Hypercomplex Algebras and Calculi Derived from Generalized Kinematics

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Abstract. The paper provides a study of the commutative algebras generated by iteration of the cross products in $\mathbb{C}^3$. Focusing on particular real forms we also consider the analytical properties of the corresponding rings of functions and relate them to different physical problems. Familiar results from the theory of holomorphic and bi-holomorphic functions appear naturally in this context, but new types of hypercomplex calculi emerge as well. The parallel transport along smooth curves in $\mathbb{E}^3$ and the associated Maurer-Cartan form are also studied with examples from kinematics and electrodynamics. Finally, the dual extension is discussed in the context of screw calculus and Galilean mechanics; a similar construction is studied also in the multi-dimensional real and complex cases.

Introduction

Hypercomplex number systems have become a major tool in mathematical physics during the past few decades. The variety of applications of the four normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ is a classical example of the advantages such an approach has to offer [1]. Then, one may introduce complexification and consider their split versions $\mathbb{C}'$, $\mathbb{H}'$ and $\mathbb{O}'$ related to hyperbolic geometry, relativity and quantum mechanics (see [2], [3] and [4]), as well as dualization, i.e., central extension with a nilpotent element, which is useful for instance in the treatment of non-homogeneous isometries (see [5]). Clifford's geometric algebras provide another strong confirmation. Here we discuss hypercomplex numbers originating in rigid body kinematics and more precisely, the iterated cross product in $\mathbb{R}^3$, extended via complexification, with a study on the split and null forms, as well as the dual setting. Thus, we rediscover some algebraic and functional properties well known from complex, bicomplex, hyperbolic and dual number analysis, but new objects appear within our construction too. Furthermore, altering the directional vector on the unit sphere we also consider the parallel transport in the so obtained fibre bundle, and finally, provide a generalization to arbitrary dimensions and signatures.

1. Preliminaries

The cross product of two vectors has no direct analogue in dimensions other than three (apart from the trivial one: k-vectors in $\mathbb{R}^{3k}$) so it is essentially related to the geometry of $\mathbb{R}^3$. From the perspective of Clifford's geometric algebras, this construction may be viewed as a composition of the exterior product $\wedge$ and the Hodge star operator $\star$ acting on monomials in Cliff_n as

$$\star: \quad \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \ldots \mathbf{e}_{i_k} \;\rightarrow\; (-1)^{[i_1, i_2, \ldots i_n]} \mathbf{e}_{i_{k+1}} \wedge \mathbf{e}_{i_{k+2}} \wedge \ldots \mathbf{e}_{i_n} \tag{1.1}$$

where $[i_1, i_2, \ldots i_n]$ denotes the parity of the permutation. The latter clearly sends a subspace of $\mathbb{R}^n$ to its orthogonal complement. For $n = 3$ in particular, it associates the 2-plane spanned by $\mathbf{e}_i$ and $\mathbf{e}_j$ with its normal direction $\mathbf{e}_k = \mathbf{e}_i \times \mathbf{e}_j$, where $[i, j, k] = 0$. Thus, we end up with a vector[1]

$$\mathbf{u} \times \mathbf{v} = (\mathbf{u} \wedge \mathbf{v})^\star \tag{1.2}$$

by coincidence: the codimension of the span equals the dimension of each factor. Note, however, that $\mathbf{u} \times \mathbf{v}$ is an image of a *bi-vector*, i.e., an even element in the $\mathbb{Z}_2$ grading of $\mathrm{Cliff}_{0,3}$. In this sense, the Cliffordian realization is much less misleading as there one begins with a bi-vector in the first place. It also gives a direct clue how the cross product endows $\mathbb{R}^3$ with the Lie algebra structure of $\mathfrak{so}(3)$. Namely, the obvious vector space isomorphism $\chi : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ maps the commutator of Lie generators to the cross product of their χ-images and vice versa. Thus, the adjoint action ad_ξ of each element $\xi \in \mathfrak{so}(3)$ is replaced in $\mathbb{R}^3$ by a linear transformation of the type $\hat{\mathbf{u}} = \mathbf{u}^\times$ with $\mathbf{u} = \chi(\xi)$, whose matrix in a suitably chosen basis may be written as

$$\hat{\mathbf{u}} \;\rightarrow\; \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \;\in\; \mathrm{End}(\mathbb{R}^3). \tag{1.3}$$

Since the adjoint action yields all derivations in a semi-simple Lie algebra, the cross product appears naturally in ODE's and PDE's related tho spatial geometry, such as the kinematic equation of rigid bodies with a fixed point

$$\dot{\mathbf{r}} = \hat{\boldsymbol{\omega}}\, \mathbf{r} \tag{1.4}$$

where $\mathbf{r}$ and $\boldsymbol{\omega}$ denote respectively the coordinate radius-vector and angular velocity in a fixed frame. Even in the simple case $\boldsymbol{\omega} = const.$ the solution

$$\mathbf{r}(t) = \mathrm{e}^{t\hat{\boldsymbol{\omega}}} \mathbf{r}_0 = \sum_{k=0}^{\infty} \frac{t^k}{k!}\, \hat{\boldsymbol{\omega}}^k \mathbf{r}_0, \qquad \mathbf{r}_0 = \mathbf{r}(0) \tag{1.5}$$

[1] or rather, a *pseudo-vector*, since it is preserved by the principal involution $\mathbf{x} \rightarrow -\mathbf{x}$.

has the form of an operator-valued exponent and one encounters the iterated cross products at this very basic level. According to the famous Hamilton-Cayley theorem, the matrix (1.3) satisfies its characteristic equation

$$\hat{\mathbf{u}}^3 + \mathbf{u}^2\hat{\mathbf{u}} = 0 \tag{1.6}$$

from which we derive a recurrent relation for the powers of $\hat{\boldsymbol{\omega}}$ and thus reduce (1.5) to a polynomial. In order to study the map (1.3) and its iterations, it is convenient to begin with the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and use homogeneity. Namely, if we set $\mathbf{x} = \lambda x \in \mathbb{R}^3$ with $x \in \mathbb{S}^2$ and $\lambda = ||\mathbf{x}|| \in \mathbb{R}^+$, this clearly yields $\hat{\mathbf{x}} = \lambda\hat{x}$ and thus, $\hat{\mathbf{x}}^n = \lambda^n\hat{x}^n$. Moreover, if we let $\mathcal{I}$, $\mathcal{P}_x^{||}$ and $\mathcal{P}_x^{\perp}$ denote respectively the identity, the parallel and the normal projector defined as

$$\mathcal{P}_x^{||} = xx^t, \qquad \mathcal{P}_x^{\perp} = \mathcal{I} - \mathcal{P}_x^{||}$$

it is not hard to see for example, either from (1.6) or by direct computation, that $\hat{x}^2 = -\mathcal{P}_x^{\perp}$. Hence, one derives the recurrent relation for $n \in \mathbb{N}$

$$\hat{x}^{n+2} = -\hat{x}^n, \qquad \hat{x}^0 = \mathcal{I}. \tag{1.7}$$

With the above construction it is straightforward to represent standard geometric operators, e.g. the parallel projector in the x-direction $\mathcal{P}_x^{||} = \hat{x}^0 + \hat{x}^2$, the half-turn $\mathcal{O}_x = \hat{x}^0 + 2\hat{x}^2$ about it or the mirror reflection $\mathcal{M}_x^{\perp} = -\mathcal{O}_x$ with respect to the normal plane $x^{\perp}$ and of course, all rotations in it, with the aid of formula (1.5). Note, however, that the commutative algebra generated by powers of $\hat{x}$ has plenty of zero divisors as none of the operators $\hat{x}$, $\mathcal{P}_x^{||}$ and $\mathcal{P}_x^{\perp}$ has maximal rank. In the following pages we study the structure of this algebra, together with various generalizations and physical implementations.

2. Algebraic Structure

So far we have a representation of a particular commutative and associative unital algebra Ω, with basis elements p, q and r corresponding to the operators $\mathcal{P}_x^{||}$, $\mathcal{P}_x^{\perp}$ and $\hat{x}$, respectively. The multiplication table

$$p^2 = p, \qquad p\,q = p\,r = 0, \qquad qr = r, \qquad q^2 = -r^2 = q \tag{2.1}$$

yields a direct sum decomposition $\Omega \cong \Omega_0 \oplus \Omega_\perp$ where $\Omega_0 \cong \mathbb{R}$ is spanned by the parallel component p and $\Omega_\perp \cong \mathbb{C}$ - by q and r. Thus, the map

$$\Omega \ni \varphi = \varphi_0 p + \varphi_1 q + \varphi_2 r \quad \longleftrightarrow \quad \{\varphi_0,\, \varphi_1 + i\varphi_2\} \in \mathbb{R} \oplus \mathbb{C}$$

is an isomorphism of algebras and formula (2.1) implies the product rule

$$\varphi\psi = \varphi_0\psi_0\, p + (\varphi_1\psi_1 - \varphi_2\psi_2)\, q + (\varphi_1\psi_2 + \varphi_2\psi_1)\, r. \tag{2.2}$$

In geometric terms, one may always choose an orthonormal coordinate frame such that x is aligned with the first axis, so φ has the matrix representation

$$\varphi \sim \begin{pmatrix} \varphi_0 & 0 & 0 \\ 0 & \varphi_1 & -\varphi_2 \\ 0 & \varphi_2 & \varphi_1 \end{pmatrix}. \tag{2.3}$$

Clearly, the element $\varphi \in \Omega$ is invertible whenever the determinant of its matrix representation is non-vanishing and the inverse has the explicit form

$$\varphi^{-1} = \frac{1}{\varphi_0}\, p + \frac{\varphi_1}{\varphi_1^2 + \varphi_2^2}\, q - \frac{\varphi_2}{\varphi_1^2 + \varphi_2^2}\, r = \frac{\varphi^*}{||\varphi||^2} \tag{2.4}$$

where the conjugation (principal involution) $\varphi \to \varphi^*$ in Ω corresponds to sign inversion of x or equivalently, transposition in (2.3)

$$* : \{p,\, q,\, r\} \to \{p,\, q,\, -r\}$$

and the norm in Ω may be introduced in the form of a product of the norms in its two orthogonal components, namely as

$$||\varphi||^2_\Omega = ||\varphi||^2_0 ||\varphi||^2_\perp = \varphi_0 \det \varphi = \varphi_0^2(\varphi_1^2 + \varphi_2^2). \tag{2.5}$$

Thus, each $\varphi \in \Omega$ is algebraically invertible if and only if its norm (2.5) is non-zero and the multiplicative inverse φ^{-1} is given by formula (2.4). Note, however, that an element φ may not be invertible in the whole algebra Ω but only in its parallel or perpendicular invariant subspace associated respectively with $\ker \mathcal{P}_x^\perp$ or $\ker \mathcal{P}_x^{||}$. The corresponding projectors are realized as

$$\langle \varphi \rangle_0 = p\, \varphi, \qquad \langle \varphi \rangle_\perp = q\varphi = \varphi_\perp.$$

