Quasilinear Schrödinger-Poisson system with exponential and logarithmic nonlinearities

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Abstract: In this paper, we consider the following quasilinear Schrödinger-Poisson system with exponential and logarithmic nonlinearities

$$\begin{cases} -\Delta u + \phi u = |u|^{p-2}u \log |u|^2 + \lambda f(u), & \text{in } \Omega, \\ -\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $4 < p < +\infty$, $\varepsilon, \lambda > 0$ are parameters, $\Delta_4 \phi = \operatorname{div}(|\nabla \phi|^2 \nabla \phi), \Omega \subset \mathbb{R}^2$ is a bounded domain and f has exponential critical growth. By adapting the reduction argument and a truncation technique, we prove for every $\varepsilon > 0$, the above system admits at least one pair of nonnegative solutions $(u_{\varepsilon,\lambda}, \phi_{\varepsilon,\lambda})$ for $\lambda > 0$ large. Furthermore, we research the asymptotical behavior of solutions with respect to the parameters ε and λ. The novelty of this system is the intersection among the quasilinear term, logarithmic term, and exponential critical term. These results are new and improve some existing results in the literature.

Keywords: Quasilinear Schrödinger-Poisson system; Logarithmic nonlinearity; Exponential critical growth; Truncation scheme; Variational methods

Mathematics Subject Classification: 35J10, 35J25, 35J60.

1 Introduction

In this article, we are devoted to studying the following quasilinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \phi u = |u|^{p-2}u \log |u|^2 + \lambda f(x, u), & \text{in } \Omega, \\ -\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

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where $4 < p < +\infty$, $\varepsilon, \lambda > 0$ are parameters, $\Delta_4\phi = \text{div}(|\nabla\phi|^2\nabla\phi), \Omega \subset \mathbb{R}^2$ is a bounded domain and f satisfies the following assumptions:
(f_1) $f(s) \in C(\mathbb{R}, \mathbb{R})$ and $f(s) \equiv 0$ for $s \leq 0$;
(f_2) $\lim\limits_{s\to 0} \frac{f(s)}{s} = 0$;
(f_3) $0 < pF(s) \leq sf(s)$ for all $s > 0$, where $F(s) := \int_0^s f(t)dt$;
(f_4) there exists $\alpha_0 > 0$ such that

$$\lim_{s\to\infty} \frac{f(s)}{\exp(\alpha s^2)} = 0 \text{ for } \alpha > \alpha_0 \text{ and } \lim_{s\to\infty} \frac{f(s)}{\exp(\alpha s^2)} = \infty \text{ for } \alpha < \alpha_0;$$

(f_5) there exist $\theta > 2$ and $M > 0$ such that

$$\inf_{|s|\geq M} f(s)s \geq |s|^\theta.$$

One example of a function that satisfies the above conditions is

$$f(s) = \begin{cases} |s|^{\theta-2}s\exp(\alpha_0 s^2), & \text{for } s \geq 0, \\ 0, & \text{for } s < 0. \end{cases}$$

We establish the existence and asymptotical behavior of the nonnegative solutions by using variational methods, reduction argument, and truncation technique.

Problem $(\mathcal{P}_\varepsilon)$ is the planar version of the following quasilinear Schrödinger-Poisson system

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u + (V + \phi(x))u, & x \in \mathbb{R}^3, \\ -\text{div}[\varepsilon(\nabla\phi)\nabla\phi] = |u|^2 - n^*, & x \in \mathbb{R}^3, \\ u(x, 0) = u(x), & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\varepsilon \geq 0$, V denotes a real effective potential function which does not depend on time t, n^* represents a dopant-density and the charge density $n(x, t)$ is arising from the Schrödinger wave function $u(x, t)$ by $n(x, t) = |u(x, u)|^2$. System (1.1) appears when to study a quantum mechanical model of extremely small devices in semiconductor nanostructures as considering quantum structure and the longitudinal field oscillations during the beam propagation. This is reflected by the fact that the dielectric permittivity depends on the electric field by

$$c_{\text{diel}}(\nabla\phi) = 1 + \varepsilon^4|\nabla\phi|^2, \ \varepsilon > 0.$$

For more physical background about this system, readers can refer to [16, 17, 24] and the references within.

The study of planar nonlocal problems $(\mathcal{P}_\varepsilon)$ is much less understood and remained for a long time an open field of investigation. The main problem here is that the nonlocal term exhibits several mathematical differences due to the presence of the quasilinear term. The methods dealing with $(\mathcal{P}_\varepsilon)$ are different from those handling the Schrödinger-Poisson system in $\mathbb{R}^3$ (see e.g. [9, 23, 29]). As far as authors' knowledge, few papers deal with the system $(\mathcal{P}_\varepsilon)$ or a similar one. Here we cite [17] where authors considered periodic boundary conditions on the unit cube $Q := [0,1]^N, N = 1,2,3$ and obtained global existence and uniqueness of solutions. Benmilh and Kavian [7] was devoted to studying the problem in the whole space $\mathbb{R}^3$ and obtained ground state solutions as well as asymptotic behavior. [11] concerned with an asymptotically linear nonlinearity in $\mathbb{R}^3$ and authors verified the existence and asymptotical behavior. Li and Yang [20] researched the initial boundary value problem under one-dimension case and obtained existence and uniqueness of a globally mild solution. For our scope, we shall mention [13], in which Figueiredo and Siciliano were concerned with the following problem under two dimensional case

$$\begin{cases} -\Delta u + \phi u = f(u), & \text{in } \Omega, \\ -\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where f is assumed to be an exponential critical nonlinearity, they obtained the existence of the solutions as well as the asymptotical behavior with respect to the parameter ε. Later, Figueiredo and Siciliano in [12] extended the bounded domain case to $\mathbb{R}^3$ and proved the existence of solutions for quasilinear Schrödinger-Poisson system. In [26], we extended these results in [12] and considered Schrödinger-Poisson system with double quasilinear terms.

Recently, the logarithmic Schrödinger equation given by

$$i\frac{\partial \Phi}{\partial t} = -\Delta\Phi - \Phi \log|\Phi|^2, \ N \geq 3 \tag{NLS}$$

where $\Phi : [0,+\infty)\times\mathbb{R}^N \to \mathbb{C}$, has also obtained special attention due to its physical influence, such as quantum mechanics, quantum optics, nuclear physics, effective quantum and Bose-Einstein condensation (see [37]). Standing wave solutions for (NLS) have the ansatz form $\Phi(t,x) = u(x)e^{-i\omega t}$, where $\omega \in \mathbb{R}$, which leads us to a system that

$$-\Delta u + \omega(x)u = u\log u^2, \ \text{in } \mathbb{R}^N. \tag{1.3}$$

From the mathematic point of view, (1.3) is very interesting because it arises many difficulties when applying the variational methods ([4, 14, 22, 28, 31, 35]) to find a solution. In fact, the associated energy functional of (1.3) would formally be the following form

$$\tilde{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \omega(x)|u|^2)dx - \int_{\mathbb{R}^N} F(u)dx,$$

where

$$F(t) = \int_0^t s\log s^2 ds = -\frac{t^2}{2} + \frac{t^2\log t^2}{2}, \ \text{for all } t \in \mathbb{R}.$$

In the last years, researchers have developed several techniques to solve Problem (1.3) or similar one. We cite [8] where authors worked in a suitable Banach space endowed with a Luxemburg type norm, in this way, the functional $\tilde{J}$ is well defined and C^1 smooth; [15] where the authors penalized the nonlinearity around the origin and tried to obtain a prior estimates to get a nontrivial solution at the limit; [5, 6] where the authors used the non-smooth critical point theory introduced in [10] to obtain the existence and multiplicity of solutions; [32] where authors tried to construct solutions of (1.3) through spatially $2L$-periodic solutions; [1, 2, 3, 18, 27, 30] where authors decomposed $\tilde{J}$ into the sum of a C^1 functional and a convex lower semicontinuous functional, and applied the minimax principles for lower semicontinuous functionals to obtain solutions; [21, 34, 36] where authors considered the sign-changing solutions with logarithmic nonlinearity in a bounded domain.

