Multiplicity of normalized solutions for p-Laplacian equation with critical growth in $\mathbb{R}^N$

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Abstract: In this paper, we consider the following p-Laplacian equation

$$\begin{cases} -\Delta_p u + |u|^{p-2}u - \lambda u = \mu |u|^{q-2}u + |u|^{p^*-2}u, & \text{in } \mathbb{R}^N, \\ u > 0, \ \displaystyle\int_{\mathbb{R}^N} u^2 dx = a^2, \end{cases}$$

where $a, \mu > 0$, $-\Delta_p u = div(|\nabla u|^{p-2}\nabla u), 1 < p < N$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier, $p < q < \frac{N+2}{N}p$ and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. By employing a minimization theorem and the truncation argument, we establish the multiplicity of normalized solutions of above equation. Our results extend and complement some existing results in the literature.

Keywords: p-Laplacian equation; normalized solutions; critical growth; variational methods; truncation argument

1 Introduction

In this paper, we are concerned with the multiplicity of normalized solutions for the following p-Laplacian equation

$$\begin{cases} -\Delta_p u + |u|^{p-2}u - \lambda u = \mu |u|^{q-2}u + |u|^{p^*-2}u, & \text{in } \mathbb{R}^N, \\ u > 0, \ \displaystyle\int_{\mathbb{R}^N} u^2 dx = a^2, \end{cases} \tag{1.1}$$

where $a > 0$ is a prescribed mass, $-\Delta_p u = div(|\nabla u|^{p-2}\nabla u), 1 < p < N$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier, $p < q < \frac{N+2}{N}p$ and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. Such types of equations have been derived as models of

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several physical phenomena and have been the subject of extensive study in recent years. In the case $N > p \neq 2$, the important contribution to p-Laplacian equation

$$-\Delta_p + \omega|u|^{p-2}u = f(x,u),\ \omega \in \mathbb{R},\ \text{in } \mathbb{R}^N \tag{1.2}$$

can be found in [25]. After that, in [27], Li and Zhou researched the p-Laplacian problem (1.2) in bounded domain where $f(x,u) = |u|^{p-1}$ at infinity. Gu et al. [18] studied the following type of p-Laplacian equation

$$-\Delta_p + V(x)|u|^{p-2}u = \lambda|u|^{p-2}u + a|u|^{s-2}u,\ \text{in } \mathbb{R}^N, \tag{1.3}$$

where $a \geq 0, \lambda \in \mathbb{R}, p \in (1,N), s = p + \frac{p^2}{N}$ and $V(x)$ is the potential function satisfying the suitable conditions. They obtained the existence of solutions of (1.3) by using variational methods. As we all know, if $p = 2$ in (1.1), the solutions of (1.1) are related to the existence of standing waves for the following Schrödinger equation of the form

$$i\partial_t\psi + \Delta\psi + g(|\psi|^2)\psi = 0,\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.4}$$

where $N \geq 1$, i denotes the imaginary unit, $\psi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ is an unknown function and g is a real function. By standing waves, we denote solutions of the form $\psi(t,x) = e^{i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ is a parameter and $u : \mathbb{R}^N \to \mathbb{R}$ is a function to be found. At this point, (1.4) can be reduced to the following form

$$-\Delta u + \lambda u = f(u), \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

If $\lambda \in \mathbb{R}$ is a given value, there are a lot of papers devoted to studying the existence and multiplicity of solutions of (1.5) (see [8, 9, 10, 11]) by using the variational methods ([3, 17, 36, 38, 41, 43]). In the last few years, mathematicians begin to find solutions of (1.5) with prescribed mass and in this case, $\lambda \in \mathbb{R}$ is part of the unknown. From the physical point of view, this approach seems to be particularly meaningful because the L^2-norm is a preserved quantity of the evolution and the variational characterization of such solutions is often a strong help to analyze their orbital stability, see [7, 11, 39] for more details in this direction. A solution of (1.5) with $||u||_2^2 = c$ can be obtained as a constrained critical point of the functional

$$\mathcal{I}(u) := \frac{1}{2}\int_{\mathbb{R}^N} u^2 dx - \int_{\mathbb{R}^N} F(u)dx$$

on the constraint

$$S(c) := \{u \in H^1(\mathbb{R}^N) |\ ||u||_2^2 = c\}.$$

Note that λ cannot longer be imposed but instead appears as a Lagrange parameter. Now we recall some existing results in this direction. The pioneer of this type of work can date back to Louis Jeanjean in [20]. After that, many researchers begin to study the normalized solutions under different forms of (1.5). We refer to [26] for Choquard equations; [45, 46] for

Kirchhoff-type equations; [14, 15, 47] for quasilinear Schrödinger equation; [7, 12, 13, 21] for Schrödinger-Poisson equation; [4, 6, 28] for Schrödinger systems; [22, 23, 39, 40] for combined nonlinearities; [32] for fractional Schrödinger equation; [33, 34, 35, 37] for bounded domain. For more detailed results concerning normalized problems, readers can refer to the above mentioned papers and the reference within.

