# Global uniqueness in an inverse problem for a class of damped stochastic plate equations

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**Abstract**

This paper deals with the global uniqueness of an inverse problem for the stochastic plate with structural damping. The key point is the Carleman estimate for the fourth order stochastic plate operators $dy_t - \rho\Delta y_t dt + \Delta^2 y dt$. To this aim, a weighted pointwise identity for a fourth order stochastic plate operator is established, via which we obtained the desired Carleman estimate for the corresponding stochastic plate equation with structural damping.

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**Key Words**. stochastic plate equation, structurally damping, inverse problem, Carleman estimate.

## 1 Introduction and main results

In this paper, we study the global uniqueness in an inverse problem for a class of damped stochastic plate equations by establishing a global Carleman estimate. To begin with, we introduce some basic spaces used in stochastic framework.

Fix a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, on which a one-dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined such that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $B(\cdot)$, augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $T > 0$ and $\mathcal{H}$ be a Banach space, and let $L^2_{\mathbf{F}}(0, T; \mathcal{H})$ be the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;\mathcal{H})}) < \infty$, with the canonical norm; by $L^\infty_{\mathbf{F}}(0, T; \mathcal{H})$ the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted essentially bounded processes; and by $L^2_{\mathbf{F}}(\Omega; C([0, T]; \mathcal{H}))$ the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0,T];\mathcal{H})}) < \infty$. Similarly, one can define $L^\infty_{\mathbf{F}}(\Omega; C^m([0, T]; \mathcal{H}))$ for any positive integer $m$.

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Let $l > 0$, $x = (x_1, x') \in \mathbb{R}^n$, $x' = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$, and $G' \subset \mathbb{R}^{n-1}$ be a bounded domain with $C^4$ boundary. Denote $G = (0, l) \times G'$, $Q = G \times (0, T)$, and $\Sigma = \Gamma \times (0, T)$, where $\Gamma$ is the boundary of $G$.

We are interesting in the following damped stochastic plate equation:

$$\begin{cases} dy_t - \rho \Delta y_t dt + \Delta^2 y dt = \Big[a_1 y + H(t, x')R(t, x)\Big]dt \\ \qquad\qquad\qquad\qquad\qquad + \Big[a_2 y + F(t, x')R(t, x)\Big]dB(t) & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = 0, \ y_t(0) = 0 & \text{in } G. \end{cases} \tag{1.1}$$

Here $\rho > 0$, $a_1 \in L^\infty_{\mathbf{F}}(0, T; W^{1,\infty}(G))$, $a_2 \in L^\infty_{\mathbf{F}}(0, T; W^{2,\infty}(G))$, $R \in C^5([0, T] \times \overline{G})$ are given functions, and $H \in L^2_{\mathbf{F}}(0, T; H^2(G'))$, $F \in L^\infty_{\mathbf{F}}(0, T; H^4(G'))$ are two unknown functions. Referring to [19, 33], it is easy to show that (1.1) admits a unique strong solution

$$y \in L^2_{\mathbf{F}}(\Omega; C([0, T]; U)) \cap L^2_{\mathbf{F}}(\Omega; C^1([0, T]; H^2(G) \cap H^1_0(G))).$$

where

$$U = \{y \in H^4(G) \cap H^1_0(G) \ \Big| \ \Delta y|_\Gamma = 0\}$$

The main purpose of this paper is to study the inverse source problem of the system (1.1), that is, to determine $H$ and $F$ simultaneously from the boundary observation $\frac{\partial y}{\partial \nu}|_{[0,t_0]\times\Gamma}$ and $\frac{\partial \Delta y}{\partial \nu}|_{[0,t_0]\times\Gamma}$, where $t_0 \in (0, T)$ and $\nu = \nu(x)$ denotes the unit outer normal vector of $G$ at $x \in \Gamma$. More precisely, we study the following problems:

**Inverse Source Problem:** Let $R$ be given and $t_0 \in (0, T)$. Does $\frac{\partial y}{\partial \nu}|_{[0,t_0]\times\Gamma} = \frac{\partial \Delta y}{\partial \nu}|_{[0,t_0]\times\Gamma} = 0$ imply $H(t, x') = F(t, x') = 0$ in $(0, t_0) \times G'$, $\mathbb{P}$-$a.s.$?

The undamped plate equation ($\rho = 0$) is presented as a linear model of a rigid vibrating body in which the potential energy contains curvature-like terms. Energy dissipation is ignored in this model and the equation has no smoothing effect ([5]). One adds damping terms to incorporate the loss of energy. Structural damping describes a situation where higher frequencies are more strongly damped than low frequencies. Structurally damped beam equations and plate equations have been studied intensively in the deterministic case (e.g [8, 17, 18, 22]). In such equations, the damping term has "half of the order" of the leading elastic term, as it was proposed in the seminal paper [5, 20].

It is well known that there are numerous studies on inverse problems for deterministic partial differential equations (e.g [3, 2, 13, 25]). In this respect, we mention [26, 31] for a study of an inverse source problem of plate equations, and the identified sources are independent of spatial variables. Stability estimates for a plate problem were studied in [10, 23], the Lipschitz stability for a Kirchhoff plate equation was obtained in [30], and inverse source problems of Euler-Bernoulli plate equations were studied in [1]. However, unlike the deterministic counterpart, the solution of a stochastic equation cannot have derivatives with respect to time variable, and the stochastic equation is time-irreversible. For the inverse problems of stochastic partial differential equations, we refer to [15, 24, 29] for the heat equations and [16, 28] for the wave equations. In the case of inverse problems for the fourth order stochastic plate equations with structural damping, to the best of our knowledge, there is no references addressing it. In this paper, we study the global uniqueness problem for stochastic plate equation with structural damping.

