On the nonlocal Schrödinger-Poisson type system in the Heisenberg group

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Abstract

This paper is concerned with the following nonlocal Schrödinger-Poisson type system:

$$\begin{cases} -\left(a - b\int_\Omega |\nabla_H u|^2 dx\right)\Delta_H u + \mu\phi u = \lambda |u|^{q-2}u, & \text{in } \Omega, \\ -\Delta_H \phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a, b > 0$ and Δ_H is the Kohn-Laplacian on the first Heisenberg group $\mathbb{H}^1$, $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $\lambda > 0$, $\mu \in \mathbb{R}$ are some real parameters and $1 < q < 2$. With the aid of the Ekeland's variational principle and the mountain pass theorem, the existence of negative energy solution, positive energy solution and positive ground state solution are obtained, respectively. Moreover, we also obtain the multiplicity of solutions by using the Clark theorem. Our result is new even in the Euclidean case.

Keywords: Heisenberg group; Schrödinger-Poisson type system; Ekeland's variational principle; Mountain pass lemma; Clark theorem.

Mathematics Subject Classification (2000) : 35J20; 35R03; 46E35

1 Introduction and Main results

Recently, Heisenberg group has attracted the attention of many scholars, and it plays an important role in quantum mechanics, harmonic analysis, partial differential equations and other branches. The expansion and contraction of Heisenberg group gives it a very rich geometric structure, which is essentially different from Euclidean space. The first mathematicians who study of subelliptic analysis on the Heisenberg group were Folland and Stein in [13], who consistently created a generalisation of the analysis for more general stratified groups [14]. And it can also be noted that Rothschild and Stein generalised these results for general vector fields satisfying the Hormander's conditions. These results were

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published in the famous book by Folland and Stein [15] which laid the anisotropic analysis. And it is worth noting that homogeneous Lie group is nilpotent.

The present study is concerned with a class of nonlocal Schrödinger-Poisson type system in the Heisenberg:

$$\begin{cases} -\left(a - b\int_\Omega |\nabla_H u|^2 dx\right)\Delta_H u + \mu\phi u = \lambda |u|^{q-2}u, & \text{in} \quad \Omega, \\ -\Delta_H \phi = u^2 & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $a, b > 0$ and Δ_H is the Kohn-Laplacian on the first Heisenberg group $\mathbb{H}^1$, $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $\lambda > 0$, $\mu \in \mathbb{R}$ are some real parameters and $1 < q \leq 2$.

The paper was motivated by some works appeared in recent years. On the one hand, Tyagi in [33] considered the following singular boundary value problem on the Heisenberg group:

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = \mu \frac{g(\xi)u}{(|z|^4+t^2)^{\frac{1}{2}}} + \lambda f(\xi, t), & \xi \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

the existence of weak solutions are obtained by using the Bonanno's three critical point theorem. In [3], the authors studied the the following Schrödinger-Poisson type system

$$\begin{cases} -\Delta_H u + \mu\phi u = \lambda |u|^{q-2}u + |u|^2 u, & \text{in} \quad \Omega, \\ -\Delta_H \phi = u^2 v & \text{in} \quad \Omega, \\ u = \phi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

where $1 < q < 2$, by the Green's representation formula and the critical point theory, they obtained at least two positive solutions and a positive ground state solution. In [23], the author proved that problem (1.3) with $q = 2$ and $\mu = 0$ admits at least one positive solution. And then, this result was extended to a critical semilinear boundary problem with singular nonlinearities, see [24]. In [27], the authors proved the concentration-compactness principles on the Heisenberg group for treating nonlinear problems involving critical nonlinearities and Hardy terms. On some recent results recovering the Heisenberg group, we refer to [6, 7, 23, 28, 29, 30] and the references therein.

On the other hand, the study of Kirchhoff-type problems, which arise in various models of physical and biological systems, have received more and more attention in recent years, we refer the reader to [2, 8, 11, 12, 19, 20, 21, 31, 35]. In this paper, we mainly consider a new type of Kirchhoff equation, that is, the form with a non-local coefficient $(a - b\int_\Omega |\nabla u|^2 dx)$. Its background is derived from the negative Young's modulus, when the atoms are pulled apart rather than compressed together, the strain is negative. Recently, the authors in [36] first studied this kind of problem

$$\begin{cases} -\left(a - b\int_\Omega |\nabla u|^2 dx\right)\Delta u = |u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $2 < p < 2^*$, they obtained the existence of solutions by using the mountain pass lemma. Furthermore, some interesting results have been obtained for this kind of Kirchhoff-type problem with sublinear terms, we refer the readers to [10, 25, 34] and the references therein.

Inspired by the works in the above references, our main purpose in this paper is to study the existence and multiplicity of solutions for problem (1.1). To the best of our knowledge, this paper is the first to deal with Schrödinger-Poisson system with the nonlocal term in the case of Kohn-Laplacian, our result is new even in the Euclidean case.

We are now in a position to state the existence result as follows.

Theorem 1.1. *Let $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ be satisfied. Then, there exists $\Lambda_* > 0$ such that $\lambda \in (0, \Lambda_*)$ problem (1.1) has a negative energy solution.*

Theorem 1.2. *Let $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ be satisfied. Then, there exists $\Lambda^* > 0$ such that $\lambda \in (0, \Lambda^*)$ problem (1.1) has a positive energy solution.*

Theorem 1.3. *Let $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ be satisfied. Then, there exists $\Lambda^* > 0$ such that $\lambda \in (0, \Lambda^*)$ problem (1.1) has a positive ground state solution.*

We also obtain the multiplicity of solutions for problem (1.1) by using the Clark theorem.

Theorem 1.4. *Let $-\infty < \mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ be satisfied. Then, there exists $\Lambda > 0$ such that $\lambda \in (0, \Lambda)$ problem (1.1) has at least $m - n$ pairs of solutions.*

We point out that problem (1.1) contains nonlocal terms ϕu and nonlocal coefficient $(a - b\int_\Omega |\nabla u|^2 dx)$, there is no doubt that we encounter serious difficulties because of the lack of compactness, it becomes more difficult to give the accurate threshold of c (see Lemma 3.2). In addition, it's different from the superlinear problem, due to the limitation of λ, it becomes more difficult to apply the mountain pass theorem (see Theorem 4.2). In order to overcome these difficulties, we will use some more accurate estimates for related expressions.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on the Heisenberg group functional setting. In Section 3, we prove the Palais-Smale compactness condition. In Section 4, we prove Theorem 1.1 and Theorem 1.2 via the Ekeland's variational principle and Mountain Pass theorem. In Section 5, we prove that problem (1.1) has a positive ground state solution. Finally, Section 6 is devoted to the proof of Theorems 1.4, that is to the proof of existence and multiplicity of solutions for problem (1.1) by using the Clark theorem.

