APPLICATIONS OF STATISTICALLY PROBABILITY CONVERGENCE TO APPROXIMATION THEOREM

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Abstract. The purpose of this article is to study and investigate statistically probability convergence for the sequence of random variables by virtue of deferred Nörlund and deferred Euler summability mean. With the aid of MATLAB software, we also graphically exhibits the statistically probability convergence for probability density function of random variables. Further, as an application of our newly form summability mean, we prove Korovkin type approximation theorem via deferred Nörlund and deferred Euler statistically probability convergence and present compelling instances to illustrate the findings.

1. Introduction and Preliminaries

The concept of statistical convergence was initially presented by Fast [5] and Schoenberg [20] independently. Later on, Rath and Tripathy [17] studied statistical convergence from sequence spaces point of view. In 1980 Gadjiev and Orhan [7] studied the order of statistical convergence for the sequence of numbers. Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denotes the set of natural and real numbers, respectively.

Let $T \subset \mathbb{N}$ such that $T_m = \{n : n \leq m \text{ and } n \in T\}$. A sequence $z = (z_m)$ is said to be statistically convergent to z_0 if the set T_m has zero natural density, (see [6]) i.e., for each $\varepsilon > 0$ we have

$$\lim_{m\to\infty} \frac{1}{m} |\{n : n \leq m \text{ and } |z_n - z_0| \geq \varepsilon\}| = 0,$$

where the vertical bar implies cardinality of the enclosed set. In such case, we write

$$St \lim_{m\to\infty} z_m = z_0.$$

For more results on statistical convergence one may refer ([3], [4], [11], [13], [19]).

By Δ, we denote the class of events and let S be any event space of Δ. Suppose that $P : \Delta \to \mathbb{R}$ be a probability function. If a random variable Z_m is determined on a given event space S for each positive integer m, then we say that $Z_1, Z_2, \cdots, Z_m$ is a sequence of random variables and is represented by (Z_m).

The analysis on the random variable Z_m will be much remarkable if there exists $a \in \mathbb{R}$ such that

$$P(|Z_m - a| < \varepsilon) = 1, \tag{1.1}$$

for given $\varepsilon > 0$ which is adequately small. This implies that the values of (Z_m) lies in a

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very small neighbourhood of a.

As m gradually increases, for $a \in \mathbb{R}$ the relation (1.1) becomes more and more distinguished for the sequence (Z_m). Further, the existence of a will be satisfied by the concept of probability convergence for a sequence (Z_m).
The sequence (Z_m) of random variables is called as statistically probability convergent to Z where $Z : S \to \mathbb{R}$ is a random variable, if for any $\varepsilon > 0$ and $\delta > 0$

$$\lim_{m\to\infty} \frac{1}{m}\left|n : n \leq m \text{ and } P(|Z_m - Z| \geq \varepsilon) \geq \delta\right| = 0$$

or

$$\lim_{m\to\infty} \frac{1}{m}\left|n : n \leq m \text{ and } 1 - P(|Z_m - Z| \leq \varepsilon) \geq \delta\right| = 0.$$

We may denote it as

$$St_P \lim_{m\to\infty} P(|Z_m - Z| \geq \varepsilon) = 0 \quad \text{or} \quad St_P \lim_{m\to\infty} P(|Z_m - Z| \leq \varepsilon) = 1.$$

We know that every statistical convergent sequence is statistically probability convergent, but the converse need not be true as shown in the following example.

Example 1: Consider a prime number p and a probability density function of Z_m as

$$f_m(z) = \begin{cases} \frac{1}{\sqrt{p}}, \text{ for } 0 < z < \sqrt{p},\ m = n^2 \\ \\ 0, \text{ otherwise,} \end{cases}$$

and

$$f_m(z) = \begin{cases} \frac{mz^{m-1}}{p^m}, \text{ for } 0 < z < p,\ m \neq n^2 \\ \\ 0, \text{ otherwise.} \end{cases}$$

For $\varepsilon > 0$ and $\delta < 1$,

$$P(|Z_m - p| \geq \varepsilon) = \begin{cases} \frac{1}{\sqrt{p}}, \text{ for } m = n^2 \\ 1 - P(|Z_m - p| < \varepsilon) = \left(1 - \frac{\varepsilon}{p}\right)^m \text{ for } m \neq n^2, \end{cases}$$

which mean that

$$\begin{aligned} \lim_{m\to\infty} \frac{1}{m}\left|\left\{n : n \leq m \text{ and } P(|Z_m - p| \geq \varepsilon) \geq \delta\right\}\right| &\leq \lim_{m\to\infty} \frac{1}{m}|\{1^2, 2^2, \cdots, n^2\}| \\ &\leq \lim_{m\to\infty} \frac{1}{m} = 0. \end{aligned}$$