The motivation of this paper is derived from [13, 36], as the fact that the quasilinear Schrödinger-Poisson system with mixed type, that is, exponential critical and logarithmic nonlinearities, has not been investigated before. The novelty of this kind of problem is the intersection among the logarithmic term, quasilinear term, and exponential term, which will influence the structure of energy functional. Our results give a better understanding on the quasilinear problem, especially under two-dimension case. The main difficulties are related to the facts: (i) the 'fourth" order term loses homogeneity property and it prevents us from using the standard procedure to prove the boundedness of (PS) sequence; (ii) the logarithmic term may be sign-changing which also brings new obstacles; (iii) the exponential critical growth will lead to a lack of compactness.

Define

$$E = H_0^1(\Omega) \cap W_0^{1,4}(\Omega)$$

which is a Banach space endowed with the norm

$$||\phi||_E = ||\nabla\phi||_2 + ||\nabla\phi||_4.$$

Moreover, one can see that $E \hookrightarrow L^\infty(\Omega)$ (see [13]).

By a solution of $(\mathcal{P}_\varepsilon)$ we mean a pair of $(u_{\varepsilon,\lambda}, \phi_{\varepsilon,\lambda}) \in H_0^1(\Omega) \times E$ such that

$$\begin{aligned}\int_\Omega \nabla u_{\varepsilon,\lambda} \nabla v dx + \int_\Omega \phi_{\varepsilon,\lambda} u_{\varepsilon,\lambda} v dx = \int_\Omega |u_{\varepsilon,\lambda}|^{p-2} u_{\varepsilon,\lambda} v \log |u_{\varepsilon,\lambda}|^2 dx \\ + \lambda \int_\Omega f(u_{\varepsilon,\lambda}) v dx, \text{ for all } v \in H_0^1(\Omega)\end{aligned} \tag{1.4}$$

and

$$\int_\Omega \nabla\phi_{\varepsilon,\lambda} \nabla\varphi dx + \varepsilon^4 \int_\Omega |\nabla\phi_{\varepsilon,\lambda}|^2 \nabla\phi_{\varepsilon,\lambda} \nabla\varphi dx = \int_\Omega u_{\varepsilon,\lambda}^2 \varphi dx, \text{ for all } \varphi \in E. \tag{1.5}$$

Now, we are in the position to state our main results.

Theorem 1.1. *Assume that $(f_1)-(f_5)$ hold. Then there exists $\Lambda > 0$ such that for all $\lambda \geq \Lambda$, $(\mathcal{P}_\varepsilon)$ admits at least a pair of nonnegative solutions $(u_{\varepsilon,\lambda}, \phi_{\varepsilon,\lambda}) \in H_0^1(\Omega) \times E$. Moreover, for each $\varepsilon > 0$, there hold*

1. $\lim\limits_{\lambda\to\infty} ||u_{\varepsilon,\lambda}|| = 0$;

2. $\lim_{\lambda\to\infty} ||\phi_{\varepsilon,\lambda}||_E = 0$;

3. $\lim_{\lambda\to\infty} ||\phi_{\varepsilon,\lambda}||_\infty = 0$.

Theorem 1.2. *Assume that $(f_1) - (f_5)$ hold. For each fixed $\tilde{\lambda} \geq \Lambda$, let $(u_{\varepsilon,\tilde{\lambda}}, \phi_{\varepsilon,\tilde{\lambda}})$ be the pair of solutions of $(\mathcal{P}_\varepsilon)$ given by Theorem 1.1. Then as $\varepsilon \to 0^+$,*

$$u_{\varepsilon,\tilde{\lambda}} \to u_{0,\tilde{\lambda}} \text{ in } H_0^1(\Omega) \text{ and } \phi_{\varepsilon,\tilde{\lambda}} \to \phi_{0,\tilde{\lambda}} \text{ in } E,$$

where $(u_{0,\tilde{\lambda}}, \phi_{0,\tilde{\lambda}}) \in H_0^1(\Omega) \times E$ is a pair of solutions of the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \phi u = |u|^{p-2}u \log |u|^2 + \tilde{\lambda} f(u), & \text{in } \Omega, \\ -\Delta \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The main procedure of proving Theorem 1.1 and 1.2 can be summarized as follows:

(i) To apply variational methods to prove the main results, we consider the energy functional $\mathcal{I}_{\varepsilon,\lambda}(u,\phi)$ of $(\mathcal{P}_\varepsilon)$ (see (2.4)). However, due to the strong indefiniteness of $\mathcal{I}_{\varepsilon,\lambda}$, we adopt a reduction method to transfer $\mathcal{I}_{\varepsilon,\lambda}(u,\phi)$ into a single variable functional $I_{\varepsilon,\lambda}(u)$.

(ii) Due to the presence of the "fourth" order term, it is not easy to verify the boundedness of (PS) sequence. To do this, we introduce a suitable truncated functional $I^T_{\varepsilon,\lambda}$ depending on $T > 0$. In such way, we can prove every (PS) sequence of $I^T_{\varepsilon,\lambda}$ is bounded and it is actually the (PS) sequence of $I_{\varepsilon,\lambda}$.

(iii) Because we consider the exponential critical growth, (PS) condition is hard to verify. The key point is to show the behavior of mountain pass level $c_{\varepsilon,\lambda}$ with respect to λ, see Lemma 3.3. With this property in hand, we can obtain the compactness results.

Remark 1.1. *Due to the specialty of the nonlinearities, it is crucial to give estimates of the nonlinearities. For the indefinite logarithmic term, we make use of an elementary inequality (see (3.8)) and for exponential critical growth term, the Trudinger-Moser inequality (see Lemma 2.3) plays an important role.*

The rest of this paper is organized as follows. In section 2, we show the variational framework and introduce the reduction procedure. In section 3, we give the proof of Theorem 1.1. Section 4 is devoted to proving Theorem 1.2.

Throughout this paper, we make use of the following notations:

- $H_0^1(\Omega)$ denotes the usual Sobolev space endowed with the scalar product and norm given by

$$(u,v) = \int_\Omega \nabla u \nabla v dx, \ ||u|| = (u,u)^{1/2}, \ \text{ for any } u, v \in H_0^1(\Omega).$$

- For $1 \leq s < \infty$, $L^s(\Omega)$ is the usual Lebesgue space with the norm

$$||u||_s = \left(\int_\Omega |u|^s dx \right)^{\frac{1}{s}}$$

and $||\cdot||_\infty$ denotes L^∞-norm.

- We use " $\rightarrow$ " and " $\rightharpoonup$ " to denote the strong and weak convergence in the related function space respectively.

- For any $x_0 \in \Omega$ and $R > 0$, $B_R(x_0)$ denotes the ball centered at x_0 with radius R and $|\Omega|$ denotes the Lebesgue measure of Ω.

- $C, C_1, C_2, \cdots$ represent positive constants which may change from lines to lines and $o_n(1)$ denotes the quantity that tends to 0 as $n \to \infty$.

2 Preliminaries

We start this section by considering the single equation

$$\begin{cases} -\Delta\phi - \beta\Delta_4\phi = \delta \ (\beta > 0), & \text{in } \Omega \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

This kind of equation involves the (p, q)-Laplacian and has been studied broadly. Set $\beta = \varepsilon^4$, $\delta = u^2 \in E^*$ (E^* denotes the dual space of E), then (2.1) will become the following form

$$\begin{cases} -\Delta\phi - \varepsilon^4\Delta_4\phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Define

$$\Psi(\phi) = \frac{1}{2}\int_\Omega |\nabla\phi|^2 dx + \frac{\varepsilon^4}{4}\int_\Omega |\nabla\phi|^4 dx - \int_\Omega \phi u^2 dx.$$

Using classical PDE theory, one can know $\Psi(\phi)$ has a unique critical point $\phi_\varepsilon := \phi_\varepsilon(u)$, that says, (2.2) possesses the unique solution ϕ_ε satisfying

$$\int_\Omega |\nabla\phi_\varepsilon|^2 dx + \varepsilon^4 \int_\Omega |\nabla\phi_\varepsilon|^4 dx = \int_\Omega \phi_\varepsilon u^2 dx. \tag{2.3}$$

The following lemmas gives the main properties of ϕ_ε.

