Controllability Analysis for Impulsive Integro-differential Equation via AB Fractional Derivative

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Abstract

In this work, we analyse the controllability for certain classes of impulsive integro - differential equations(IIDE) of fractional order via Atangana Baleanu derivative involving finite delay with initial and nonlocal conditions using Banach fixed point theorem.

Keywords: Fractional derivatives, Integro-differential equations; Controllability; Differential equations with impulses; Contraction map and fixed-point theories.

1. Introduction

Fractional differential equations(FDE) have attracted many researchers in applied mathematics. It has been treated as another model to nonlinear differential equations [11]. FDE's have been used for several models in engineering and also to progress the accuracy of models in natural sciences [29]. The theory of FDE's been widely discussed by Delbosco and Rodino [14], Diethelm [15], Kilbas et al [20], Lakshmikantham et al [22, 23, 24, 25], Michalski [26], Miller and Ross [27], Podlubny [30] and Tarasov [36].

Controllability is the basic concept in control theory initiated by Kalman in 1960.It has been significant area for several authors in applied problems. Impulsive differential equations(IDE) is a differential equations with impulse effect and its first paper is related to A. D. Mishkis and V. D. Mil'man in 1960 and 1963 [28]. IDE's been grown rapidly during the last decades [5, 6, 7, 32, 33]. Both IDE and FDE have gained the attention among the researchers due to its vast applications in various field of science. These combination may yield an object of investigation, viz., impulsive FDEs made several papers [8, 9, 10, 11, 13, 16, 17, 21, 37].

The Atangana - Baleanu (AB) derivative [4] is an commonly used fractional derivative with no singularities. The Caputo version of AB derivative(ABC) shows more efficient and useful in enormous works. Many authors have covers their theory in the sense of AB or ABC derivatives to comprehensive their work [1, 2, 3, 18, 31, 34].

We analyse the controllability for IIDE of fractional order with finite delay via ABC derivative in our present work,

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$$
{}^{ABC}\mathbb{D}^{\lambda} z(v) = \mathbb{A}z(v) + \mathbb{B}\mathfrak{u}(v) + \phi\left(v, z_v, \int_0^v \mathscr{P}(v,s,z_s)ds\right), v \in \mathbb{I} = [0,\mathfrak{T}], \tag{1.1}
$$

$$
\Delta z(v)|_{v=v_k} = I_k(z(v_k)), \quad k = 1,2,...,l, \tag{1.2}
$$

$$
z(v) = \psi(v) \in \mathscr{B},\ v \in [-\eta, 0], \tag{1.3}
$$

where ${}^{ABC}\mathbb{D}^{\lambda}$ is the ABC derivative of fractional order λ. $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{M} \to \mathbb{M}$ is the infinitesimal generator of an $\lambda-$ resolvent family $(S_{\lambda}(v))_{v\geq 0}$, $(T_{\lambda}(v))_{v\geq 0}$ is defined on a complex Banach space $(\mathbb{M}, \|\cdot\|)$, and $\mathfrak{u}(\cdot) \in L^2([0,\mathfrak{T}],\mathbb{U})$ is the control, where $\mathbb{U}$ is a Banach space. $\mathbb{B} : \mathbb{U} \to \mathbb{M}$ is linear and bounded. Let $\mathscr{B} = \{\mathscr{P} : [-\eta, 0] \to \mathbb{M},\ \mathscr{P}$ is continuous every where except for a finite number of s at which $\mathscr{P}(s^-), \mathscr{P}(s^+)$ exist and $\mathscr{P}(s^-) = \mathscr{P}(s)\}, \phi \in C(\mathbb{I}\times S\times S, S), \mathscr{P} : \mathbb{I}\times\mathbb{I}\times D \to S$ is continuous on $\mathbb{I}\times\mathbb{I}, \mathbb{I}_k \in C(\mathbb{U},\mathbb{U}), \Delta z(v)|_{v=v_k} = z(v_k^+) - z(v_k^-), 0 = v_0 < v_1 < v_2 < \ldots < v_m < v_{m+1} = T$ for $k = 1,2,...l$, and $z(v_k^-), z(v_k^+)$ are the left - hand and right - hand limits of $z(t)$ at $v = v_k$. $z_v \in \mathscr{B}$ satisfies $z_v(\delta) = z(v+\delta),\ \delta \in [-\eta, 0]$, $z_v(\cdot)$ have time from $v-\eta$ to v.

To the best of our knowledge, the study of the controllability for certain class of the form $(1.1)-(1.3)$, is an untreated content and it motivates our work.

The structure of this article as follows: In section 2, we present some preliminary results. Section 3 establishes the result for a class of Fractional Impulsive Integro-differential Equation(FIIE) with initial conditions, while the fourth section covers the respective results of the system with initial condition. Section 5 establishes the result for a class of FIIE with nonlocal condition and section 6 contains the respective results of the system with non local condition.