This implies that Z_m is neither convergent nor statistically convergent but it is statistically probability convergent to p. The graphical representation of the statistically probability convergence is presented in Figure 1 below.
The hypothesis of deferred Cesaro statistically probability convergence was introduced

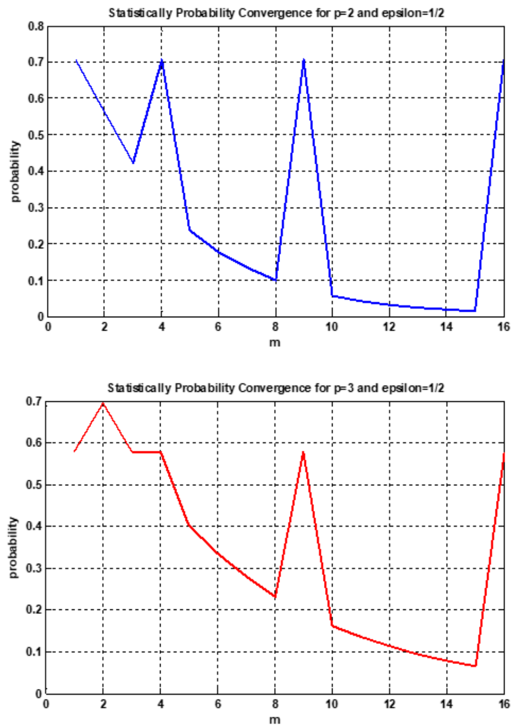

FIGURE 1. Statistically probability convergence of the sequence of random variables for $\epsilon = \frac{1}{2}$ and $p = 2$ and 3, respectively.

recently by Srivastava et al. in [22]. For detailed analysis on deferred Cesaro statistically probability convergence one may refer ([2], [10], [21], [23]).

Suppose that (x_m) and (y_m) are the sequences of non-negative integers fulfilling

$$x_m < y_m, \ \forall \ m \ \in \mathbb{N} \qquad \text{and} \qquad \lim_{x \to \infty} y_m = \infty. \tag{1.2}$$

Further, let (e_m) and (f_m) be two sequences of non-negative real numbers such that

$$\mathcal{E}_m = \sum_{n=x_m+1}^{y_m} e_n \quad \text{and} \quad \mathcal{F}_m = \sum_{n=x_m+1}^{y_m} f_n. \tag{1.3}$$

The convolution of (1.3) is defined as

$$\mathcal{R}_m = \sum_{v=x_m+1}^{y_m} e_v f_{y_m - v}.$$

As introduced by Srivastava et al. in [23], the deferred Nörlund (DN) mean is defined as

$$t_m = \frac{1}{\mathcal{R}_m} \sum_{n=x_m+1}^{y_m} e_{y_m - n} f_n z_n.$$

Next, the deferred Euler (DE) mean of r^{th} order is given as (see [16])

$$s_m = \frac{1}{(1+r)^{y_m}} \sum_{n=x_m+1}^{y_m} \binom{y_m}{n} r^{y_m-n} z_n,$$

$\forall\, m \in \mathbb{N}$ and $r > 0$.

Definition 1.1. A sequence $z = (z_m)$ is said to be deferred Euler statistically convergent to z, if for each $\varepsilon > 0$

$$E_m = \{n : n \leq (1+r)^{y_m} \text{ and } r^{y_m-n}|z_m - z| \geq \varepsilon\}$$

has zero natural density, i.e.,

$$\lim_{m\to\infty} \frac{|E_m|}{(1+r)^{y_m}} = 0.$$

For detailed study on Euler mean one may refer [8].
Now, we define the product of means obtained by deferred Nörlund (DN) and deferred Euler (DE) as follows

$$\omega_m \quad = \quad (t_m s_m) = (ts)_m = \frac{1}{\mathcal{R}_m(1+r)^{y_m}} \sum_{n=x_m+1}^{y_m} \binom{y_m}{n} e_{y_m-n} f_n r^{y_m-n} z_n.$$

Further, the sequence (ω_m) is said to be summable to z by the product $D(NE)$ summability mean if

$$\lim_{m\to\infty} \omega_m = z.$$

2. Main result

Definition 2.1. Suppose that (x_m) and (y_m) are the sequences fulfilling conditions (1.2) and $(e_m), (f_m)$ are sequences satisfying (1.3). A real sequence (z_m) is said to be deferred Nörlund and deferred Euler statistically or $(D(NE)S)-$convergent to z if $\forall\, \varepsilon > 0$, the set

$$\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \;\text{ and }\; e_{y_m-n} f_n r^{y_m-n}|z_m - z| \geq \varepsilon\}$$

has deferred Nörlund Euler density zero, i.e.