On the other hand, the conjugation naturally fixes the even subalgebra $\Omega_{ev} \cong \mathbb{R} \oplus \mathbb{R}$ spanned by p and q, represented by diagonal matrices in (2.3). It is convenient to introduce cylindrical coordinates in Ω in the form

$$\varphi_1 + i\varphi_2 = \rho\, \mathrm{e}^{i\vartheta}, \qquad \rho = ||\varphi||_\perp, \qquad \vartheta = \arg\langle \varphi \rangle_\perp = \mathrm{atan}_2\, [\varphi_2, \varphi_1] \tag{2.6}$$

where $\mathrm{atan}_2\, [\varphi_2, \varphi_1]$ stands for the so-called *proper quadrant inverse tangent.* Here we recover all standard facts about complex numbers, e.g., even elements of Ω act as scale transformations in the x direction (with a possible reflection if $\varphi_0 < 0$) and spiral motions in the plane $x^\perp$. Moreover, the following holds

Theorem 2.1. *For all $n \in \mathbb{N}$ (or $n \in \mathbb{Z}$ as long as $||\varphi||_\Omega \neq 0$) one has*

$$\varphi^n = \varphi_0^n\, p + \rho^n \left(\cos n\vartheta\, q + \sin n\vartheta\, r\right) = \varphi_0^n\, p + \rho^n \langle \mathrm{e}^{n\vartheta r} \rangle_\perp .$$

Similarly, the n-th root of an arbitrary $\varphi \in \Omega$ is given as

$$(\sqrt[n]{\varphi})_{jk} = (\sqrt[n]{\varphi_0})_j\, p + \rho^{\frac{1}{n}} \left(\cos \frac{\vartheta + 2k\pi}{n}\, q + \sin \frac{\vartheta + 2k\pi}{n}\, r \right)$$

where $(\sqrt[n]{\varphi})_{jk} = \psi_{jk}$ denotes the jk-th of the n^2 solution to the equation $\psi^n = \varphi$ with $j, k = 0, \ldots, n-1$, while $\rho^{\frac{1}{n}}$ is the unique non-negative real root.

This somewhat classical result is a direct consequence of the algebraic structure of Ω and some elementary trigonometry provided in the next section. Note also that demanding only real values for $\sqrt[n]{\varphi_0}$ one clearly ends up with either 0, n or $2n$ solutions depending on the parity of n and the sign of φ_0.

3. Analytical Properties

The decomposition $\Omega \cong \mathbb{R} \oplus \mathbb{C}$ allows for introducing functions $f : \Omega \to \Omega$ in the following manner: let $\varphi \in \Omega$ be represented by the pair $\{\varphi_0,\, z\}$ where $\varphi_0 \in \mathbb{R}$, $z = \varphi_1 + i\varphi_2 \in \mathbb{C}$ and consider a function $f(\varphi) = \langle f\rangle_0 + \langle f\rangle_\perp$ with

$$\langle f\rangle_0 = f_0(\varphi_0)\, p, \qquad \langle f\rangle_\perp = f_1(\varphi_\perp)\, q + f_2(\varphi_\perp)\, r.$$

Using the $\mathbb{R} \oplus \mathbb{C}$ representation, the above may also be given in the form

$$\{\varphi_0,\, z\} \xrightarrow{f} \{f_0(\varphi_0),\, \tilde{f}(z)\}, \qquad \tilde{f}(z) = \tilde{f}_1(z) + i\tilde{f}_2(z) \tag{3.1}$$

where $\tilde{f}$ might be taken as the complex extension of f_0, but generally one only demands $\tilde{f}(0) = f_0(0)$ since $\varphi^0 = p + q$. Take for example the exponent

$$\mathrm{e}^{\varphi} = \mathrm{e}^{\varphi_0} p + \mathrm{e}^{\varphi_1}\left(\cos\varphi_2\, q + \sin\varphi_2\, r\right) \tag{3.2}$$

that is globally invertible (in the algebraic sense) as $||\exp\varphi||_\Omega = \mathrm{e}^{\varphi_0+\varphi_1} \neq 0$ and it is not hard to see that $\mathrm{e}^{\varphi}\mathrm{e}^{\psi} = \mathrm{e}^{\varphi+\psi}$. It is an example of an analytic (holomorphic) function in Ω. More generally, we work with the following

Definition 3.1. A function $f : \Omega \to \Omega$ is said to be *analytic* in a subset $D \subset \Omega$ if f_0 is real-analytic in $D_0 = D \cap \langle\Omega\rangle_0$ and $\langle f\rangle_\perp$ satisfies the *Cauchy-Riemann conditions* in $D_\perp = D \cap \langle\Omega\rangle_\perp$, i.e.,

$$\frac{\partial f_1}{\partial\varphi_1} = \frac{\partial f_2}{\partial\varphi_2}, \qquad \frac{\partial f_2}{\partial\varphi_1} = -\frac{\partial f_1}{\partial\varphi_2}. \tag{3.3}$$

This is equivalent to the algebraic definition of analyticity demanding the tangent map to be in Ω and due to the mutual annihilation of p and q also means that f has a power series expansion (we choose the origin for convenience) $f(\varphi) = \sum\limits_{n=0}^{\infty} a_n\varphi^n$, in which both the variable φ and the coefficients $\{a_k\}$ are $\Omega-$valued. Separating the term $a_0\varphi^0 = a_0(p+q)$ allows for writing

$$f(\varphi) = \sum_{n=0}^{\infty}\langle a_n\rangle_0\, \varphi_0^n + \sum_{n=0}^{\infty}\langle a_n\rangle_\perp \varphi_\perp^n. \tag{3.4}$$

Since the above splitting is applicable also to inverse powers, as can be seen from Moivre's formula, we may express an arbitrary meromorphic function in a similar way. Consider for example the familiar geometric series

$$\sum_{n=0}^{\infty}\varphi^n = \frac{p}{1-\varphi_0} + \frac{(1-\varphi_1)\, q + \varphi_2 r}{(1-\varphi_1)^2 + \varphi_2^2}, \qquad ||\varphi||_0, ||\varphi||_\perp < 1$$

or the Cayley map $\varphi \xrightarrow{\mathrm{Cay}} (1+\varphi)(1-\varphi)^{-1}$ which takes the explicit form

$$\mathrm{Cay}(\varphi) = \frac{1+\varphi_0}{1-\varphi_0}\, p + \frac{(1-||\varphi||_\perp^2)\, q + 2\varphi_2 r}{(1-\varphi_1)^2 + \varphi_2^2}. \tag{3.5}$$

Naturally, this construction allows for a straightforward generalization of the mean value and residue theorems as both differentiation and integration are performed separately in Ω_0 and $\Omega_\perp$, which also simplifies calculations greatly.

Note, however, that the principal part of the Laurent series is generally defined only in the restriction $\Omega_\perp$, e.g. $r^{-1} = -r$ in $\Omega_\perp \cong \mathbb{C}$, while $\nexists\, r^{-1}$ in Ω.

Next, we shall study the invertibility of functions over Ω. In the analytic case an inverse function at φ exists as long as the Jacobian is non-vanishing:

$$\exists\, f^{-1} \quad \Leftrightarrow \quad ||f'(\varphi)||^2_\Omega = |f_0'(\varphi_0)|^2 |\tilde{f}'(z)|^2 \neq 0$$

and the inversion is being performed separately on Ω_0 and $\Omega_\perp$. If the above condition holds, f defines a *biholomorphic map* on Ω. Consider for instance the exponent (3.2) and its inverse - the logarithm defined (modulo $2k\pi r$) as

$$\ln\varphi = \ln\varphi_0\, p + \ln||\varphi||_\perp\, q + \arg\langle\varphi\rangle_\perp r. \tag{3.6}$$

A particular case would be a map f with the property $f_0'(\varphi_0)\, |\tilde{f}'(z)| = const.$, which is easily illustrated by the restriction $\varphi = \lambda r \in \Omega$, i.e., f has a purely imaginary argument. For example, the exponent now takes the simple form

$$\mathrm{e}^{\lambda r} = p + \cos\lambda\, q + \sin\lambda\, r \tag{3.7}$$

that yields the famous *Rodrigues' rotation formula*, where $x \in \mathbb{S}^2$ determines the invariant axis and $\lambda = ||\mathbf{x}||$ is the corresponding angle with counterclockwise orientation. Similarly, the geometric series here is reduced to

$$\sum_{n=0}^{\infty} (\lambda r)^n = p + \frac{q + \lambda r}{1+\lambda^2}, \qquad |\lambda| < 1 \tag{3.8}$$

and finally, the Cayley map (3.5) takes the form

$$\mathrm{Cay}(\lambda r) = p + \frac{1-\lambda^2}{1+\lambda^2}\, q + \frac{2\lambda}{1+\lambda^2}\, r \tag{3.9}$$

which is easily seen to represent a rotation about x by an angle $\phi = 2\arctan\lambda$, so the group structure on SO(3) allows for analytic continuation to $\lambda \in \mathbb{RP}^1$.

Finally, let us point out that the algebraic equivalence between r and the imaginary unit i yields alternating power series in the case $\varphi = \lambda r$, namely

$$\sum_{n=0}^{\infty} a_n (\lambda r)^n = a_0 p + \sum_{k=0}^{\infty} (-1)^k \langle a_{2k}\rangle_\perp \lambda^{2k}\, q + \sum_{k=0}^{\infty} (-1)^k \langle a_{2k+1}\rangle_\perp \lambda^{2k+1}\, r$$

so we may use the Leibniz convergence test to show that if $||a_n||_\perp R^n \to 0$, the above series converges at least in a ball of radius R, e.g. $f(\mathbf{x}) = \sum_{n=1}^{\infty} n^{-s}\hat{\mathbf{x}}^n$ is well defined in the unit ball for $s > 0$, while the exponent exists in all $\mathbb{R}^3$. Note the geometric interpretation of analyticity in this context, namely as preservation of $x \in \mathbb{S}^2$ which is equivalent to a choice of basis. We also see that poles of meromorphic functions $\tilde{f}(i\lambda)$ correspond to radii of singular spheres for $\mathbf{x} \in \mathbb{R}^3$. Similarly, critical points of $f(\lambda r)$, i.e., zeroes of the equation $||f'||_\Omega = 0$, are associated with spheres, on which f is non-invertible, etc.