Lemma 2.1. *([13]) If $u_n \to u$ in $L^q(\Omega)$ for $q \in [1, +\infty)$, then there hold*

(i) $\lim\limits_{n\to+\infty} \int_\Omega |\nabla\phi_\varepsilon(u_n)|^2 dx = \int_\Omega |\nabla\phi_\varepsilon(u)|^2 dx;$

(ii) $\lim\limits_{n\to+\infty} \int_\Omega |\nabla\phi_\varepsilon(u_n)|^4 dx = \int_\Omega |\nabla\phi_\varepsilon(u)|^4 dx;$

(iii) $\lim_{n\to+\infty} \phi_\varepsilon(u_n) = \phi_\varepsilon(u)$ *in* $L^\infty(\Omega)$;

(iv) $\lim_{n\to+\infty} \int_\Omega \phi_\varepsilon(u_n)u_n^2 dx = \int_\Omega \phi_\varepsilon(u)u^2 dx.$

Lemma 2.2. *([7]) If $u_\varepsilon \to u$ in $L^{\frac{6}{5}}(\Omega)$ as $\varepsilon \to 0^+$, then*

$$\phi_\varepsilon(u_\varepsilon) \to \phi_0(u) \text{ in } H_0^1(\Omega) \text{ and } \varepsilon\phi_\varepsilon(u_\varepsilon) \to 0 \text{ in } W_0^{1,4}(\Omega).$$

It is standard to see the energy functional of $(\mathcal{P}_\varepsilon)$ can be defined by

$$\begin{aligned}\mathcal{I}_{\varepsilon,\lambda}(u,\phi) =& \frac{1}{2}||u||^2 + \frac{1}{2}\int_\Omega \phi u^2 dx + \frac{2}{p^2}\int_\Omega |u|^p dx - \frac{1}{p}\int_\Omega |u|^p \log|u|^2 dx \\ &- \lambda\int_\Omega F(u)dx - \frac{1}{4}\int_\Omega |\nabla\phi|^2 dx - \frac{\varepsilon^4}{8}\int_\Omega |\nabla\phi|^4 dx.\end{aligned} \tag{2.4}$$

Moreover, one can check $\mathcal{I}_{\varepsilon,\lambda} \in C^1(H_0^1(\Omega)\times E, \mathbb{R})$. Note that $\mathcal{I}_{\varepsilon,\lambda}(u,\phi)$ possesses two variables which bring more obstacles. Inspired by [12, 13], in the sequel, we will adopt a reduction method which can transfer $\mathcal{I}_{\varepsilon,\lambda}$ to a single variable functional.

Define the following map

$$\Phi_\varepsilon : u \in H_0^1(\Omega) \to \phi_\varepsilon(u) \in E \tag{2.5}$$

and $\mathcal{G}(\Phi_\varepsilon)$ denotes the graph of the map Φ_ε. According to the classical implicit function theorem, we have

$$\mathcal{G}(\Phi_\varepsilon) = \{(u,\phi) \in H_0^1(\Omega) \times E : \partial_\phi \mathcal{I}_{\varepsilon,\lambda}(u,\phi) = 0\}.$$

Consequently, the functional $\mathcal{I}_\varepsilon(u,\phi)$ can be reduced to

$$\begin{aligned}I_{\varepsilon,\lambda}(u) :=& \mathcal{I}_{\varepsilon,\lambda}(u,\Phi_\varepsilon(u)) \\ =& \frac{1}{2}||u||^2 + \frac{2}{p^2}\int_\Omega |u|^p dx + \frac{1}{4}\int_\Omega |\nabla\phi_\varepsilon|^2 dx + \frac{3\varepsilon^4}{8}\int_\Omega |\nabla\phi_\varepsilon|^4 dx \\ &- \frac{1}{p}\int_\Omega |u|^p \log|u|^2 dx - \lambda\int_\Omega F(u)dx.\end{aligned}$$

Note that $I_{\varepsilon,\lambda} \in C^1(H_0^1(\Omega), \mathbb{R})$ and for any $v \in C_0^\infty(\Omega)$,

$$\begin{aligned}I'_{\varepsilon,\lambda}(u)v &= \partial_u \mathcal{I}_{\varepsilon,\lambda}(u,\phi_\varepsilon(u))v + \partial_\phi \mathcal{I}_{\varepsilon,\lambda}(u,\phi_\varepsilon(u)) \circ \Phi'_\varepsilon(u)v \\ &= \partial_u \mathcal{I}_{\varepsilon,\lambda}(u,\phi_\varepsilon(u))v,\end{aligned}$$

which yields,

$$I'_{\varepsilon,\lambda}(u)v = \int_\Omega \nabla u\nabla v dx + \int_\Omega \phi_\varepsilon(u)uv dx - \int_\Omega |u|^{p-2}uv\log|u|^2 dx - \lambda\int_\Omega f(u)v dx.$$

As a consequence, $(\mathcal{P}_\varepsilon)$ can be reduced to the following problem

$$\begin{cases} -\Delta u + \phi_\varepsilon u = |u|^{p-2}u\log|u|^2 + \lambda f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{Q}$$

Remark 2.1. *Let λ, ε be fixed, then the following two statements are equivalent:*

(i) $(u_{\varepsilon,\lambda}, \phi_{\varepsilon,\lambda}) \in H_0^1(\Omega) \times E$ *is a pair of critical points of* $\mathcal{I}_{\varepsilon,\lambda}$,

(ii) $u_{\varepsilon,\lambda}$ *is a critical point of* $I_{\varepsilon,\lambda}$ *and* $\phi_{\varepsilon,\lambda} = \phi_\varepsilon(u_{\varepsilon,\lambda})$.

For the sake of simplicity in the presentation, we introduce the following functional

$$J_\varepsilon : u \in H^1(\Omega) \to \frac{1}{4}\int_\Omega |\nabla \phi_\varepsilon(u)|^2 dx + \frac{3\varepsilon^4}{8}\int_\Omega |\nabla \phi_\varepsilon(u)|^4 dx.$$

Thus we get

$$I_{\varepsilon,\lambda}(u) = \frac{1}{2}||u||^2 + \frac{2}{p^2}\int_\Omega |u|^p dx + J_\varepsilon(u) - \frac{1}{p}\int_\Omega |u|^p \log |u|^2 dx - \lambda \int_\Omega F(u) dx.$$

Remark 2.2. *By above analysis, we know that* $J_\varepsilon \in C^1(H_0^1(\Omega), \mathbb{R})$ *and for any* $v \in H_0^1(\Omega)$,

$$\frac{d}{dt} J_\varepsilon(tv) = J_\varepsilon'(tv)v = t\int_\Omega \phi_\varepsilon(tv) v^2 dx.$$

We shall end this section by introducing the Trudinger-Moser inequality which plays a crucial role in the proofs of main results.

Lemma 2.3. *([25, 33]) If* $\alpha > 0$ *and* $u \in H_0^1(\Omega)$, *then*

$$\int_\Omega e^{\alpha |u|^2} dx < \infty.$$

Moreover, if $\alpha < 4\pi$, *then there exists* $C = C(\alpha, \Omega) > 0$ *such that*

$$\sup_{||u|| \leq 1} \int_\Omega e^{\alpha |u|^2} dx \leq C.$$

3 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. Firstly, we shall use a truncation method to overcome the growth of "fourth" order in $I_{\varepsilon,\lambda}$.

