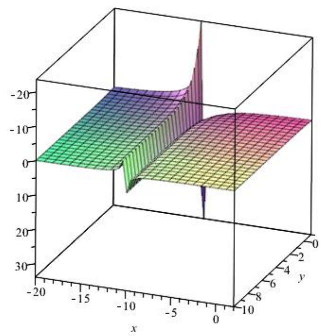
5. Graphical Interpretation and Discussion



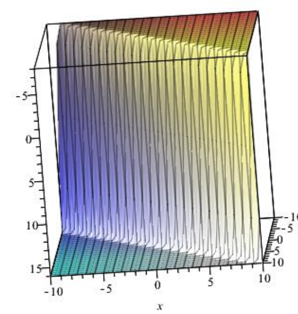
(A) (B) (C) (D)



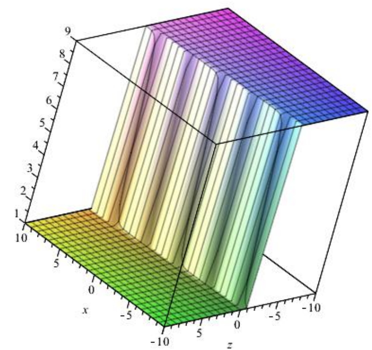
(E)



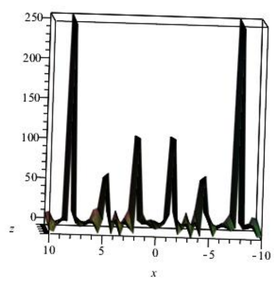
(F)



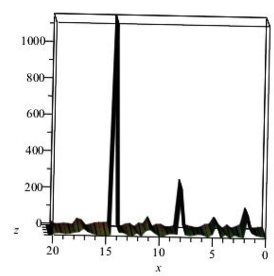
(G)



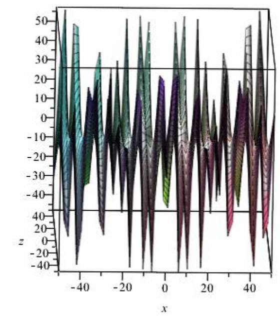
(H)



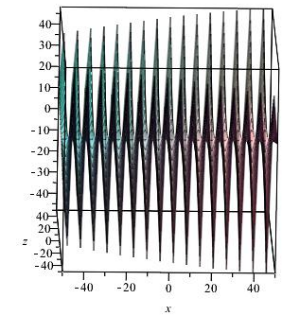
(I)



(J)



(K)



(L)

(A) In Eq. (4.9), $t = 10.358,\ K_1 = 50,\ K_2 = 2,\ K_3 = 3,\ K_4 = 4$ and $x, z \in [20, 30]$,
(B) In Eq. (4.12), $x = 1.024,\ K_1 = 1.0950,\ K_2 = 2,\ K_3 = 3,\ K_4 = 4$ and $z \in [-20, 20],\ t \in [-10, 0]$,
(C) In Eq. (4.18), $x = 1.025,\ K_1 = 11,\ K_2 = 2,\ K_3 = 3,\ K_5 = 1$ and $z \in [-2, 2],\ y \in [-1, 1]$,
(D) In Eq. (4.18), $x = 0.025,\ K_1 = 11,\ K_2 = 2,\ K_3 = 3,\ K_5 = 1$ and $y \in [-10, 1],\ z \in [-10, 5]$,
(E) In Eq. (4.18), $x = 1.025,\ K_1 = 11,\ K_2 = 2,\ K_3 = 3,\ K_5 = 1$ and $y \in [-1, 1],\ z \in [-5, 5]$,
(F) In Eq. (4.25), $K_1 = 1.98,\ K_2 = 10.105,\ K_3 = 1.508$ and $x \in [-20, 2],\ y \in [-1, 10]$,
(G) In Eq. (4.29), $K_1 = 1, K_3 = 3, K_4 = 4, K_5 = 5, K_7 = 7$ and $x, y \in [-10, 10]$,
(H) In Eq. (4.33), $y = 0.025,\ K_1 = 1,\ K_2 = 2,\ K_3 = 3,\ K_5 = 1$ and $x, z \in [-10, 10]$,
(I) In Eq. (4.59), $\alpha(x) = \sin(x^2),\ \beta(x) = \sec^2(x)$ and $x, z \in [-10, 10]$,
(J) In Eq. (4.59), $\alpha(x) = \sin(x^2),\ \beta(x) = \sec^2(x)$ and $x, z \in [-20, 20]$,
(K) In Eq. (4.59), $\alpha(x) = \sin(x^2),\ \beta(x) = \sec^2(x)$ and $x, z \in [-50, 50]$,
(L) In Eq. (4.59), $\alpha(x) = \sin(x),\ \beta(x) = \mathrm{sech}(x^2)$ and $x, z \in [-50, 50]$.