4. The Lorentz Setting

The complex version of the above construction yields a linear map $\tilde{\chi} : \mathbb{C}^3 \to \mathrm{End}(\mathbb{C}^3)$ given in matrix terms again by formula (1.3), this time with $\mathbf{u} \in \mathbb{C}^3$. The natural physical context here would be relativistic as $\tilde{\chi}$ takes values in the complex Lie algebra $\mathfrak{so}_3$ that is isomorphic to $\mathfrak{so}(3,1)$ and respectively

$$\mathrm{SO}(3,\mathbb{C}) \cong \mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}^+(3,1) \tag{4.1}$$

where $\mathrm{SO}^+(3,1)$ denotes the *proper Lorentz group*, i.e., the connected component of identity in $\mathrm{SO}(3,1)$, which preserves causality in Minkowski spacetime (cf [7]). Unlike in the real case, here normalization is not always possible due to the presence of isotropic directions. However, one still has the relation

$$\hat{\mathbf{x}}^2 = \mathbf{x}\mathbf{x}^t - \mathbf{x}^2\mathcal{I}$$

and may use it to obtain a recurrent formula for $\hat{\mathbf{x}}^n$ similar to, or in the particular case $\mathbf{x}^2 \neq 0$, identical with (1.7), with the same notation $\mathbf{x} = \lambda x$ with $\lambda = \sqrt{\mathbf{x}^2}$ and $x^2 = 1$ as before. The multiplication rule (2.2) and the matrix representation (2.3) also remain valid in a suitably chosen, although with $\varphi_k \in \mathbb{C}$. Hence, for $\mathbf{x}^2 \neq 0$ we end up with a complexification of Ω

$$\Omega^{\mathbb{C}} \cong \mathbb{C} \oplus \mathbb{B}$$

where $\mathbb{B} \cong \mathbb{C}^{\mathbb{C}}$ stands for the ring of *bicomplex numbers* (see [6]), also referred to as *tessarines*, obtained by introducing a second imaginary init, which, unlike in the Cayley-Dickson doubling process that yields $\mathbb{H}$, commutes with the initial one. Thus, we may choose a basis $\{1,\, i,\, j,\, k\}$ satisfying

$$i^2 = j^2 = -1, \qquad ij = ji = k$$

and the presence of zero divisors allows for constructing the two idempotents

$$\tau_\pm = \frac{1}{2}\,(1 \pm k), \qquad \tau_\pm^2 = \tau_\pm, \qquad \tau_+\tau_- = 0$$

such that every element $\psi_\perp \in \mathbb{B}$ may be decomposed as

$$\psi_\perp = \psi_-\tau_- + \psi_+\tau_+, \qquad \psi_\pm = \psi_1 \mp i\psi_2 \in \mathbb{C} \tag{4.2}$$

which yields the splitting $\mathbb{B} \cong \mathbb{C} \oplus \mathbb{C}$. Note also that the expressions (3.7) and (3.9) still hold for non-isotropic directions but the equivalence between the two does so only under the Plücker condition $\mathbf{x}^2 \in \mathbb{R}$. More precisely, one has $\mathrm{Cay}(\lambda r) = \mathrm{e}^{\phi r}$ with $\phi = 2\arctan\lambda$ where the parameters λ and ϕ are real in the positive case $x^2 = 1$ and purely imaginary in the negative one $x^2 = -1$.

As far as analyticity is concerned, f_0 is bound to satisfy the classical Cauchy-Riemann conditions in $\mathbb{C}$, while the bicomplex counterpart $f_\perp$ satisfies a generalized set of twelve relations for the partial derivatives $\{\partial_1,\, \partial_i,\, \partial_j,\, \partial_k\}$ of the real components of $f_{1,2}$ imposed by demanding that in the 4×4 real matrix representation the differential of $f_\perp$ takes values in $\Omega_\perp^{\mathbb{C}} \cong \mathbb{B}$. However, eight of them hold automatically if we restrict to the 2×2 complex matrix realization and demand that $f_{1,2}$ are analytic in both their arguments, namely

$$f_\perp \;\leftrightarrow\; \tilde{f} \;\in\; \mathcal{C}^1(\mathbb{B}): \quad \psi_1 + j\psi_2 \;\to\; f_1 + jf_2, \qquad \bar{\partial}_{\psi_m} f_n = 0 \tag{4.3}$$

where $m,n\in\{1,2\}$ and $\bar\partial_{\psi_m}=\partial_{\bar\psi_m}$ stands for the complex conjugates of

$$\partial_{\psi_1}=\frac{1}{2}\left(\partial_1-i\partial_i\right),\qquad \partial_{\psi_2}=\frac{1}{2}\left(\partial_j-i\partial_k\right)\cdot$$

The remaining four arise from the j-complex structure in the familiar form

$$\partial_{\psi_1}f_1=\partial_{\psi_2}f_2,\qquad \partial_{\psi_1}f_2=-\partial_{\psi_2}f_1 \tag{4.4}$$

Note that all twelve analyticity conditions can be represented in a rather compact form using the canonical conjugations of the bicomplex derivative

$$\partial=\frac{1}{2}\left(\partial_{\psi_1}-j\partial_{\psi_2}\right)$$

namely, the usual $4\bar\partial=\partial_1+i\partial_i-j\partial_j-k\partial_k$ inverting the sign or i and the one with respect to j which yields $4\partial^*=\partial_1-i\partial_i+j\partial_j-k\partial_k$. Adding their composition $4\bar\partial^*=\partial_1+i\partial_i+j\partial_j+k\partial_k$ we obtain the following (see [6])

Theorem 4.1. *A function $\tilde f\in\mathcal{C}^1(D\subset\mathbb{B})$ is bicomplex holomorphic in D if and only if it lies simultaneously in the kernels of the operators $\bar\partial$, ∂^* and $\bar\partial^*$.*

Note that the equalities (4.4) come from ∂^* while $\bar\partial$ and $\bar\partial^*$ combined yield (4.3). Typically, one would expect the Dirac-type operator $\bar\partial^*$ to provide the non-trivial part but $\mathbb{B}\cong\mathrm{Cliff}_2(\mathbb{C})$ is not a real Clifford algebra although it is a real form of $\mathrm{Cliff}_{0,3}\cong\mathbb{H}\oplus\mathbb{H}$. Classical results of complex calculus, such as the Cauchy theorem and integral representation, extend naturally to $\mathbb{B}$. We refer to [6] for details while [4] provides insight for the physical applications.

Isotropic Directions

Next, we consider the isotropic case $x^2=0$, $\mathbb{C}^3\ni x\neq 0$. This yields a nilpotent element $\ell\leftrightarrow\hat x$ in our algebra since $\hat x^2=xx^t$, i.e., we have the relation $\ell^2=\varepsilon$ and respectively, $\varepsilon\ell=\varepsilon^2=0$. Then, the algebra $\Omega^{\mathbb{C}}_{null}$ generated by the nilpotent element ℓ of order three and the identity can be represented as

$$\Omega^{\mathbb{C}}_{null}\ \ni\ \psi=\psi_0+\psi_1\ell+\psi_2\varepsilon\ \sim\ \begin{pmatrix}\psi_0&\psi_1&\psi_2\\0&\psi_0&\psi_1\\0&0&\psi_0\end{pmatrix}\cdot \tag{4.5}$$

It is straightforward to derive the multiplication there in the form[2]

$$\begin{aligned}\varphi\,\psi&=\varphi_0\psi_0+(\psi_0\varphi_1+\varphi_0\psi_1)\ell+(\varphi_1\psi_1+\psi_0\varphi_2+\varphi_0\psi_2)\,\varepsilon\\ \psi^n&=\psi_0^n+n\psi_0^{n-1}\psi_1\ell+n\psi_0^{n-2}\left(\frac{n-1}{2}\,\psi_1^2+\psi_0\psi_2\right)\varepsilon\end{aligned} \tag{4.6}$$

where the latter certainly holds also for negative (not necessarily integer) powers. The nilpotent counterpart ($\psi_0=0$) is mapped to strict upper triangular matrices, while the even subalgebra generated by $\{1,\varepsilon\}$ is isomorphic to the well-known *dual complex numbers* $\mathbb{C}[\varepsilon]$ with the multiplication rule

$$\varphi\,\psi=\varphi_0\psi_0+(\psi_0\varphi_2+\varphi_0\psi_2)\,\varepsilon.$$

[2]in projective coordinates $\tilde\psi=\psi_0^{-1}\psi$ this yields $\tilde\varphi\,\tilde\psi=1+(\tilde\varphi_1+\tilde\psi_1)\ell+(\tilde\varphi_1\tilde\psi_1+\tilde\varphi_2+\tilde\psi_2)\,\varepsilon$.

For functions we use power series expansion and formula (4.6) to show that

Theorem 4.2. *If $f(\psi_0)$ is analytic in $\mathbb{C}$, then $f(\psi)$ is analytic in $\Omega^{\mathbb{C}}_{null}$ and*

$$f(\psi) = f(\psi_0) + f'(\psi_0)\,[\psi_1\ell + \psi_2\,\varepsilon] + \frac{1}{2}f''(\psi_0)\psi_1^2\,\varepsilon. \qquad (4.7)$$

Moreover, f is invertible if and only if $f'(\psi_0) \neq 0$ and the inverse is given by (4.7) *but the coefficients $f_0^{(k)}$ are replaced with $h_0^{(k)}$ where $h[f(\psi_0)] = \psi_0$.*

In particular, f is completely determined by its two-jet at ψ_0, e.g. we have

$$\mathrm{e}^{\psi} = \mathrm{e}^{\psi_0}\left[1 + \psi_1\ell + \left(\psi_2 + \frac{\psi_1^2}{2}\right)\varepsilon\right]$$

while for the inversion of power functions, one may use formula (4.6). Note also that (4.7) provides a convenient way to calculate Lie derivatives as

$$\dot{f}(\psi) = f_0'\dot{\psi}_0 + f_0''\left[(\dot{\psi}_0\psi_1 + \dot{\psi}_1)\ell + (\dot{\psi}_0\psi_2 + \psi_1\dot{\psi}_1 + \dot{\psi}_2)\,\varepsilon\right] + \frac{1}{2}f_0'''\dot{\psi}_0\psi_1^2\,\varepsilon$$

with the notation $f_0^{(k)} = f^{(k)}(\psi_0)$. The general case is studied in Section 5.

Real Hyperbolic Transformations

As pointed out in [7], there are different ways to construct the isomorphism (4.1) explicitly and only the Plücker setting $x^2 \in \mathbb{R}$ guarantees the existence of an invariant plane in the $\mathbb{R}^{3,1}$ representation as this keeps the real and imaginary parts of $\hat{x}^n$ separated, which yields a restriction to one of the Wigner little groups. After studying the Euclidean and isotropic (front form) cases, we focus on the hyperbolic one, where a vector $\mathbf{x} \in \mathbb{R}^{2,1}$ qualifies as either *space-like*, *time-like* or *null* (*light-like*) depending on whether its square

$$\mathbf{x}^2 = (\mathbf{x}, \eta\mathbf{x}), \qquad \eta = \mathrm{diag}\,(1, 1, -1)$$

is positive, negative or vanishing, respectively. For the first two types one may introduce pseudo-normalization $\mathbf{x}^2 = \alpha|\mathbf{x}|^2$ with $\alpha = x^2 = \mathrm{sgn}\,\mathbf{x}^2$, while the isotropic (null) case $\mathbf{x}^2 = 0$ is scale invariant. The analogue of the map (1.3) in the pseudo-Euclidean setting is given with the aid of the metric η as

$$\mathbf{u} \in \mathbb{R}^{2,1} \quad \to \quad \hat{\mathbf{u}} \sim \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ u_2 & -u_1 & 0 \end{pmatrix} \in \mathfrak{so}(2,1)$$

and for a normalized regular vector $x \in \mathbb{R}^{2,1}$ ($x^2 = \pm 1$) one easily shows that

$$\hat{x}^2 = \alpha\mathcal{I} - x(\eta x)^t \qquad (4.8)$$

which yields in the time-like case $\hat{x}^2 = -\mathcal{P}_x^{\perp}$ and thus, we recover the algebra $\Omega \cong \mathbb{R}\otimes\mathbb{C}$ corresponding to the Euclidean construction in $\mathbb{R}^3$. For space-like vectors, on the other hand, one ends up with an algebra generated by the $\{p,\, q,\, r\}$ basis with $r^2 = q$ that is isomorphic to $\mathbb{R}\otimes\mathbb{C}'$ where $\mathbb{C}'$ denotes the set of *split-complex* (or *hyperbolic*) numbers. Those are just the *real tessarines* with basis $\{1,\, k\}$ where $k^2 = 1$, which therefore have zero-divisors in the form $\tau_\pm = \frac{1}{2}(1 \pm k)$ that may be used for the direct sum decomposition $\mathbb{C}' \cong \mathbb{R}\oplus\mathbb{R}$.