3.1 The truncated functional

Let $\eta \in C_0^\infty(\mathbb{R}^+, [0,1])$ be a cut-off function defined by

$$\begin{cases} \eta(t) = 1, \ \text{if } t \in [0,1], \\ 0 \leq \eta(t) \leq 1, \ \text{if } t \in (1,2), \\ \eta(t) = 0, \ \text{if } t \in [2, +\infty), \\ \eta' \leq 0, \ |\eta'|_\infty \leq 2. \end{cases} \tag{3.1}$$

For every $T > 0$, we define $k_T(u) = \eta(\frac{||u||^2}{T^2})$ and the truncated functional $I^T_{\varepsilon,\lambda} : H^1_0(\Omega) \to \mathbb{R}$ is given by

$$I^T_{\varepsilon,\lambda}(u) = \frac{1}{2}||u||^2 + \frac{2}{p^2}\int_\Omega |u|^p dx + k_T(u)J_\varepsilon(u) - \frac{1}{p}\int_\Omega |u|^p \log|u|^2 dx - \lambda\int_\Omega F(u)dx. \quad (3.2)$$

Observe that $I^T_{\varepsilon,\lambda}(u) \in C^1(H^1_0(\Omega), \mathbb{R})$ and for $v \in C^\infty_0(\Omega)$,

$$\begin{aligned}(I^T_{\varepsilon,\lambda})'(u)v = &\int_\Omega \nabla u \nabla v dx + k_T(u)\int_\Omega \phi_\varepsilon(u)uv dx + \frac{2}{T^2}\eta'(\frac{||u||^2}{T^2})J_\varepsilon(u)\int_\Omega \nabla u\nabla v dx \\ &- \int_\Omega |u|^{p-2}uv\log|u|^2 dx - \lambda\int_\Omega f(u)v dx.\end{aligned} \quad (3.3)$$

In the sequel, we prove that the functional $I^T_{\varepsilon,\lambda}$ satisfies the Mountain Pass geometry uniformly in ε and λ.

Lemma 3.1. *Assume that $(f_1) - (f_5)$ hold. Then*

(i) *for every $\lambda > 0$ fixed, there exist $\delta_\lambda, \rho_\lambda > 0$ such that for any $T > 0$ and $\varepsilon > 0$,*

$$I^T_{\varepsilon,\lambda}(u) \geq \rho_\lambda, \text{ for all } ||u|| = \delta_\lambda;$$

(ii) *for every $T > 0$ fixed, there exists $e_T \in H^1_0(\Omega)$ with $||e_T|| > \delta_\lambda$ (given in* **(i)***) such that for any $\varepsilon > 0$ and $\lambda > 1$,*

$$I^T_{\varepsilon,\lambda}(e_T) < 0.$$

Proof. **(i)** From $(f_1), (f_2)$ and (f_4), for each $\epsilon > 0$ and $\alpha > \alpha_0$, there exists $C_\epsilon > 0$ such that for all $t \in \mathbb{R}$ and $q > 0$,

$$f(t)t \leq \epsilon|t|^2 + C_\epsilon|t|^q \exp(\alpha|t|^2). \quad (3.4)$$

Moreover, since $4 < p < \infty$, for given $\epsilon > 0$, there exists $\tilde{C}_\epsilon > 0$ such that for all $r > p$ and $t \in \mathbb{R}$,

$$|t|^{p-1}\log|t|^2 \leq \epsilon|t|^3 + \tilde{C}_\epsilon|t|^{r-1}. \quad (3.5)$$

For each $\lambda > 0$, choose $\epsilon > 0$ sufficiently small. Letting $q > 2$ in (3.4) and gathering (3.5), Hölder inequality with Sobolev embedding theorem, we obtain

$$\begin{aligned}I^T_{\varepsilon,\lambda}(u) \geq &\frac{1}{2}\int_\Omega |\nabla u|^2 dx - \epsilon\int_\Omega |u|^4 dx - \tilde{C}_\epsilon\int_\Omega |u|^r dx \\ &- \lambda\epsilon\int_\Omega |u|^2 dx - C_\epsilon\lambda\int_\Omega |u|^q \exp(\alpha|u|^2)dx \\ \geq &C_1||u||^2 - C_2||u||^4 - C_3||u||^r \\ &- C_\epsilon\lambda\left(\int_\Omega |u|^{2q}dx\right)^{\frac{1}{2}}\left(\int_\Omega \exp(2\alpha||u||^2\frac{u^2}{||u||^2})dx\right)^{\frac{1}{2}} \\ \geq &C_1||u||^2 - C_2||u||^4 - C_3||u||^r \\ &- C_4\lambda||u||^q\left(\int_\Omega \exp(2\alpha||u||^2\frac{u^2}{||u||^2})dx\right)^{\frac{1}{2}}.\end{aligned} \quad (3.6)$$

Taking $||u|| = \tilde{\delta}_\lambda > 0$ small such that $2\alpha\tilde{\delta}_\lambda^2 < 4\pi$, then from Lemma 2.3, one can yield

$$I^T_{\varepsilon,\lambda}(u) \geq C_1\tilde{\delta}_\lambda^2 - C_2\tilde{\delta}_\lambda^4 - C_3\tilde{\delta}_\lambda^r - \tilde{C}_4\lambda\tilde{\delta}_\lambda^q > 0.$$

Therefore, there exists $\rho_\lambda > 0$ such that for all $||u|| = \delta_\lambda \in (0, \tilde{\delta}_\lambda)$,

$$I^T_{\varepsilon,\lambda}(u) \geq \rho_\lambda.$$

(ii) Let $T > 0$ be fixed. For any $v \in H_0^1(\Omega)$ with $v \neq 0$ and $t \geq 0$, by the definition of k_T, we know that

$$\lim_{t\to+\infty} k_T(tv) = 0. \tag{3.7}$$

Recall the element inequality

$$2t^p - pt^p \log t^2 \leq 2 \text{ for all } t \in (0, \infty). \tag{3.8}$$

By (f_5), one can easily verify that there exists $\bar{C} > 0$ such that for some $\theta > 2$ and $M > 0$,

$$\inf_{|t|\geq M} F(t) \geq \bar{C}|t|^\theta. \tag{3.9}$$

Then for each $\varepsilon > 0$ and $\lambda > 1$, it follows from (3.7)-(3.9) that as $t \to +\infty$,

$$\begin{aligned} I^T_{\varepsilon,\lambda}(tv) &= \frac{t^2}{2}||v||^2 + \frac{2}{p^2}\int_\Omega |tv|^p dx + k_T(tv)J_\varepsilon(tv) - \frac{1}{p}\int_\Omega |tv|^p \log|tv|^2 dx - \lambda\int_\Omega F(tv)dx \\ &\leq \frac{t^2}{2}||v||^2 + \frac{2}{p^2}|\Omega| - \bar{C}t^\theta||v||_\theta^\theta \to -\infty, \end{aligned} \tag{3.10}$$

where we use the fact $\theta > 2$. Let $t^* > 0$ and define a path $h : [0, 1] \to H_0^1(\Omega)$ by $h(t) = t(t^*v)$. For $t^* > 0$ large enough, we can have

$$\int_\Omega |\nabla h(1)|^2 dx > \delta_\lambda^2 \text{ and } I^T_{\varepsilon,\lambda}(h(1)) < 0,$$

where δ_λ is given in **(i)**. Setting $e^T_\lambda = h(1)$, then the result follows. □

Remark 3.1. *Here we want to emphasize the independence of parameters, that is, δ_λ and ρ_λ in* **(i)** *do not depend on T, ε and e_T in* **(ii)** *does not depend on λ, ε.*

Applying the Mountain Pass Theorem ([35]), we know that for every $T > 0, \lambda > 1$ and $\varepsilon > 0$, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$, which depends on T, λ and ε, such that as $n \to \infty$,

$$I^T_{\varepsilon,\lambda}(u_n) \to c^T_{\varepsilon,\lambda} > 0 \text{ and } (I^T_{\varepsilon,\lambda})'(u_n) \to 0,$$

where

$$c^T_{\varepsilon,\lambda} := \inf_{\gamma\in\Gamma^T_\lambda} \max_{t\in[0,1]} I^T_{\varepsilon,\lambda}(\gamma(t)) > 0 \tag{3.11}$$

and

$$\Gamma^T_\lambda := \{\gamma \in C([0, 1], H_0^1(\Omega))|\ \gamma(0) = 0, \gamma(1) = e_T\}.$$

Define the path

$$\gamma_* : t \in [0, 1] \to te_T \in H_0^1(\Omega). \tag{3.12}$$

It is easy to see that $\gamma_* \in \bigcap_{\lambda>1,\varepsilon>0} \Gamma^T_\lambda$.