2. Preliminaries

Let $(I, \|\cdot\|)$ and $(U, \|\cdot\|)$ be two Banach spaces. $\mathscr{L}(I,U)$ denotes the space of bounded linear operators from J into U with uniform operator topology, and $C(I,U)$ indicates the Banach space of all continuous functions from I to U.

The norm of $\mathscr{P} \in \mathscr{B}$ is

$$
||\mathscr{P}||_{\mathscr{B}} = \sup\{||\mathscr{P}(\rho)|| : \rho \in [-\eta, 0]\},
$$

$\mathscr{B}_1 = PC(I^+, U) = \{S : I^+ \to U : z(\cdot)$ is continuous except for a finite number of point t_k at which $z(v_k^+), z(v_k^-)$ exist and $z(v_k^-) = z(v_k), k = 1, \ldots, l\}$,

$\mathscr{B}_1$ with $\|z\|_{\mathscr{B}_\infty} = \sup\{\|z(v)\| : v \in [-\eta, T]\}$ is a Banach space, for any $v \in J$ and $z \in \mathscr{B}_1$, we have $z_v \in \mathscr{B}$.

In this part, we give some basic definition of the fractional calculus.

Definition 2.1. [18, 30] For $\lambda > 0$, the integral

$$
(\,{}_0I^{\lambda} z)(a) = \frac{1}{\Gamma(\lambda)} \int_0^t (a-s)^{\lambda-1} z(s)ds, \tag{2.1}
$$

is defined as the R-L integral of order λ.

Definition 2.2. [18, 30] For $0 < \lambda < 1$, the fractional derivative

$$
(\,{}_0\mathbb{D}^{\lambda} z)(a) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dt} \int_0^t (a-s)^{-\lambda} z(s)ds, \tag{2.2}
$$

is defined as the left R-L derivative of order λ.

Definition 2.3. [18, 30] For $0 < \lambda < 1$, the fractional derivative

$$({}_0\mathbb{D}^{\lambda} z)(a) = \frac{1}{\Gamma(\lambda)} \int_0^t (a-s)^{\lambda-1} z'(s) ds, \tag{2.3}$$

is defined as the Caputo derivative of order λ.

Definition 2.4. [4] For $\mathscr{P} \in H'(c,d)$, and $0 \leq \lambda \leq 1$

$$({}_0^{ABC}\mathbb{D}^{\lambda})(\mathscr{P}(a)) = \frac{B(\lambda)}{1-\lambda} \int_0^t \mathscr{P}'(s) E_{\lambda} \left[\frac{(a-s)^{\lambda}}{\lambda - 1} \right] ds, \tag{2.4}$$

where $E(\lambda)$ is the Mittag-Leffler function, $B(\lambda)$ is a normalizing positive function $\ni B(0) = B(1) = 1$ is defined as the fractional AB derivative of order λ in the sense of Caputo .

Definition 2.5. [4] For $0 \leq \lambda \leq 1$ and $\mathscr{P} \in H'(c,d)$,

$$({}_0^{ABR}\mathbb{D}^{\lambda})(\mathscr{P}(a)) = \frac{B(\lambda)}{1-\lambda} \frac{d}{da} \int_0^a \mathscr{P}(s) E_{\lambda} \left[\frac{(a-z)^{\delta}}{\lambda - 1} \right] ds, \tag{2.5}$$

is defined as the AB derivative in the sense Riemann Liouville .

The AB integral is [19, 27, 35]

$$({}_0^{AB}I^{\lambda} z)(a) = \frac{1-\lambda}{B(\lambda)} z(a) + \frac{\lambda}{B(\lambda)} ({}_0I^{\lambda} z)(a),$$

where ${}_0I^{\lambda}$ is defined in (2.1).

3. Controllability result

Definition 3.1. A function $z(\cdot) \in \mathscr{B}_1$ is said to be solution of system $(1.1)-(1.3)$ with restriction of $z(\cdot)$ to the interval K, $k = 1,2,...,l$. is continuous and satisfies the following:

$$z(v) = \begin{cases} \phi(v), & v \in [-\eta, 0] \\ \mathscr{G}T_{\lambda}(v)\phi(0) + \frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda} \int_{v_k}^{v} (v-s)^{\lambda-1} [Bu_{\mathfrak{a}}(s) + \phi(v, z_v, \int_0^v \psi(v,s,z_s)ds)]ds & \\ \quad + \frac{\lambda\mathscr{G}^2}{B(\lambda)} \int_{v_k}^{v} S_p(v-s)[Bu_{\mathfrak{a}}(s) + \phi(v, \mathfrak{a}_v, \int_0^v \psi(v,s,\mathfrak{a}_s)ds)]ds & v \in [0, v_1] \\ \mathscr{G}T_{\lambda}(v-v_k)(z(v_k^-)) + I_k(z(v_k^-)) + \frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda} \int_{v_k}^{v} (v-s)^{\lambda-1} [Bu_z(s) + & \\ \quad \phi(v, z_v, \int_0^v \psi(v,s,z_s)ds)]ds + \frac{\lambda\mathscr{G}^2}{B(\lambda)} \int_{v_k}^{v} S_p(v-s)[Bu_z(s) & \\ \quad + \phi(v, z_v, \int_0^v \psi(v,s,z_s)ds)]ds & v \in (v_k, v_{k+1}], \ \text{k=1,2,...,l} \end{cases} \tag{3.1}$$