2 Preliminaries

In this section, we briefly recall the relevant definitions and notations related to the Heisenberg group functional setting. For a complete treatment, we refer to [16, 17, 22, 26]. The Heisenberg group $\mathbb{H}^1$ is identified with $\mathbb{R}^3$ under the following group composition: if $\xi = (x, y, t) \in \mathbb{H}^1$, then the group law is defined by

$$\tau : \mathbb{H}^1 \to \mathbb{H}^1, \; \tau_\xi(\xi') = \xi \circ \xi',$$

where

$$\xi \circ \xi' = (x + x', y + y', t + t' + 2(x'y - y'x)), \quad \forall\, \xi, \xi' \in \mathbb{H}^1.$$

For $s>0$, a natural group of dilation on $\mathbb{H}^1$ is defined by $\delta_s(\xi)=(sx,sy,s^2t)$. Hence, $\delta_s(\xi_0\circ\xi)=\delta_s(\xi_0)\circ\delta_s(\xi)$. The homogeneous dimension of $\mathbb{H}^1$ is $Q=4$. For $\xi\in\mathbb{H}^1$, $|\xi|_H$ is the intrinsic distance of the point ξ to the origin, namely

$$|\xi|_H=\left[(x^2+y^2)^2+t^2\right]^{\frac{1}{4}},\quad \xi\in\mathbb{H}^1.$$

The Kohn Laplacian Δ_H on $\mathbb{H}^1$ is defined as

$$\Delta_H u=\mathrm{div}_H(\nabla_H u),$$

where

$$\nabla_H u=(X,Y),\quad X=\frac{\partial}{\partial x}+2y\frac{\partial}{\partial t},\quad Y=\frac{\partial}{\partial y}-2x\frac{\partial}{\partial t},$$

and ∇_H is the horizontal gradient, X and Y is a basis for Lie algebra of left-invariant vector fields on $\mathbb{H}^1$. The left-invariant distance d_H on $\mathbb{H}^1$ is accordingly defined by

$$d_H(\xi_0,\xi)=|\xi^{-1}\circ\xi_0|_H.$$

It is well known that Δ_H is a very degenerate elliptic operator and Bony's maximum principle is satisfied (see [5]).

In the paper, statements involving measure theory are always understood to be with respect to Haar measure on $\mathbb{H}^1$, which coincides with the L^3 dimensional Lebesgue measure (see [27]); then

$$|B_H(\xi_0,r)|=\alpha_Q r^Q,$$

where $B_H(\xi_0,r)$ is the Heisenberg ball of radius r centered at ξ_0, that is

$$B_H(\xi_0,r)=\left\{\xi\in H^1: d_H(\xi_0,\xi)<r\right\}$$

and $\alpha_Q=|B_H(0,1)|$.

The Folland-Stein space $S_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^2=\|u\|^2_{S_0^1(\Omega)}=\int_\Omega|\nabla_H u|^2d\xi.$$

Let

$$\|u\|_p^p=\int_\Omega|u|^pd\xi,\quad u\in\ L^p(\Omega),$$

denotes the usual L^p-norm. We denote by B_ρ the closed ball of radius ρ centered at zero in the Folland-Stein space $S_0^1(\Omega)$, and by S_ρ its relative boundary, that is,

$$B_\rho=\{u\in S_0^1(\Omega):\|u\|\le\rho\},\ S_\rho=\{u\in S_0^1(\Omega):\|u\|=\rho\}.$$

By [13], we know the Folland-Stein space is a Hilbert space and the embedding

$$S_0^1(\Omega)\hookrightarrow L^\theta(\Omega)\ \text{ for }\theta\in[1,Q^*)$$

is compact, where $Q^*:=\frac{2Q}{Q-2}$. While it is only continuous if $\theta=Q^*$, the best Sobolev constant

$$S=\inf_{u\in S_0^1(\mathbb{H}^1)}\frac{\int_{\mathbb{H}^1}|\nabla_H u|^2d\xi}{(\int_{\mathbb{H}^1}|u|^{Q^*}d\xi)^{\frac{2}{Q^*}}}\tag{2.1}$$

is achieved by the C^∞ function

$$U(x,y,t) = \frac{c_0}{\sqrt{(1+x^2+y^2)^2+t^2}},$$

where c_0 is a suitable positive constant (see [18]). On the other hand, the function U is a positive solution of the following equation:

$$-\Delta_H u = u^3, \quad u \in S_0^1(\mathbb{H}^1) \tag{2.2}$$

and satisfies

$$\int_{\mathbb{H}^1} |\nabla_H U|^2 d\xi = \int_{\mathbb{H}^1} |U|^4 d\xi = S^2.$$

Set

$$u_\varepsilon(\xi) = \varphi(\xi) U_\varepsilon(\xi) = \frac{c_0 \varepsilon \varphi(\xi)}{\sqrt{(\varepsilon + x^2+y^2)^2+t^2}} \tag{2.3}$$

where $\varphi \in C_0^\infty(B_H(0,r_0))$, $0 \le \varphi \le 1$ and $\varphi = 1$ in $B_H(0, \frac{r_0}{2})$ (see [3]).

Let us first consider the following problem:

$$\begin{cases} -\Delta \phi = u^2 & in\ \ \Omega \\ \phi = 0 & on\ \ \partial\Omega. \end{cases} \tag{2.4}$$

It follows from the Lax-Milgram theorem that, for every $u \in S_0^1(\Omega)$, problem (2.4) has a unique solution $\phi_u \in S_0^1(\Omega)$. In addition, from the maximum principle, we have $\phi_u \ge 0$. Moreover, $\phi_u > 0$ if $u \neq 0$. We give some properties of the solution ϕ_u, the detailed proof process can be found in [3].

Proposition 2.1. *For each solution $\phi_u \in S_0^1(\Omega)$ of problem* (2.4)*, we have*

(i) $\phi_{tu} = t^2 \phi_u$ *for all* $t \neq 0$*;*

(ii) For all $u \in S_0^1(\Omega)$, there holds that

$$\int_\Omega \phi_u u^2 d\xi = \int_\Omega |\nabla_H \phi_u|^2 d\xi \le S^{-1} \|u\|_{\frac{8}{3}}^4 \le S^{-3} |\Omega|^{\frac{1}{2}} \|u\|^4; \tag{2.5}$$

(iii) Let $u_n \rightharpoonup u$ in $S_0^1(\Omega)$. Then $\phi_{u_n} \to \phi_u$ in $S_0^1(\Omega)$ and

$$\int_\Omega \phi_{u_n} u_n v d\xi \to \int_\Omega \phi_u u v d\xi, \quad \forall\ v \in S_0^1(\Omega).$$

Now, we define a weak solution to problem (1.1).