The main result of this paper is the following uniqueness result:

**Theorem 1.1** *Assume that $t_0 \in (0,T)$, and $|R(t,x)| \neq 0$ for all $(t,x) \in [0,t_0] \times \overline{G}$. Let $y \in L^2_{\mathbf{F}}(\Omega; C([0,T];U)) \cap L^2_{\mathbf{F}}(\Omega; C^1([0,T]; H^2(G) \cap H^1_0(G)))$ satisfies (1.1). If*

$$\frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = 0 \quad \text{on} \quad [0,t_0] \times \Gamma, \ \mathbb{P}\text{-a.s.}, \tag{1.2}$$

*then*

$$H(t,x') = F(t,x') = 0 \quad \text{for all} \quad (t,x') \in [0,t_0] \times G', \ \mathbb{P}\text{-a.s.}$$

**Remark 1.1** *In this paper, we consider stochastic plate equation with hinged boundary conditions. It would be interesting to consider other kind of boundary conditions for the stochastic plate equation. For example, the boundary conditions in (1.1) is changed to the clamped boundary conditions $y = \frac{\partial y}{\partial \nu} = 0$ on $\Sigma$. The method developed in this paper also can deal with the inverse source problem in this situation, i.e. if $\Delta y = \frac{\partial \Delta y}{\partial \nu} = 0$ on $[0,t_0] \times \Gamma, \mathbb{P}$-a.s., then $H(t,x') = F(t,x') = 0$ in $[0,t_0] \times G'$, $\mathbb{P}$-a.s.*

In order to prove Theorem 1.1, the tool we employed is the global Carleman estimate, which can be regarded as weighted energy estimates. To this aim, we first give the choice of our weight functions.

Let $\lambda, \mu > 0$ and $\psi(t) \in C^3(\mathbb{R})$ with $|\psi_t| > 0$. Write

$$\xi(t) = \lambda \varphi(t), \ \varphi(t) = e^{\mu \psi(t)}, \ \text{and} \ \ \theta = e^{\xi}. \tag{1.3}$$

Throughout of this paper, $C = C(a_1, a_2, R, T, G)$ is used to denote a generic positive constant which may vary from line to line. The global Carleman estimate for (1.1) is the following:

**Theorem 1.2** *Let $\rho > 0$, $\psi(t) = -t$, and $\theta, \varphi$ be given in (1.3). There exists a constant $\mu_1 > 0$ such that for all $\mu \geq \mu_1$, one can find two positive constants $C$ and $\lambda_1 = \lambda_1(\mu)$, such that for all $y$ satisfying*

$$dy_t - \rho \Delta y_t dt + \Delta^2 y dt = f dt + g dB(t) \tag{1.4}$$

*with $f \in L^2_{\mathbf{F}}(0,T; L^2(G))$, $g \in L^2_{\mathbf{F}}(0,T; H^1(G))$ and $y = \Delta y = 0$ on $\Gamma$, for all $\lambda \geq \lambda_1$, it holds that*

$$\begin{aligned}
&\mathbb{E}\int_Q \theta^2 \Big[\lambda^3\mu^4\varphi^3|y|^2 + \lambda^2\mu^3\varphi^2|\nabla y|^2 + \lambda\mu^2\varphi\Big(|\Delta y|^2 + |y_t|^2\Big) \\
&\qquad + \mu\Big(|\nabla \Delta y|^2 + |\nabla y_t|^2\Big)\Big] dxdt \\
&\leq C(\rho)\Big[\mathbb{E}\int_Q \theta^2 |f|^2 dxdt + \mathbb{E}\int_Q \theta^2\Big(\lambda\mu\varphi|g|^2 + |\nabla g|^2\Big) dxdt \\
&\qquad + e^{C\lambda}\Big(\mathbb{E}|(y(0), y_t(0))|^2_{H^3(G)\times L^2(G)} + \mathbb{E}|\nabla y_t(T)|^2_{L^2(G)}\Big)\Big],
\end{aligned} \tag{1.5}$$

*where*

$$C(\rho) = C\Big(1 + \rho^2 + \frac{1}{\rho^2}\Big). \tag{1.6}$$

**Remark 1.2** *It is obviously that blow-up phenomena for constant $C(\rho)$ could occur when $\rho = 0$, which means that the same result cannot be obtained with our method for the undamped plate equation, even in one-dimensional case. In our forthcoming paper, by adding extra observational information, we can obtain the global uniqueness of the following stochastic beam equation:*

$$\begin{cases} dy_t + y_{xxxx}dt = \Big[a_1 y + H(t)R(t,x)\Big]dt + \Big[a_2 y + F(t)R(t,x)\Big]dB(t) & \text{in } (0,1)\times(0,T), \\ y(0,t) = y(1,t) = y_{xx}(0,t) = y_{xx}(1,t) = 0 & \text{on } (0,T), \\ y(0) = 0, \ y_t(0) = 0 & \text{in } (0,1). \end{cases}$$

*More precisely, by $y_x(0,t) = y_x(1,t) = y_{xxx}(0,t) = y_{xxx}(1,t) = 0$ on $(0,T)$ and $y = 0$ in $G_0 \times (0,T)$, where $G_0$ be a given nonempty open subset of $(0,1)$, we can also determine $H$ and $F$ at the same time. As far as we know, multi-dimensional case (i.e.(1.1) in the case of $\rho = 0$) is an unsolved problem.*

**Remark 1.3** *Although this paper deals with stochastic plate equations with structural damping, it is easy to know that all results in this paper cover the deterministic case.*

**Remark 1.4** *The Carleman estimate also can be established by decomposing the stochastic plate operator. Put*

$$w = y_t - (a - ib)\Delta y, \quad a = \frac{\rho}{2}, \quad b^2 = \frac{4-\rho^2}{4},$$

*where $i = \sqrt{-1}$, $a > 0$, $b \in \mathbb{R}$ and $a^2 + b^2 = 1$. Therefore, (1.4) can be written equivalently as the following system*

$$\begin{cases} dw - (a+ib)\Delta w dt = f dt + g dB(t) & \text{in } Q, \\ y_t - (a-ib)\Delta y = w & \text{in } Q. \end{cases}$$

*Then, the global Carleman estimate for the damped plate operator can be proved by using the Carleman estimate for stochastic complex Ginzburg-Landau operator in an iteration manner. But this method only work for $\rho \in (0,2)$, so we establish the estimate directly from the operator itself.*

**Remark 1.5** *The reason why we choose the general weight function in (1.3) instead of $\psi(t) = -t$ is that other kind choice of $\psi(t)$ can be used to study other inverse problems. For example, $\psi(t) = t$ is chosen to consider backward uniqueness.*

The rest of this paper is organized as follows. In Section 2, we devote to establishing the global Carleman estimates for structurally damped stochastic plate equation. Finally, we give the detailed proof of the inverse problem results in Section 3.

## 2 Proof of Theorem 1.2

This section is devoted to establishing the global Carleman estimate for the fourth stochastic plate operator $dy_t - \rho\Delta y_t dt + \Delta^2 y dt$.