Functions over $\mathbb{C}'$ have peculiar properties, e.g. their singularities supported by infinite lines (rather than points) associated with $\tau_\pm$ (the "light cone" in the hyperbolic plane). We refer to a function $f_\perp(\varphi_1 + k\varphi_2) = f_1 + kf_2$ as *analytic* in $\mathbb{C}'$ if it satisfies the modified Cauchy-Riemann conditions (cf [6])

$$\frac{\partial f_1}{\partial \varphi_1} = \frac{\partial f_2}{\partial \varphi_2}, \qquad \frac{\partial f_1}{\partial \varphi_2} = \frac{\partial f_2}{\partial \varphi_1} \tag{4.9}$$

and the equality of mixed second derivatives ensures that for such functions we have $\Box f_{1,2} = 0$, where $\Box$ represents the $1+1$ dimensional wave operator. Using the well-known Green identities we can show that the famous Cauchy theorem applies to the hyperbolic case of analytic functions as well. One example of a globally analytic function (beside polynomials) is the exponent

$$\mathrm{e}^{\varphi_1 + k\varphi_2} = \mathrm{e}^{\varphi_1}(\cosh\varphi_2 + k\sinh\varphi_2) = \mathrm{e}^{\varphi_1 - \varphi_2}\tau_- + \mathrm{e}^{\varphi_1+\varphi_2}\tau_+.$$

Note, however, that due to the zero divisors $\tau_\pm$ neither algebraic, nor functional invertibility is guaranteed for generic non-vanishing elements of $\mathbb{C}'$. Whenever $\varphi_1 = \pm\varphi_2$ or $f_1 = \pm f_2$ one comes across that problem. Thus, we may define the split-complex conjugate $\mathbb{C}' \ni \varphi \to \varphi^* = \varphi_1 - k\varphi_2$ and the pseudo-norm as $||\varphi||^2_\Omega = \varphi^*\varphi$, which take place in the algebraic inversion

$$||\varphi||^2_\Omega \neq 0 \quad \to \quad \varphi^{-1} = \frac{\varphi^*}{||\varphi||^2}.$$

Note also that in $\mathbb{C}'$ the two idempotents $\tau_\pm = \frac{1}{2}(1 \pm k)$ yield a decomposition into two orthogonal ideals, hence, each element $\varphi \in \Omega$ can be expressed as[3]

$$\varphi = \varphi_0 p + (\varphi_- \tau_- + \varphi_+\tau_+)\, q, \qquad \varphi_\pm = \varphi_1 \pm \varphi_2$$

and similarly, for a function $f : \Omega \to \Omega$ one has

$$f = f_0 p + (f_-\tau_- + f_+\tau_+)\, q, \qquad f_\pm = f(\varphi_\pm) \tag{4.10}$$

as illustrated above for the exponent. Moreover, $f(\varphi)$ has an inverse if and only if all its projections on the ideals defined by p and $\tau_\pm q$ are invertible, i.e., we demand $||f'||_\Omega \neq 0$ just as in the Euclidean case. Note that for analytic functions this means non-vanishing partial derivative with respect to φ_1 since

$$\mathrm{d}h(z) = \frac{\partial h}{\partial z}\,\mathrm{d}z = \frac{\partial h}{\partial z_1}\,\mathrm{d}z, \qquad z = z_1 + kz_2 \in \mathbb{C}'$$

as long as $h : \mathbb{C}' \to \mathbb{C}'$ satisfies the Cauchy-Riemann equations (4.9) that may be written also in the form $\dfrac{\partial h}{\partial z^*} = 0$ familiar from complex analysis. Finally, in the isotropic (null) case $x^2 = 0$ one encounters the real version of the algebra $\Omega^{\mathbb{C}}_{null}$ described above we shall refer to simply as Ω_{null}. In particular, formulas (4.5), (4.6) and (4.7) still apply, but restricted to real coefficients.

[3]the pseudo-norm in this representation is given as $||\varphi||_\Omega = |\varphi_0\varphi_-\varphi_+|$.

5. Further Extensions and Applications

In this section we consider applications in various physical contexts: form rigid body kinematics to electrodynamics and special relativity. Then, we let the unit vector x vary on $\mathbb{S}^2$ according to Frenet's equations for the moving frame along a smooth curve γ, thus obtaining a bundle with fibre Ω, for which one may consider the parallel transport and the associated affine connection. We also discuss the dual extension in mechanical (screw-theoretic) context. Before we proceed, let us note that the algebra Ω appears naturally in some kinematical problems, e.g. a similar idea has been used in [8] for the derivation of a convenient system of differential equations on $\Omega \cong \mathbb{R} \otimes \mathbb{C}$. It has been developed further in [9] in the projective (vector-parametric) setting for $\bar{\Omega} \cong \mathbb{S}^1 \otimes \mathbb{S}^2$ (see also [10]) and in [11] it has been used for the derivation of a large class of solutions for rigid body kinematics and quantum mechanics.

The Multi-Variable Setting

Let us now consider a setting, in which we do not have a preferred direction in space, but rather, allow different directions to interact. Namely, let $\chi: \mathbf{x}_i \in \mathbb{R}^3 \to \hat{\mathbf{x}}_i \in \mathrm{End}(\mathbb{R}^3)$ as before and we shall attempt to derive polynomial expressions spanned by monomials in the form $\Pi_n = \hat{\mathbf{x}}_n \hat{\mathbf{x}}_{n-1} \cdots \hat{\mathbf{x}}_1$. Since our operations are linear, however, we can always choose a basis of three vectors, e.g. $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and restrict our considerations to it. Let $g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j$ be the corresponding Gram matrix and λ^k - the contravariant components of an arbitrary vector $\mathbf{x} \in \mathbb{R}^3$ in the basis $\{\mathbf{a}_i\}$. Then, one obviously has

$$\hat{\mathbf{a}}_i \hat{\mathbf{a}}_j \, \mathbf{x} = \lambda^k \hat{\mathbf{a}}_i \times (\hat{\mathbf{a}}_j \times \hat{\mathbf{a}}_k) = -\lambda^k g_{i[j} \mathbf{a}_{k]} = -g_{i[j} \mathbf{a}_{k]} \mathbf{a}^k \cdot \mathbf{x} \tag{5.1}$$

where $a_{[i} b_{j]} = a_i b_j - a_j b_i$ stands for the usual alternator, $\mathbf{a}^i = g^{ij} \mathbf{a}_j$ denotes the corresponding vector in the dual basis and Einstein summation over repeated indices is assumed. Then, one may show by induction that

Theorem 5.1. *With the above notation and all indices varying from 1 to 3*

$$\hat{\mathbf{x}}_{2n} \hat{\mathbf{x}}_{2n-1} \cdots \hat{\mathbf{x}}_1 = (-1)^n \lambda_{2n}^{i_{2n}} \lambda_{2n-1}^{i_{2n-1}} \cdots \lambda_1^{i_1} g_{i_{2n}[i_{2n-1}} \cdots g_{i_2[i_1} \mathbf{a}_{k]]\ldots]} \mathbf{a}^k \cdot \tag{5.2}$$

while monomials of odd degree are then given as $\Pi_{2n+1} = \hat{\mathbf{x}}_{2n+1} \Pi_{2n}$.

In particular, if we choose an orthonormal basis $\{\mathbf{e}_i\}$ formula (5.2) simplifies greatly as $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ and the components of $\mathbf{e}^k$ are the same as those of $\mathbf{e}_k$. Moreover, it suffices to begin with only two directions as we clearly have

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2, \qquad \hat{\mathbf{e}}_3 = [\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2]$$

and thus, introducing a second direction in Ω breaks the symmetry providing a non-commutative extension of $\Omega \cong \mathbb{R} \oplus \mathbb{C} \subset \mathbb{H}$. Next, we show how this process may be realized as a deformation and related to a bundle connection.

Parallel Transport

After studying the fixed axis setting and briefly discussing multiple directions, we are ready to consider deformations. Let $\gamma = \gamma(s)$ be a smooth curve in $\mathbb{R}^3$ parameterized with arc length and $\{\mathbf{e}_k\}$ represent the corresponding Frenet moving frame spanned by the tangent, normal and bi-normal direction at each point on γ. Kinematics involves time-derivatives of the $\mathbf{e}_k$'s

$$\dot{\mathbf{e}}_i = v\,\Gamma_i^j \mathbf{e}_j, \qquad \Gamma \sim \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \tag{5.3}$$

where κ and τ are respectively the curvature and torsion on γ and $v = \dot{s}$ is the velocity magnitude. Fibres of the tangent bundle T_γ may then be linked to copies of Ω associated with the tangent or another preferred direction in the local frame, which defines a bundle of algebras Ω_γ and the time flow on γ determines the parallel transport on Ω_γ via formula (5.3). More precisely, let us consider a Ω-valued differentiable $t-$dependent function on γ in the form

$$\varphi(t) = \varphi_0(t)p(t) + \varphi_1(t)q(t) + \varphi_2(t)r(t)$$

where r is associated with $\hat{\mathbf{e}}_1$ and the relations (2.1) hold at each instant t. Then, a straightforward differentiation yields for the moving frame in Ω_γ

$$\mathrm{d}\varphi = \mathrm{d}\varphi_i\omega^i + \varphi_i\mathrm{d}\omega^i$$

where $\{\omega^i\} = \{p, q, r\}$ denotes the basis on each fibre of Ω_γ. Next, it is not difficult to show that $\dot{p} = -\dot{q} = \{r, \dot{r}\} = \dot{r}r + r\dot{r}$, so one has

$$\dot{\varphi} = \dot{\varphi}_0 p + \dot{\varphi}_1 q + \dot{\varphi}_2 r + (\varphi_0 - \varphi_1)(\dot{r}r + r\dot{r}) + \varphi_2\dot{r} \tag{5.4}$$

where the last two terms are due to differentiation of the basis and may be obtained explicitly with the aid of (5.3). This way we introduce two more elements in the algebra, namely r' and r'' corresponding to the transformations $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$, respectively. This breaks the commutativity, so one may work with commutators and anti-commutators to complete the multiplication table, e.g.

$$[r, r'] = r'', \qquad [r', r''] = r, \qquad [r'', r] = r'$$

thus $\mathfrak{so}(3)$ is embedded in the deformation of Ω. Moreover, we may associate $\{r, r'\}$ with the symmetric operator[4] $\mathcal{P}_{12} = \mathbf{e}_1\mathbf{e}_2^t + \mathbf{e}_2\mathbf{e}_1^t$ and discover that

$$[p, r'] = -[q, r'] = [r, \{r, r'\}] = \{r, r''\}, \qquad \{p, r'\} = r'.$$

With this in mind, the derivative (5.4) can be written also as

$$\dot{\varphi} = \dot{\varphi}_0 p + \dot{\varphi}_1 q + \dot{\varphi}_2 r + \kappa v\left[\varphi_2 r' + (\varphi_0 - \varphi_1)\{r, r'\}\right]. \tag{5.5}$$

In particular, if $\varphi(t) = r(t)$, then one clearly has $\dot{\varphi} = \kappa v r'$. Similarly, for $\varphi(t) = r^2(t) = -q(t)$, the derivative is $\dot{\varphi} = \kappa v\{r, r'\}$. Next, we consider a slightly more general setting related to rigid body kinematics, namely let

$$g(\phi, r) = \mathrm{e}^{\phi \mathrm{r}} = \mathrm{p} + \cos\phi\,\mathrm{q} + \sin\phi\,\mathrm{r} \tag{5.6}$$

[4] clearly, one also has for the other anti-commutators $\{r', r''\} \leftrightarrow \mathcal{P}_{23}$ and $\{r, r''\} \leftrightarrow \mathcal{P}_{13}$.