Definition 3.2. If for every initial function $\psi \in \mathscr{B}_1$ and $z_1 \in S$, $\exists$ a control $\mathfrak{u} \in L^2(\mathbb{I}, U) \ni z(\cdot)$ of $(1.1)-(1.3)$ satisfies $z(\mathfrak{T}) = z_1$, then the system $(1.1)-(1.3)$ is said to be controllable on $\mathbb{I}$.

4. Main Result:

We analyse the controllability of mild solutions-type (3.1) for the problem $(1.1)-(1.3)$.If $A \in A^{\lambda}(\beta_0, \omega_0)$ then $||T_\lambda(v)|| \leq Me^{wv}$ and $||S_\lambda(v)|| \leq \mathscr{C} e^{wv}(1+v^{\lambda-1})$, for every $v > 0$, $w > w_0$. Hence, $\tilde{\tau} = \sup_{v\geq 0} ||T_\lambda(v)||$, $\tilde{\tau}_1 = \sup_{v\geq 0} \mathscr{C} e^{wv}(1+v^{\lambda-1})$. So we get $||T_\lambda(v)|| \leq \tilde{\tau}$; $||S_\lambda(v)|| \leq v^{\lambda-1}\tilde{\tau}_1$.
Let us assume that,

H₁ Let $\phi : \mathbb{J} \times S \times S \to S$ be a continuous function and $\exists\ M_\phi > 0$ for which:

$$||\phi(v,u,s)|| \leq M_\phi, for\ v \in \mathbb{J},\ u,s \in S$$

.

H₂ $\mathbb{W} : L^2(J,U) \to S$, the linear operator defined by

$$\mathbb{W}_u = \frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda} \int_{t_k}^{t} (T-S)^{\lambda-1} Bu(s)ds + \frac{\lambda \mathscr{G}^2}{B(\lambda)} \int_{t_k}^{t} S_p(T-S)Bu_z(s)ds,\ k = 0,1,...,l$$

admits $\mathbb{W}^{-1} : S \to L^2(J,U)/ker(\mathbb{W})$, $\mathbb{W}^{-1}$ is bounded and $\exists\ \tilde{\tau} > 0$ and $\tilde{\tau}_1 > 0 \ni ||B|| \leq \tilde{\tau}$ and $||\mathbb{W}^{-1}|| \leq \tilde{\tau}_1$.

H₃ $\mathbb{F}_k \in C(S,S)$, $\exists\ \omega > 0$ such that $||\mathbb{F}_k(\mathfrak{a})|| < \omega$ for every $\mathfrak{a} \in S$

H₄ $\phi : \mathbb{J} \times \mathbb{D} \times S \to S$ is continuous and $\exists$ a function $\mathfrak{G} \in \mathbb{L}'(\mathbb{J}, R^+) \ni$

$$\left|\left|\phi\left(v, \mathfrak{a}_v, \int_0^v \mathscr{P}(v,s,\mathfrak{a}_s)ds\right) - \phi\left(v, \mathfrak{b}_v, \int_0^v \mathscr{P}(v,s,\mathfrak{b}_s)ds\right)\right|\right| \leq \mathfrak{G}\Big[||\mathfrak{a}-\mathfrak{b}|| + \left|\left|\int_0^v [\mathscr{P}(v,s,\mathfrak{a}_s) - \mathscr{P}(v,s,\mathfrak{b}_s)]ds\right|\right|\Big]$$

for $\mathfrak{a}, \mathfrak{b} \in PC$.

H₅ $\mathscr{P} : \mathbb{J} \times \mathbb{J} \times \mathbb{D} \to S$ is continuous and $\exists\ \tau_1 > 0 \ni$ for every $(t,s) \in J \times J$,

$$\left|\left|\int_0^t [\mathscr{P}(v,s,\mathfrak{a}_s) - \mathscr{P}(v,s,\mathfrak{b}_s)]ds\right|\right|_S \leq \tau_1 ||\mathfrak{a}-\mathfrak{b}||_{PC}.$$

H₆ The function $I_k : S \to S$ are continuous and $\exists\ \lambda_k > 0$ such that

$$||I_k(\mathfrak{a}) - I_k(\mathfrak{b})||_S \leq \lambda_k ||\mathfrak{a}-\mathfrak{b}||, \mathfrak{a}, \mathfrak{b} \in S,\ k = 1,2,\ldots,l$$