Carleman estimate (1.5) is derived by a weighted identity for a structurally damped stochastic plate operator. Carleman-type estimates were introduced by T. Carleman in 1939 to prove the unique continuation property for some elliptic equations in [4]. Up to now, they have become one of the useful tools for studying deterministic partial differential equations and the related control and inverse problems. The weighted identity method is one of the the significant ways in establishing global Carleman estimates for deterministic partial differential equations (see [9, 7] for example). Recently, Carleman estimates for stochastic partial differential equations are getting more and more attention. We refer to [14, 21, 27] for stochastic parabolic equation, [32] for stochastic hyperbolic equation, [6] for stochastic complex Ginzburg-Landau equations, [11, 12] for stochastic Kuramoto-Sivashinsky equation, [33] for stochastic beam equation and so on. The main idea of this approach is to multiply the partial differential operator by an appropriate multiplier. Then, the product is transformed into a sum of divergence terms, energy terms, and some good terms. By integrating this identity and choosing parameters large enough to absorb the undesired terms, which is the key idea of Carleman estimates.

### 2.1 A pointwise weighted identify

Let

$$\mathcal{L}y = dy_t - \rho\Delta y_t dt + \Delta^2 y dt.$$

Set $z = \theta y$, recalling the definition of $\theta$, it is easy to show that

$$\theta\mathcal{L}y = dz_t - 2\xi_t z_t dt + (\xi_t^2 - \xi_{tt})z dt - \rho\Delta z_t dt + \rho\xi_t\Delta z dt + \Delta^2 z dt = I_1 dt + I_2 - I_3, \quad (2.1)$$

where

$$\begin{cases} I_1 = -2\xi_t z_t - \rho\Delta z_t + c_1\mu\Delta z \\ I_2 = dz_t + \Delta^2 z dt + (\xi_t^2 - \xi_{tt})z dt + \rho\xi_t\Delta z dt + c_2\mu z_t dt, \\ I_3 = c_1\mu\Delta z dt + c_2\mu z_t dt. \end{cases} \quad (2.2)$$

Here, two auxiliary functions with coefficients $c_1$ and $c_2$ in this weighted identity are introduced to make sure that the coefficients in front of the energy terms are positive. We have the following weighted identity, which will play a fundamental role in what follows.

**Lemma 2.1** *Let $c_1, c_2$ are given constants. Assume that $y$ is an $H^4(\mathbb{R}^n)$-valued continuous* *semi-martingale. Set $\theta = e^{\xi}$ and $z = \theta y$. Then, one has the following equality:*

$$
\begin{aligned}
&I_1\Big(\theta\mathcal{L}y + c_1\mu\Delta z dt + c_2\mu z_t dt\Big)\\
&= I_1^2 dt + dM + \nabla\cdot V + \Big(3\xi_{tt}\xi_t^2 - \xi_{tt}^2 - \xi_t\xi_{ttt}\Big)|z|^2 dt + \Big(\xi_{tt} - 2c_2\mu\xi_t\Big)|z_t|^2 dt\\
&\quad + \Big[-3\rho\xi_t\xi_{tt} + \frac{\rho}{2}\xi_{ttt} - c_1\mu(\xi_t^2 - \xi_{tt})\Big]|\nabla z|^2 dt + \Big(c_1\mu + c_2\rho\mu\Big)|\nabla z_t|^2 dt\\
&\quad + \Big(\xi_{tt} + \frac{\rho^2}{2}\xi_{tt} + c_1\rho\mu\xi_t\Big)|\Delta z|^2 dt - c_1\mu|\nabla\Delta z|^2 dt + \xi_t(dz_t)^2 - \frac{\rho}{2}|\nabla dz_t|^2,
\end{aligned}
\tag{2.3}
$$

*where $I_1$ is given by* (2.2) *and*

$$
\left\{
\begin{aligned}
&V = -\rho\nabla z_t dz_t + \Big[-c_1\mu z_t\nabla z_t + (-2\xi_t z_t - \rho\Delta z_t + c_1\mu\Delta z)\nabla\Delta z + 2\xi_t\Delta z\nabla z_t\\
&\qquad + (\xi_t^2 - \xi_{tt})(-\rho\nabla z_t + c_1\mu\nabla z)z - 2\rho\xi_t^2 z_t\nabla z + c_2\mu(-\rho\nabla z_t + c_1\mu\nabla z)z_t\Big]dt,\\
&M = (-\xi_t z_t + c_1\mu\Delta z)z_t - \frac{\rho}{2}|\nabla z_t|^2 + \frac{\rho}{2}|\nabla\Delta z|^2 - \Big(1 + \frac{\rho^2}{2}\Big)\xi_t|\Delta z|^2\\
&\qquad - \xi_t(\xi_t^2 - \xi_{tt})z^2 + \Big(\frac{3\rho}{2}\xi_t^2 - \frac{\rho^2}{2}\xi_{tt} - \frac{c_1c_2}{2}\mu^2\Big)|\nabla z|^2.
\end{aligned}
\right.
\tag{2.4}
$$

**Proof.** The proof is long, we divide it into several steps.

**Step 1.** By (2.1) and (2.2), it is easy to see that

$$
I_1\Big(\theta\mathcal{L}y + I_3\Big) = I_1^2 dt + I_1 I_2, \tag{2.5}
$$

and

$$
I_1 I_2 = I_1 dz_t + I_1\Delta^2 z dt + (\xi_t^2 - \xi_{tt})I_1 z dt + \rho\xi_t I_1\Delta z dt + c_2\mu I_1 z_t dt. \tag{2.6}
$$

Now, we compute every term in the right side of the sign of equality in (2.6), respectively. First, let us compute $I_1 dz_t$. On the one hand,

$$
\begin{aligned}
&(-2\xi_t z_t - \rho\Delta z_t)dz_t\\
&= d(-\xi_t z_t^2) + \xi_{tt}z_t^2 dt + \xi_t(dz_t)^2 - \nabla\cdot(\rho\nabla z_t dz_t) + d\Big(\frac{\rho}{2}|\nabla z_t|^2\Big) - \frac{\rho}{2}|\nabla dz_t|^2.
\end{aligned}
$$

On the other hand,

$$
\Delta z dz_t = d(\Delta z z_t) - \Delta z_t z_t dt = d(\Delta z z_t) - \nabla\cdot(z_t\nabla z_t)dt + |\nabla z_t|^2 dt.
$$