$\otimes$	p	q	r	r'	$\{r, r'\}$	r''
p	$1 + rr$	0	0	$r''r$	$r'r$	$-r'r$
q	0	$-rr$	r	$-rr''$	rr'	rr'
r	0	r	rr	rr'	rr''	rr''
r'	$-rr''$	$r''r$	$r'r$	$r'r'$	$-r'r''$	$r'r''$
$\{r, r'\}$	rr'	$r'r$	$-r''r$	$r''r'$	$rr + r'r'$	$rr - r'r'$
r''	rr'	$-r'r$	$r''r$	$r''r'$	$r'r' - rr$	$r''r''$

Table 1. Multiplication table of the non-commutative extension to Ω expressed in the minimal basis $\{1, r, r', r''\}$.

represent an element of S$O(3)$ in Ω. Then, from (5.5) one obtains directly

$$\mathrm{d}g = \mathrm{d}\phi\,(\cos\phi\, r - \sin\phi\, q) + \sin\phi\,\mathrm{d}r + (1 - \cos\phi)\,\{r, \mathrm{d}r\}$$

which yields for the Maurer-Cartan forms $\omega_- = g^{-1}\mathrm{d}g$ and $\omega_+ = g\,\omega_- g^{-1}$

Theorem 5.2. *The* SO(3) *action* (5.6) *on* Ω *yields the Maurer-Cartan forms*

$$\omega_\pm = \mathrm{d}\phi\, r + \sin\phi\,\mathrm{d}r \pm (1 - \cos\phi)[\,r, \mathrm{d}r]. \tag{5.7}$$

We may also consider the pull-back $\tilde{\omega}_\pm$ associated with restriction to a smooth spatial curve γ with a lift $t \in \mathbb{R} \to g(t)$ to S$O(3)$, i.e., $\phi = \phi(t)$ and $r = r(t)$. Then, using the connection (5.3) we obtain the $1-$forms $\tilde{\omega}_\pm$ in the $\Omega-$image

$$\tilde{\omega}_\pm = \mathrm{d}\phi\, r + \kappa v \sin\phi\,\mathrm{d}t\, r' \pm \kappa v(1 - \cos\phi)\mathrm{d}t\, r''.$$

The above expressions define the angular velocity of a screw motion along γ ($\mathbf{e}_1(t)$ represents the rotation axis) in the body and inertial frame respectively

$$\boldsymbol{\omega}_\pm = \Big(\dot{\phi},\ \kappa v \sin\phi,\ \pm\kappa v(1-\cos\phi)\Big)^t. \tag{5.8}$$

Note that the latter implies both dilation and precession with rate $\dfrac{\phi}{2}$ about the instantaneous velocity and the integrability condition pointed out in [11] is satisfied (one may find explicit solutions there). Moreover, the rotation axis does not need to be aligned with the tangent to γ. We may consider for example a rigid body with helical trajectory $\gamma : \{a\cos t, a\sin t, bt\}$ parameterized with time, spinning about its instantaneous velocity and formula (5.8) yields (in our notation $\kappa v = \sin\vartheta$ and $\tau v = \cos\vartheta$ with $\vartheta = \mathit{const}$ for the helix)

$$\boldsymbol{\omega}_\pm = \Big(\dot{\phi},\ \sin\vartheta\sin\phi,\ \pm\sin\vartheta(1-\cos\phi)\Big)^t, \qquad \vartheta = \operatorname{arccot}\frac{|a|}{b}$$

but if we want to consider a different type of motion, e.g. rotation about the normal, then the torsion comes into play and one ends up with the expression

$$\boldsymbol{\omega}_\pm = \mathcal{R}_2(\vartheta)\Big(\pm(1-\cos\phi),\ \dot{\phi},\ \sin\phi\Big)^t$$

where $\mathcal{R}_2(\vartheta)$ denotes a rotation by an angle ϑ in the rectifying plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_3$, e.g. for $\vartheta = \dfrac{\pi}{2}$ the spin axis precesses in a circular orbit.

Note also that (5.7) clearly holds in the non-isotropic case and moreover, ϕ is real as long as $\mathbf{x}^2$ is real positive. As for the null setting, it is not hard to verify for example that $\dot{\ell}$ anti-commutes with ε and their product is proportional to ε. The same refers to ℓ and $\dot{\varepsilon}$ due to the property $\ell\dot{\varepsilon} = \varepsilon\dot{\ell}$, while $\varepsilon\dot{\varepsilon} = \dot{\varepsilon}\varepsilon = 0$. In order to obtain the Maurer-Cartan form of the exponential map

$$g = \exp(\lambda\ell) = 1 + \lambda\ell + \frac{\lambda^2}{2}\,\varepsilon \tag{5.9}$$

one can either use direct differentiation together with the above relations or consider the isotropic limit of formula (5.7), both leading to the result

Corollary 5.3. *The Maurer-Cartan forms associated with* (5.9) *are given by*

$$\omega_\pm = \mathrm{d}\lambda\,\ell + \lambda\,\mathrm{d}\ell \pm \frac{\lambda^2}{2}[\ell, \mathrm{d}\ell]. \tag{5.10}$$

Although formula (5.3) has no such direct analogue in $\mathbb{C}^3$, one may impose various restrictions. For example, consider a null vector field $x = \mathbf{e}_3 + i\mathbf{e}_1$ defined on the tangent bundle of the helix with the above parametrization. Using (5.10) and the Frenet's equations (5.3) we easily obtain

$$\boldsymbol{\omega}_\pm = \left(\dot{\lambda} \pm \frac{i\lambda}{2}\,\mathrm{e}^{-i\vartheta}\right) x - \lambda\mathrm{e}^{-i\vartheta}\mathbf{e}_2 \tag{5.11}$$

so x is an eigenvector of $\boldsymbol{\omega}_\pm^\times$ with eigenvalue $i\lambda\mathrm{e}^{-i\vartheta}$. This solution may be related to the physical setting of an electromagnetic wave with circular polarization traveling through a helical coil. The oscillation rate of the induced current depends on the coil's curvature and torsion in a rather simple manner. A thorough treatment of this subject would demand extensive volume and divert the focus of the present study, so we leave it for future research.

Finally, let us point out that as the Cayley map (3.9) provides an alternative representation of rotations and Lorentz boosts, it may also be used to define the angular velocity $\boldsymbol{\omega}$ via differentiation which yields (see [10] for details)

$$\boldsymbol{\omega}_\pm = \frac{2}{1+\boldsymbol{\xi}^2}\,(\mathcal{I} \pm \boldsymbol{\xi}^\times)\,\dot{\boldsymbol{\xi}} \tag{5.12}$$

where the relation between the projective Rodrigues' parameter $\boldsymbol{\xi} = \lambda x$ and the spherical vector $\mathbf{x} = \phi\, x$ in the exponential map (3.7) is given as $\lambda = 2\arctan\phi$. Interpreted in the context of Ω and its derivations, this leads to

Corollary 5.4. *The Maurer-Cartan forms associated with* (3.9) *are given as*

$$\omega_\pm = \frac{2}{1+\lambda^2}\left(\mathrm{d}\lambda\, r + \lambda\mathrm{d}r \pm \lambda^2[r, \mathrm{d}r]\right). \tag{5.13}$$

This approach works both in the real and complex cases, providing a unified geometric description of well-known kinematic effects in classical mechanics and special relativity, due to Coriolis, Thomas, Sagnac and Hall (see [3, 10]). For a thorough investigation of the physical applications, however, one needs to develop calculus on Ω bundles, which goes beyond the scope of this paper.

Some Explicit Calculations

Let us consider a complex vector $\mathbf{x} = \mathbf{x}_1 + i\mathbf{x}_2$ satisfying[5] $\mathbf{x}^2 > 0$ and the Plücker relation $\mathbf{x}_1 \perp \mathbf{x}_2$ that provide an invariant plane, then normalize as

$$x = \cosh\psi\, x_1 + i\sinh\psi\, x_2, \qquad x_{1,2} \in \mathbb{S}^2.$$

Clearly, this yields for the associated skew operator $r = \cosh\psi\, r_1 + i\sinh\psi\, r_2$ and thus, the corresponding parallel and normal projectors are given by

$$p = \frac{p_1 - \beta^2 p_2 + i\beta p_{12}}{1-\beta^2}, \qquad q = \frac{q_1 - \beta^2 q_2 - i\beta p_{12}}{1-\beta^2} \tag{5.14}$$

with $\beta = \tanh\psi$ and $p_{12} = \{r_1, r_2\} = r_1 r_2 + r_2 r_1$. The algebraic properties of the r_k's are the same as the ones of r, r' and r'' introduced above. Now we may represent an element of $\mathrm{SO}(3,\mathbb{C})$ as an exponential in $\Omega^{\mathbb{C}}$, namely

$$\mathrm{e}^{\phi r} = \frac{p_1 - \beta^2 p_2}{1-\beta^2} + \cos\phi\,\frac{q_1 - \beta^2 q_2}{1-\beta^2} + \frac{\sin\phi}{\sqrt{1-\beta^2}}\, r_1 + (1-\cos\phi)\frac{i\beta p_{12}}{1-\beta^2} + \frac{i\sin\phi\,\beta}{\sqrt{1-\beta^2}}\, r_2$$

where the real part is associated to the rotational component and the imaginary one to a boost transformation, as usual. Of course, there is some mixing due to well-known relativistic effects of Thomas, Sagnac, Lorentz and others.