Existence of multiple normalized solutions has also been studied extensively in the latest few years. In [31], Luo established the multiplicity of radial normalized solutions for Schrödinger-Poisson-Slater equations inspired by [5]. Later, Xie and Chen [44] considered the nonlinear Kirchhoff type problem and proved infinitely many radial normalized solutions. By applying the symmetric mountain pass approaches, Hirata and Tanaka in [19] obtained the existence of infinitely many normalized solutions for nonlinear scalar field equations with L^2 constraint. After that, Jeanjean and Lu [24] proved multiple nonradial normalized solutions and these solutions are sign-changing. Very recently, in [2], Alves et al. considered the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + \lambda u = \mu |u|^{q-2}u + f(u), \text{ in } \mathbb{R}^N, \\ u > 0, \quad \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.6}$$

where $a, \mu > 0, \lambda < 0$, $q \in (2, 2 + \frac{4}{N})$, f has an exponential critical growth when $N = 2$ and $f(u) = |u|^{2^*-2}u$ when $N \geq 3$. By applying a minimax theorem found in [24] and the truncation argument introduced in [16], they obtained the existence of multiple normalized solutions of (1.6). As far as we know, [2] is the first paper devoted to establishing the existence of multiple solutions when combined power nonlinearity is of mixed type.

Motivated by [2], the aim of this paper is to extend the results in [2] and find the multiple prescribed norm solutions of (1.1). To authors' knowledge, there is no work concerning p-Laplacian equation with L^2-norm except [42] that constructs normalized solutions under the L^2-supercritical and Sobolev subcritical cases by a mountain pass argument. In our work, we consider the p-Laplacian equation with critical growth which makes the problem more complicated since the lack of the compactness caused by the critical exponent.

A solution u to the problem (1.1) with $\int_{\mathbb{R}^N} |u|^2 dx = a^2$ corresponds to a critical point of the functional

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx \tag{1.7}$$

on the constraint

$$S(a) = \{u \in X : \int_{\mathbb{R}^N} |u|^2 dx = a^2\},$$

where $X := W^{1,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ with the norm

$$||u|| = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{1}{2}}$$

and $W^{1,p}(\mathbb{R}^N)$ is the usual Sobolev space with the norm

$$||u||_{W^{1,p}} = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx\right)^{\frac{1}{p}}.$$

The main results of this paper can be presented as follows.

Theorem 1.1. *For given $a > 0$ and $n \in \mathbb{N}$, there exist $\mu^* > 0$ independent of n and $\mu_* = \mu(n) > 0$ such that (1.1) admits at least n couples $(u_j, \lambda_j) \in X \times \mathbb{R}$ $(j = 1, 2, \cdots, n)$ of weak solutions possessing negative energy for $\mu_* < \mu < \mu^*$ with $\int_{\mathbb{R}^2} |u_j|^2 dx = a^2$ and $\lambda_j < 0$.*

The above theorem extends some existing results found in the literature for the nonlinear elliptic equation, because we consider the p-Laplacian operator which is more general than the case $p = 2$. Moreover, it is very important to point out that a similar result can be obtained if we consider Sobolev subcritical case, i.e.,

$$\begin{cases} -\Delta_p u + |u|^{p-2}u - \lambda u = \mu |u|^{q-2}u + |u|^{r-2}u, & \text{in } \mathbb{R}^N, \\ u > 0, \ \int_{\mathbb{R}^N} u^2 dx = a^2, \end{cases} \tag{1.8}$$

where $p < q < \frac{N+2}{N}p < r < p^*$. Hence we have the following corollary:

Corollary 1.1. *For given $a > 0$ and $n \in \mathbb{N}$, there exist $\mu^* > 0$ independent of n and $\mu_* = \mu(n) > 0$ such that (1.8) admits at least n couples $(u_j, \lambda_j) \in X \times \mathbb{R}$ $(j = 1, 2, \cdots, n)$ of weak solutions possessing negative energy for $\mu_* < \mu < \mu^*$ with $\int_{\mathbb{R}^2} |u_j|^2 dx = a^2$ and $\lambda_j < 0$.*

Remark 1.1. *It is noteworthy that as [42] mentioned, we cannot investigate the problem*

$$-\Delta_p u - \lambda |u|^{p-2}u = \mu |u|^{q-2}u + |u|^{p^*-2}u, \ \text{in } \mathbb{R}^N, \tag{1.9}$$

for $q \in (p, p + \frac{p^2}{N})$, which is L^p-supercritical growth problem. In fact, if we consider the functional of (1.9)

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx$$

on the constraint

$$\mathcal{S}_a = \{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^p dx = a^2\},$$

it fails to prove the weak limit u is a solution of some equation because $W^{1,p}(\mathbb{R}^N)$ cannot embed into a Hilbert space, which is an essential condition, see [8] for more detailed analysis. In our setting, it is easy to obtain $X \hookrightarrow H$, where H is a Hilbert space, then one can define the inner product (see (2.1)). Thus the minimax argument can be formulated (see Sect.2.1).