H₇ $\mathscr{P} : \mathbb{J} \times \mathbb{J} \times D \to S$ is continuous and $\exists\ \tau_1 > 0 \ni$ for all $(v,s) \in \mathbb{J} \times \mathbb{J}$

$$\left|\left|\int_0^v [\mathscr{P}(v,s,\mathfrak{a}_s) - \mathscr{P}(v,s,\mathfrak{b}_s)]ds\right|\right|_S \leq \tau_1 ||\mathfrak{a}-\mathfrak{b}||_{PC.}$$

H₈ Let us assume that the value of

$$\max_{1\leq i\leq \tau} [v^{2-\alpha} \tilde{\tau}_T(\tilde{\tau}\lambda_k + \tau_1 + 1) + \tilde{\tau}_S(\tilde{\tau}\lambda_k + L(\tau_1+1))] < 1$$

for our result.

Consider the mild solution as

$$\begin{aligned} z(v) &= \mathscr{G}T_\lambda(v-v_k)[z(v_k^-)+I_kz(v_k^-)] + \frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda}\int_{v_k}^{v}(v-s)^{\lambda-1}\Big[Bu_z(s) \\ &+ \phi\Big(v,z_v,\int_0^v \mathscr{P}(v,s,z_s)ds\Big)\Big]ds + \frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_{v_k}^{v} S_p(v-s)\Big[Bu_z(s) \\ &+ \phi\Big(v,z_v,\int_0^v \mathscr{P}(v,s,z_s)ds\Big)\Big]ds, v\in(v_k,v_{k+1}]; k=1,2,\ldots,l. \end{aligned}$$

The linear operator corresponding to control term u is

$$\mathbb{W}_u = \frac{k\mathscr{G}(1-\lambda)}{B(\lambda)\Gamma\lambda}\int_{v_k}^{v}(v-s)^{\lambda-1}Bu_z(s)ds + \frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_{v_k}^{v}S_\lambda(v-s)Bu_z(s)ds, v\in(v_k,v_{k+1}].$$

Transfer the system from it intial state to final state $u\to v_{k+1}$

$$\mathbb{W}_u = \frac{k\mathscr{G}(1-\lambda)}{B(\lambda)\Gamma\lambda}\int_{v_k}^{v_{k+1}}(v-s)^{\lambda-1}Bu_z(s)ds + \frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_{v_k}^{v_{k+1}}S_\lambda(v-s)Bu_z(s)ds.$$

Now in solution, the corresponding term is replaced by $\mathbb{W}_u$

$$\begin{aligned} z(v) &= \mathscr{G}T_\lambda(v-v_k)[z(v_k^-)+I_k(z(v_k^-))] + \mathbb{W}_u \\ &+ \frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda}\int_{v_k}^{v}(v-s)^{\lambda-1}\phi\Big(v,z_v,\int_0^v \mathscr{P}(v,s,z_s)ds\Big)ds \\ &+ \frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_{v_k}^{v}S_p(v-s)\phi\Big(v,z_v,\int_0^v \mathscr{P}(v,s,z_s)ds\Big)ds, v\in(v_k,v_{k+1}]. \end{aligned}$$

The control is defined by

$$\begin{aligned} u(v) &= \mathbb{W}^{-1}\Big[z_1 - \mathscr{G}T_\lambda(v-v_k)[z(v_k^-)+I_k(z(v_k^-))] \\ &- \frac{k\mathscr{G}(1-\lambda)}{B(\lambda)\Gamma\lambda}\int_{v_k}^{v}(v-s)^{\lambda-1}\phi\Big(v,z_v,\int_0^v \mathscr{P}(v,s,z_s)ds\Big)ds \\ &- \frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_{v_k}^{v}S_p(v-s)\phi\Big(v,z_v,\int_0^v \mathscr{P}(v,s,z_s)ds\Big)\Big]ds,\ v\in(v_k,v_{k+1}]. \end{aligned}$$

The solution of the system $(1.1)-(1.3)$ is (3.1)

We define the operator $N : PC([-\eta,\mathfrak{T}],S)\to PC([-\eta,\mathfrak{T}],S)$ by

$$N(z(v)) = \begin{cases} \psi(v),\ v\in[-\eta,0] \\ \mathscr{G}T_\lambda(v)\psi(0) + \frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda}\int_0^v(v-s)^{\lambda-1}[Bu_z(s)+\phi(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds)]ds \\ +\frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_0^v S_p(v-s)[Bu_z(s)+\phi(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds)]ds,\ v\in[0,v_1] \\ \mathscr{G}T_\lambda(v-v_k)(z(v_k^-)+I_k(z(v_k^-)+\frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda}\int_{v_k}^v(v-s)^{\lambda-1}[Bu_z(s) \\ +\phi(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds)]ds + \frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_{v_k}^v S_p(v-s)[Bu_z(s) \\ +\phi(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds)]ds,\ v\in(v_k,v_{k+1}],\ k=1,...,l. \end{cases}$$