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m(1+r)^{y_m}} \left|\left\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \;\text{ and }\; e_{y_m-n} f_n r^{y_m-n}|z_m - z| \geq \varepsilon\right\}\right| \quad = \quad 0.$$

We may write it as

$$St_{D(NE)} \lim z_m = l.$$

Definition 2.2. Suppose that (x_m) and (y_m) are the sequences fulfilling conditions (1.2) and $(e_m), (f_m)$ are sequences satisfying (1.3). A sequence (Z_m) of random variables is called as deferred Nörlund and deferred Euler statistically probability $(D(NE)P)$ convergent to Z if $\forall\, \varepsilon > 0$, the set

$$\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \;\text{ and }\; e_{y_m-n} f_n r^{y_m-n} P(|Z_m - Z| \geq \varepsilon) \geq \delta\}$$

has $D(NE)$ density zero, i.e. if

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m(1+r)^{y_m}} \Big|\Big\{n : n \le (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n} P(|Z_m - Z| \ge \varepsilon) \ge \delta\Big\}\Big| = 0$$

or

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m(1+r)^{y_m}} \Big|\Big\{n : n \le (1+r)^{y_m}\mathcal{R}_m \text{ and } 1 - e_{y_m-n} f_n r^{y_m-n} P(|Z_m - Z| \le \varepsilon) \ge \delta\Big\}\Big| = 0,$$

and it is denoted as

$$St_{(D(NE)P)} \lim_{m\to\infty} e_{y_m-n} f_n r^{y_m-n} P(|Z_m - Z| \ge \varepsilon) = 0$$

or

$$St_{(D(NE)P)} \lim_{m\to\infty} e_{y_m-n} f_n r^{y_m-n} P(|Z_m - Z| \le \varepsilon) = 1.$$

Theorem 2.3. *Let (z_m) be the sequence of constants such that $St_{D(NE)} z_m \to z$. Then for a sequence (Z_m) with one point distribution at that point*

$$St_{(D(NE)P)} Z_m \to Z.$$

Proof. From definition 2.1, we have

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m(1+r)^{y_m}} \Big|\Big\{n : n \le (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n} |z_m - z| \ge \varepsilon\Big\}\Big| = 0,$$

for $\varepsilon > 0$ be arbitrary small positive real number. Now, for $\delta > 0$

$$\Big\{n : n \le (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n} P(|Z_m - Z| \ge \varepsilon) \ge \delta\Big\}$$
$$\subseteq \Big\{n : n \le (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n} |z_m - z| \ge \varepsilon\Big\}.$$

Thus, from definition 2.2 we have

$$St_{(D(NE)P)} Z_m \to Z.$$

□

Example 2: Suppose that $(\frac{r}{s})$ is a rational number and $x_m = 2m-1, y_m = 4m-1$. Suppose that $e_{y_m-m} = 2m, f_m = 1$ and $r = 1$. Further, consider a probability density function of Z_m as

$$f_m(z) = \begin{cases} 1, \text{ for } 0 < z < 1 \text{ and } \forall\, m = n^{\frac{r}{s}} \\ 0, \text{ otherwise,} \end{cases}$$

and

$$f_m(z) = \begin{cases} \frac{(m+1)z^m}{2^{m+1}}, \text{ for } 0 < z < 2 \text{ and } \forall \ m \neq n^{\frac{r}{s}}, \ n \in \mathbb{N} \\ \\ 0, \text{ otherwise.} \end{cases}$$

For $\varepsilon > 0$ and $\delta > 1$,

$$P(|Z_m - 2| \geq \varepsilon) = \begin{cases} 1, \text{ for } 0 < z < 1 \text{ and } m = n^{\frac{r}{s}} \\ \\ 1 - P(|Z_m - 2| < \varepsilon) = (1 - \frac{\varepsilon}{2})^m, \text{ for } m \neq n^{\frac{r}{s}}. \end{cases}$$

Hence, we have

$$\lim_{m\to\infty} \frac{1}{2m \ 2^{4m-1}} \Big| \Big\{ n : n \leq (1+r)^{y_m} \mathcal{R}_m \text{ and } 2mP(|Z_m - 2| \geq \varepsilon) \geq \delta \Big\} \Big| \quad = \quad 0.$$

We see that Z_m is neither convergent nor $(D(NE)S)-$convergent. However, it is $(D(NE)$ $P)-$convergent to 2. The graphical representation of the statistically probability convergence for particular values of ϵ and $\frac{r}{s}$ is given in Figure 2.