Remark 1.2. *Recall the Gagliardo-Nirenberg inequality (see [1]) which is often used in the study of normalized solutions:*

For any $u \in L^s(\mathbb{R}^N)$ and $s \geq 2$, there exists $C_{N,s} > 0$ such that

$$||u||_s \leq C_{N,s} ||\nabla u||_p^{\frac{Np(s-2)}{s[Np-2(N-p)]}} ||u||_2^{\frac{2Np-2s(N-p)}{s[Np-2(N-p)]}}. \tag{1.10}$$

Remark 1.3. *The proof of Theorem 1.1 bases on an abstract theorem involving genus theory developed by Jeanjean and Lu in [24], we will show more details in Sect.2. In the proofs of Theorem 1.1, we shall work on a radial space X_{rad} since it supplies the delicate compact embedding. Moreover, it follows from Palais' principle of symmetric criticality (see [36]), that the critical points of I in X_{rad} are actually the critical points in the whole space X.*

The remaining part of this paper is organized as follows. In section 2, we introduce some preliminaries and establish some technical lemmas used in the proofs of Theorem 1.1. In section 3, we give the proof of Theorem 1.1.

Throughout this paper, we make use of the following notations:

- For $1 \le s < \infty$, $L^s(\mathbb{R}^2)$ is the usual Lebesgue space endowed with the norm
$$||u||_s = \left(\int_{\mathbb{R}^2} |u|^s dx\right)^{\frac{1}{s}}$$
and $||\cdot||_\infty$ denotes the L^∞-norm;
- X^* is the dual space of X;
- We use " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence in the related function space respectively;
- For any $x_0 \in \mathbb{R}^2$ and $R > 0$, $B_R(x_0)$ denotes the ball centered at x_0 with radius R;
- Letters $C, C_1, C_2, \cdots$ represent positive constants which may change from lines to lines and $o_n(1)$ denotes the quantity that tends to 0 as $n \to \infty$.

2 Preliminaries

In this section, we show the minimax theorem and some crucial lemmas which are important in the proofs of our main results.

2.1 A minimax theorem

Now we present a minimax theorem for a class of constrained even functionals developed in [24]. First of all, we give some useful notations.

Let E be a real Banach space with the norm $||\cdot||_E$ and H be a real Hilbert space with inner product $(\cdot,\cdot)_H$. For any $c > 0$, define the following manifold

$$\mathcal{N} = \{u \in E|\ (u,u)_H = c\}, \tag{2.1}$$

which is endowed with the topology inherited from E. And the tangent space of $\mathcal{N}$ at a point $u \in \mathcal{N}$ is defined by

$$T_u\mathcal{N} = \{v \in E|\ (u, v)_H = 0\}.$$

Let $\tilde{I} \in C^1(E, \mathbb{R})$, then $\tilde{I}_{|\mathcal{N}}$ belongs to C^1 on $\mathcal{N}$. The norm of the derivative of $I_{|\mathcal{N}}$ at any point $u \in \mathcal{N}$ is given by

$$||\tilde{I}'_{|\mathcal{N}}(u)|| = \sup_{||v||_E \leq 1, v \in T_u\mathcal{N}} |\langle \tilde{I}'(u), v\rangle|.$$

Note that $\mathcal{N}$ is symmetric with respect to $0 \in E$ and $0 \notin \mathcal{N}$. Let $\Sigma(\mathcal{N})$ be the family of closed symmetric subsets of $\mathcal{N}$. In order to formulate the minimax argument, in the following, we introduce some concepts of Z_2 genus, see [30] for more description. For any nonempty set $A \in \Sigma(\mathcal{N})$, the Z_2 genus $\gamma(A)$ of A is defined by

$$\gamma(A) = \begin{cases} 0, \text{ if } A = \emptyset, \\ \inf\{n : \text{there exists an odd, continuous } \phi : A \to \mathbb{R}^n \backslash \{0\}\}, \\ +\infty, \text{ if it does not exist odd, continuous } h : A \to \mathbb{R}^n \backslash \{0\}. \end{cases}$$

In the following lemma, we give the main properties of genus.