3.2 Estimation of $c^T_{\varepsilon,\lambda}$ and Boundedness of $(PS)_{c^T_{\varepsilon,\lambda}}$ sequence

In the sequel, we shall prove that every $(PS)_{c^T_{\varepsilon,\lambda}}$ sequence of $I^T_{\varepsilon,\lambda}$ is bounded in $H^1_0(\Omega)$. To do this, we need to estimate the value of $c^T_{\varepsilon,\lambda}$. Firstly, we give the following crucial lemma.

Lemma 3.2. *Suppose that $(f_1)-(f_5)$ hold. Let $T>0, \lambda>1$ and $\varepsilon>0$ be fixed, then for every $w\in H^1_0(\Omega)\setminus\{0\}$, the function*

$$t\in[0,+\infty)\to I^T_{\varepsilon,\lambda}(tw)\in\mathbb{R}$$

has a positive global maximum point, denoted by $t^T_\lambda(w)$, which is independent of ε . Moreover, for any $T>0$, there holds

$$\lim_{\lambda\to+\infty} t^T_\lambda(w)=0.$$

Proof. Let $\lambda>1$ and $\varepsilon>0$ be fixed. For each $w\in H^1_0(\Omega)\setminus\{0\}$, similar to (3.6), let $q>2$ in (3.4) and we have

$$\begin{aligned} I^T_{\varepsilon,\lambda}(tw)\geq &C_1t^2||w||^2-C_2t^4||w||^4-C_3t^r||w||^r\\ &-C_4\lambda t^q||w||^q\left(\int_\Omega \exp(2\alpha||tw||^2\frac{|tw|^2}{||tw||^2})dx\right)^{1/2}. \end{aligned}$$

Let $t>0$ small enough such that $2\alpha||tw||^2<4\pi$, Lemma 2.3 yields that

$$I^T_{\varepsilon,\lambda}(tw)\geq C_1t^2||w||^2-C_2t^4||w||^4-C_3t^r||w||^r-\tilde{C}_4\lambda t^q||w||^q. \tag{3.13}$$

Since $r>p>4, q>2$, then we get $I^T_{\varepsilon,\lambda}(tw)>0$ for t sufficiently small.

On the other hand, for fixed $T>0$, we know that

$$\lim_{t\to+\infty} k_T(tw)=0.$$

Similar to (3.10), we can choose $t^*>0$ large such that for all $t>t^*$,

$$\begin{aligned} I^T_{\varepsilon,\lambda}(tw)&\leq \frac{t^2}{2}||w||^2+\frac{2}{p^2}|\Omega|-\bar{C}\lambda t^\theta||w||^\theta_\theta\\ &<0. \end{aligned} \tag{3.14}$$

Gathering (3.13) with (3.14), we verify the existence of $t^T_\lambda(w)$.

Now we prove $\lim_{\lambda\to+\infty} t^T_\lambda(w)=0$. For the sake of simplicity, we set $t_\lambda:=t^T_\lambda(w)$. Without loss of generality, we set $||w||=1$. Moreover, we know that for any $v\in C^\infty_0(\Omega)$,

$$(I^T_{\varepsilon,\lambda})'(t_\lambda w)v=0. \tag{3.15}$$

Taking $v=t_\lambda w$ in (3.15), then it arrives at

$$\begin{aligned} \int_\Omega |t_\lambda w|^p\log|t_\lambda w|^2dx+\lambda\int_\Omega f(t_\lambda w)t_\lambda wdx=&t^2_\lambda+t^2_\lambda\eta(\frac{||t_\lambda w||^2}{T^2})\int_\Omega \phi_\varepsilon(t_\lambda w)w^2dx\\ &+\frac{2t^2_\lambda}{T^2}\eta'(\frac{||t_\lambda w||^2}{T^2})J_\varepsilon(t_\lambda w). \end{aligned} \tag{3.16}$$

Due to $\eta' < 0$, from (3.8) and (f_5), one can deduce

$$t_\lambda^2 + t_\lambda^2 \eta(\frac{||t_\lambda w||^2}{T^2}) \int_\Omega \phi_\varepsilon(t_\lambda w) w^2 dx \geq \frac{2t_\lambda^p}{p} \int_\Omega |w|^p dx - \frac{2}{p}|\Omega| + \lambda t_\lambda^\theta ||w||_\theta^\theta. \tag{3.17}$$

Assuming that t_λ is unbounded, then there is $\lambda_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} t_{\lambda_n} = \infty$. Thus, for n large enough, from the definition of η, we obtain

$$t_\lambda^2 \geq \frac{2t_\lambda^p}{p} \int_\Omega |w|^p dx - \frac{2}{p}|\Omega| + \lambda t_\lambda^\theta ||w||_\theta^\theta,$$

which is impossible since $\theta > 2$ and $p > 4$. Therefore, up to a subsequence, there exists $\tilde{t} \geq 0$ such that

$$t_\lambda \to \tilde{t} \text{ as } \lambda \to \infty.$$

We claim $\tilde{t} = 0$. Arguing by contradiction that $\tilde{t} > 0$, from Lemma 2.1, we deduce that

$$J_\varepsilon(t_\lambda w) \to J_\varepsilon(\tilde{t}w)$$

and

$$\phi_\varepsilon(t_\lambda w) \to \phi_\varepsilon(\tilde{t}w) \text{ in } L^\infty(\mathbb{R}^3) \text{ as } \lambda \to +\infty.$$

Using (3.5) and Lebesgue theorem, we have

$$\int_\Omega |t_\lambda w|^p \log |t_\lambda w|^2 dx \to \int_\Omega |\tilde{t}w|^p \log |\tilde{t}w|^2 dx \text{ as } \lambda \to +\infty.$$

Passing to the limit as $\lambda \to \infty$ in (3.16), the above analysis yields that

$$\begin{aligned} &\int_\Omega |\tilde{t}w|^p \log |\tilde{t}w|^2 dx + \lim_{\lambda\to\infty} \lambda \int_\Omega f(t_\lambda w) t_\lambda w dx \\ &= \tilde{t}^2 + \tilde{t}^2 \eta(\frac{||\tilde{t}w||^2}{T^2}) \int_\Omega \phi_\varepsilon(\tilde{t}w) w^2 dx + \frac{2\tilde{t}^2}{T^2} \eta'(\frac{||\tilde{t}w||^2}{T^2}) J_\varepsilon(\tilde{t}w). \end{aligned} \tag{3.18}$$

which is impossible. Hence $\tilde{t} = 0$ and the proof is completed. □

In the sequel, we devote to estimating the value of $c_{\varepsilon,\lambda}^T$ defined by (3.11).

Lemma 3.3. *Assume that $(f_1) - (f_5)$ hold. Then for any $T > 0$, there holds*

$$\lim_{\lambda\to\infty} \sup_{\varepsilon>0} c_{\varepsilon,\lambda}^T = 0.$$

Proof. Let $T > 0$ be fixed. It is sufficient to verify that for any $\tau > 0$, there exists $\lambda_0 > 1$ such that for all $\lambda > \lambda_0$,

$$0 < \max_{t\in[0,1]} I_\lambda^T(\gamma_*(t)) < \tau,$$

where γ_* is given in (3.12).