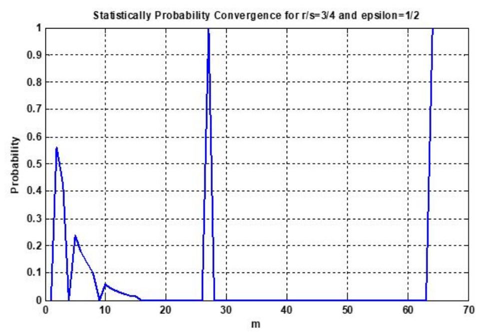

FIGURE 2. Statistically probability convergence of the sequence of random variables for $\epsilon = \frac{1}{2}$ and $\frac{r}{s} = \frac{3}{4}$, respectively.

3. Korovkin-Type Theorem

Korovkin-type approximation theorems have been investigated by many mathematicians under various background, involving function spaces, Banach spaces, and so on. Recently, Mohiuddine studied Korovkin and Voronovskaya type approximation theorems in [12]. Further, Hazarika et al. [9] studied Korovkin approximation theorem for Bernstein operator of rough statistical convergence of triple sequence. For detailed study on Korovkin approximation theorem one may refer ([2], [14], [15], [18]).

By $\mathcal{C}(Z)$, we denote the space of all continuous probability functions defined on a compact subset $Z \subset \mathbb{R}$. The space $\mathcal{C}(Z)$ is a Banach space with respect to the norm

$$||f||_\infty = \sup_{z\in Z}\{|f(z)|\}, \qquad f \in \mathcal{C}(Z).$$

We say that $\mathcal{Y}$ is a positive linear operator of sequence of random variables if

$$\mathcal{Y}(f, z) \geq 0 \text{ whenever } f \geq 0.$$

Theorem 3.1. *Let $\mathcal{Y}_n : \mathcal{C}(Z) \to \mathcal{C}(Z)$ be a sequence of random variables of positive linear operators. Then for all $f \in \mathcal{C}(Z)$, we have*

$$St_{(D(NE)P)} \lim_{n\to\infty} ||\mathcal{Y}_n(f,z) - f(z)||_\infty = 0 \tag{3.1}$$

iff

$$St_{(D(NE)P)} \lim_{n\to\infty} ||\mathcal{Y}_n(1,z) - 1||_\infty = 0, \tag{3.2}$$

$$St_{(D(NE)P)} \lim_{n\to\infty} ||\mathcal{Y}_n(z,z) - z||_\infty = 0, \tag{3.3}$$

$$St_{(D(NE)P)} \lim_{n\to\infty} ||\mathcal{Y}_n(z^2,z) - z^2||_\infty = 0. \tag{3.4}$$

Proof. We know that $f_0(z) = 1, f_1(z) = z, f_2(z) = z^2$ belongs to $\mathcal{C}(Z)$ and are continuous. Therefore, (3.1) $\Rightarrow$ (3.2) to (3.4) is obvious. We show that (3.2), (3.3) and (3.4) are true. For $f \in \mathcal{C}(Z)$, $\exists$ a constant $\mathcal{M} > 0$ such that

$$|f(z)| \leq \mathcal{M} \ \ (\forall\, z \in Z).$$

This implies that

$$|f(u) - f(z)| \leq 2\mathcal{M}, \ (u, z \in Z). \tag{3.5}$$

Clearly f is continuous, i.e. for given $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(u) - f(z)| < \varepsilon, \tag{3.6}$$

whenever $|u - z| < \delta$, $\forall\, u$ and $z \in Z$. Let us choose $\Phi_1 = \Phi_1(u,z) = (u-z)^2$.
If $|u - z| \geq \delta$, then we have

$$|f(u) - f(z)| < \frac{2\mathcal{M}}{\delta^2}\Phi_1(u,z). \tag{3.7}$$

From (3.6) and (3.7), we get

$$|f(u) - f(z)| < \varepsilon + \frac{2\mathcal{M}}{\delta^2}\Phi_1(u,z),$$

which means that

$$-\varepsilon - \frac{2\mathcal{M}}{\delta^2}\Phi_1(u,z) \leq f(u) - f(z) \leq \varepsilon + \frac{2\mathcal{M}}{\delta^2}\Phi_1(u,z). \tag{3.8}$$