Proposition 2.1. ([38]) *Let $A, B \in \Sigma(\mathcal{N})$. Then*
(1) *If there exists an odd map $f \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$.*
(2) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
(3) *If there exists an odd homeomorphism between A and B, then $\gamma(A) = \gamma(B)$.*
(4) *If S^{N-1} is the sphere in $\mathbb{R}^N$, then $\gamma(S^{N-1}) = N$.*
(5) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
(6) *If $\gamma(A) < \infty$, then $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.*
(7) *If A is compact, then $\gamma(A) < \infty$, and there exists $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$, where $N_\delta(A) = \{x \in Y|\ dist(x, A) \leq \delta\}$.*
(8) *If Y_0 is a subspace of Y with codimension k, and $\gamma(A) > k$, then $A \cap Y_0 \neq \emptyset$.*

For each $k \in \mathbb{N}$, let

$$\Gamma_k = \{A \in \Sigma(\mathcal{N})|\ \gamma(A) \geq k\}.$$

Now we state the minimax theorem and the detailed proof can refer to [24], here we omit.

Proposition 2.2. *Let $\tilde{I} \in C^1(E, \mathbb{R})$ be an even functional. Assume that $\Gamma_k \neq \emptyset$ for each $k \in \mathbb{N}$, $\tilde{I}_{|\mathcal{N}}$ is bounded from below and satisfies the $(PS)_d$ condition for all $d < 0$. Define*

$$d_k = \inf_{A \in \Gamma_k} \sup_{u \in A} \tilde{I}(u), \ k \geq 1.$$

Then the following statements hold:

(i) $-\infty < d_1 \leq d_2 \leq \cdots \leq d_k \leq \cdots$ *and d_k is a critical value of $\tilde{I}_{|\mathcal{N}}$ if $d_k < 0$.*

(ii) *If $d_k = d_{k+1} = \cdots = d_{k+l-1} =: d < 0$ for some $k, l \geq 1$, then $\gamma(K_d) \geq l$, where K_d denotes the set of critical points of $\tilde{I}_{|\mathcal{N}}$ at the level d. In particular, $\tilde{I}_{|\mathcal{N}}$ admits infinitely many critical points at the level d if $l \geq 2$.*

In the sequel, we restrict our study to the space X_{rad} which supplies the delicate compactness results. Precisely, if $\{u_n\}$ is bounded in X_{rad}, then for $\hat{p} \in [p, p^*)$, one has

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{\hat{p}} dx = \int_{\mathbb{R}^N} |u|^{\hat{p}} dx. \tag{2.2}$$

2.2 The truncation argument

To proceed our proof, we shall employ the truncation argument introduced in [16]. By Sobolev embedding and the Gagliardo-Nirenberg inequality (1.10), we have

$$\begin{aligned} I(u) &\geq \frac{1}{p}||\nabla u||_p^p - \frac{C_1\mu}{q}||\nabla u||_p^{\frac{Np(q-2)}{Np-2(N-p)}}||u||_2^{\frac{2Np-2q(N-p)}{Np-2(N-p)}} - \frac{1}{p^* S^{\frac{p^*}{p}}}||\nabla u||_p^{p^*} \\ &=: g(||\nabla u||_p), \end{aligned} \tag{2.3}$$

where

$$g(s) = \frac{1}{p}s^p - \frac{C_1\mu a^{\frac{2Np-2q(N-p)}{Np-2(N-p)}}}{q} s^{\frac{Np(q-2)}{Np-2(N-p)}} - \frac{1}{p^* S^{\frac{p^*}{p}}} s^{p^*}$$

and S is the best constant of the Sobolev embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ given by

$$S = \inf_{u\in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{||\nabla u||_p^p}{||u||_{p^*}^p}.$$

Recalling that $q \in (p, \frac{N+2}{N}p)$, which gives that $\frac{Np(q-2)}{Np-2(N-p)} < p$, then one can infer that g attains its negative local minimum. Moreover, for given $a > 0$, there exists $\mu^* > 0$ such that for $0 < \mu < \mu^*$, $\mu a^{\frac{2Np-2q(N-p)}{Np-2(N-p)}}$ can take small value, which can ensure that the function g reaches its positive local maximum. Therefore, the function g possesses two positive zero points, denoted by R_0 and R_1 ($R_0 < R_1$) respectively.