Combining the above two equalities with (2.2), we find that

$$
\begin{aligned}
&I_1 dz_t\\
&= d\Big(-\xi_t z_t^2 + c_1\mu\Delta z z_t + \frac{\rho}{2}|\nabla z_t|^2\Big) - \nabla\cdot(\rho\nabla z_t dz_t + c_1\mu z_t\nabla z_t dt)\\
&\quad + \xi_{tt}z_t^2 dt + c_1\mu|\nabla z_t|^2 dt + \xi_t(dz_t)^2 - \frac{\rho}{2}|\nabla dz_t|^2.
\end{aligned}
\tag{2.7}
$$

**Step 2.** Let us compute "$I_1\Delta^2 z dt$". Notice that

$$
\begin{aligned}
&(-\rho\Delta z_t + c_1\mu\Delta z)\Delta^2 z dt\\
&= \nabla\cdot\Big[(-\rho\Delta z_t + c_1\mu\Delta z)\nabla\Delta z\Big]dt + d\Big(\frac{\rho}{2}|\nabla\Delta z|^2\Big) - c_1\mu|\nabla\Delta z|^2 dt.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&-2\xi_t z_t\Delta^2 z dt = \nabla\cdot(-2\xi_t z_t\nabla\Delta z)dt + 2\xi_t\nabla z_t\cdot\nabla\Delta z dt\\
&= \nabla\cdot\Big(-2\xi_t z_t\nabla\Delta z + 2\xi_t\nabla z_t\Delta z\Big)dt + d\Big(-\xi_t|\Delta z|^2\Big) + \xi_{tt}|\Delta z|^2 dt.
\end{aligned}
$$

Then the second term of (2.6) reads as

$$
\begin{aligned}
&I_1\Delta^2 z dt\\
&= \nabla\cdot\Big[(-2\xi_t z_t - \rho\Delta z_t + c_1\mu\Delta z)\nabla\Delta z + 2\xi_t\nabla z_t\Delta z\Big]dt + d\Big(\frac{\rho}{2}|\nabla\Delta z|^2 - \xi_t|\Delta z|^2\Big)\\
&\quad+\xi_{tt}|\Delta z|^2 dt - c_1\mu|\nabla\Delta z|^2 dt.
\end{aligned}
\tag{2.8}
$$

**Step 3.** Compute "$(\xi_t^2 - \xi_{tt})I_1 z dt$". First, we have

$$
\begin{aligned}
&-2\xi_t(\xi_t^2 - \xi_{tt})z_t z dt\\
&= d\Big[-\xi_t(\xi_t^2 - \xi_{tt})z^2\Big] + \xi_{tt}(\xi_t^2 - \xi_{tt})z^2 dt + \xi_t(2\xi_t\xi_{tt} - \xi_{ttt})z^2 dt.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&(\xi_t^2 - \xi_{tt})(-\rho\Delta z_t + c_1\mu\Delta z)z dt\\
&= \nabla\cdot\Big[(\xi_t^2 - \xi_{tt})(-\rho\nabla z_t + c_1\mu\nabla z)z\Big]dt + (\xi_t^2 - \xi_{tt})\Big(\rho\nabla z_t\cdot\nabla z - c_1\mu|\nabla z|^2\Big)dt\\
&= \nabla\cdot\Big[(\xi_t^2 - \xi_{tt})(-\rho\nabla z_t + c_1\mu\nabla z)z\Big]dt + d\Big[\frac{\rho}{2}(\xi_t^2 - \xi_{tt})|\nabla z|^2\Big]\\
&\quad-\frac{\rho}{2}(2\xi_t\xi_{tt} - \xi_{ttt})|\nabla z|^2 dt - c_1\mu(\xi_t^2 - \xi_{tt})|\nabla z|^2 dt.
\end{aligned}
$$

Then, by the definition of $I_1$, we see that

$$
\begin{aligned}
&(\xi_t^2 - \xi_{tt})I_1 z dt\\
&= \nabla\cdot\Big[(\xi_t^2 - \xi_{tt})(-\rho\nabla z_t + c_1\mu\nabla z)z\Big]dt + d\Big[-\xi_t(\xi_t^2 - \xi_{tt})z^2 + \frac{\rho}{2}(\xi_t^2 - \xi_{tt})|\nabla z|^2\Big]\\
&\quad+\Big(3\xi_{tt}\xi_t^2 - \xi_{tt}^2 - \xi_t\xi_{ttt}\Big)z^2 dt - \Big[\frac{\rho}{2}(2\xi_t\xi_{tt} - \xi_{ttt}) + c_1\mu(\xi_t^2 - \xi_{tt})\Big]|\nabla z|^2 dt.
\end{aligned}
\tag{2.9}
$$

**Step 4.** Let us compute "$\rho\xi_t I_1\Delta z dt$". Notice that

$$
\rho\xi_t(-\rho\Delta z_t + c_1\mu\Delta z)\Delta z dt = d\Big(-\frac{\rho^2}{2}\xi_t|\Delta z|^2\Big) + \frac{\rho^2}{2}\xi_{tt}|\Delta z|^2 dt + c_1\rho\mu\xi_t|\Delta z|^2 dt.
$$

Further,

$$-2\rho\xi_t^2 z_t \Delta z dt = \nabla\cdot\Big(-2\rho\xi_t^2 z_t\nabla z\Big)dt + d\Big(\rho\xi_t^2|\nabla z|^2\Big) - 2\rho\xi_t\xi_{tt}|\nabla z|^2 dt.$$

Then, one can know that

$$\begin{aligned}&\rho\xi_t I_1\Delta z dt\\ &= \nabla\cdot\Big(-2\rho\xi_t^2 z_t\nabla z\Big)dt + d\Big(-\frac{\rho^2}{2}\xi_t|\Delta z|^2 + \rho\xi_t^2|\nabla z|^2\Big)\\ &\quad+\Big(\frac{\rho^2}{2}\xi_{tt} + c_1\rho\mu\xi_t\Big)|\Delta z|^2 dt - 2\rho\xi_t\xi_{tt}|\nabla z|^2 dt.\end{aligned} \tag{2.10}$$