Let $w \in C_0^\infty(\Omega)$ and $w \geq 0$ with $||w|| = 1$. By Lemma 3.2, there exists $t_\lambda := t_\lambda^T(w) > 0$ (independent of ε) such that $I_{\varepsilon,\lambda}^T(t_\lambda w) = \max_{t\geq 0} I_{\varepsilon,\lambda}^T(tw)$ and $\lim_{\lambda\to\infty} t_\lambda = 0$. In this perspective, we can conclude that

$$\lim_{\lambda\to\infty} \sup_{\varepsilon>0} t_\lambda = 0.$$

On the other hand, since k_T and J_ε are both continuous, then for any $\varepsilon > 0$, we have

$$\lim_{\lambda\to\infty} k_T(t_\lambda w) = 1 \text{ and } \lim_{\lambda\to\infty} J_\varepsilon(t_\lambda w) = 0.$$

Therefore for any $\tau > 0$, there exists $\lambda_0 > 1$ large such that for all $\lambda > \lambda_0$,

$$\begin{aligned} 0 < \max_{t\in[0,1]} I_{\varepsilon,\lambda}^T(\gamma_*(t)) &= I_{\varepsilon,\lambda}^T(t_\lambda w) \\ &= \frac{t_\lambda^2}{2}||w||^2 + \frac{2}{p^2}\int_\Omega |t_\lambda w|^p dx + k_T(t_\lambda w) J_\varepsilon(t_\lambda w) \\ &\quad - \frac{1}{p}\int_\Omega |t_\lambda w|^p \log |t_\lambda w|^2 dx - \lambda \int_\Omega F(t_\lambda w) dx \\ &< \tau. \end{aligned} \tag{3.19}$$

Thus the proof of this lemma is finished. □

Next we will show that every $(PS)_{c_{\varepsilon,\lambda}^T}$ sequence of $I_{\varepsilon,\lambda}^T$ is bounded in $H_0^1(\Omega)$.

Lemma 3.4. *Assume that $(f_1) - (f_5)$ hold. Then for fixed $T > 0$, there exists $\lambda(T) > 0$ large enough such that for all $\lambda > \lambda(T)$,*

$$\sup_{\varepsilon>0} c_{\varepsilon,\lambda}^T \leq \frac{p-2}{2p} T^2. \tag{3.20}$$

Furthermore, for given $\varepsilon > 0$, if $\{u_n\}$ is a $(PS)_{c_{\varepsilon,\lambda}^T}$ sequence of $I_{\varepsilon,\lambda}^T$, then

$$||u_n|| \leq T.$$

Proof. In view of Lemma 3.3, we can easily deduce that (3.20) holds. Since $\{u_n\}$ is a $(PS)_{c_{\varepsilon,\lambda}^T}$ sequence of $I_{\varepsilon,\lambda}^T$, that is, as $n \to \infty$,

$$I_{\varepsilon,\lambda}^T(u_n) \to c_{\varepsilon,\lambda}^T, \ (I_{\varepsilon,\lambda}^T)'(u_n) \to 0. \tag{3.21}$$

Firstly, we show that $||u_n||^2 \leq 2T^2$. If the conclusion does not hold, then there exists a subsequence of $\{u_n\} \subset H_0^1(\Omega)$, still denoted by $\{u_n\}$, satisfying

$$||u_n||^2 > 2T^2. \tag{3.22}$$

From (3.21), (3.22) and (f_3), we have

$$\begin{aligned} c^T_{\varepsilon,\lambda} + o_n(1) &= I^T_{\varepsilon,\lambda}(u_n) - \frac{1}{p}(I^T_{\varepsilon,\lambda})'(u_n)u_n \\ &\geq \frac{p-2}{2p}||u_n||^2 + \eta(\frac{||u_n||^2}{T^2})\left(J_\varepsilon(u_n) - \frac{1}{p}\int_\Omega \phi_\varepsilon(u_n)u_n^2 dx\right) \\ &\quad - \frac{2}{pT^2}\eta'(\frac{||u_n||^2}{T^2})||u_n||^2 J_\varepsilon(u_n) \\ &\geq \frac{p-2}{2p}||u_n||^2 \\ &> \frac{p-2}{p}T^2, \end{aligned}$$

where we use the facts that $\eta' \leq 0$ and $\eta(t) = 0$ for $t > 2$, which is a contradiction to (3.20).

Now we prove $||u_n||^2 \leq T^2$. Note that for $p > 4$,

$$J_\varepsilon(u_n) - \frac{1}{p}\int_\Omega \phi_\varepsilon(u_n)u_n^2 dx > 0. \tag{3.23}$$

By contradiction, if $T^2 < ||u_n||^2 \leq 2T^2$, since $\eta \geq 0$ and $\eta' \leq 0$, from (3.23) and (f_3), we can obtain

$$\begin{aligned} c^T_{\varepsilon,\lambda} + o_n(1) &= I^T_{\varepsilon,\lambda}(u_n) - \frac{1}{p}(I^T_{\varepsilon,\lambda})'(u_n)u_n \\ &\geq \frac{p-2}{2p}||u_n||^2 + \eta(2)\left(J_\varepsilon(u_n) - \frac{1}{p}\int_\Omega \phi_\varepsilon(u_n)u_n^2 dx\right) \\ &\quad - \frac{2}{pT^2}\eta'(\frac{||u_n||^2}{T^2})||u_n||^2 J_\varepsilon(u_n) \\ &\geq \frac{p-2}{2p}||u_n||^2 \\ &> \frac{p-2}{2p}T^2, \end{aligned}$$

which contradicts to (3.20). At this point, we conclude that $||u|| \leq T$. ☐

Remark 3.2. *It follows from Lemma 3.4 that for any $T > 0$ and $\varepsilon > 0$ given, there exists $\lambda(T) > 0$ such that for all $\lambda > \lambda(T)$, every $(PS)_{c^T_{\varepsilon,\lambda}}$ sequence of $I^T_{\varepsilon,\lambda}$ is bounded in $H^1_0(\Omega)$ and $||u_n|| \leq T$, thus it is actually a (PS) sequence of $I_{\varepsilon,\lambda}$ at level $c_{\varepsilon,\lambda} = c^T_{\varepsilon,\lambda}$.*

3.3 Compactness results of the functional $I_{\varepsilon,\lambda}$

Lemma 3.5. *Assume that $(f_1) - (f_5)$ hold. Then for given $T > 0$ and $\varepsilon > 0$, there exists $\tilde{\lambda}(T) > \lambda(T)$ such that for all $\lambda > \tilde{\lambda}(T)$, the functional $I_{\varepsilon,\lambda}$ satisfies the (PS) condition at the level $c_{\varepsilon,\lambda}$, where $\lambda(T)$ is given in Lemma 3.4.*

Proof. In view of Remark 3.2, we know that for all $\lambda > \lambda(T)$, $I_{\varepsilon,\lambda}$ possesses a bounded $(PS)_{c_{\varepsilon,\lambda}}$ sequence $\{u_n\} \subset H^1_0(\Omega)$, that is,

$$I_{\varepsilon,\lambda}(u_n) \to c_{\varepsilon,\lambda},\ I'_{\varepsilon,\lambda}(u_n) \to 0 \text{ as } n \to \infty \tag{3.24}$$

and $||u_n|| \leq T$ uniformly in ε. We shall prove that $\{u_n\}$ admits a convergent subsequence in $H_0^1(\Omega)$. Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, up to a subsequence, there exists $u_{\varepsilon,\lambda} \in H_0^1(\Omega)$ with $||u_{\varepsilon,\lambda}|| \leq T$ such that

$$
\begin{aligned}
&u_n \rightharpoonup u_{\varepsilon,\lambda} \text{ in } H_0^1(\Omega),\\
&u_n \to u_{\varepsilon,\lambda} \text{ in } L^s(\Omega) \text{ for } s \in [1,\infty),\\
&u_n \to u_{\varepsilon,\lambda} \text{ a.e. } x \in \Omega.
\end{aligned}
$$

We claim that

$$\int_\Omega f(u_n)u_n dx \to \int_\Omega f(u_{\varepsilon,\lambda})u_{\varepsilon,\lambda} dx. \tag{3.25}$$

It is easy to verify that $f(u_n)u_n \to f(u_{\varepsilon,\lambda})u_{\varepsilon,\lambda}$ a.e. $x \in \Omega$. To verify (3.25), it is sufficient to prove that there exists $g(x) \in L^1(\Omega)$ such that

$$|f(u_n)(x)u_n(x)| \leq g(x) \text{ a.e. } x \in \Omega.$$

Then by Lebesgue theorem, the claim holds.