For each $0 < \mu < \mu^*$ and $0 < R_0 < R_1$ given above, now we define the nonincreasing cut-off function $\eta \in C^\infty(\mathbb{R}^+, [0,1])$ by

$$\eta(t) = \begin{cases} 1, & \text{if } t \leq R_0, \\ 0, & \text{if } t \geq R_1. \end{cases}$$

In the sequel, we define the truncation functional

$$I^T(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + |u|^p)dx - \frac{\mu}{q}\int_{\mathbb{R}^N}|u|^q dx - \frac{\eta(||\nabla u||_p)}{p^*}\int_{\mathbb{R}^N}|u|^{p^*}dx.$$

Then similar above, we know that $I^T(u) \geq \tilde{g}(||\nabla u||_p)$, where

$$\tilde{g}(s) = \frac{1}{p}s^p - \frac{C_1\mu a^{\frac{2Np-2q(N-p)}{Np-2(N-p)}}}{q} s^{\frac{Np(q-2)}{Np-2(N-p)}} - \frac{\eta(s)}{p^* S^{\frac{p^*}{p}}} s^{p^*}.$$

From above analysis, it is not difficult to observe that $\tilde{g}$ attains its negative local minimum and has unique zero point R_0. Furthermore, we know that $\tilde{g}$ is coercive, bounded from below and $\tilde{g}(s) < 0$ for $s \in (0, R_0)$. Without loss of generality, hereafter we will assume that

$$\frac{1}{p}s^p - \frac{1}{p^* S^{\frac{p^*}{p}}} s^{p^*} \geq 0, \ \forall \ s \in [0, R_0] \text{ and } R_0 < S^{\frac{N}{p}}. \tag{2.4}$$

The following lemma gives some important properties of the truncation functional I^T.

Lemma 2.1. **(i)** *The functional $I^T \in C^1(X, \mathbb{R})$.*

(ii) *If $I^T(u) \leq 0$, then $||\nabla u||_p < R_0$ and $I(v) = I^T(v)$ for all $v \in N_\sigma(u)$, where*

$$N_\sigma(u) = \{v : \ |v - u| \leq \sigma\}.$$

(iii) *For all $\mu \in (0, \mu^*)$, I^T satisfies the Palais-Smale condition for the level $c < 0$, where μ^* is given above.*

Proof. **(i)** and **(ii)** are trivial. Now we prove **(iii)**. Let $\{u_n\} \subset X_{rad} \cap S(a)$ be a $(PS)_c$ sequence of I^T with $c < 0$, then we have $||\nabla u||_p < R_0$ for n large enough. At this point, $\{u_n\}$ is in fact a $(PS)_c$ sequence of I, that is

$$I(u_n) \to c \text{ and } ||I'(u_n)|_{S(a)}|| \to 0 \text{ as } n \to \infty. \tag{2.5}$$

On one hand, in view of the definition of I^T, one can see that I^T is coercive on $S(a)$, from where it follows that $\{u_n\}$ is bounded in X_{rad}. Up to a subsequence if necessary, there exists $u \in X_{rad}$ satisfying

$$u_n \rightharpoonup u \text{ in } X, \ u_n \to u \text{ a.e. in } \mathbb{R}^N,$$

$$u_n \to u \text{ in } L^r(\mathbb{R}^N) \text{ for } r \in [p, p^*).$$

By adapting the arguments used in Lemma 2.7 in [42], we know that there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

(a) $-\Delta_p u_n + |u_n|^{p-2}u_n - \lambda_n u_n - \mu|u_n|^{q-2}u_n - |u_n|^{p^*-2}u_n \to 0$ in X^*;

(b) there exists $\lambda_a \in \mathbb{R}$ such that $\lambda_n \to \lambda_a$;

(c) $-\Delta_p u_n + |u_n|^{p-2}u_n - \lambda_a u_n - \mu|u_n|^{q-2}u_n - |u_n|^{p^*-2}u_n \to 0$ in X^*;

(d) $-\Delta_p u + |u|^{p-2}u - \lambda_a u - \mu|u|^{q-2}u - |u|^{p^*-2}u = 0$ in X^*.

We claim $u \neq 0$. Arguing by contradiction that $u = 0$, then we have $||u_n||_p = ||u_n||_q = o_n(1)$. Then it follows from Sobolev inequality and (2.4) that

$$\begin{aligned} I(u_n) &\geq \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx - \frac{1}{p^* S^{\frac{p^*}{p}}}\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{p^*}{p}} \\ &\geq -\frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx. \end{aligned}$$

Hence one concludes that

$$0 > c + o_n(1) = I(u_n) \geq -\frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx = 0,$$

which is impossible. So the claim holds.