Note that N is well defined on $\mathscr{PC}([-\eta,\mathbb{T}],S)$. Let us take $v\in(0,v_1]$ & $\mathfrak{a},\mathfrak{b}\in PC([-\eta,\mathbb{T}],S)$

$$
\begin{aligned}
||N(\mathfrak{a}(v))-N(\mathfrak{b}(v))|| \leq & \Big|\Big|\mathscr{G}T_\lambda(v)\psi(0)+\frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda}\int_0^v (v-s)^{\lambda-1}\Big[Bu_{\mathfrak{a}}(s) \\
& +\phi\Big(v,\mathfrak{a}_v,\int_0^v \mathscr{P}(v,s,\mathfrak{a}_s)ds\Big)\Big]ds \\
& +\frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_0^v S_p(v-s)\Big[Bu_{\mathfrak{a}}(s) \\
& +\phi\Big(v,\mathfrak{a}_v,\int_0^v \mathscr{P}(v,s,\mathfrak{a}_s)ds\Big)\Big]ds-\mathscr{G}T_\lambda(v)\psi(0) \\
& -\frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda}\int_0^v (v-s)^{\lambda-1}\Big[Bu_{\mathfrak{b}}(s)+\phi\Big(v,\mathfrak{b}_v,\int_0^v \mathscr{P}(v,s,\mathfrak{b}_s)ds\Big)\Big]ds \\
& -\frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_0^v S_p(v-s)\Big[Bu_{\mathfrak{b}}(s)+\phi\Big(v,\mathfrak{b}_v,\int_0^v \mathscr{P}(v,s,\mathfrak{b}_s)ds\Big)\Big]ds\Big|\Big| \\
\leq & \Big|\Big|\frac{k\mathscr{G}(1-\lambda)}{B(\lambda)\Gamma\lambda}\Big[\int_0^v (v-s)^{\lambda-1}B[u_{\mathfrak{a}}(s)-u_{\mathfrak{b}}(s)]ds \\
& +\int_0^v (v-s)^{\lambda-1}\Big[\phi\Big(v,\mathfrak{a}_v,\int_0^v \mathscr{P}(v,s,\mathfrak{a}_s)ds\Big) \\
& -\phi\Big(v,\mathfrak{b}_v,\int_0^v \mathscr{P}(v,s,\mathfrak{b}_s)ds\Big)\Big]ds]+\frac{\lambda\mathscr{G}^2}{B(\lambda)}\Big[\int_0^v S_\lambda(v-s)B[u_{\mathfrak{a}}(s)-u_{\mathfrak{b}}(s)]ds \\
& +\int_0^v S_\lambda(v-s)\Big[\phi\Big(v,\mathfrak{a}_v,\int_0^v \mathscr{P}(v,s,\mathfrak{a}_s)ds\Big)-\phi\Big(v,b_v,\int_0^v \mathscr{P}(v,s,\mathfrak{b}_s)ds\Big)\Big]ds\Big]\Big|\Big| \\
\leq & v^{1-\alpha}\tilde{\tau}_T\tilde{\tau}\lambda_k||\mathfrak{a}-\mathfrak{b}||_S v+v^{1-\alpha}\tilde{\tau}_T L\Big[||\mathfrak{a}-\mathfrak{b}|| \\
& +\Big|\Big|\int_0^v \mathscr{P}(v,s,\mathfrak{a}_s)ds-\int_0^v \mathscr{P}(v,s,\mathfrak{b}_s)ds\Big|\Big|\Big]v+\tilde{\tau}_s\tilde{\tau}\lambda_k||\mathfrak{a}-\mathfrak{b}||_S v \\
& +\tilde{\tau}_s L\Big[||\mathfrak{a}-\mathfrak{b}||+\Big|\Big|\int_0^v \mathscr{P}(v,s,\mathfrak{a}_s)ds-\int_0^v \mathscr{P}(v,s,\mathfrak{b}_s)ds\Big|\Big|\Big]v \\
\leq & v^{1-\alpha}\tilde{\tau}_T\tilde{\tau}\lambda_k|\mathfrak{a}-\mathfrak{b}||_S v+v^{1-\alpha}\tilde{\tau}_T L\Big[||\mathfrak{a}-\mathfrak{b}||+\tau_1||\mathfrak{a}-\mathfrak{b}||_{PC}\Big]v \\
& +\tilde{\tau}_S\tilde{\tau}\lambda_k||\mathfrak{a}-\mathfrak{b}||_S v+\tilde{\tau}_S L\Big[||\mathfrak{a}-\mathfrak{b}||+\tau_1||\mathfrak{a}-\mathfrak{b}||_{PC}]v \\
\leq & v^{2-\alpha}\tilde{\tau}_T\tilde{\tau}\lambda_k||\mathfrak{a}-\mathfrak{b}||_{PC}v+v^{1-\alpha}\tilde{\tau}_T\Big[||\mathfrak{a}-\mathfrak{b}||_{PC}+\tau_1||\mathfrak{a}-\mathfrak{b}||_{PC}\Big]v \\
& +\tilde{\tau}_S\tilde{\tau}\lambda_k||\mathfrak{a}-\mathfrak{b}||_{PC}+\tilde{\tau}_S L\Big[||\mathfrak{a}-\mathfrak{b}||_{PC}+\tau_1||\mathfrak{a}-\mathfrak{b}||_{PC}\Big]v \\
\leq & \Big(v^{2-\alpha}\tilde{\tau}_T\tilde{\tau}\lambda_k+v^{1-\alpha+1}\tilde{\tau}_T+v^{1-\alpha+1}\tilde{\tau}_T\tau_1\Big)||\mathfrak{a}-\mathfrak{b}||_{PC} \\
& +\Big(\tilde{\tau}_S\tilde{\tau}\lambda_k+\tilde{\tau}_S L(\tau_1+1)\Big)||\mathfrak{a}-\mathfrak{b}||_{PC} \\
\leq & v^{2-\alpha}\Big(\tilde{\tau}_T\tilde{\tau}\lambda_k+\tilde{\tau}_T+\tilde{\tau}_T\tau_1\Big)||\mathfrak{a}-\mathfrak{b}||_{PC} \\
& +\tilde{\tau}_S\Big(\tilde{\tau}\lambda_k+L(\tau_1+1)\Big)||\mathfrak{a}-\mathfrak{b}||_{PC} \\
\leq & v^{2-\alpha}\tilde{\tau}_T\Big(\tilde{\tau}\lambda_k+\tau_1+1\Big)||\mathfrak{a}-\mathfrak{b}||_{PC}+\tilde{\tau}_S\Big(\tilde{\tau}\lambda_k+L(\tau_1+1)\Big)]||\mathfrak{a}-\mathfrak{b}||_{PC} \\
\leq & \Big[v^{2-\alpha}\tilde{\tau}_T\Big(\tilde{\tau}\lambda_k+\tau_1+1\Big)+\tilde{\tau}_S\Big(\tilde{\tau}\lambda_k+L(\tau_1+1)\Big)\Big]||\mathfrak{a}-\mathfrak{b}||_{PC} \\
||N\mathfrak{a}(v)-N\mathfrak{b}(v)|| \leq & ||\mathfrak{a}-\mathfrak{b}||_{PC}.
\end{aligned}
$$