Next, we let our frame move along a smooth complex curve and calculate the associated generalized angular velocity, for which we need to express

$$\dot r = \dot\psi(\sinh\psi\, r_1 + i\cosh\psi\, r_2) + \cosh\psi\,\dot r_1 + i\sinh\psi\,\dot r_2$$

which yields for the commutator

$$[r, \dot r] = i\dot\psi r_3 + \cosh^2\psi\,[r_1, \dot r_1] - \sinh^2\psi\,[r_2, \dot r_2] + i\sinh\psi\cosh\psi[r_{(1}, \dot r_{2)}]$$

where we denote $[r_{(1}, \dot r_{2)}] = [r_1, \dot r_2] + [r_2, \dot r_1]$. Suppose that the orthogonality condition $x_1 \perp x_2$ holds at each instant of time and let x_1 be the unit tangent to a smooth curve in $\mathbb{R}^3$, then x_2 will be restricted to the normal plane, so if we denote the unit normal and binormal to the curve respectively x_1' and x_1'', the corresponding $\Omega^{\mathbb{C}}$ representation takes the form

$$r_2 = \cos\theta\, r_1' - \sin\theta\, r_1'', \qquad r_3 = \sin\theta\, r_1' + \cos\theta\, r_1'' \tag{5.15}$$

for some real parameter $\theta = \theta(t)$. Differentiating and taking into account the Frenet equations, we easily obtain $\dot r_2 = (\tau - \dot\theta)\, r_3 - \kappa\cos\theta\, r_1$ which yields

$$[r_1, \dot r_1] = \kappa(\cos\theta\, r_3 - \sin\theta\, r_2), \qquad [r_2, \dot r_2] = (\tau - \dot\theta)\, r_1 + \kappa\cos\theta\, r_3$$

hence $[r_{(1}, \dot r_{2)}] = \kappa\cos\theta\, r_1 + (\dot\theta - \tau)\, r_2$ and we may express in the $\{r_i\}$ basis

$$\boldsymbol{\Omega} = L_\psi T(\theta, \psi)\boldsymbol{\Omega}_0 \tag{5.16}$$

where $\boldsymbol{\Omega}_0 = (\dot\phi,\, \sin\phi,\, 1 - \cos\phi)^t$ is familiar from the Euclidean case and the transformations in the LT-decomposition above are given explicitly as

[5]the case $\mathbf{x}^2 < 0$ is treated similarly, one only needs to multiply with $i = \sqrt{-1}$.

follows: L_ψ is simply a rotation by an angle $i\psi$ in the $x_{1,2}$ plane, i.e., one has

$$L_\psi = \begin{pmatrix} \cosh\psi & -i\sinh\psi & 0 \\ i\sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.17}$$

while $T(\theta,\psi)$ is bit a more complex, so we decompose for convenience

$$\begin{aligned} T(\theta,\psi) = &\begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa\cos\theta & -\kappa\sin\theta\cosh\psi \\ 0 & \kappa\sin\theta\cosh\psi & \kappa\cos\theta \end{pmatrix} + \\ &\qquad i\begin{pmatrix} 0 & 0 & 0 \\ 0 & \dot\psi & (\dot\theta-\tau)\sinh\psi \\ 0 & -(\dot\theta-\tau)\sinh\psi & \dot\psi \end{pmatrix}. \end{aligned} \tag{5.18}$$

Note that in the case of pure rotation $\psi = 0$ the real part of T becomes proportional also to a rotation $R_1(\theta)$ by an angle θ about the x_1 axis, while the off-diagonal elements of the imaginary part vanish, so one obtains simply

$$L_\psi^{-1}\mathbf{\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} \dot\phi \\ \kappa\sin\phi \\ \kappa(1-\cos\phi) \end{bmatrix} + i\dot\psi \begin{bmatrix} 0 \\ \sin\phi \\ 1-\cos\phi \end{bmatrix}$$

and in particular, for $\theta = 0$, the real part coincides with the one we know from the Euclidean case. On the other hand, $\theta = \pi/2$, that is the case when x_2 is constantly aligned with the bi-normal of the curve, guarantrees $\mathbf{\Omega}^2 \in \mathbb{R}$.

As already pointed out, formula (5.7) is valid in the complex case but we do not have the simple set of Frenet equations (5.3) for the moving frame. However, the property $x \cdot \dot x = 0$ remains valid for an arbitrary normalized vector x, which leads to a Cauchy-Riemann type conditions in the plane determined by $x_{1,2}$ as long as we have the Plücker relation $x^2 \in \mathbb{R}$ satisfied. This is quite easy to see in the isotropic case $x^2 = 0$ where $x = x_1 + ix_2$ and the flow preserves the complex structure in the plane determined by $x_{1,2}$, i.e., acts as a multiplication by a complex number, while adding a shift in the normal direction ς. In other words, one has $\dot x = \mu x + \nu\varsigma$ and we can see that

$$\dot\varepsilon = \dot\ell\ell + \ell\dot\ell, \qquad \varepsilon\dot\ell = \dot\ell\varepsilon = -\ell\dot\varepsilon = -\dot\varepsilon\ell, \qquad \varepsilon\dot\varepsilon = \dot\varepsilon\varepsilon = 0. \tag{5.19}$$

From kinematical perspective, it suffices to consider only the first group of relations given by formula (5.19) as there is a small parameter $\omega\,\mathrm{d}t$ attached to the unit vector ς, where $\omega \in \mathbb{C}$ may be interpreted as a generalized angular velocity. Considering an infinitesimal shift of a null vector $x \xrightarrow{\chi} \ell \in \Omega^{\mathbb{C}}_{null}$ linked to a proper Lorentz transformation via the exponential map (5.9) we apply the above definition of $\omega_\pm$ and using either (5.19) easily obtain (5.10). Note that the isotropic case may be obtained as a contraction $\psi \to \infty$ or $\beta \to 1$ of either the space-like or the time-like Wigner little group. In the following section we use dual extension to incorporate generic rigid motions.

The Dual Setting

Dual numbers are defined as a central extension of $\mathbb{R}$ with a nilpotent element

$$t, u \in \mathbb{R} \quad \longrightarrow \quad \underline{t} = t + \varepsilon u \in \mathbb{R}[\varepsilon], \qquad \varepsilon^2 = 0$$

that yields for differentiable functions $f\colon \mathbb{R}[\varepsilon] \to \mathbb{R}[\varepsilon]$ the Taylor expansion

$$f(t + \varepsilon u) = f(t) + \varepsilon f'(t) u \tag{5.20}$$

thus making analysis in $\mathbb{R}[\varepsilon]$ quite manageable. One may introduce similarly dual vectors, quaternions or Lie groups, keeping in mind that formula (5.20) demands commutativity. We shall consider dualization of Ω starting with

$$\underline{\mathbf{x}} = \mathbf{x} + \varepsilon \mathbf{h}, \qquad \mathbf{x}, \mathbf{h} \in \mathbb{R}^3$$

and decomposing $\mathbf{h}$ into a parallel and normal to $\mathbf{x}$ parts, i.e., $\mathbf{h} = \mathbf{h}^{\|} + \mathbf{h}^{\perp}$ determined via the projectors $\mathcal{P}_x^{\|}$ and $\mathcal{P}_x^{\perp}$, e.g. $\mathbf{h}^{\perp} = 0$ yields upon normalization $\mathbf{x} = \lambda x$, $x \in \mathbb{S}^2$ the operator $\hat{\underline{\mathbf{x}}} = \underline{\lambda} \hat{x}$. Contrary to the usual $\mathbb{R}^3$ case, the vector $\underline{\mathbf{x}}$ cannot be normalized with a real constant λ, so we have instead

$$\hat{\underline{\mathbf{x}}} = \underline{\lambda} \hat{x}, \qquad \underline{\lambda} = \lambda + \varepsilon \mu = \lambda(1 + \varepsilon \nu).$$

Therefore, the operator $\hat{\underline{x}}$ is mapped to $\underline{\Omega} = \Omega[\varepsilon]$ as

$$\underline{r} = (1 + \varepsilon \nu) r \quad \Rightarrow \quad \underline{r}^k = (1 + \varepsilon k \nu)\, r^k$$

so the dual extension of Ω in the parallel case has the simple form

$$\Omega^{\|}[\varepsilon] \cong \mathbb{R}[\varepsilon] \otimes \Omega$$

and we may apply formula (5.20) to functions in $\Omega^{\|}[\varepsilon]$ thus obtaining

$$f(\underline{r}) = f(r) + \varepsilon \nu r f'(r).$$

Similarly, $\mathbf{h}^{\|} = 0$ yields $\underline{\mathbf{x}}^2 \in \mathbb{R}$, so we normalize $\underline{\mathbf{x}} = \lambda \underline{x}$ with$\lambda \in \mathbb{R}$ and[6] $\underline{x} = x + \varepsilon x' \in \mathbb{S}^2[\varepsilon]$, so $\hat{\underline{x}}$ is directly associated with $\underline{r} = r + \varepsilon r' \in \Omega[\varepsilon]$, i.e.

$$\underline{r}^2 = -\underline{q}, \qquad \underline{q} = q - \varepsilon\{r, r'\}.$$

Defining $\underline{p} = 1 - \underline{q}$ and using Table 5 we see that $\underline{p}$, $\underline{q}$ and $\underline{r}$ satisfy the relations (2.1) of $\{p, q, r\}$, so as a real algebra $\Omega^{\perp}[\varepsilon]$ is isomorphic to $\Omega \cong \mathbb{R} \oplus \mathbb{C}$, but with a hypercomplex basis, and one has $\underline{r}^{k+2} = -\underline{r}^k$ for $k \in \mathbb{N}$ which yields

$$\underline{r}^{2k+1} = (-1)^k \underline{r}, \qquad \underline{r}^{2k} = (-1)^k \underline{q}. \tag{5.21}$$

Finally, the generic case $\mathbf{h} = \mathbf{h}^{\|} + \mathbf{h}^{\perp}$ may be represented as a superposition

$$\underline{\mathbf{x}} = \mathbf{x} + \varepsilon \mathbf{h} = \underline{\lambda x}, \qquad \underline{x} = x + \varepsilon x' \in \mathbb{S}^2[\varepsilon]$$

where we use the notation

$$\underline{\lambda} = |\mathbf{x}| + \varepsilon x \cdot \mathbf{h}, \qquad x' = |\mathbf{x}|^{-1} \mathbf{h}^{\perp}.$$

This certainly preserves the validity of (5.21) and allows for interpreting $\Omega[\varepsilon]$ as a dualization of $\underline{\Omega}^{\perp} \cong \Omega$. Hence, it is straightforward to see that the main results laid out so far, for instance Theorem 2.1, apply in the dual case as

[6] the dual sphere $\mathbb{S}^2[\varepsilon] \cong \{\mathbf{u} + \varepsilon \mathbf{v} :\ \mathbf{u}^2 = 1,\ \mathbf{u} \cdot \mathbf{v} = 0\}$ parameterizes ruled surfaces in $\mathbb{R}^3$.

a manifestation of the so-called *transfer principle* (see [5]). Trigonometric functions of the dual angle $\underline{\phi} = \phi + \varepsilon d$ are obtained with the aid of (5.20) as

$$\sin\underline{\phi} = \sin\phi + \varepsilon d\cos\phi, \qquad \cos\underline{\phi} = \cos\phi - \varepsilon d\sin\phi.$$

Similarly, the matrix exponent (3.7) and the Cayley map (3.9) take the form

$$\mathrm{e}^{\underline{\phi r}} = \underline{p} + \cos\underline{\phi}\,\underline{q} + \sin\underline{\phi}\,\underline{r} \tag{5.22}$$

and respectively

$$\mathrm{Cay}(\underline{\lambda r}) = \underline{p} + \frac{1-\underline{\lambda}^2}{1+\underline{\lambda}^2}\,\underline{q} + \frac{2\underline{\lambda}}{1+\underline{\lambda}^2}\,\underline{r} \tag{5.23}$$

to which the expressions for $\underline{\Omega}^{\|,\perp}$ considered above appear as particular cases. However, since usual trigonometry applies to the dual setting too, the representations (5.22) and (5.23) are equivalent under the relation $\underline{\lambda} = 2\arctan\underline{\phi}$.