Definition 2.1. *We say that $u \in S_0^1(\Omega)$ is a weak solution of problem* (1.1) *if and only if*

$$a\int_\Omega \nabla u \nabla v d\xi - b\int_\Omega |\nabla u|^2 d\xi \int_\Omega \nabla u \nabla v d\xi + \mu \int_\Omega \phi_u u v d\xi - \lambda \int_\Omega |u|^{q-2} u v d\xi = 0$$

for any $v \in S_0^1(\Omega)$.

The corresponding energy functional $I_\lambda(u) : S_0^1(\Omega) \to \mathbb{R}$ to problem (1.1) is defined by

$$I_\lambda(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 + \frac{\mu}{4}\int_\Omega \phi_u u^2 d\xi - \frac{\lambda}{q}\int_\Omega |u|^q d\xi. \tag{2.6}$$

From Proposition 2.1, we know that the functional I_λ is well defined and $I_\lambda \in C^1(S_0^1(\Omega), \mathbb{R})$ (see [32]). Moreover, the Fréchet derivative of I_λ is given by

$$\begin{aligned} \langle I'_\lambda(u), v\rangle = &\int_\Omega \nabla_H u \nabla_H v d\xi - b\int_\Omega |\nabla_H u|^2 d\xi \int_\Omega \nabla_H u \nabla_H v d\xi \\ &+ \mu\int_\Omega \phi_u u v d\xi - \lambda\int_\Omega |u|^{q-2} u v d\xi, \end{aligned} \tag{2.7}$$

for all $u, v \in S_0^1(\Omega)$. Thus, the (weak) solutions of problem (1.1) coincide with the critical points of I_λ.

3 $(PS)_c$ condition

In this section, we first begin giving the following general mountain pass theorem (see [1]).

Theorem 3.1. *Let E be a Banach space and $I_\lambda \in C^1(E, \mathbb{R})$, with $I_\lambda(0) = 0$. Suppose that*
(I_1) there exist $\rho, \alpha > 0$ such that $I_\lambda(u) \geq \alpha$ for all $u \in E$, with $\|u\| = \rho$;
(I_2) there exists $e \in E$ satisfying $\|e\|_E > \rho$ such that $I_\lambda(e) < 0$.
Define $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$.

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I_\lambda(\gamma(t)) \geq \alpha$$

and there exists a $(PS)_c$ sequence $\{u_n\}_n \subset E$.

Now, we begin proving that I_λ satisfies the assumptions of the mountain pass theorem.

Lemma 3.1. *Assume that $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ are satisfied. Then there exists $\Lambda_* > 0$ such that for each $\lambda \in (0, \Lambda_*)$, the functional I_λ satisfies the mountain pass geometry, that is,*

(i) there exist $\alpha, \rho > 0$ such that $I_\lambda(u) \geq \alpha$ for any $u \in S_0^1(\Omega)$ such that $\|u\| = \rho$;

(ii) there exists $e \in S_0^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.

Proof. We will prove the main conclusion in two cases:
Case I: $\mu \leq 0$.
If $1 < q < 2$, we have

$$I_\lambda(u) \geq \|u\|^q \left[\frac{a}{2}\|u\|^{2-q} - d_0\|u\|^{4-q} - \frac{\lambda}{q} S^{-\frac{q}{2}} |\Omega|^{\frac{4-q}{4}}\right],$$

where $d_0 = \frac{b - \mu S^{-3}|\Omega|^{\frac{1}{2}}}{4}$.

Let

$$\eta(s) = \frac{a}{2}s^{2-q} - d_0 s^{4-q}.$$

Then

$$\rho_0 = \left(\frac{a(2-q)}{2d_0(4-q)}\right)^{\frac{1}{2}}$$

is the maximum value point of $\eta(s)$, that is,

$$\eta(\rho_0) = \max_{s>0}\eta(s) = \frac{a}{4-q}\left(\frac{a(2-q)}{2d_0(4-q)}\right)^{\frac{2-q}{2}} > 0.$$

Set

$$\Lambda_* = \frac{1}{2}qS^{\frac{q}{2}}|\Omega|^{\frac{q-4}{4}}\eta(\rho_0).$$

Then, for all $\lambda \in (0, \Lambda_*)$, we have

$$I_\lambda(u) \geq \frac{\eta(\rho_0)}{2}\rho_0^q > 0, \quad \text{for all } u \in S_0^1(\Omega) \text{ with } \|u\| = \rho, .$$

where ρ small enough.

Case II: $0 < \mu < bS^3|\Omega|^{-\frac{1}{2}}$.

In this case, it is easy to prove that condition (i) of Lemma 3.1 holds.

On the other hand, let $u \in S_0^1(\Omega) \setminus \{0\}$, it follows from the Hölider inequality, $0 < \mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ that

$$\begin{aligned}
I_\lambda(tu) &= \frac{a}{2}\|u\|^2t^2 - \frac{b}{4}\|u\|^4t^4 + \frac{\mu t^4}{4}\int_\Omega \phi_u u^2 d\xi - \frac{t^q\lambda}{q}\int_\Omega |u|^q d\xi \\
&\leq \frac{a}{2}\|u\|^2t^2 - \frac{b}{4}\|u\|^4t^4 + \frac{\mu t^4}{4}S^{-3}|\Omega|^{\frac{1}{2}}\|u\|^4 d\xi - \frac{t^q\lambda}{q}\int_\Omega |u|^q d\xi \\
&= \frac{a}{2}\|u\|^2t^2 - \left(\frac{b - \mu S^{-3}|\Omega|^{\frac{1}{2}}}{4}\right)\|u\|^4t^4 - \frac{t^q\lambda}{q}\int_\Omega |u|^q d\xi.
\end{aligned}$$

So, we have

$$I_\lambda(tu) \to -\infty \quad \text{as } t \to \infty.$$

Furthermore, if $\mu \leq 0$, it is obvious that $I_\lambda(tu) \to -\infty$ as $t \to +\infty$. Thus, for $\mu < bS^3|\Omega|^{-\frac{1}{2}}$, there exists $e \in S_0^1(\Omega) \setminus B_{\rho_0}$ such that $I_\lambda(e) < 0$. This completes the proof of Lemma 3.1. □

We recall that a C^1 functional I_λ on Banach space $S_0^1(\Omega)$ is said to satisfy the Palais-Smale condition at level c ($(PS)_c$ in short) if every sequence $\{u_n\} \subset S_0^1(\Omega)$ satisfying $\lim_{n\to\infty} I(u_n) = c$ and $\lim_{n\to\infty} \|I(u_n)\|_{S_0^1(\Omega)^*} = 0$ has a convergent subsequence.