Recall inequality (3.4) with $q = 1$ and $\alpha = \alpha_0 + 1$, then one gets

$$f(u_n)u_n \leq \epsilon u_n^2 + C_\epsilon |u_n| \exp\left((\alpha_0+1)u_n^2\right).$$

It is clear that $\{u_n^2\}$ is convergent in $L^1(\Omega)$, thus there exists $g_1(x) \in L^1(\Omega)$ such that, up to a subsequence,

$$u_n^2(x) \leq g_1(x) \text{ a.e. } x \in \Omega.$$

Set $h_n = |u_n| \exp\left((\alpha_0+1)u_n^2\right)$, then

$$h_n(x) \to |u_{\varepsilon,\lambda}| \exp\left((\alpha_0+1)u_{\varepsilon,\lambda}^2\right) \text{ a.e. } x \in \Omega. \tag{3.26}$$

On the other hand, by (3.23), (3.24) and (f_3), we can obtain

$$
\begin{aligned}
c_{\varepsilon,\lambda} + o_n(1) &= I_{\varepsilon,\lambda}(u_n) - \frac{1}{p} I'_{\varepsilon,\lambda}(u_n)u_n\\
&\geq \frac{p-2}{2p}||u_n||^2 + J_\varepsilon(u_n) - \frac{1}{p}\int_\Omega \phi_\varepsilon(u_n)u_n^2 dx\\
&\geq \frac{p-2}{2p}||u_n||^2.
\end{aligned} \tag{3.27}
$$

Then it follows from Lemma 3.3 and (3.27), one can choose $\tilde{\lambda}(T) > \lambda(T)$ such that for some $s \in (1,2)$,

$$\limsup_{n\to\infty} ||u_n||^2 \leq \frac{4\pi}{s(\alpha_0+1)}. \tag{3.28}$$

Lemma 2.3 and (3.28) give that

$$\int_\Omega \exp\left(s(\alpha_0+1)u_n^2\right) dx = \int_\Omega \exp\left(s(\alpha_0+1)||u_n||^2 \frac{u_n^2}{||u_n||^2}\right) dx \leq \hat{C},$$

where $\hat{C}$ does not depend on n. Note that

$$\exp\left(s(\alpha_0+1)u_n^2\right) \to \exp\left(s(\alpha_0+1)u_{\varepsilon,\lambda}^2\right) \ \text{ a.e. } x \in \Omega.$$

Then by Lemma 4.8 in [19], it holds that

$$\exp\left((\alpha_0+1)u_n^2\right) \to \exp\left((\alpha_0+1)u_{\varepsilon,\lambda}^2\right) \ \text{ in } L^s(\Omega). \tag{3.29}$$

Obviously, it also holds

$$|u_n| \to |u_{\varepsilon,\lambda}| \text{ in } L^{s'}(\Omega), \text{ where } \frac{1}{s}+\frac{1}{s'}=1. \tag{3.30}$$

In view of Hölder inequality, for above $s \in (1,2)$, we have

$$\begin{aligned}\int_\Omega h_n dx &= \int_\Omega |u_n| \exp\left((\alpha_0+1)u_n^2\right) dx \\ &\le \left(\int_\Omega |u_n|^{s'}\right)^{1/s'} \left(\int_\Omega \exp\left(s(\alpha_0+1)u_n^2\right) dx\right)^{1/s}.\end{aligned} \tag{3.31}$$

Combining (3.26), (3.29)-(3.31) with Lebegue theorem, we can obtain

$$\int_\Omega h_n dx = \int_\Omega |u_n| \exp\left((\alpha_0+1)u_n^2\right) dx \to \int_\Omega |u_{\varepsilon,\lambda}| \exp\left((\alpha_0+1)u_{\varepsilon,\lambda}^2\right) dx,$$

which yields

$$h_n \to |u_{\varepsilon,\lambda}| \exp\left((\alpha_0+1)u_{\varepsilon,\lambda}^2\right) \ \text{ in } L^1(\Omega).$$

Hence, up to a subsequence, there exists $g_2(x) \in L^1(\Omega)$ such that

$$h_n(x) = |u_n| \exp\left((\alpha_0+1)u_n^2\right) \le g_2(x) \text{ a.e. } x \in \Omega.$$

Taking $g(x) = g_1(x) + g_2(x)$, then the claim is proved.

From (3.5) and Lebesgue theorem, we know

$$\int_\Omega |u_n|^p \log |u_n|^2 dx \to \int_\Omega |u_{\varepsilon,\lambda}|^p \log |u_{\varepsilon,\lambda}|^2 dx. \tag{3.32}$$

Since $I'_{\varepsilon,\lambda}(u_n)u_n = o_n(1)$ and $I'_{\varepsilon,\lambda}(u_{\varepsilon,\lambda})u_{\varepsilon,\lambda} = 0$, then

$$\begin{aligned}&\int_\Omega |\nabla u_n|^2 dx + \int_\Omega \phi_\varepsilon(u_n)u_n^2 dx - \int_\Omega |u_n|^p \log |u_n|^2 dx - \lambda \int_\Omega f(u_n)u_n dx \\ &= \int_\Omega |\nabla u_{\varepsilon,\lambda}|^2 dx + \int_\Omega \phi_\varepsilon(u_{\varepsilon,\lambda})u_{\varepsilon,\lambda}^2 dx - \int_\Omega |u_{\varepsilon,\lambda}|^p \log |u_{\varepsilon,\lambda}|^2 dx - \lambda \int_\Omega f(u_{\varepsilon,\lambda})u_{\varepsilon,\lambda} dx + o_n(1).\end{aligned}$$

From Lemma 2.1, (3.25) and (3.32), we get

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla u_{\varepsilon,\lambda}|^2 dx + o_n(1),$$

which implies $u_n \to u_{\varepsilon,\lambda}$ in $H_0^1(\Omega)$.

Up to now, we finish the proof of this lemma. □

Proof of Theorem 1.1 From Remark 3.2 and Lemma 3.5, we know that for all $\lambda \geq \Lambda := \tilde{\lambda}(T)+1$, $I_{\varepsilon,\lambda}$ possesses one critical point $u_{\varepsilon,\lambda} \in H_0^1(\Omega)$ and $I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) = c_{\varepsilon,\lambda}$. Remark 2.1 implies that $(u_{\varepsilon,\lambda}, \phi_{\varepsilon,\lambda}) \in H_0^1(\Omega) \times E$ is a pair of critical points of $\mathcal{I}_{\varepsilon,\lambda}$, that is, $(u_{\varepsilon,\lambda}, \phi_{\varepsilon,\lambda})$ is a pair of solutions of $(\mathcal{P}_\varepsilon)$. Set $\phi_{\varepsilon,\lambda}^- := \max\{-\phi_{\varepsilon,\lambda}, 0\}$ and then by multiplying the second equation of $(\mathcal{P}_\varepsilon)$ and integrating, we reach to

$$\int_\Omega |\phi_{\varepsilon,\lambda}^-|^2 dx + \int_\Omega |\phi_{\varepsilon,\lambda}^-|^4 dx \leq 0,$$

which implies that $\phi_{\varepsilon,\lambda}^- \equiv 0$. So $\phi_{\varepsilon,\lambda} \geq 0$. Similarly, we can prove $u_{\varepsilon,\lambda} \geq 0$.