Let us note that in the applications dual vectors and quaternion typically do not appear in a polarized form, i.e., with vanishing $\mathbf{h}^{\|}$ or $\mathbf{h}^{\perp}$. For instance, in the dual quaternion representation of the spatial Euclidean group one has

$$\underline{\mathbf{n}} = \mathbf{n} + \varepsilon\mathbf{m} \ \in\ \mathbb{S}^2[\varepsilon], \text{ i.e.,} \qquad \mathbf{n}^2 = 1, \quad \mathbf{m}\perp\mathbf{n}$$

representing in Plücker coordinates the instantaneous screw line appearing in the Mozzi-Chasles theorem. Here $\mathbf{n}$ is the unit vector along the rotation axis and the moment $\mathbf{m}$ is expressed in terms of the translation vector $\mathbf{t}$ as

$$\mathbf{m} = \mathbf{r}\times\mathbf{n} = \frac{1}{2}\left(\cot\frac{\phi}{2}\,\mathcal{I} - \mathbf{n}^{\times}\right)\mathcal{P}_{\mathbf{n}}^{\perp}\mathbf{t} \tag{5.24}$$

with $\mathbf{r} = \mathbf{n}\times\mathbf{m}$ denoting the radius-vector of the screw line and $\mathcal{P}_{\mathbf{n}}^{\perp}$ - the orthogonal projector with respect to $\mathbf{n}$. Similarly, one has for the dual angle

$$\underline{\phi} = \phi + \varepsilon d, \qquad d = \mathbf{t}\cdot\mathbf{n}$$

where d is referred to as *screw displacement* while $\tilde{p} = d\csc\phi$ is the so-called *screw pitch* that takes place in the definition of the dual Rodrigues' vector[7]

$$\underline{\mathbf{c}} = \tan\frac{\phi+\varepsilon d}{2}\,(\mathbf{n}+\varepsilon\mathbf{m}) = \tan\frac{\phi}{2}\left(1+\frac{\varepsilon\, d}{\sin\phi}\right)\underline{\mathbf{n}}. \tag{5.25}$$

Moreover, the transfer principle allows for defining the dual angular velocity $\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon\mathbf{v}$ (with $\mathbf{v}$ being the linear one) in $\underline{\Omega}$ by extending Theorem 5.2 as

Theorem 5.5. *The* $\underline{\mathrm{SO}}(3)$ *action* (5.22) *on* $\underline{\Omega}$ *yields the Maurer-Cartan forms*

$$\underline{\omega}_{\pm} = \mathrm{d}\underline{\phi}\,\underline{r} + \sin\underline{\phi}\,\mathrm{d}\underline{r} \pm (1-\cos\underline{\phi})[\,\underline{r},\mathrm{d}\underline{r}] \tag{5.26}$$

while (5.23) *yields respectively the representation*

$$\underline{\omega}_{\pm} = \frac{2}{1+\underline{\lambda}^2}\left(\mathrm{d}\underline{\lambda}\,\underline{r} + \underline{\lambda}\mathrm{d}\underline{r} \pm \underline{\lambda}^2[\underline{r},\mathrm{d}\underline{r}]\right). \tag{5.27}$$

The transfer principle allows for considering $\mathbb{C}[\varepsilon]$ in a similar manner although no such straightforward physical application of this algebra is known so far.

[7] for $\phi = 0$ both $\tilde{p}$ and $\mathbf{m}$ are ill-defined and one has instead $\underline{\mathbf{c}} = \frac{1}{2}\varepsilon\,\mathbf{t}$ with $\mathbf{m} = \mathbf{n}\times\mathbf{t} = 0$.

6. Higher-Dimensional Extension

We extend the cross-product (1.2) to higher dimensions as a linear mapping

$$\hat{\sigma}: \quad \xi \longrightarrow (\sigma \wedge \xi)^\star \tag{6.1}$$

where ξ and σ are multi-vectors in $\mathrm{Cliff}(\mathbb{R}^n)$ of grade 1 and n–2, respectively. In particular, if σ is a blade (decomposable multi-vector), one has complete analogy with three-dimensional cross-product and formula (6.1) is linked to rotations in the plane determined by $\sigma^\star$. In this case we may assume $\sigma^2 = \pm 1$ since one can always choose an orthonormal basis $\{\mathbf{e}_k\}$ in $\mathbb{R}^n$ such that after proper normalization $\sigma = \mathbf{e}_1\mathbf{e}_2 \cdots \mathbf{e}_{n-2}$, hence the sign of σ^2 is given as

$$\sigma^2 = (-1)^N, \qquad N = \binom{n-2}{2} = \dim \mathrm{SO}(n-2). \tag{6.2}$$

Note also that the Plücker relations for a multi-vector ψ may be written as

$$\psi^2 \in \mathbb{R} \;\Rightarrow\; \langle \psi^2 \rangle_k = 0, \quad k \neq 0$$

where $\langle \cdot \rangle_k$ denotes the so-called grade projector onto the corresponding subspace of $k-$vectors. In our case, however, due to the low codimension, the Hodge duality allows for writing the above as a single quadric

$$\sigma^2 \in \mathbb{R} \quad \Leftrightarrow \quad \sigma^\star \wedge \sigma^\star = 0. \tag{6.3}$$

The Clifford product of a $k-$blade ϑ and a $m-$blade ψ with $k + m \leq n$ expands as a sum of blades with grades from $k{+}m$ to $|k{-}m|$ in the form

$$\vartheta\, \psi = \vartheta \wedge \psi + \;\ldots\; + \vartheta \cdot \psi \tag{6.4}$$

where $\cdot$ denotes right or left contraction, depending on the context. This allows us to define parallel and normal projections of a vector ξ with respect to a blade ϑ using the properties $\vartheta \cdot \xi = \vartheta \xi_\|$ and $\vartheta \wedge \xi = \vartheta \xi_\perp$ which yield

$$\xi_\| = \vartheta^{-1}(\vartheta \cdot \xi), \qquad \xi_\perp = \vartheta^{-1}(\vartheta \wedge \xi). \tag{6.5}$$

Moreover, when Hodge duality comes into play one has the de Morgan laws

$$(\vartheta \wedge \psi)^\star = \vartheta \cdot \psi^\star, \qquad (\vartheta \cdot \psi)^\star = \vartheta \wedge \psi^\star \tag{6.6}$$

for every pair of multi-vectors ϑ and ψ. Now, for the iterations of the map (6.1) it is convenient to use the above properties and the fact that since $\star$ is associated with the action of the volume form (pseudo-scalar) I, namely as

$$\psi^\star = \psi I^{-1}, \qquad I = \mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n$$

for a generic multi-vector ψ, one obviously has

$$\star^2 = I^2 = (-1)^{N'}, \qquad N' = \binom{n}{2} = \dim \mathrm{SO}(n). \tag{6.7}$$

Taking all this into consideration, we finally obtain

$$\hat{\sigma}^2: \quad \xi \longrightarrow (\sigma \wedge (\sigma \wedge \xi)^\star)^\star = (-1)^{N'} \sigma \cdot (\sigma \wedge \xi) = (-1)^{N+N'} \sigma^{-1} \sigma\, \xi_\perp = -\xi_\perp$$

as it is not hard to see that the sum $N + N'$ is always an odd number and $\sigma \cdot (\sigma \wedge \xi) = \sigma(\sigma \wedge \xi)$ since all terms of higher grade involve exterior product of intersecting blades and thus vanish. In this way we have shown the following

Lemma 6.1. *The linear map* (6.1) *for a normalized blade σ squares as*[8]

$$\hat{\sigma}^2 = -\mathcal{P}_\sigma^\perp \tag{6.8}$$

where $\mathcal{P}_\sigma^\perp$ is the orthogonal projector (6.5) *with respect to σ. Moreover, the operators $\hat{\sigma}$, $\mathcal{P}_\sigma^\perp$ and $\mathcal{P}_\sigma^\parallel = \mathcal{I} - \mathcal{P}_\sigma^\perp$ close an algebra isomorphic to $\mathbb{R} \oplus \mathbb{C}$.*

For instance, the decomposable setting yields a reduction $\mathrm{SO}(n) \to \mathrm{SO}(3)$ where the rotation plane is associated with $\sigma^\star$ (we refer to [7] for details). A generic multi-vector $\sigma \in \mathrm{Cliff}(\mathbb{R}^n)$, on the other hand, can always be written as a sum of blades, e.g. for $n = 4$ one has $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1 \cdot \sigma_2 = 0$, which yields $\mathcal{P}_1^\parallel = \mathcal{P}_2^\perp$ and $\mathcal{P}_2^\parallel = \mathcal{P}_1^\perp$. Moreover, if $\sigma_k^2 = -\lambda_k^2$ with $\lambda_k \in \mathbb{R}$, then $\sigma_1 \wedge \sigma_2 = \lambda_1 \lambda_2 I$. Now, going back to formula (6.1) we easily see that

$$\xi \quad \xrightarrow{\hat{\sigma}^2} \quad [(\sigma_1 + \sigma_2) \wedge (\sigma_1 \wedge \xi + \sigma_2 \wedge \xi)^\star]^\star =$$

$$(\sigma_1 + \sigma_2) \cdot (\sigma_1 \xi_\perp^{(1)} + \sigma_2 \xi_\perp^{(2)}) = -\lambda_1^2 \xi_\perp^{(1)} - \lambda_2^2 \xi_\perp^{(2)}$$

where $\xi_\perp^{(1,2)}$ denotes the normal projection of ξ with respect to $\sigma^{1,2}$ and since we have a direct sum decomposition ($\sigma_1^\star \sim \sigma_2$ and $\sigma_2^\star \sim \sigma_1$), they may be written simply as $\xi_\perp$ and $\xi_\parallel$. Hence, we end up with a commutative algebra (spanned by $\hat{\sigma}_{1,2}$ and $\mathcal{P}_{1,2}^\perp$) isomorphic to $\mathbb{B} \cong \mathbb{C} \oplus \mathbb{C}$. For the higher-dimensional case it is useful to point out that each two-vector in $\mathbb{R}^n$ or $\mathbb{R}^{p,q}$ with $p + q = n$ may be expressed as a sum of $\left[\frac{n}{2}\right]$ blades and if σ is a multi-vector of grade $n - 2$, then such decomposition clearly applies to $\sigma^\star$ and by Hodge duality, to σ itself. Let us consider first the even-dimensional case $n = 2k$, in which one may write in complete analogy with $\mathbb{R}^4$

$$\sigma = \sigma_1 + \sigma_2 + \ldots + \sigma_k, \qquad \sigma_i^\star \cdot \sigma_j^\star = -\delta_{ij} \lambda_j^2, \qquad \sigma_1^\star \wedge \sigma_2^\star \wedge \ldots \wedge \sigma_k^\star = \lambda_1 \lambda_2 \ldots \lambda_k I$$

where δ_{ij} stands for the Kronecker symbol. This justifies the following result

Lemma 6.2. *Let $\sigma = \sum_{i=1}^k \sigma_i$ $(n = 2k)$ with $\sigma_i \wedge \sigma_i = 0$ and $\mathcal{P}_i^\perp = \mathcal{P}_{\sigma_i}^\perp$, then*

$$\hat{\sigma}^2 = -\lambda_1^2 \mathcal{P}_1^\perp - \lambda_2^2 \mathcal{P}_2^\perp - \ldots - \lambda_k^2 \mathcal{P}_k^\perp$$

and since we have

$$\mathcal{P}_i^\perp \mathcal{P}_j^\perp = \mathcal{P}_j^\perp \mathcal{P}_i^\perp = \delta_{ij} \mathcal{P}_j^\perp, \qquad \hat{\sigma}_i \mathcal{P}_j^\perp = \mathcal{P}_i^\perp \hat{\sigma}_j = \delta_{ij} \hat{\sigma}_j$$

the algebra closed by all $\hat{\sigma}_i$, $\mathcal{P}_i^\parallel$ and $\mathcal{P}_i^\perp$ is isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C} \cong \mathbb{C}^k$.