Lemma 3.2. *Assume that $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ are satisfied. Then for each $\lambda > 0$, there exists a positive constant M which is independent of λ such that*

$$\limsup_{n\to\infty} \|u_n\| \leq M.$$

Proof. Let $\{u_n\}$ be a sequence in $S_0^1(\Omega)$ such that

$$c+o(1)=I_\lambda(u_n)=\frac{a}{2}\|u_n\|^2-\frac{b}{4}\|u_n\|^4+\frac{\mu}{4}\int_\Omega \phi_{u_n}{u_n}^2d\xi-\frac{\lambda}{q}\int_\Omega |u_n|^q d\xi \tag{3.1}$$

and

$$\begin{aligned} o(1)\|u_n\| \;&=\; \langle I'_\lambda(u_n),v\rangle=\int_\Omega \nabla_H u_n\nabla_H v d\xi-b\int_\Omega |\nabla_H u_n|^2 d\xi\int_\Omega \nabla_H u_n\nabla_H v d\xi\\ &\quad+\mu\int_\Omega \phi_{u_n}u_n v d\xi-\lambda\int_\Omega |u_n|^{q-2}u_n v d\xi. \end{aligned}\tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned} c+o(1)\|u_n\| \;&=\; I_\lambda(u_n)-\frac{1}{q}\langle I'_\lambda(u_n),u_n\rangle\\ &=\; \frac{(q-2)a}{2q}\|u_n\|^2+\frac{(4-q)b}{4q}\|u_n\|^4-\frac{\mu(4-q)}{4q}\int_\Omega \phi_{u_n}{u_n}^2d\xi\\ &\geq\; -\frac{(2-q)a}{2q}\|u_n\|^2+\frac{(4-q)}{4q}(b-\mu S^{-3}|\Omega|^{\frac{1}{2}})\|u_n\|^4. \end{aligned}\tag{3.3}$$

From the Young's inequality, we have

$$\frac{(2-q)a}{2q}\|u_n\|^2\leq \frac{1}{2\varepsilon}\left[\frac{(2-q)a}{2q}\right]^2+\frac{\varepsilon}{2}\|u_n\|^4.$$

Let $\varepsilon=\frac{(4-q)}{4q}(b-\mu S^{-3}|\Omega|^{\frac{1}{2}})>0$. It follows from (3.3) that

$$\begin{aligned} c+o(1)\|u_n\| \;&=\; I_\lambda(u_n)-\frac{1}{q}\langle I'_\lambda(u_n),u_n\rangle\\ &\geq\; \frac{(4-q)}{8q}(b-\mu S^{-3}|\Omega|^{\frac{1}{2}})\|u_n\|^4-\frac{(2-q)^2a^2}{2q(4-q)}\left(b-\mu S^{-3}|\Omega|^{\frac{1}{2}}\right)^{-1}. \end{aligned}$$

This means that $\{u_n\}$ is also bounded in $S_0^1(\Omega)$ since $\mu<bS^3|\Omega|^{-\frac{1}{2}}$ and $1<q<2$. Thus for each $\lambda>0$, there exists a positive constant M which is independent of λ such that

$$\limsup_{n\to\infty}\|u_n\|\leq M.$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume that $\mu<bS^3|\Omega|^{-\frac{1}{2}}$ and $1<q<2$ are satisfied. Then for each $\lambda>0$, the functional I_λ satisfies the $(PS)_c$ condition with $c<\frac{a^2}{4b}-D\lambda$, where*

$$D=\left(\frac{4-q}{4q}\right)S^{-\frac{q}{2}}|\Omega|^{\frac{4-q}{4}}M^q,$$

M is given by Lemma 3.2.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence, by Lemma 3.2, $\{u_n\}$ is bounded in $S_0^1(\Omega)$. Since $S_0^1(\Omega)$ is reflexible. Therefore, we still may assume that $u_n \rightharpoonup u_0$ weakly in $S_0^1(\Omega)$ and $u_n \to u_0$ strongly in $L^\theta(\Omega)$ with $1 \le \theta < 4$.

Set $w_n = u_n - u_0$, then $\|w_n\| \to 0$. Otherwise, there exists a subsequence (still denoted by w_n) such that

$$\lim_{n\to\infty} \|w_n\| = l. \tag{3.4}$$

For every $v \in S_0^1(\Omega)$, it holds

$$\langle I_\lambda(u_n), v\rangle = (a - b\|u_n\|^2)\int_\Omega \nabla u_n \nabla v d\xi + \mu \int_\Omega \phi_{u_n} u_n v d\xi - \lambda \int_\Omega |u_n|^{q-2} u_n v d\xi = o(1). \tag{3.5}$$

Using the Brézis-Lieb lemma (see [4]) and let $n \to \infty$, we have

$$[a - (bl^2 + b\|u_0\|^2)]\int_\Omega \nabla u_0 \nabla v d\xi + \mu \int_\Omega \phi_{u_0} u_0 v d\xi - \lambda \int_\Omega |u_0|^{q-2} u_0 v d\xi = o(1). \tag{3.6}$$

Particularly, take $v = u_0$ in (3.6), there is

$$[a - (bl^2 + b\|u_0\|^2)]\int_\Omega |\nabla u_0|^2 d\xi + \mu \int_\Omega \phi_{u_0} {u_0}^2 d\xi - \lambda \int_\Omega |u_0|^q d\xi = o(1). \tag{3.7}$$

Furthermore, as $n \to \infty$, it holds

$$\langle I'_\lambda(u_n), u_n\rangle = a\|u_n\|^2 - b\|u_n\|^4 + \mu \int_\Omega \phi_{u_n} {u_n}^2 d\xi - \lambda \int_\Omega |u_n|^q d\xi. \tag{3.8}$$

Using the Brézis-Lieb lemma again, we get

$$\begin{aligned} o(1) \;\; = \;\; & a\|w_n\|^2 + a\|u_0\|^2 - 2b\|w_n\|^2\|u_0\|^2 - b\|u_0\|^4 - b\|w_n\|^4 \\ & + \mu \int_\Omega \phi_{u_n} {u_n}^2 d\xi - \lambda \int_\Omega |u_n|^q d\xi. \end{aligned} \tag{3.9}$$