On the other hand, since $u \neq 0$, then from this observation, by Pohozaev identity, similar to the method used in Lemma 2.9 in [42], we can prove that $\lambda_a < 0$. On account of $\{u_n\}$ is bounded in X_{rad}, applying the concentration-compactness principle due to Lions [29], it follows that

$$|\nabla u_n|^p \rightharpoonup \kappa \text{ and } |u_n|^{p^*} \rightharpoonup \nu \text{ in the sense of measure.}$$

Moreover, for an at most countable index set $\mathcal{J}$, we have

$$\begin{cases} (1)\ \nu = |u|^{p^*} + \sum\limits_{j\in\mathcal{J}} \nu_i \delta_{x_j},\ \ \nu_j > 0, \\ (2)\ \kappa \geq |\nabla u|^p + \sum\limits_{j\in\mathcal{J}} \kappa_i \delta_{x_j},\ \ \kappa_j > 0, \\ (3)\ S\nu_j^{p/p^*} \leq \kappa_j,\ \ \forall j \in \mathcal{J}. \end{cases}$$

where δ_{x_j} is the Dirac mass at x_j. In view of **(a)**, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, we derive that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla\phi dx + \int_{\mathbb{R}^N} |u_n|^{p-2}u_n\phi dx - \lambda_n \int_{\mathbb{R}^N} u_n\phi dx \\ &\quad = \mu\int_{\mathbb{R}^N} |u_n|^{q-2}u_n\phi dx + \int_{\mathbb{R}^N} |u_n|^{p^*-2}u_n\phi dx + o_n(1). \end{aligned} \tag{2.6}$$

In the sequel, for any $\epsilon > 0$, we consider the function $\chi \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$\chi \equiv 1 \text{ on } B_\epsilon(x_j),\ \chi \equiv 0 \text{ on } \mathbb{R}^N \setminus B_{2\epsilon}(x_j) \text{ and } |\nabla\chi| \leq \tfrac{2}{\epsilon}.$$

Let $\phi = \chi u_n$ in (2.6), one can obtain that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u_n|^p\chi dx + \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\chi u_n dx + \int_{\mathbb{R}^N} |u_n|^p\chi dx - \lambda_n\int_{\mathbb{R}^N} |u_n|^2\chi dx \\ &\quad = \mu\int_{\mathbb{R}^N} |u_n|^q\chi dx + \int_{\mathbb{R}^N} |u_n|^{p^*}\chi dx + o_n(1). \end{aligned} \tag{2.7}$$

Passing to the limit as $n \to \infty$ in (2.7), we arrive at

$$\begin{aligned} &\int_{\mathbb{R}^N} \chi d\kappa + \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\chi u_n dx + \int_{\mathbb{R}^N} |u_n|^p\chi dx - \lambda_a\int_{\mathbb{R}^N} |u_n|^2\chi dx \\ &\quad = \mu\int_{\mathbb{R}^N} |u_n|^q\chi dx + \int_{\mathbb{R}^N} \chi d\nu + o_n(1). \end{aligned} \tag{2.8}$$

Observe that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^N} |u_n|^p\chi dx = \lim_{\epsilon\to 0}\int_{\mathbb{R}^N} |u_n|^2\chi dx = \lim_{\epsilon\to 0}\int_{\mathbb{R}^N} |u_n|^q\chi dx = 0 \tag{2.9}$$

and by Hölder inequality,

$$0 \leq \lim_{\epsilon \to 0} \left| \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla \chi u_n dx \right| \leq \lim_{\epsilon \to 0} C \left(\int_{B_{2\epsilon}(x_j)} |u_n|^{p^*} dx \right)^{1/p^*} = 0. \tag{2.10}$$

In view of (2.8)-(2.10), one obtains

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \chi d\kappa = \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \chi d\nu,$$

which gives that

$$\kappa_j = \nu_j, \text{ for each } j \in \mathcal{J}. \tag{2.11}$$

Recalling that

$$S \nu_j^{p/p^*} \leq \kappa_j, \ \forall \ j \in \mathcal{J},$$

then we have

$$\nu_j = 0 \text{ or } \nu_j \geq S^{\frac{N}{p}}, \ \forall \ j \in \mathcal{J}.$$

If $\nu_j \geq S^{\frac{N}{p}}$, then $R_0 \geq \limsup_{n \to +\infty} |\nabla u_n|^p \geq S^{\frac{N}{p}}$, which contradicts to (2.4). Thus $\nu_j = 0$ and $\mathcal{J} = \emptyset$. So we arrive at

$$u_n \to u \text{ in } L^{p^*}_{loc}(\mathbb{R}^N).$$

Next we prove for each $R > 0$, $u_n \to u$ in $L^{p^*}(\mathbb{R}^N \setminus B_R(0))$. Indeed, for each $n \in \mathbb{N}$, $u_n \in X_{rad}$ is a radial function, we know that

$$|u_n(x)| \leq \frac{||u_n||}{|x|^{\frac{N-1}{p}}} \text{ a.e. in } \mathbb{R}^N.$$