As for the odd-dimensional case, the Euler invariant axis theorem asserts that there is a common normal direction (a vector $\sigma_0^\star$) to all the planes $\{\sigma_i^\star\}$. Then, the planes $\{\sigma_i\}$ intersect along $\sigma_0^\star$ and we may express

$$\sigma = \sigma_0^\star \sum_{i=1}^k \sigma_i, \qquad \sigma_i^\star \cdot \sigma_j^\star = -\delta_{ij} \lambda_j^2, \qquad \sigma_0^\star \wedge \sigma_1^\star \wedge \sigma_2^\star \wedge \ldots \wedge \sigma_k^\star = \lambda_0 \lambda_1 \lambda_2 \ldots \lambda_k I$$

[8]one may use the identity $\hat{\sigma} = -\sigma^\star \cdot$ in order to simplify the proof.

so in the case $n = 2k+1$ there is an additional projector $\mathcal{P}_0^\perp$ associated with $\{\sigma_0^\star\}$, which annihilates all other operators in the algebra, thus leading to an additional (real) degree of freedom, i.e., one has for the two cases

$$\Omega^{2k} \cong \mathbb{C}^k, \qquad \Omega^{2k+1} \cong \mathbb{R} \oplus \mathbb{C}^k \tag{6.9}$$

where Ω^n denotes the associative commutative algebra generated by the $\hat{\sigma}_k$'s in $\mathbb{R}^n$. For example, in the case of $\mathbb{R}^7$ the generalized cross-product yields $\Omega^7 \cong \mathbb{R} \oplus \mathbb{C}^3$, while in $\mathbb{R}^8$ one has $\Omega^8 \cong \mathbb{C}^4 \cong \mathbb{B}^2$ instead, and so on. Here one may complexify similarly to the three-dimensional case and find out that the above direct sum decomposition has several different complex analogues according to the particular choice of σ. The simplest one replaces $\mathbb{R}$ with $\mathbb{C}$ and $\mathbb{C}$ with $\mathbb{B}$. It is realized when all blade components σ_i are time-like, while space-like blades lead to split complex numbers and isotropic ones generate nilpotent components as described in Section 4 and discussed below in detail.

Next, we consider non-trivial signatures of the type $\mathbb{R}^{p,q}$ with $p+q=n$. To gain some intuition first, we begin with the four-dimensional Lorentzian $(+,-,-,-)$ and the ultra-hyperbolic $(+,+,-,-)$ metrics. The signature determines the square of the Hodge star as

$$\star^2 = (-1)^{M'}, \qquad M' = \binom{n}{2} + q$$

while the sign $(-1)^{M_i}$ of σ_i^2 for a given blade σ_i depends on the number of vectors q_i with negative square that take part in its decomposition as

$$M_i = \binom{n-2}{2} + q_i$$

e.g. if $\sigma_1 = \mathbf{e}_0\mathbf{e}_1$ and $\sigma_2 = \mathbf{e}_2\mathbf{e}_3$ in the Lorentzian signature, then $\sigma_1^2 = 1$ and $\sigma_2^2 = -1$. Unlike in the Euclidean case, in which $N + N'$ is always a negative number, here the parity of $M' + M_i$ is equal to the one of $q - q_i + 1$, i.e.,

$$\hat{\sigma}_i^2 = (-1)^{q-q_i+1}\mathcal{P}_i^\perp \tag{6.10}$$

and when the sign is positive one ends up with a copy of $\mathbb{C}' \cong \mathbb{R}\oplus\mathbb{R}$ instead of $\mathbb{C}$ in the above decomposition. Note also that it is easier to work with the irreducible plane $\{\sigma_i^\star\}$, in which the infinitesimal rotation generated by $\hat{\sigma}_i$ takes place. In particular, it is straightforward to see that

$$(-1)^{q-q_i+1} = \operatorname{sgn}(\sigma_i^{\star 2})$$

and we may classify these irreducible planes as *space-like*, for which the above sign is positive and *time-like*, for which it is negative. For instance, in the case of $\mathbb{R}^{1,3}$ considered above and $\sigma = \lambda_1\sigma_1 + \lambda_2\sigma_2$, one has $\sigma_1^\star \sim \sigma_2$, hence

$$\hat{\sigma}^2 = -\lambda_1^2\mathcal{P}_1^\perp + \lambda_2^2\mathcal{P}_2^\perp \quad \Rightarrow \quad \Omega_\sigma^{1,3} \cong \mathbb{C}\oplus\mathbb{C}' \cong \mathbb{C}\oplus\mathbb{R}^2.$$

Note that if the signature is non-trivial, the decomposition of Ω depends on the particular choice of σ, e.g. in the case of $\mathbb{R}^{2,2}$ if we choose $\sigma_1 = \mathbf{e}_1\mathbf{e}_2$, we end up with $\Omega_\sigma^{2,2} \cong \mathbb{B}$, while $\sigma_1 = \mathbf{e}_1\mathbf{e}_3$ yields $\Omega_\sigma^{2,2} \cong \mathbb{C}'\oplus\mathbb{C}' \cong \mathbb{R}^4$ instead.

Finally, we consider the isotropic case pointing out that the Lorentzian signature $(1, n-1)$ allows for only one null vector in each blade. Indeed, denoting $\sigma_{ij} = (\mathbf{e}_0 + \mathbf{e}_i)(\mathbf{e}_0 + \mathbf{e}_j)$ one finds that $\sigma_{ij}^2 \in \mathbb{R} \Leftrightarrow i = j$. Moreover, in the orthogonal decomposition of a bivector into blades, only one of the components may have a null direction, otherwise the sum is reducible. Similarly, in $\mathbb{R}^{p,q}$ one cannot have more than q null directions in each blade and no null direction is repeated in the same blade expansion. For instance, in the usual Minkowski space $\mathbb{R}^{1,3}$, we may choose $\sigma = (\mathbf{e}_0 + \mathbf{e}_1)\mathbf{e}_2$ which yields $\sigma^\star = (\mathbf{e}_0 + \mathbf{e}_1)\mathbf{e}_3$ and since the two mutually perpendicular planes share a common direction, we can no longer have an orthogonal decomposition in the form $\mathbb{R}^{1,3} \cong \{\sigma\}\times\{\sigma^\star\}$ that leads to an effect known as *light cone singularity*. On the other hand, the sum of σ and $\sigma^\star$ is also decomposable[9] and therefore

$$\begin{aligned} \hat{\sigma}^2 : \quad \xi \longrightarrow (\sigma \wedge (\sigma \wedge \xi)^\star)^\star &= (-1)^{M'} \sigma \cdot (\sigma \wedge \xi) = \\ (-1)^{M'} \sigma (\sigma\, \xi - \sigma \cdot \xi) &= (-1)^{M'+1} \mathcal{P}_{\parallel}^{\circ} \xi \end{aligned} \tag{6.11}$$

where we use the fact that $\sigma^2 = 0$ and let $\mathcal{P}_{\parallel}^{\circ}$ denote the linear operator that extracts the parallel to σ component of the vector ξ with scale quadratic with respect to σ. We cannot use the term projector here, as $\mathcal{P}_{\parallel}^{\circ}$ is nilpotent and moreover, mutually annihilates with $\hat{\sigma}$. Thus, each isotropic direction in the blade expansion for σ yields an algebra $\mathcal{N}_i$ of the type $\Omega_{null}^{\mathbb{C}}$ described in Section 4 with $\ell_i^3 = 0$. These algebras share a common identity and their nilpotent parts commute - they come in the form of a cluster, rather than a direct sum. The process may be seen as "contraction" of Wigner little groups, i.e., a transition between the complex (time-like) and the split-complex (space-like) state. Such an approach is widely used in the physical literature [3]. Thus, formula (6.9) extends to (here we use that $\mathbb{C}' \cong \mathbb{R} \oplus \mathbb{R}$)

Theorem 6.3. *The generalized cross product in $\mathbb{R}^{p,q}$ yields for $n = p + q$ odd*

$$\Omega^{p+q} \cong \mathcal{N}_{2k+1} \oplus \mathbb{R}^{2l} \oplus \mathbb{C}^m, \qquad 2(k + l + m) = p + q - 1 \tag{6.12}$$

and for the even one, respectively

$$\Omega^{p+q} \cong \mathcal{N}_{2k} \oplus \mathbb{R}^{2l} \oplus \mathbb{C}^m, \qquad 2(k + l + m) = p + q \tag{6.13}$$

where $\mathcal{N}_{2k+1}$ denotes the unital algebra generated by $k \leq q \leq p$ mutually commuting elements ℓ_i satisfying $\ell_i^3 = 0$ associated with isotropic directions.

For instance, we have $\mathcal{N}_3 \cong \Omega_{null}$, while $\mathcal{N}_{2k} \cong \mathcal{N}_2^k$ where $\mathcal{N}_2 \cong \{\ell, \ell^2 = \varepsilon\}$ with $\ell^3 = 0$ is purely nilpotent and decomposable. In other words, the above hypercomplex algebras correspond to a decomposition with respect to l space-like, m time-like and k null directions in $\mathbb{R}^{p,q}$. In particular, for $k = 0$ one has $\mathcal{N}_1 \cong \mathbb{R}$ and $\mathcal{N}_0 \cong \emptyset$, so we finally end up with a regular decomposition of generalized boost and rotation components, while for $k = l = 0$ one obtains formula (6.9). In the complex setting we substitute $\mathbb{R}$ with $\mathbb{C}$ and $\mathbb{C}$ with $\mathbb{B}$, while the $\mathcal{N}$-components emerge naturally for isotropic vectors and have complex dimension $2k+1$ and $2k$, respectively (e.g. $\mathcal{N}_1^{\mathbb{C}} \cong \mathbb{C}$ and $\mathcal{N}_3^{\mathbb{C}} \cong \Omega_{null}^{\mathbb{C}}$).

[9] allowing rotation in $\{\mathbf{e}_2, \mathbf{e}_3\}$ yields an action of the *front form Wigner little group*.

Final Remarks

The study presented above is inspired by an idea proposed in [12] but has evolved in a completely different direction. It is proposed mainly as a tool for more advanced research in geometry and physics via hypercomplex analysis.

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