By (3.7) and (3.9), we have

$$al^2 - bl^4 - bl^2\|u_0\|^2 = 0, \quad l > 0. \tag{3.10}$$

So that

$$l^2 = \frac{a}{b} - \|u_0\|^2. \tag{3.11}$$

On the one hand, from (3.9) we have

$$b\|u_0\|^4 = a\|u_0\|^2 - bl^2\|u_0\|^2 - b\|u_0\|^4 + \mu \int_\Omega \phi_{u_0} {u_0}^2 d\xi - \lambda \int_\Omega |u_0|^q d\xi. \tag{3.12}$$

Recalling that $\|u_n\| \leq M$, using (3.12) and the Hölider inequality, it follows that

$$\begin{aligned} I_\lambda(u_0) &= \frac{a}{2}\|u_0\|^2 - \frac{a}{4}\|u_0\|^2 + \frac{bl^2}{4}\|u_0\|^2 - \frac{\mu}{4}\int_\Omega \phi_{u_0} {u_0}^2 d\xi + \frac{\lambda}{4}\int_\Omega |u_0|^q d\xi \\ &\quad + \frac{\mu}{4}\int_\Omega \phi_{u_0} {u_0}^2 d\xi - \frac{\lambda}{q}\int_\Omega |u_0|^q d\xi \\ &= \frac{a}{4}\|u_0\|^2 + \frac{bl^2}{4}\|u_0\|^2 - \left(\frac{4-q}{4q}\right)\lambda \int_\Omega |u_0|^q d\xi \\ &\geq \frac{a}{4}\|u_0\|^2 + \frac{bl^2}{4}\|u_0\|^2 - \left(\frac{4-q}{4q}\right)\lambda S^{-\frac{q}{2}}|\Omega|^{\frac{4-q}{4}} M^q \\ &= \frac{a}{4}\|u_0\|^2 + \frac{bl^2}{4}\|u_0\|^2 - D\lambda, \end{aligned} \tag{3.13}$$

where $D = \left(\frac{4-q}{4q}\right) S^{-\frac{q}{2}}|\Omega|^{\frac{4-q}{4}} M^q$.

On the other hand, from (3.10) and (3.11), we have

$$\begin{aligned} I_\lambda(u_0) &= \frac{a}{2}\|u_0\|^2 - \frac{b}{4}\|u_0\|^4 + \frac{\mu}{4}\int_\Omega \phi_{u_0} {u_0}^2 d\xi - \frac{\lambda}{q}\int_\Omega |u_0|^q d\xi \\ &= I_\lambda(u_n) - \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u_0\|^2 + o(1) \\ &= c - \frac{a}{4}\|w_n\|^2 + \frac{b}{4}\|w_n\|^2\|u_0\|^2 + o(1) \\ &< \frac{a^2}{4b} - D\lambda - \frac{a}{4}\left(\frac{a}{b} - \|u_0\|^2\right) + \frac{b}{4}\|w_n\|^2\|u_0\|^2 \\ &= \frac{a}{4}\|u_0\|^2 + \frac{b}{4}l^2\|u_0\|^2 - D\lambda, \end{aligned} \tag{3.14}$$

which is a contradiction by comparing the calculations from (3.13) with (3.14). Therefore, $l = 0$, it implies that $u_n \to u_0$ in $S_0^1(\Omega)$. This completes the proof of Lemma 3.3. □

4 Proof of Theorem 1.1 and Theorem 1.2

In this section, we apply Ekeland's variational principle and mountain pass theorem to prove that Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. On the one hand, for any $u \in S_0^1(\Omega) \setminus \{0\}$, we have

$$\lim_{s \to 0^+} \frac{I_\lambda(su)}{s^q} = -\frac{\lambda}{q}\int_\Omega |u|^q d\xi < 0 \tag{4.1}$$

which means that there exists $u \in B_{\rho_0}$ such that $I_\lambda(u) < 0$, that is,

$$\inf_{u \in B_{\rho_0}} I_\lambda(u) < 0.$$

On the other hand, from Lemma 3.1, we have

$$\inf_{u \in B_{\rho_0}} I_\lambda(u) < 0 < \inf_{u \in S_{\rho_0}} I_\lambda(u).$$

Noting that $I_\lambda(|u|) = I_\lambda(u)$, by applying Ekeland's variational principle in B_{ρ_0}, there exists a minimizing sequence $u_n \subset B_{\rho_0}$ such that

$$I_\lambda(u_n) < \inf_{u\in B_{\rho_0}} I_\lambda(u) + \frac{1}{n} \quad \text{and} \quad I_\lambda(v) \geq I_\lambda(u_n) - \frac{1}{n}\|v-u_n\| \quad \text{for all } v \in B_{\rho_0}.$$

Therefore

$$I_\lambda(u_n) \to c \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{ as } n \to \infty.$$

Since $u_n \leq \rho_0$ and $u_n \geq 0$, there exists $u_\lambda \in B_{\rho_0}$ with $u_\lambda \geq 0$ such that $u_n \rightharpoonup u_\lambda$ in $S_0^1(\Omega)$ as $n \to \infty$. By Lemma 3.3, we can obtain $u_n \to u_\lambda$ in $S_0^1(\Omega)$ and

$$d = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u_\lambda) < 0.$$

Hence, we have $u_\lambda \geq 0$ with $u_\lambda \neq 0$ is a solution of problem (1.1). By the maximum principle [5], we can know that $u_\lambda > 0$ in Ω. The proof of Theorem 1.1 is complete. □

Lemma 4.1. *Let $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ be satisfied. Then, there exists $\Lambda_2 > 0$ such that if $\lambda \in (0, \Lambda_2)$,*

$$\sup_{t\geq 0} I_\lambda(u_\lambda + tu_\varepsilon) < \frac{a^2}{4b} - D\lambda. \tag{4.2}$$