**Step 5.** Now, we compute the last term "$c_2\mu I_1 z_t dt$" in the right side of the sign of equality in (2.6). First,

$$\begin{aligned}&(-\rho\Delta z_t + c_1\mu\Delta z)z_t dt\\ &= \nabla\cdot\Big[(-\rho\nabla z_t + c_1\mu\nabla z)z_t\Big]dt + \rho|\nabla z_t|^2 dt + d\Big(-\frac{c_1}{2}\mu|\nabla z|^2\Big).\end{aligned}$$

Then, from the above equality and (2.2), it follows that

$$\begin{aligned}&c_2\mu I_1 z_t dt\\ &= \nabla\cdot\Big[c_2\mu(-\rho\nabla z_t + c_1\mu\nabla z)z_t\Big]dt - d\Big(\frac{c_1c_2}{2}\mu^2|\nabla z|^2\Big)\\ &\quad+c_2\rho\mu|\nabla z_t|^2 dt - 2c_2\mu\xi_t z_t^2 dt.\end{aligned} \tag{2.11}$$

Combining (2.6)-(2.11) with (2.5), one can get the desired weighted identity (2.3). □

## 2.2 Proof of Theorem 1.2

This subsection is devoted to proving the Carleman estimate (1.5) by means of the weighted identity (2.3).

**Proof of Theorem 1.2.** By $\psi(t) = -t$, it is easy to check that

$$\xi_t = -\lambda\mu\varphi, \quad \xi_{tt} = \lambda\mu^2\varphi, \quad \xi_{ttt} = -\lambda\mu^3\varphi, \quad \varphi_t = -\mu\varphi, \quad \varphi_{tt} = \mu^2\varphi. \tag{2.12}$$

Integrating (2.3) in $Q$ and taking expectation. Choosing $c_1 = -\rho$, $c_2 = 2$, there exists a $\mu_0 > 0$ such that for all $\mu \geq \mu_0$, it holds that

$$\begin{aligned}&\mathbb{E}\int_G M\Big|_0^T dx + \mathbb{E}\int_Q \nabla\cdot V dxdt + \mathbb{E}\int_Q\Big(-\lambda\mu\varphi|dz_t|^2 - \frac{\rho}{2}|\nabla dz_t|^2\Big)dx\\ &\quad+C\mathbb{E}\int_Q\Big(\lambda^3\mu^4\varphi^3|z|^2 + \rho\lambda^2\mu^3\varphi^2|\nabla z|^2 + \lambda\mu^2\varphi|z_t|^2 + \rho^2\lambda\mu^2\varphi|\Delta z|^2\\ &\quad+\rho\mu|\nabla z_t|^2 + \rho\mu|\nabla\Delta z|^2\Big)dxdt + \mathbb{E}\int_Q|I_1|^2 dxdt\\ &\leq \mathbb{E}\int_Q|I_1|^2 dxdt + \mathbb{E}\int_Q\theta^2|f|^2 dxdt.\end{aligned} \tag{2.13}$$

In the following, we estimate the terms in the left hand side of inequality (2.13) one by one. Since $\xi_t, \varphi_t < 0$, it is easy to check that

$$
\begin{aligned}
&\mathbb{E}\int_G M\Big|_0^T dx \\
&= \mathbb{E}\int_G \Big[(-\xi_t z_t + c_1\mu\Delta z)z_t - \frac{\rho}{2}|\nabla z_t|^2 + \frac{\rho}{2}|\nabla\Delta z|^2 - (1+\frac{\rho^2}{2})\xi_t|\Delta z|^2 \\
&\qquad -\xi_t(\xi_t^2 - \xi_{tt})z^2 + \Big(\frac{3\rho}{2}\xi_t^2 - \frac{\rho^2}{2}\xi_{tt} - \frac{c_1c_2}{2}\mu^2\Big)|\nabla z|^2\Big]\Big|_0^T dx \\
&\geq -Ce^{C\lambda}(1+\rho+\rho^2)\Big(\mathbb{E}|(y(0), y_t(0))|^2_{H^3(G)\times L^2(G)} + \mathbb{E}|\nabla y_t(T)|^2_{L^2(G)}\Big).
\end{aligned}
\tag{2.14}
$$

Further, by $y = \Delta y = 0$ on $\Gamma$ and $z = \theta(t)y$, we have that

$$
\begin{aligned}
&\mathbb{E}\int_Q \nabla\cdot V dxdt \\
&= \mathbb{E}\int_\Sigma \Big[(-\rho\Delta z_t + c_1\mu\Delta z)\nabla\Delta z + 2\xi_t\Delta z\nabla z_t\Big]\cdot\nu dSdt = 0.
\end{aligned}
\tag{2.15}
$$

By Itô's formula, one knows that

$$
\begin{aligned}
&\mathbb{E}\int_Q \Big(-\lambda\mu\varphi|dz_t|^2 - \frac{\rho}{2}|\nabla dz_t|^2\Big)dx \\
&\geq -C(1+\rho)\mathbb{E}\int_Q \theta^2\Big[\lambda\mu\varphi(|g|^2+y^2) + |\nabla g|^2 + |\nabla y|^2\Big]dxdt.
\end{aligned}
\tag{2.16}
$$

Combining (2.13)-(2.16), it follows that there exists a $\mu_1 > 0$ such that for all $\mu \geq \mu_1$, one can find a $\lambda_1 = \lambda_1(\mu) > 0$, so that for any $\lambda \geq \lambda_1$, it holds that

$$
\begin{aligned}
&\mathbb{E}\int_Q \Big[\lambda^2\mu^3\varphi^2\Big(\lambda\mu\varphi z^2 + |\nabla z|^2\Big) + \lambda\mu^2\varphi\Big(|\Delta z|^2 + |z_t|^2\Big) + \mu\Big(|\nabla\Delta z|^2 + |\nabla z_t|^2\Big)\Big]dxdt \\
&\leq C(\rho)\Big[\mathbb{E}\int_Q \theta^2\Big|f|^2dxdt + e^{C\lambda}\Big(\mathbb{E}|(y(0), y_t(0))|^2_{H^3(G)\times L^2(G)} + \mathbb{E}|\nabla y_t(T)|^2_{L^2(G)}\Big) \\
&\qquad + \mathbb{E}\int_Q \theta^2\Big(\lambda\mu\varphi|g|^2 + |\nabla g|^2\Big)dxdt\Big],
\end{aligned}
$$

where $C(\rho)$ is given in (1.6). Noting that $z = \theta y$, one can get the desired result immediately. $\square$

# 3 Proof of Theorem 1.1

In this section, we will give the proof of global uniqueness result.