holds for $v \in (0, v_1)$, $v \in (v_1, v_2]$, $v \in (v_i, v_{i+1}]$ & $v \in (v_{i+1}, T]$. Thus for all $v \in [0, \mathbb{T}]$, we have,

$$||N\mathfrak{a}(v) - N\mathfrak{b}(v)||_{PC} \leq \max_{1 \leq i \leq \tau} [v^{2-\alpha} \tilde{\tau}_T (\tilde{\tau}\lambda_k + \tau_1 + 1) + \tilde{\tau}_S(\tilde{\tau}\lambda_k + L(\tau_1 + 1))]||\mathfrak{a} - \mathfrak{b}||_{PC}.$$

Since

$$\max_{1 \leq i \leq \tau} [v^{2-\alpha} \tilde{\tau}_T (\tilde{\tau}\lambda_k + \tau_1 + 1) + \tilde{\tau}_S(\tilde{\tau}\lambda_k + L(\tau_1 + 1))] < 1$$

Thus N is a Contraction map.
$\therefore N$ has a unique fixed point by Banach contraction principle.
This completes the proof.

5. Nonlocal controllability result

Here, we investigate the controllability system with a nonlocal condition

$$z(0) + \mathscr{P}(z) = z_0 \tag{5.1}$$

where $\mathscr{P} : PC(\mathbb{J}, S) \to S$ is a given function. Nonlocal conditions were initiated by Byszewski and Lakshmikantham when they works on the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems [12]. As remarked by them, the nonlocal condition is better applicable than the initial condition $z(0) = z_0$.

For example,

$$\mathscr{P}(z) = \sum_{i=1}^{n} c_i z(v_i),$$

where $c_i (i = 1, 2, \ldots, l)$ are constants and $0 < v_1 < v_2 < \ldots < v_l < b$.