Now we prove the asymptotical behavior of $(u_{\varepsilon,\lambda}, \phi_{\varepsilon,\lambda})$ with respect to λ. Note that

$$I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) = c_{\varepsilon,\lambda} \text{ and } I'_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) = 0.$$

Similar to (3.27), we have

$$\begin{aligned} c_{\varepsilon,\lambda} &= I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) - \frac{1}{p} I'_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) u_{\varepsilon,\lambda} \\ &\geq \frac{p-2}{2p} ||u_{\varepsilon,\lambda}||^2 + J_\varepsilon(u_{\varepsilon,\lambda}) - \frac{1}{p} \int_\Omega \phi_\varepsilon(u_{\varepsilon,\lambda}) u_{\varepsilon,\lambda}^2 dx \\ &\geq \frac{p-2}{2p} ||u_{\varepsilon,\lambda}||^2. \end{aligned} \tag{3.33}$$

Recall Lemma 3.3, we know that

$$\lim_{\lambda \to \infty} \sup_{\varepsilon > 0} c_{\varepsilon,\lambda} = 0. \tag{3.34}$$

Then (3.33) and (3.34) give that for each $\varepsilon > 0$,

$$\lim_{\lambda \to \infty} u_{\varepsilon,\lambda} = 0 \text{ in } H_0^1(\Omega).$$

As Φ_ε (see (2.5)) is continuous, we know

$$\lim_{\lambda \to \infty} \phi_{\varepsilon,\lambda} = 0 \text{ in } E.$$

Recalling the fact $E \hookrightarrow L^\infty(\Omega)$, one gets $\lim\limits_{\lambda \to \infty} ||\phi_{\varepsilon,\lambda}||_\infty = 0$.

The above analysis concludes the proof of Theorem 1.1.

□

4 Proof of Theorem 1.2

In the sequel, we fix the parameter $\tilde{\lambda} \geq \Lambda$ and consider the asymptotical behavior of the pair of solutions $(u_{\varepsilon,\tilde{\lambda}}, \phi_{\varepsilon,\tilde{\lambda}})$ with respect to the parameter ε, where Λ is given above.

Lemma 4.1. *Assume that* $(f_1) - (f_5)$ *hold. Let* $0 < \varepsilon < 1$ *and* $\{(u_{\varepsilon,\tilde{\lambda}}, \phi_{\varepsilon,\tilde{\lambda}})\} \subset H_0^1(\Omega) \times E$ *be a sequence of solutions of* $(\mathcal{P}_\varepsilon)$. *Then there hold*

$$\lim_{\varepsilon \to 0^+} u_{\varepsilon,\tilde{\lambda}} = u_{0,\tilde{\lambda}} \ \textit{in} \ H_0^1(\Omega) \ \textit{and} \ \lim_{\varepsilon \to 0^+} \phi_{\varepsilon,\tilde{\lambda}} = \phi_{0,\tilde{\lambda}} \ \textit{in} \ E,$$

where $(u_{0,\tilde{\lambda}}, \phi_{0,\tilde{\lambda}})$ *is a pair of solutions of the following Schrödinger-Poisson system*

$$\begin{cases} -\Delta u + \phi u = |u|^{p-2} u \log |u|^2 + \tilde{\lambda} f(u), & \text{in } \Omega, \\ -\Delta \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Proof. Since $\{(u_{\varepsilon,\tilde{\lambda}}, \phi_{\varepsilon,\tilde{\lambda}})\} \subset H_0^1(\Omega) \times E$ is a sequence of solutions of $(\mathcal{P}_\varepsilon)$, then it follows from Remark 2.1 that $\{u_{\varepsilon,\tilde{\lambda}}\}$ is a sequence of solutions of (Q) and for each $\varepsilon > 0$, there hold

$$I_{\varepsilon,\tilde{\lambda}}(u_{\varepsilon,\tilde{\lambda}}) = c_{\varepsilon,\tilde{\lambda}}, \ I'_{\varepsilon,\tilde{\lambda}}(u_{\varepsilon,\tilde{\lambda}}) = 0,$$

where $c_{\varepsilon,\tilde{\lambda}}$ is the mountain pass level of $I_{\varepsilon,\tilde{\lambda}}$. Note that for any $0 < \varepsilon < 1$,

$$0 < c_{\varepsilon,\tilde{\lambda}} < c_{1,\tilde{\lambda}}.$$

Similar to (3.33), one yields that

$$\begin{aligned} c_{1,\tilde{\lambda}} > c_{\varepsilon,\tilde{\lambda}} &= I_{\varepsilon,\tilde{\lambda}}(u_{\varepsilon,\tilde{\lambda}}) - \frac{1}{p} I'_{\varepsilon,\tilde{\lambda}}(u_{\varepsilon,\tilde{\lambda}}) u_{\varepsilon,\tilde{\lambda}} \\ &\geq \frac{p-2}{2p} ||u_{\varepsilon,\tilde{\lambda}}||^2 + J_\varepsilon(u_{\varepsilon,\tilde{\lambda}}) - \frac{1}{p} \int_\Omega \phi_\varepsilon(u_{\varepsilon,\tilde{\lambda}}) u^2_{\varepsilon,\tilde{\lambda}} dx \\ &\geq \frac{p-2}{2p} ||u_{\varepsilon,\tilde{\lambda}}||^2, \end{aligned} \tag{4.2}$$

which implies that $\{u_{\varepsilon,\tilde{\lambda}}\}$ is bounded uniformly in $H_0^1(\Omega)$. Then, up to s subsequence if necessary, there exists $u_{0,\tilde{\lambda}} \in H_0^1(\Omega)$ such that as $\varepsilon \to 0^+$,

$$\begin{aligned} & u_{\varepsilon,\tilde{\lambda}} \rightharpoonup u_{0,\tilde{\lambda}} \text{ in } H_0^1(\Omega), \\ & u_{\varepsilon,\tilde{\lambda}} \to u_{0,\tilde{\lambda}} \text{ in } L^s(\Omega) \text{ for } s \in [1, \infty), \\ & u_{\varepsilon,\tilde{\lambda}} \to u_{0,\tilde{\lambda}} \text{ a.e. } x \in \Omega. \end{aligned}$$

Due to $\tilde{\lambda} \geq \Lambda$, then repeating the method in Lemma 3.5, we can prove

$$u_{\varepsilon,\tilde{\lambda}} \to u_{0,\tilde{\lambda}} \ \text{ in } H_0^1(\Omega).$$

Observe that as $\varepsilon \to 0^+$,

$$u^2_{\varepsilon,\tilde{\lambda}} \to u^2_{0,\tilde{\lambda}} \text{ in } L^{\frac{6}{5}}(\Omega).$$

In view of Lemma 2.2, we can obtain as $\varepsilon \to 0^+$,

$$\phi_\varepsilon(u_{\varepsilon,\tilde{\lambda}}) \to \phi_0(u_{0,\tilde{\lambda}}) \text{ in } H_0^1(\Omega),\ \ \varepsilon\phi_\varepsilon(u_{\varepsilon,\tilde{\lambda}}) \to 0 \text{ in } W_0^{1,4}(\Omega). \tag{4.3}$$

On the other hand, since $u_{\varepsilon,\tilde{\lambda}}$ is a solution of (Q), then for any $v \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \nabla u_{\varepsilon,\tilde{\lambda}} \nabla v dx + \int_\Omega \phi_\varepsilon(u_{\varepsilon,\tilde{\lambda}}) u_{\varepsilon,\tilde{\lambda}} v dx = \int_\Omega |u_{\varepsilon,\tilde{\lambda}}|^{p-2} u_{\varepsilon,\tilde{\lambda}} v \log |u_{\varepsilon,\tilde{\lambda}}|^2 dx + \tilde{\lambda} \int_\Omega f(u_{\varepsilon,\tilde{\lambda}}) v dx.$$

Using (4.3), by standard argument, we can prove

$$\int_\Omega \nabla u_{0,\tilde{\lambda}} \nabla v dx + \int_\Omega \phi_0(u_{0,\tilde{\lambda}}) u_{0,\tilde{\lambda}} v dx = \int_\Omega |u_{0,\tilde{\lambda}}|^{p-2} u_{0,\tilde{\lambda}} v \log |u_{0,\tilde{\lambda}}|^2 dx + \tilde{\lambda} \int_\Omega f(u_{0,\tilde{\lambda}}) v dx.$$

which implies that $(u_{0,\tilde{\lambda}}, \phi_0(u_{0,\tilde{\lambda}}))$ gives rise to a pair of solutions of the Schrödinger-Poisson system (4.1).

The proof of this lemma is completed. □

Remark 4.1. *Theorem 1.2 follows from Lemma 4.1.*

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