**Proof of Theorem 1.1.** For arbitrary small $\epsilon > 0$, choose $t_1$ and $t_2$ such that $0 < t_0 - \epsilon < t_1 < t_2 < t_0$. Also, let $\chi \in C^\infty(\mathbb{R}; [0,1])$ satisfy that $\chi(t) = 1$ for $t \leq t_1$ and $\chi(t) = 0$ for

$t \geq t_2$. For any strong solution $y$ of (1.1), put $y = Rh$. It is easy to check that $h$ satisfies that

$$\begin{cases} dh_t - \rho\Delta h_t dt + \Delta^2 h dt = \Big[F_1(h) + H(t,x')\Big]dt + \Big[a_2 h + F(t,x')\Big]dB(t) & \text{in } [0,t_0]\times G, \\ h = \dfrac{\partial h}{\partial \nu} = \Delta h = \dfrac{\partial \Delta h}{\partial \nu} = 0 & \text{on } [0,t_0]\times \Gamma, \\ h(0) = 0,\ h_t(0) = 0 & \text{in } G, \end{cases} \tag{3.1}$$

where

$$F_1(h) = f_1 h + \mathbf{f_2}\cdot\nabla h + f_3 h_t + f_4\Delta h + \mathbf{f_5}\cdot\nabla h_t + \mathbf{f_6}\cdot\nabla\Delta h - 4\sum_{j,k=1}^{n}\frac{R_{x_jx_k}}{R}h_{x_jx_k},$$

and

$$\begin{cases} f_1 = a_1 - \dfrac{R_{tt}}{R} + \dfrac{\rho\Delta R_t}{R} - \dfrac{\Delta^2 R}{R}, \quad \mathbf{f_2} = \dfrac{2\rho\nabla R_t}{R} - \dfrac{4\nabla\Delta R}{R}, \\ f_3 = \dfrac{\rho\Delta R - 2R_t}{R}, \quad f_4 = \dfrac{\rho R_t - 2\Delta R}{R}, \quad \mathbf{f_5} = \dfrac{2\rho\nabla R}{R}, \quad \mathbf{f_6} = -\dfrac{4\nabla R}{R}. \end{cases}$$

Differential both sides of (3.1) with respect to $x_1$ and set $u = h_{x_1}$. Noting that $h_{x_1} = \frac{\partial h}{\partial \nu} = 0$ on $(\{0\}\times G')\cup(\{l\}\times G')$, we get that

$$\begin{cases} du_t - \rho\Delta u_t dt + \Delta^2 u dt = \Big[F_1(u) + P_1(h)\Big]dt + \Big(a_2 u + a_{2,x_1}h\Big)dB(t) & \text{in } [0,t_0]\times G, \\ u = \dfrac{\partial u}{\partial \nu} = \Delta u = 0 & \text{on } [0,t_0]\times \Gamma, \\ u(0) = 0,\ u_t(0) = 0 & \text{in } G, \end{cases}$$

where

$$\begin{aligned} P_1(h) = {} & f_{1,x_1}h + (\mathbf{f_2})_{x_1}\cdot\nabla h + f_{3,x_1}h_t + f_{4,x_1}\Delta h + (\mathbf{f_5})_{x_1}\cdot\nabla h_t \\ & + (\mathbf{f_6})_{x_1}\cdot\nabla\Delta h - 4\sum_{j,k=1}^{n}\frac{R_{x_jx_kx_1}}{R}h_{x_jx_k}. \end{aligned}$$

Set $w = \chi u$, we know that $w$ solves

$$\begin{cases} dw_t - \rho\Delta w_t dt + \Delta^2 w dt = \Big[E_1(u) + F_1(w) + \chi P_1(h)\Big]dt \\ \qquad\qquad + \Big(a_2 w + a_{2,x_1}\chi h\Big)dB(t) & \text{in } [0,t_0]\times G, \\ w = \dfrac{\partial w}{\partial \nu} = \Delta w = 0 & \text{on } [0,t_0]\times \Gamma, \\ w(0) = 0,\ w_t(0) = 0 & \text{in } G, \end{cases}$$

where

$$E_1(u) = \chi'' u + 2\chi' u_t - \rho\chi'\Delta u - \chi' f_3 u - \chi'\mathbf{f_5}\cdot\nabla u.$$

By means of $u = h_{x_1}$ and $h(t,0,x') = 0$ for $(t,x') \in (0,t_0) \times G'$, then we obtain that

$$\chi h = \chi \int_0^{x_1} h_{x_1}(t,\eta,x')d\eta = \chi \int_0^{x_1} u(t,\eta,x')d\eta = \int_0^{x_1} w(t,\eta,x')d\eta, \tag{3.2}$$

and

$$\chi h_t = \chi \int_0^{x_1} u_t(t,\eta,x')d\eta = \int_0^{x_1} w_t(t,\eta,x')d\eta - \chi' \int_0^{x_1} u(t,\eta,x')d\eta. \tag{3.3}$$

Therefore, $w$ satisfies

$$\begin{cases} dw_t - \rho\Delta w_t dt + \Delta^2 w dt = \Big[E_2(u) + F_2(w)\Big]dt + \Big[a_2 w \\ \qquad\qquad + a_{2,x_1}\displaystyle\int_0^{x_1} w(t,\eta,x')d\eta\Big]dB(t) & \text{in } [0,t_0]\times G, \\ w = \dfrac{\partial w}{\partial \nu} = \Delta w = 0 & \text{on } [0,t_0]\times\Gamma, \\ w(0) = 0,\ w_t(0) = 0 & \text{in } G, \end{cases} \tag{3.4}$$

where

$$\begin{cases} E_2(u) = E_1(u) - \chi' f_{3,x_1}\displaystyle\int_0^{x_1} u(t,\eta,x')d\eta - \chi'(\mathbf{f_5})_{x_1}\cdot\nabla\int_0^{x_1} u(t,\eta,x')d\eta, \\ F_2(w) = F_1(w) + f_{1,x_1}\displaystyle\int_0^{x_1} w(t,\eta,x')d\eta + (\mathbf{f_2})_{x_1}\cdot\nabla\int_0^{x_1} w(t,\eta,x')d\eta \\ \qquad + f_{3,x_1}\displaystyle\int_0^{x_1} w_t(t,\eta,x')d\eta + f_{4,x_1}\Delta\int_0^{x_1} w(t,\eta,x')d\eta \\ \qquad + (\mathbf{f_5})_{x_1}\cdot\nabla\displaystyle\int_0^{x_1} w_t(t,\eta,x')d\eta + (\mathbf{f_6})_{x_1}\cdot\nabla\Delta\int_0^{x_1} w(t,\eta,x')d\eta \\ \qquad -4\displaystyle\sum_{j,k=1}^{n}\Big(\frac{R_{x_jx_k}}{R}\Big)_{x_1}\Big(\int_0^{x_1} w(t,\eta,x')d\eta\Big)_{x_jx_k}. \end{cases} \tag{3.5}$$