Proof. Let u_ε as defined in (2.3). Since u_λ is a positive solution of (1.1) and $I_\lambda(u_\lambda) < 0$, if $0 < \mu < bS^3|\Omega|^{-\frac{1}{2}}$, it holds that

$$\begin{aligned}
I_\lambda(u_\lambda + tu_\varepsilon) &= \frac{a}{2}\|u_\lambda + tu_\varepsilon\|^2 - \frac{b}{4}\|u_\lambda + tu_\varepsilon\|^4 - \frac{\lambda}{q}\int_\Omega |u_\lambda + tu_\varepsilon|^q d\xi + \frac{\mu}{4}\int_\Omega \phi_{u+tu_\varepsilon}(u_\lambda + tu_\varepsilon)^2 d\xi \\
&= \frac{a}{2}\|u_\lambda\|^2 + at\int_\Omega (\nabla u_\lambda \nabla u_\varepsilon)d\xi + \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{b}{4}\|u_\lambda\|^4 - \frac{bt^4}{4}\|u_\varepsilon\|^4 \\
&\quad -bt\|u_\lambda\|^2\int_\Omega (\nabla u_\lambda \nabla u_\varepsilon)d\xi - bt^2\left(\int_\Omega (\nabla u_\lambda \nabla u_\varepsilon)d\xi\right)^2 - \frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2 \\
&\quad -bt^3\|u_\varepsilon\|^2\int_\Omega (\nabla u_\lambda \nabla u_\varepsilon)d\xi - \frac{\lambda}{q}\int_\Omega |u_\lambda + tu_\varepsilon|^q d\xi + \frac{\mu}{4}\int_\Omega \phi_{u_\lambda+tu_\varepsilon}(u_\lambda + tu_\varepsilon)^2 d\xi \\
&\leq I_\lambda(u_\lambda) + \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2 \\
&\quad +\frac{\mu}{4}\left(\int_\Omega \phi_{u_\lambda+tu_\varepsilon}(u_\lambda + tu_\varepsilon)^2 - \phi_{u_\lambda}{u_\lambda}^2 - 4t\phi_{u_\lambda}u_\lambda u_\varepsilon d\xi\right) \\
&\quad -\frac{\lambda}{q}\left(\int_\Omega |u_\lambda + tu_\varepsilon|^q - |u_\lambda|^q d\xi\right) + \lambda t\int_\Omega {u_\lambda}^{q-1}u_\varepsilon d\xi \\
&\leq \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2 + \lambda t\left(\int_\Omega {u_\lambda}^3 d\xi\right)^{\frac{q-1}{3}}\left(\int_\Omega {u_\varepsilon}^{\frac{3}{4-q}}d\xi\right)^{\frac{4-q}{3}} \\
&\quad +\frac{\mu}{4}\left(\int_\Omega \phi_{u_\lambda+tu_\varepsilon}(u_\lambda + tu_\varepsilon)^2 - \phi_{u_\lambda}{u_\lambda}^2 - 4t\phi_{u_\lambda}u_\lambda u_\varepsilon d\xi\right) \\
&= \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2 + \lambda t\|u_\lambda\|_3^{q-1}\|u_\varepsilon\|_{\frac{4-q}{3}} \\
&\quad +\frac{\mu}{4}\left(\int_\Omega \phi_{u_\lambda+tu_\varepsilon}(u_\lambda + tu_\varepsilon)^2 - \phi_{u_\lambda}{u_\lambda}^2 - 4t\phi_{u_\lambda}u_\lambda u_\varepsilon d\xi\right).
\end{aligned} \tag{4.3}$$

On the other hand, we have

$$\begin{aligned}
&\int_\Omega (\phi_{u_\lambda+tu_\varepsilon}(u_\lambda+tu_\varepsilon)^2-\phi_{u_\lambda}{u_\lambda}^2-4t\phi_{u_\lambda}u_\lambda u_\varepsilon)d\xi\\
&\le \int_\Omega (6t^2\phi_{u_\lambda}{u_\varepsilon}^2+4t^3\phi_{u_\varepsilon}u_\lambda u_\varepsilon+t^4\phi_{u_\varepsilon}{u_\varepsilon}^2)d\xi. \qquad (4.4)
\end{aligned}$$

From (4.3) and (4.4), one has

$$\begin{aligned}
I_\lambda(u_\lambda+tu_\varepsilon) \ \le\ & \frac{at^2}{2}\|u_\varepsilon\|^2-\frac{bt^4}{4}\|u_\varepsilon\|^4-\frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2 \qquad (4.5)\\
&+\frac{\mu}{4}\left(\int_\Omega (6t^2\phi_{u_\lambda}{u_\varepsilon}^2+4t^3\phi_{u_\varepsilon}u_\lambda u_\varepsilon+t^4\phi_{u_\varepsilon}{u_\varepsilon}^2)d\xi\right)+\lambda t\|u_\lambda\|_2^{q-1}\|u_\varepsilon\|_{\frac{3-q}{2}}.
\end{aligned}$$

It follows from the Hölder inequality and Lemma 2.1 of [3] that

$$\begin{aligned}
&\|u_\varepsilon\|^2=S^2+O(\varepsilon^2), \quad \int_\Omega \phi_{u_\lambda}{u_\varepsilon}^2d\xi\le\|\phi_{u_\lambda}\|_4\|u_\varepsilon\|^2_{\frac{8}{3}}\le C\varepsilon,\\
&\int_\Omega \phi_{u_\varepsilon}u_\lambda u_\varepsilon d\xi\le\|\phi_{u_\lambda}\|_4\|u_\lambda\|_{\frac{8}{3}}\|u_\varepsilon\|_{\frac{8}{3}}\le C\varepsilon^{\frac{3}{2}},\\
&\int_\Omega \phi_{u_\varepsilon}{u_\varepsilon}^2d\xi\le\|\phi_{u_\varepsilon}\|_4\|u_\lambda\|^2_{\frac{8}{3}}\le C\varepsilon^2.
\end{aligned}$$

Noting that

$$\int_\Omega \phi_{u_\varepsilon}{u_\varepsilon}^2\le S^{-3}|\Omega|^{\frac{1}{2}}\|u_\varepsilon\|^4.$$

Thus, it follows from (4.5) that

$$\begin{aligned}
I_\lambda(u_\lambda+tu_\varepsilon) \ \le\ & at^2\left(\frac{S^2}{2}+C\varepsilon\right)-\frac{a_0t^4(S^2+O(\varepsilon^4))}{4}\\
&-\frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2+Ct^3\varepsilon^{\frac{3}{2}}+\lambda t\|u_\lambda\|_2^{q-1}\|u_\varepsilon\|_{\frac{3-q}{2}}, \qquad (4.6)
\end{aligned}$$

where

$$a_0=b-\mu S^{-3}|\Omega|^{\frac{1}{2}}>0.$$

Define

$$g(t)=at^2\left(\frac{S^2}{2}+C\varepsilon\right)-\frac{a_0t^4(S^2+O(\varepsilon^4))}{4}-\frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2+Ct^3\varepsilon^{\frac{3}{2}}+\lambda t\|u_\lambda\|_2^{q-1}\|u_\varepsilon\|_{\frac{3-q}{2}}.$$