On account of $\{u_n\}$ is bounded in X_{rad}, we have

$$|u_n(x)| \leq \frac{C}{|x|^{\frac{N-1}{p}}} \text{ a.e. in } \mathbb{R}^N,$$

so one gets

$$|u_n(x)|^{p^*} \leq \frac{C}{|x|^{\frac{N(N-1)}{N-p}}} \text{ a.e. in } \mathbb{R}^N.$$

Note that

$$\frac{C}{|x|^{\frac{N(N-1)}{N-p}}} \in L^1(\mathbb{R}^N \setminus B_R(0)).$$

Then it follows from Lebesgue's Dominated Convergence Theorem that

$$u_n \to u \text{ in } L^{p^*}(\mathbb{R}^N \setminus B_R(0)).$$

Therefore, we obtain

$$u_n \to u \text{ in } L^{p^*}(\mathbb{R}^N).$$

To complete the proof, it remains to show that

$$u_n \to u \text{ in } X_{rad}.$$

In view of **(c)** and **(d)**, choosing u_n and u as the testing functions respectively, one gets

$$\int_{\mathbb{R}^N}(|\nabla u_n|^p+|u_n|^p)dx-\lambda_a\int_{\mathbb{R}^N}|u_n|^2dx-\mu\int_{\mathbb{R}^N}|u_n|^qdx-\int_{\mathbb{R}^N}|u_n|^{p^*}dx=o_n(1) \tag{2.12}$$

and

$$\int_{\mathbb{R}^N}(|\nabla u|^p+|u|^p)dx-\lambda_a\int_{\mathbb{R}^N}|u|^2dx-\mu\int_{\mathbb{R}^N}|u|^qdx-\int_{\mathbb{R}^N}|u|^{p^*}dx=0. \tag{2.13}$$

Then using the fact that $u_n\to u$ in $L^s(\mathbb{R}^N)$ for $s\in[p,p^*]$, gathering (2.12) with (2.13), one can deduce that

$$\lim_{n\to\infty}(||\nabla u_n||_p^p-\lambda_a||u_n||_2^2)=||\nabla u||_p^p-\lambda_a||u||_2^2.$$

Due to $\lambda_a<0$, we have

$$\lim_{n\to\infty}||\nabla u_n||_p^p=||\nabla u||_p^p \text{ and } \lim_{n\to\infty}||u_n||_2^2=||u||_2^2=a^2,$$

which give that $u_n\to u$ in X_{rad}.

Thus we finish the proof of this lemma. □

3 Proof of Theorem 1.1

The aim of this section is to give the proof of Theorem 1.1. In the following, we will construct an appropriate minimax sequence possessing negative critical values for the functional I^T.

For given $\varepsilon>0$, define

$$A=\{u\in X_{rad}\cap S(a):I^T(u)\le-\varepsilon\}\subset X_{rad},$$

which is a closed symmetric subset of $S(a)$ since I^T is even and continuous.

Lemma 3.1. *For given $n\in\mathbb{N}$, there are $\varepsilon_*=\varepsilon(n)>0$ and $\mu_*=\mu(n)>0$ such that $\gamma(A)\ge n$ for $0<\varepsilon<\varepsilon_*$ and $\mu>\mu_*$.*

Proof. To prove this lemma, we borrow some arguments from [16]. For fixed $n\in\mathbb{N}$ and each $\varrho>0$, let E_n be an n-dimensional subspace of X_{rad} possessing a basis of form

$$\mathfrak{B}=\{u_1,u_2,\cdots,u_n\}$$

such that $||\nabla u_i||_p^{\frac{p}{2}}=\varrho$ and $||u_i||_2=a$ for $i=1,2,\cdots,n$. Then it yields that

$$||\nabla u_i||_p^p+||u_i||_2^2=\varrho^2+a^2.$$

Set

$$\Upsilon_n:=\{t_1u_1+t_2u_2+\cdots+t_nu_n:t_1^1+t_2^2+\cdots+t_n^2=1\}.$$

As [2], we know that there exists a homomorphism between Υ_n and the sphere $\mathcal{B} \subset \mathbb{R}^N$ defined as

$$\mathcal{B} = \{(z_1, z_2, \cdots, z_n) : z_1^2 + z_2^2 + \cdots + z_n^2 = \varrho^2 + a^2\}.$$