Discussion: Fig.(A), Fig.(B), Fig(C), and Fig.(E) exhibit a standing wave with multiple breathers. The undulating array of breathers with opposite phases can be regarded as a standing wave. The initial oscillatory dynamics of multi-breathers with undulating amplitude is suggestive of the most unstable interaction among solitons. Gradually, the wave loses coherence

and the chaotic regime prevails. Note that such multi-breathers interactions are not perfectly elastic.
Fig.(D) shows a topological defect with a one-dimensional kink. The presence of at least two discontinuous fibro-fatty plaques may not be entirely discrete for each other because a one-dimensional kink connects these fibro-fatty plaques with distinct spatial localization. In higher dimensions, multiple topological defects can be connected by higher-dimensional kinks. It means that intra-arterial micro- and macro-transport is done through kink solitons. Note that the solitary wave for intra-vascular transport is kink soliton.
Fig. (F) shows how the standing wave interacts with bright and dark solitons. The phase transition interacts with an array of multiple breathers, which appears as a standing wave, due to the beaming correspondence among solitons. As a result, nonlinear wave *i.e.* partially standing and partially traveling waves with changing amplitudes, propagates. Such solutions are unstable due to oscillatory instabilities. Fig. (G) and Fig. (H) exhibit a traveling wave solution that characterizes the flows in the forward direction..
Fig. (I) and Fig. (J) exhibit three bright solitons with progressively decreasing solitary wave amplitude, whereas Fig.(I) has symmetry in three bright solitons at $x = 0$. Cumulative accretion of potential energy contributes to the creation of bright soliton. The temporo-spatial localization of its energy and narrowing of time duration contributes to increment in wave speed. As the wave propagates further, its speed is progressively retarded because bright soliton vanishes after the peak. In this case, the slower wave comes in contact with another soliton of almost the same height; again, a bright soliton emerges.
Fig. (K) and Fig. (L) depict multiple breathers. While discussing Fig.(C), we have seen that multiple breathers with varying amplitude are associated with more unstable interactions than other solitons. Fig.(K) exhibits quasi-periodic oscillatory dynamics associated with increasing wave incoherence and spatial inhomogeneity. Fig. (L) exhibits periodic oscillatory dynamics with progressively growing amplitude.

5.1. **Conclusion.** The (3+1)-dimensional gKPB equation widely appears to model physical phenomena in fluid dynamics. In fact, the gKPB equation provides more accurate approximations to dynamics of water under a fewer number of constraints than the KP equation. We have investigated various exact solutions of the (3+1)-dimensional gKPB equation using the Lie group of transformations method in the present work. The generators of infinitesimal transformations are obtained by utilizing Lie symmetry group analysis,. These generators rely on a number of parameters, and from these generators, we obtained a set of Lie algebras. Finally, using the property of invariance of the Lie group of transformations, we obtained various exact solutions for the gKPB equation in (3+1)-dimensions. The solutions we have derived are represented by the equations (4.9), (4.12), (4.18), (4.25), (4.29), (4.33), (4.40), (4.51), (4.59) etc. A geometrical profile of these solutions in 3-dimensional plots (by giving appropriate values to arbitrary constants) with an analytical discussion is provided. Traveling wave solutions, bright and dark soliton solutions, kink, and standing waves with multiple breathers profiles of solutions are presented. Our results show that the symmetry method is relevant to solve nonlinear evolution equations associated with modeling nonlinear phenomena analytically.

6. Availability of Data and Material

All data generated or analysed during this study is included in this article.

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