Since

$$\lim_{t\to 0^+}g(t)>0 \quad \text{and} \quad \lim_{t\to\infty}g(t)=-\infty.$$

Thus, there exists $t_\varepsilon>0$ such that

$$\sup_{t\ge 0}g(t)=g(t_\varepsilon), \qquad \frac{dg}{dt}\ |_{t=t_\varepsilon}=0.$$

Therefore, there exist two positive constants t_1, t_2 independent of ε, λ such that

$$0 < t_1 < t_\varepsilon < t_2 < \infty.$$

Moreover, it holds from (4.6) that

$$\begin{aligned}
\sup_{t\geq 0} I_\lambda(u_\lambda + tu_\varepsilon) &\leq \sup_{t\geq 0} g(t) \\
&\leq \sup_{t\geq 0}\left(\frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{bt^4}{4}\|u_\varepsilon\|^4\right) - C_2S^2 + C_1\varepsilon + \lambda C_3\varepsilon \\
&< \frac{a^2}{4b} - C_2S^2 + C_1\varepsilon + \lambda C_3\varepsilon. \qquad (4.7)
\end{aligned}$$

Let

$$\varepsilon = \lambda \quad \text{and} \quad \Lambda_2 := \left(\frac{C_2S^2}{2C_1 + D}\right).$$

Then

$$C\varepsilon \geq \lambda C\varepsilon.$$

Furthermore,

$$-C_2S^2 + C_1\varepsilon^{\frac{3}{2}} + \lambda C_3\varepsilon \leq -C_2S^2 + 2C_1\varepsilon < \lambda(2C_1) - \lambda(2C_1 + D) = -D\lambda$$

for all $0 < \lambda < \Lambda_2$. Combining this with (4.7) implies that

$$\sup_{t\geq 0} I_\lambda(u_\lambda + tu_\varepsilon) < \frac{a^2}{4b} - D\lambda.$$

On the other hand, it is easy to verify that

$$\sup_{t\geq 0} I_\lambda(u_\lambda + tu_\varepsilon) < \frac{a^2}{4b} - D\lambda \quad \text{for } \mu \leq 0.$$

In either case, we have

$$\sup_{t\geq 0} I_\lambda(u_\lambda + tu_\varepsilon) < \frac{a^2}{4b} - D\lambda$$

for $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q \leq 2$. The proof of Lemma 4.1 is complete. □

Proof of Theorem 1.2. Let

$$\Lambda^* = \min\left\{\Lambda_*, \Lambda_2, \frac{a^2}{4bD}\right\}.$$

Applying the mountain pass lemma (see [1]), there exists $u_n \in S_0^1(\Omega)$ such that

$$I_\lambda(u_n) \to c \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty.$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I_\lambda(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = u_\lambda, \gamma(1) = e\}.$$

It follows from Lemma 3.3 that $\{u_n\}$ has a convergent subsequence (still denoted by $\{u_n\}$) such that $u_n \to u_1$ in $S_0^1(\Omega)$. Moreover, we obtain u_1 is a non-negative weak solution of problem (1.1) and

$$I_\lambda(u_1) = \lim_{n\to\infty} I_\lambda(u_n) = c > 0.$$

It follows that $u_\lambda \neq u_1$ and $u_1 \neq 0$. In fact, similar to the proof of Theorem 1.1, we also have $u_1 > 0$ and hence u_1 is a second positive solution of problem (1.1). □

5 Proof of Theorem 1.3

In this section, we will prove that problem (1.1) has a positive ground state solution, where $1 < q < 2$. To this end, we define

$$\psi = \inf_{u\in N} I_\lambda(u), \quad \mathcal{N} = \left\{u \in S_0^1(\Omega) : u \neq 0, \langle I_\lambda'(u), u\rangle = 0\right\}. \tag{5.1}$$

Proof of Theorem 1.3. Obviously, if $u \in \mathcal{N}$, one has $I_\lambda(|u|) = I_\lambda(u)$, therefore we can consider a nonnegative minimizing sequence $\{u_n\} \subset \mathcal{N}$ and such that

$$I_\lambda(u_n) \to \psi \quad \text{as } n \to \infty. \tag{5.2}$$

By $I_\lambda(u_\lambda) < 0$ and Lemma 3.2, we can see that $\psi < 0$ and $\{u_n\}$ is bounded in $S_0^1(\Omega)$. We may assume that $u_n \rightharpoonup u_2$ weakly in $S_0^1(\Omega)$ and $u_n \to u_2$ strongly in $L^\theta(\Omega)$ with $1 \le \theta < 4$, Then $u_2 \neq 0$. In fact, if $u_2 \equiv 0$ and $\lim_{n\to\infty} \|u_n\|^2 = l$, we have

$$\langle I_\lambda'(u_n), u_n\rangle = a\|u_n\|^2 - b\|u_n\|^4 + \mu \int_\Omega \phi_{u_n} {u_n}^2 d\xi - \lambda \int_\Omega |u_n|^q d\xi. \tag{5.3}$$

So, we get

$$al - bl^2 = 0.$$

From this fact, we obtain $l = 0$ or $l = \frac{a}{b}$.

If $l = 0$, we have $I_\lambda(u_n) \to 0$ as $n \to \infty$. This is a contradiction from (5.2).

If $l = \frac{a}{b}$, we have $I_\lambda(u_n) \to \frac{a^2}{4b}$. This is a contradiction from Lemma 4.1. Therefore, we must have $u_2 \neq 0$ in $S_0^1(\Omega)$.

It follows from Lemma 3.3 that $u_n \to u_2$ in $S_0^1(\Omega)$. It means that u_2 is a positive solution of problem (1.1) and $I_\lambda(u_2) \ge \psi$.

Next, we will prove $I_\lambda(u_2) \le \psi$. By the Fatou's Lemma, we get

$$\begin{aligned} \psi &= \lim_{n\to\infty} \left\{ I_\lambda(u_n) - \frac{1}{q}\langle I_\lambda'(u_n), u_n\rangle \right\} \\ &= \lim_{n\to\infty} \left\{ \frac{q-2}{2q}\|u_n\|^2 - \frac{b(4-q)}{4q}\|u_n\|^4 - \frac{\mu(4-q)}{4q}\int_\Omega \phi_{u_n} {u_n}^2 d\xi \right\} \\ &\ge \frac{q-2}{2q}\|u_2\|^2 - \frac{b(4-q)}{4q}\|u_2\|^4 - \frac{\mu(4-q)}{4q}\int_\Omega \phi_{u_2} {u_2}^2 d\xi. \end{aligned} \tag{5.4}$$