Then it follows from Proposition 2.1 that $\gamma(\Upsilon_n) = n$. For any $v \in \Upsilon_n$ and $0 < \varrho < R_0^{\frac{p}{2}}$, we can deduce that $||\nabla v||_p = \varrho^{\frac{2}{p}} < R_0$, thus

$$I^T(v) = I(v) = \frac{1}{p}\varrho^2 + \frac{\varrho^p}{p}\int_{\mathbb{R}^N} |\frac{v}{\varrho}|^p dx - \frac{\mu\varrho^q}{q}\int_{\mathbb{R}^N} |\frac{v}{\varrho}|^q dx - \frac{\varrho^{p^*}}{p^*}\int_{\mathbb{R}^N} |\frac{v}{\varrho}|^{p^*} dx,$$

where R_0 is given in Sect.2. Since E_n is a finite dimension space, all the norms are equivalent. Then we define

$$\alpha_n = \sup\{\int_{\mathbb{R}^N} |w|^p dx : w \in S(\frac{a}{\varrho}) \cap E_n, ||\nabla w||_p^{\frac{p}{2}} = 1\} > 0,$$

$$\beta_n = \inf\{\int_{\mathbb{R}^N} |w|^{p^*} dx : w \in S(\frac{a}{\varrho}) \cap E_n, ||\nabla w||_p^{\frac{p}{2}} = 1\} > 0$$

and

$$\theta_n = \inf\{\int_{\mathbb{R}^N} |w|^q dx : w \in S(\frac{a}{\varrho}) \cap E_n, ||\nabla w||_p^{\frac{p}{2}} = 1\} > 0,$$

where

$$S(\frac{a}{\varrho}) = \{u \in X_{rad} : ||u||_2 = \frac{a}{\varrho}\}.$$

Therefore, we obtain that

$$I^T(v) \leq \frac{1}{p}\varrho^2 + \frac{\varrho^p}{p}\alpha_n - \frac{\mu\varrho^q}{q}\theta_n - \frac{\varrho^{p^*}}{p^*}\beta_n. \tag{3.1}$$

It follows from (3.1) that for given $0 < \varrho < R_0^{\frac{p}{2}}$, we can choose $\mu_* = \mu(n) > 0$ and $\varepsilon_* = \varepsilon(n) > 0$ such that for $\mu > \mu_*$ and $0 < \varepsilon < \varepsilon_*$,

$$I^T(v) \leq -\varepsilon, \text{ for all } v \in \Upsilon_n.$$

From this, we can conclude that $\Upsilon_n \subset A$ and $\gamma(A) \geq \gamma(\Upsilon_n) = n$. The proof of this lemma is completed. □

Proof of Theorem 1.1 For each $k \in \mathbb{N}$, define

$$\Sigma_k = \{\mathcal{A} \subset X_{rad} \cap S(a) : \mathcal{A} \text{ is closed, } \mathcal{A} = -\mathcal{A} \text{ and } \gamma(\mathcal{A}) \geq k\}$$

and

$$c_k = \inf_{\mathcal{A}\in\Sigma_k} \sup_{u\in\mathcal{A}} I^T(u).$$

Let $0 < \mu < \mu^*$, where μ^* is given above. For $\varepsilon > 0$, define

$$\mathcal{A}_{-\varepsilon} = \{u \in X_{rad} \cap S(a) : I^T(u) \leq -\varepsilon\} \subset X_{rad}.$$

On one hand, regarding from Lemma 3.1, for each $k \in \mathbb{N}$, there exist $\varepsilon_* = \varepsilon(k) > 0$ and $\mu_* = \mu(k) > 0$ such that $\gamma(\mathcal{A}_{-\varepsilon}) \geq k$ for all $0 < \varepsilon < \varepsilon_*$ and $\mu > \mu_*$. Since I^T is continuous, even and bounded from below, thus we obtain that

$$\mathcal{A}_{-\varepsilon} \in \Sigma_k \text{ and } -\infty < c_k \leq -\varepsilon < 0 \text{ for any } k \in \mathbb{N}.$$

Furthermore, if $c = c_k = c_{k+1} = \cdots = c_{k+r} < 0$, then by Lemma 3.1, I^T satisfies the $(PS)_c$ condition for $c < 0$, from where it follows that K_c is a compact set, where

$$K_c = \{u \in X_{rad} \cap S(a) : (I^T)'(u) = 0, I^T(u) = c\}.$$

On the other hand, by the definition of the truncated functional, we know that $I_T(u) = I(u)$ for the level $c < 0$. Therefore, by applying Proposition 2.2, the existence of multiple solutions of (1.1) is obtained.

Thus we finish the proof of Theorem 1.1. □

Remark 3.1. *As mentioned above, the argument employed in this paper can also deal with the following Sobolev subcritical case,*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u - \lambda u = \mu|u|^{q-2}u + |u|^{r-2}u, & \text{in } \mathbb{R}^N, \\ u > 0, \ \displaystyle\int_{\mathbb{R}^N} u^2 dx = a^2, \end{cases}$$

where $p < q < \frac{N+2}{N}p < r < p^*$. *The outline of the proof is similar to the Sobolev critical case, here we do not give the details.*

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