Applying the Carleman estimate (1.5) in Theorem 1.2 to (3.4), noting that $w = \frac{\partial w}{\partial \nu} = \Delta w = 0$ on $[0,t_0]\times\Gamma$ and $w(t_0) = w_t(t_0) = 0$, we have that

$$\begin{aligned} &\mathbb{E}\int_0^{t_0}\int_G \mu\theta^2\Big[\lambda^3\mu^3\varphi^3 w^2 + \lambda^2\mu^2\varphi^2|\nabla w|^2 + \lambda\mu\varphi\Big(|\Delta w|^2 + |w_t|^2\Big) + |\nabla\Delta w|^2 + |\nabla w_t|^2\Big]dxdt \\ &\le C\mathbb{E}\int_0^{t_0}\int_G \theta^2|E_2(u)|^2 dxdt + C\mathbb{E}\int_0^{t_0}\int_G \theta^2|F_2(w)|^2 dxdt \\ &\quad + C\mathbb{E}\int_0^{t_0}\int_G \theta^2\Big[\lambda\mu\varphi\Big(\Big|\int_0^{x_1} w(t,\eta,x')d\eta\Big|^2 + w^2\Big) + \Big|\nabla\int_0^{x_1} w(t,\eta,x')d\eta\Big|^2 + |\nabla w|^2\Big]dxdt. \end{aligned} \tag{3.6}$$

Since

$$\Big|\int_0^{x_1} w(t,\eta,x')d\eta\Big|^2 \le l\int_0^l |w(t,\eta,x')|^2 d\eta,$$

it is easy to see that

$$\int_0^{t_0}\int_G \theta^2\varphi\Big|\int_0^{x_1} w(t,\eta,x')d\eta\Big|^2 dxdt \le l^2\int_0^{t_0}\int_G \theta^2\varphi|w(t,\eta,x')|^2 dxdt. \tag{3.7}$$

By means of $w(t,0,x')=0$, we find that

$$\begin{aligned}\Big|\nabla\int_0^{x_1} w(t,\eta,x')d\eta\Big|^2 &= \Big|\Big(w(t,x_1,x'),\int_0^{x_1}\nabla_{x'}w(t,\eta,x')d\eta\Big)\Big|^2\\ &= \Big|\Big(w(t,x_1,x')-w(t,0,x'),\int_0^{x_1}\nabla_{x'}w(t,\eta,x')d\eta\Big)\Big|^2\\ &= \Big|\int_0^{x_1}\nabla w(t,\eta,x')d\eta\Big|^2 \le l\int_0^l |\nabla w(t,\eta,x')|^2 d\eta.\end{aligned}$$

Then,

$$\mathbb{E}\int_0^{t_0}\int_G \theta^2\Big|\nabla\int_0^{x_1} w(t,\eta,x')d\eta\Big|^2 dxdt \le l^2\mathbb{E}\int_0^{t_0}\int_G \theta^2|\nabla w(t,\eta,x')|^2 dxdt. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), there exists $\mu_2>0$ such that for all $\mu\ge\mu_2$, it holds that

$$\begin{aligned}&\mathbb{E}\int_0^{t_0}\int_G \mu\theta^2\Big[\lambda^3\mu^3\varphi^3w^2+\lambda^2\mu^2\varphi^2|\nabla w|^2+\lambda\mu\varphi\Big(|\Delta w|^2+|w_t|^2\Big)+|\nabla\Delta w|^2+|\nabla w_t|^2\Big]dxdt\\ &\le C\mathbb{E}\int_0^{t_0}\int_G \theta^2|E_2(u)|^2dxdt + C\mathbb{E}\int_0^{t_0}\int_G \theta^2|F_2(w)|^2 dxdt.\end{aligned} \tag{3.9}$$

Similarly with (3.7) and (3.8), one can get that

$$\begin{aligned}&\mathbb{E}\int_0^{t_0}\int_G \theta^2\Big(\Big|\int_0^{x_1} w_t(t,\eta,x')d\eta\Big|^2+\Big|\nabla\int_0^{x_1} w_t(t,\eta,x')d\eta\Big|^2\Big)dxdt\\ &\le l^2\mathbb{E}\int_0^{t_0}\int_G \theta^2\Big(|w_t(t,\eta,x')|^2+|\nabla w_t(t,\eta,x')|^2\Big)dxdt.\end{aligned} \tag{3.10}$$

By

$$\Big|\Delta\int_0^{x_1} w(t,\eta,x')d\eta\Big|^2 = \Big|\int_0^{x_1}\Delta w(t,\eta,x')d\eta + w_{x_1}(t,0,x')\Big|^2,$$

and

$$\Big|\nabla\Delta\int_0^{x_1} w(t,\eta,x')d\eta\Big|^2 = \Big|\int_0^{x_1}\nabla\Delta w(t,\eta,x')d\eta + \mathbf{n}\Big|^2,$$

where $\mathbf{n}=\Big(\Delta w(t,0,x'), w_{x_1x_2}(t,0,x'),\cdots,w_{x_1x_n}(t,0,x')\Big)$, and recalling that $w_{x_1}=\frac{\partial w}{\partial\nu}=0$ on $(\{0\}\times G')\cup(\{l\}\times G')$, we have that

$$\begin{aligned}&\mathbb{E}\int_0^{t_0}\int_G \theta^2\Big(\Big|\Delta\int_0^{x_1} w(t,\eta,x')d\eta\Big|^2+\Big|\nabla\Delta\int_0^{x_1} w(t,\eta,x')d\eta\Big|^2\Big)dxdt\\ &\le l^2\mathbb{E}\int_0^{t_0}\int_G \theta^2\Big(|\Delta w(t,\eta,x')|^2+|\nabla\Delta w(t,\eta,x')|^2\Big)dxdt.\end{aligned} \tag{3.11}$$