In addition, since u_2 is a positive solution of problem (1.1), one has

$$I_\lambda(u_2) = I_\lambda(u_2) - \frac{1}{q}\langle I'_\lambda(u_2), u_2\rangle = \frac{q-2}{2q}\|u_2\|^2 - \frac{b(4-q)}{4q}\|u_2\|^4 - \frac{\mu(4-q)}{4q}\int_\Omega \phi_{u_2} {u_2}^2 d\xi. \quad (5.5)$$

It follows from (5.4) that $I_\lambda(u_2) \leq \psi$. Thus $I_\lambda(u_2) = \psi$ and $\psi \neq 0$. This proves that u_2 is a positive ground state solution of problem (1.1). □

6 Proof of Theorem 1.4

In this section, we apply the Clark critical point theorem (see [9]) to prove that problem (1.1) has at least $m - n$ pairs of negative energy solutions. Note that Υ is a closed, symmetric subset of $X \setminus \{0\}$ at the origin. If $A \in \Upsilon$ define genus

$$\gamma(A) = \min\{k \in \mathbb{N} | \exists \phi \in C(A, \mathbb{R}^k \setminus \{0\}), \phi(-x) = \phi(x)\}$$

and

$$\Gamma(A) = \{A \in \Upsilon; \gamma(A) \geq k\}.$$

Theorem 6.1. *If $I_\lambda \in C^1(S_0^1(\Omega), \mathbb{R})$ is even, $I_\lambda(0) = 0$, I_λ satisfies $(PS)_c$ condition and the following conditions:*
(I_1) There is an m dimensional subspace X_m and a constant $r > 0$, $S_r(0) = \{u \in S_0^1(\Omega) | \|u\| = r\}$ such that $\sup\limits_{u \in S_r(0) \bigcap X_m} I_\lambda(u) < 0$.
(I_2) If there exist an n dimensional subspace $X_n (n < m)$ such that $\inf\limits_{u \in X_n^\perp} I_\lambda(u) > -\infty$.
Then, $I_\lambda(u)$ has at least $m - n$ pairs of critical points with negative critical value, where

$$c_j = \inf_{A \in \Gamma_j} \max_{u \in A} I_\lambda(u).$$

Remark 6.1. *Assume that $1 < q < 2$, $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $c < 0$, then for each $\lambda \in \left(0, \frac{a^2}{4b}D^{-1}\right)$, the functional I_λ satisfies the $(PS)_c$ condition, where D is given by Lemma 3.3.*

Proof. If $\lambda \in \left(0, \frac{a^2}{4b}D^{-1}\right)$, we have $\frac{a^2}{4b} - D\lambda > 0 > c$. Then, the conclusion of Remark 6.1 is the corollary of Lemma 3.3. □

Proof of Theorem 1.4. To prove Theorem 1.4, we shall use Theorem 6.1. Note that $S_0^1(\Omega)$ is a Banach space, $I_\lambda \in C^1$ be an even functional and $I_\lambda(0) = 0$.

Next, we divide the proof process into two steps.

Step I: Let $0 \neq u \in X_m \subset S_0^1(\Omega)$, where X_m is an m dimensional subspace, we define $u = \tau_m v$ with $\|v\| = 1$ and $\tau_m = \|u\|$.

If $0 < \mu \leq bS^3|\Omega|^{-\frac{1}{2}}$

$$\begin{aligned} I_\lambda(u) &= \frac{\tau_m^2 a}{2} - \frac{\tau_m^4 b}{4} + \frac{\tau_m^4 \mu}{4}\int_\Omega \phi_v v^2 d\xi - \frac{\tau_m^q \lambda}{q}\int_\Omega |v|^q d\xi \\ &\leq \frac{\tau_m^2 a}{2} - \frac{\tau_m^q \lambda}{q}\int_\Omega |v|^q d\xi. \end{aligned}$$

Since $1 < q < 2$ and the finite dimensions space all norms are equivalent, so there exists $\tau_0 > 0$ small enough such that

$$I_\lambda(u) \le \frac{\tau_m^2 a}{2} - \frac{\tau_m^q \lambda}{q} \int_\Omega |v|^q d\xi < 0 \quad \text{for all } \tau \in (0, \tau_0). \tag{6.1}$$

Besides, if $\mu \le 0$, it is obvious that there (6.1) still holds. Thus

$$\sup_{u \in S_r(0) \bigcap X_m} I_\lambda(u) < 0,$$

where $S_r(0) = \{u \in S_0^1(\Omega) | \|u\|_{S_0^1(\Omega)} = r\}$. This fact means that (I_1) in Theorem 6.1 holds true.

Step II: If $-\infty < \mu \le 0$, by (2.5) and the Hölider inequality, we get

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 + \frac{\mu}{4}\int_\Omega \phi_u u^2 d\xi - \frac{\lambda}{q}\int_\Omega |u|^q d\xi \\ &\ge \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 + \frac{\mu}{4} S^{-3}|\Omega|^{\frac{1}{2}}\|u\|^4 - \lambda S^{-\frac{q}{2}}|\Omega|^{\frac{4-q}{4}}\|u\|^q \\ &> \frac{a}{2}\|u\|^2 - C_1\|u\|^4 - C_1\|u\|^q, \end{aligned} \tag{6.2}$$

where $C_1 = \frac{b}{2}$ and $C_2 = \lambda S^{-\frac{q}{2}}|\Omega|^{\frac{4-q}{4}}$. Since $1 < q < 2$, we can choose small $\|u\|$ and an n dimensional subspace $X_n \subset X_m (n < m)$ such that

$$\inf_{u \in X_n^\perp} I_\lambda(u) > -\infty.$$

Besides, if $0 < \mu \le bS^3|\Omega|^{-\frac{1}{2}}$, it is easy to verify that (6.2) still holds. Hence, (I_2) in Theorem 6.1 is valid.

Let

$$\Lambda := \left\{\Lambda_*, \ \frac{a^2}{4b}D^{-1}\right\}.$$

It follows from Remark 6.1 and Lemma 3.3 that I_λ satisfies the $(PS)_c$ condition at all levels $c < 0$. Consequently, by theorem 6.1, we know that the problem (1.1) has at least $m - n$ pairs of negative solutions. □

Acknowledgements

Z. liu was supported by the Graduate Scientific Research Project of Changchun Normal University(SGSRPCNU, NO. 050). D. Zhang was supported by the National Natural Science Fund of China (No.11271062) and the Natural Science Fund of Jilin Province (No. 20190201014JC). S. Liang was supported by the Foundation for China Postdoctoral Science Foundation (No. 2019M662220), Scientific research projects for Department of Education of Jilin Province, China (No. JJKH20210874KJ).

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