Definition 5.1. A function $z(\cdot) \in \mathscr{B}_1$ is said to be solution of system $(1.1) - (1.2)$ with (5.1) if the impulsive condition $\Delta z(v)|_{v = v_k^-} = I_k(z(v_k^-)),\ k = 1, 2, .., l.$ is verified the restriction of $z(\cdot)$ to K, $k = 1, 2, ..., l$ is continuous and satisfies the following :

$$z(v) = \begin{cases} \psi(v),\ v \in [-\eta, 0] \\ \mathscr{G}T_\lambda(v)(\psi_0 - \mathscr{P}(\psi)) + \frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda} \int_{v_k}^{v} (v-s)^{\lambda-1} \left[Bu_z(s) + \phi\left(v, z_v, \int_0^v \mathscr{P}(v, s, z_s) ds \right) \right] ds \\ \qquad + \frac{\lambda \mathscr{G}^2}{B(\lambda)} \int_{v_k}^{v} S_p(v-s) \left[Bu_z(s) + \phi\left(v, z_v, \int_0^v \mathscr{P}(v, s, z_s) ds \right) \right] ds,\ v \in [0, v_1] \\ \mathscr{G}T_\lambda(v - v_k)(z(v_k^-)) + I_k(z(v_k^-)) + \frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda} \int_{v_k}^{v} (v-s)^{\lambda-1} \Big[Bu_z(s) + \\ \qquad \phi\left(v, z_v, \int_0^v \mathscr{P}(v, s, z_s) ds \right) \Big] ds + \frac{\lambda \mathscr{G}^2}{B(\lambda)} \int_{v_k}^{v} S_p(v-s) \Big[Bu_z(s) \\ \qquad + \phi\left(v, z_v, \int_0^v \mathscr{P}(v, s, z_s \right) ds) \Big] ds\ v \in [v_k, v_{k+1}], \\ k = 1, 2, ..., l. \end{cases} \tag{5.2}$$

Definition 5.2. If for every initial function $\psi \in \mathscr{B}_1$ and $z_0, z_1 \in S$, $\exists$ a control $\mathfrak{u} \in L^2(\mathbb{I}, U)$ such that $z(\cdot)$ of $(1.1) - (1.2)$ with (5.1) satisfies $z(0) + \mathscr{P}(z) = z_0$ and $z(\mathfrak{T}) = z_1$, then the system $(1.1) - (1.2)$ with (5.1) is said to be controllable on $\mathbb{I}$.

6. Main Result:

Let us assume that the Hypothesis **(H1)** to **(H8)**,

The solution of the system $(1.1)-(1.2)$ with (5.1) is

$$z(v)=\begin{cases}\psi(v),\ v\in[-\eta,0]\\ \mathscr{G}T_\lambda(v)(\psi_0-\mathscr{P}(\psi))+\frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda}\int_0^v(v-s)^{\lambda-1}\Big[Bu_z(s)+\phi\Big(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds\Big)\Big]ds\\ \qquad+\frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_0^v S_p(v-s)\Big[Bu_z(s)+\phi\Big(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds\Big)\Big]ds,\ v\in[0,v_1]\\ \mathscr{G}T_\lambda(v-v_k)(z(v_k^-)+I_k(z(v_k^-))+\frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda}\int_{v_k}^v(v-s)^{\lambda-1}\Big[Bu_z(s)\\ \qquad+\phi\Big(v,z_v,\int_0^v\mathscr{P}(v,s,z_s\Big)ds)\Big]ds+\frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_{v_k}^v S_p(v-s)\Big[Bu_z(s)\\ \qquad+\phi\Big(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds\Big)\Big]ds,\ v\in(v_k,v_{k+1}],\ k=1,...,l.\end{cases}$$

We define the operator $N:PC([-\eta,\mathbb{T}],S)\to PC([-\eta,\mathbb{T}],S)$ by

$$Nz(v)=\begin{cases}\psi(v),\qquad v\in[-\eta,0]\\ \mathscr{G}T_\lambda(v)(\psi_0-\mathscr{P}(\psi))+\frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda}\int_0^v(v-s)^{\lambda-1}\Big[Bu_z(s)+\phi\Big(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds\Big)\Big]ds\\ \qquad+\frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_0^v S_p(v-s)\Big[Bu_z(s)+\phi\Big(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds\Big)\Big]ds\qquad v\in[0,v_1]\\ \mathscr{G}T_\lambda(v-v_k)(z(v_k^-)+I_k(z(v_k^-))+\frac{k\mathscr{G}(1-\lambda)}{B(P)\Gamma\lambda}\int_{v_k}^v(v-s)^{\lambda-1}\Big[Bu_z(s)+\phi\Big(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds\Big)\Big]ds\\ \qquad+\frac{\lambda\mathscr{G}^2}{B(\lambda)}\int_{v_k}^v S_p(v-s)\Big[Bu_z(s)+\phi\Big(v,z_v,\int_0^v\mathscr{P}(v,s,z_s)ds\Big)\Big]ds\qquad v\in[v_k,v_{k+1}],\\ k=1,...,l.\end{cases}$$

has a fixed point. This fixed point is then a solution of the control problem $(1.1)-(1.2)$ with (5.1). Clearly, $Nz(\mathfrak{T})=z_1$, which means that the control u steers the system $(1.1)-(1.2)$ with (5.1) from the initial state z_0 to z_1 in time $\mathfrak{T}$ provided, we can obtain a fixed point of the operator N. The rest of the proof is similar to section 4, hence omitted.