A short calculation shows that

$$\sum_{j,k=1}^{n}\Big(\int_0^{x_1} w(t,\eta,x')d\eta\Big)_{x_jx_k} = \int_0^{x_1}\sum_{j,k=1}^{n} w_{x_jx_k}(t,\eta,x')d\eta.$$

This imples that

$$\mathbb{E}\int_0^{t_0}\int_G \theta^2\Big|\sum_{j,k=1}^{n}\Big(\int_0^{x_1} w(t,\eta,x')d\eta\Big)_{x_jx_k}\Big|^2 dxdt \le l^2\mathbb{E}\int_0^{t_0}\int_G \theta^2\Big|\sum_{j,k=1}^{n} w_{x_jx_k}(t,\eta,x')\Big|^2 dxdt. \tag{3.12}$$

On the other hand, it is easy to see that

$$\begin{aligned}
&\mathbb{E}\int_0^{t_0}\int_G \theta^2\Big|\sum_{j,k=1}^{n} w_{x_jx_k}\Big|^2 dxdt \le C\mathbb{E}\int_0^{t_0}\int_G \theta^2\sum_{j,k=1}^{n}|w_{x_jx_k}|^2 dxdt\\
&= C\mathbb{E}\int_0^{t_0}\int_G \theta^2\sum_{j,k=1}^{n}\Big[\Big(w_{x_j}w_{x_jx_k}\Big)_{x_k} - \Big(w_{x_j}w_{x_kx_k}\Big)_{x_j} + w_{x_jx_j}w_{x_kx_k}\Big]dxdt\\
&= C\mathbb{E}\int_0^{t_0}\int_G \theta^2|\Delta w|^2 dxdt.
\end{aligned} \tag{3.13}$$

Combing (3.6)-(3.13) with (3.5), we know that there is a $\mu_3 > 0$ such that for all $\mu \ge \mu_3$, there exists a $\lambda_1(\mu_3) > 0$ so that for all $\lambda \ge \lambda_1(\mu_3)$, it holds that

$$\begin{aligned}
&\mathbb{E}\int_0^{t_0}\int_G \mu\theta^2\Big[\lambda^3\mu^3\varphi^3 w^2 + \lambda^2\mu^2\varphi^2|\nabla w|^2 + \lambda\mu\varphi\Big(|\Delta w|^2 + |w_t|^2\Big) + |\nabla\Delta w|^2 + |\nabla w_t|^2\Big]dxdt\\
&\le C\mathbb{E}\int_0^{t_0}\int_G \theta^2|E_2(u)|^2 dxdt.
\end{aligned} \tag{3.14}$$

By the definition of $E_2(u)$ and $\chi$, fix $\mu = \mu_3$, we see that

$$\begin{aligned}
\mathbb{E}\int_0^{t_0}\int_G \theta^2|E_2(u)|^2 dxdt &\le Ce^{2\lambda e^{-\mu_3 t_1}}\mathbb{E}\int_{t_1}^{t_2}\int_G \Big(|u|^2 + |u_t|^2 + |\nabla u|^2 + |\Delta u|^2\\
&\quad + \Big|\int_0^{x_1} u(t,\eta,x')d\eta\Big|^2 + \Big|\int_0^{x_1}\nabla u(t,\eta,x')d\eta\Big|^2\Big)dxdt\\
&\le Ce^{2\lambda e^{-\mu_3 t_1}}\mathbb{E}\int_Q \Big(|u|^2 + |u_t|^2 + |\nabla u|^2 + |\Delta u|^2\Big)dxdt\\
&= Ce^{2\lambda e^{-\mu_3 t_1}}|(h_{x_1}, h_{t,x_1})|^2_{L^2_{\mathbf{F}}(0,T;H^2(G)\times L^2(G))}.
\end{aligned} \tag{3.15}$$

By virtue of (3.11) and (3.15), for all $\lambda \ge \lambda_1(\mu_3)$, this implies that

$$\begin{aligned}
&e^{2\lambda e^{-\mu_3(t_0-\epsilon)}}\mathbb{E}\int_0^{t_0-\epsilon}\int_G \Big(|w|^2 + |w_t|^2 + |\nabla w|^2 + |\Delta w|^2\Big)dxdt\\
&\le \mathbb{E}\int_0^{t_0-\epsilon}\int_G \theta^2\Big(|w|^2 + |w_t|^2 + |\nabla w|^2 + |\Delta w|^2\Big)dxdt\\
&\le Ce^{2\lambda e^{-\mu_3 t_1}}|(h_{x_1}, h_{t,x_1})|^2_{L^2_{\mathbf{F}}(0,T;H^2(G)\times L^2(G))}.
\end{aligned}$$

Therefore, by a simple calculation, the above inequality indicates that

$$|(w, w_t)|^2_{L^2_{\mathbb{F}}(0,t_0-\epsilon;H^2(G)\times L^2(G))} \leq Ce^{2\lambda(e^{-\mu_3 t_1}-e^{-\mu_3(t_0-\epsilon)})}|(h_{x_1}, h_{t,x_1})|^2_{L^2_{\mathbb{F}}(0,T;H^2(G)\times L^2(G))}.$$

Noting that $t_0 - \epsilon < t_1$, then $e^{-\mu_3 t_1} - e^{-\mu_3(t_0-\epsilon)} < 0$. Letting $\lambda \to +\infty$, it follows that

$$w(t, x) = 0 \ \text{ in } \ (0, t_0 - \epsilon) \times G, \ \mathbb{P}\text{-}a.s.$$

Together with equality (3.2) and recalling the definition of $\chi$, implies that

$$h(t, x) = 0 \ \text{ in } \ (0, t_0 - \epsilon) \times G, \ \mathbb{P}\text{-}a.s.,$$

which means

$$y(t, x) = 0 \ \text{ in } \ (0, t_0 - \epsilon) \times G, \ \mathbb{P}\text{-}a.s.$$

Then, by equation (1.1), we deduce that

$$H(t, x') = F(t, x') = 0 \ \text{ in } \ (0, t_0 - \epsilon) \times G', \ \mathbb{P}\text{-}a.s.$$

By the arbitrariness of $\epsilon > 0$, the proof of Theorem 1.1 is completed. $\square$

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