7. Example

Take the following semilinear differential equation of fractional order:

$$\begin{cases}{}^{ABC}\mathbb{D}_y^\lambda\nu(y,\delta)=\frac{\partial^2\nu}{\partial\delta^2}(y,\delta)+\zeta(y,\delta)+\frac{1}{10}\left(\frac{e^{-y}}{1+e^y}\right)\left(\frac{|\nu(y-\tau,\delta)|}{1+|\nu(y-\tau,\delta)|}\right)\qquad y\in[0,1],y\neq\frac{1}{2}\\ \nu(y,0)=\nu(y,\pi)=0,\qquad y\in[0,1]\\ \nu(y,\delta)=\psi(y,\delta),\qquad y\in[-s,0],\ \delta\in[0,\pi]\\ \Delta\nu(\frac{1}{2},\delta)=\frac{|\nu(\frac{1}{2}^-,\delta)|}{25+|\nu(\frac{1}{2}^-,\delta)|}\qquad\delta\in[0,\pi].\end{cases}\tag{7.1}$$

Where $\gamma\in(0,1),s>0$ and $\phi\in\mathbb{B}=\{h:[-s,0]\times[0,\pi]\to\mathbb{R}$, h is continuous everywhere except for a countable number of points at which $h(s^-),h(s^+)$ exist with $h(s^-)=h(s)\},\nu(\frac{1}{2}^+,\delta)=$ $\lim_{(h,\delta)\to(0^+,\delta)}\nu(\frac{1}{2}+h,\delta),\ \nu(\frac{1}{2}^-,\delta)=\lim_{(h,\delta)\to(0^-,\delta)}\nu(\frac{1}{2}+h,\delta)$.
Set $E=L^2[0,\pi]$ and $A:D(A)\subset E\to E$ an operator defined by $A\varphi=\varphi"$, $\varphi\in D(A)$, with domain

$D(A)=\{\varphi \in E; \varphi, \varphi^{'}$ are absolutely continuous, $\varphi^{"} \in E, \varphi(0)=\varphi(1)=0\}$,Then $A\varphi=\sum_{n=1}^{\infty} n^2(\varphi,\varphi_n)\varphi_n$, $\varphi \in D(A)$. Here $\varphi_n(s)=\sqrt{\frac{2}{\pi}}sin(ns), n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is a generator of an analytic semigroup $(T(y))_{y\geq 0}$ in E given by $T(y)\varphi=\sum_{n=1}^{\infty} e^{-n^2 y}(\varphi,\varphi_n)\varphi_n$, $\varphi \in E,\ y>0$ Hence $(T(y))_{y\geq 0}$ is a uniformly bounded compact semigroup, in order that $R(\kappa,A)=(\kappa-A)^{-1}$ is a compact operator for all $\kappa \in \rho(A)$, which means that $A \in A^{\lambda}(\beta_0,\varphi_0)$.In addition, the subordination principle of solution operator $(T_{\lambda}(y))_{y\geq 0}$ such that $\|T_{\lambda}(y)\| \leq \widehat{M}$ for $y \in [0,1]$. Therefore, for $(y,\delta) \in [0,1]\times[0,\pi], \beta \in [-s,0]$ and $\phi \in C([0,1],[-s,1])$, where

$$
\begin{aligned}
z(y)(\delta) &= \nu(y,\delta),\\
g(y,\psi,\nu(\phi(y)))(\delta) &= \frac{1}{10}\left(\frac{e^{-y}}{1+e^{y}}\right)\left(\frac{|\nu(y-\tau,\delta)|}{1+|\nu(y-\tau,\delta)|}\right)\\
\Delta\nu(\frac{1}{2})(\delta) &= \frac{|\nu(\frac{1}{2}^{-},\delta)|}{25+|\nu(\frac{1}{2}^{-},\delta)|}\\
Bu(y)(\delta) &= \nu(y,\delta)\\
\psi(y)(\delta) &= \nu(y,\delta), \quad y \in [-s,0]
\end{aligned}
$$

The system (7.1) is the abstract form of $(1.1)-(1.3)$. Further, the conditions $(H1)-(H3)$ are satisfied. Hence by our Main result, the fractional semilinear system (7.1) is controllable on I.

8. Conclusion

Controllability is one of the key area where the researchers have actively working for various classes of fractional differential equation due to its wide applications. Here, we have discussed and analysed the controllablility of IIDE with initial and nonlocal conditions.

We suggest that it would be very useful if someone works on qualitative results for the mentioned class in the framework of the other well-known additional conditions.

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