Since $\mathcal{Y}_n(1,z)$ is linear and monotonic, so by applying $\mathcal{Y}_n(1,z)$ to (3.8) we obtain

$$\begin{aligned} \mathcal{Y}_n(1,z)\Big(-\varepsilon - \frac{2\mathcal{M}}{\delta^2}\Phi_1(u,z)\Big) \ &\leq\ \mathcal{Y}_n(1,z)(f(u) - f(z)) \\ &\leq \mathcal{Y}_n(1,z)\Big(\varepsilon + \frac{2\mathcal{M}}{\delta^2}\Phi_1(u,z)\Big). \end{aligned}$$

We first see that z is fixed and $f(z)$ is constant. Thus, we have

$$\begin{aligned} -\varepsilon\mathcal{Y}_n(1,z) - \frac{2\mathcal{M}}{\delta^2}\mathcal{Y}_n(\Phi_1,z) \ &\leq\ \mathcal{Y}_n(f,z) - f(z)\mathcal{Y}_n(1,z) \\ &\leq \varepsilon\mathcal{Y}_n(1,z) + \frac{2\mathcal{M}}{\delta^2}\mathcal{Y}_n(\Phi_1,z). \end{aligned} \tag{3.9}$$

Also, we know that

$$\mathcal{Y}_n(f,z) - f(z) = [\mathcal{Y}_n(f,z) - f(z)\mathcal{Y}_n(1,z)] + f(z)[\mathcal{Y}_n(1,z) - 1] \tag{3.10}$$

using (3.9) and (3.10), we obtain

$$\mathcal{Y}_n(f,z)-f(z)<\varepsilon\mathcal{Y}_n(1,z)+\frac{2\mathcal{M}}{\delta^2}\mathcal{Y}_n(\Phi_1,z)+f(z)[\mathcal{Y}_n(1,z)-1]. \tag{3.11}$$

We now evaluate $\mathcal{Y}_n(\Phi_1,z)$ as

$$\begin{aligned}\mathcal{Y}_n(\Phi_1,z) &= \mathcal{Y}_n((u-z)^2,z)=\mathcal{Y}_n(u^2-2zu+z^2,z)\\ &= \mathcal{Y}_n(u^2,z)-2z\mathcal{Y}_n(u,z)+z^2\mathcal{Y}_n(1,z)\\ &= [\mathcal{Y}_n(u^2,z)-z^2]-2z[\mathcal{Y}_n(u,z)-z]+z^2[\mathcal{Y}_n(1,z)-1].\end{aligned}$$

From (3.11), we have

$$\begin{aligned}\mathcal{Y}_n(f,z)-f(z) &< \varepsilon\mathcal{Y}_n(1,z)+\frac{2\mathcal{M}}{\delta^2}\{[\mathcal{Y}_n(u^2,z)-z^2]\\ &\quad -2z[\mathcal{Y}_n(u,z)-z]+z^2[\mathcal{Y}_n(1,z)-1]\}\\ &\quad +f(z)[\mathcal{Y}_n(1,z)-1]\\ &= \varepsilon[\mathcal{Y}_n(1,z)-1]+\varepsilon+\frac{2\mathcal{M}}{\delta^2}\{[\mathcal{Y}_n(u^2,z)-z^2]\\ &\quad -2z[\mathcal{Y}_n(u,z)-z]+z^2[\mathcal{Y}_n(1,z)-1]\}\\ &\quad +f(z)[\mathcal{Y}_n(1,z)-1].\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we say

$$\begin{aligned}|\mathcal{Y}_n(f,z)-f(z)| &\leq \varepsilon+\Big(\varepsilon+\frac{2\mathcal{M}}{\delta^2}+\mathcal{M}\Big)|\mathcal{Y}_n(1,z)-1|\\ &\quad +\frac{4\mathcal{M}}{\delta^2}|\mathcal{Y}_n(u,z)-z|+\frac{2\mathcal{M}}{\delta^2}|\mathcal{Y}_n(u^2,z)-z^2|\\ &\leq \mathcal{K}(|\mathcal{Y}_n(1,z)-1|+|\mathcal{Y}_n(u,z)-z|\\ &\quad +|\mathcal{Y}_n(u^2,z)-z^2|) \qquad (3.12)\end{aligned}$$

where

$$\mathcal{K}=\max\Big(\varepsilon+\frac{2\mathcal{M}}{\delta^2}+\mathcal{M},\frac{4\mathcal{M}}{\delta^2},\frac{2\mathcal{M}}{\delta^2}\Big).$$

Now for given $r>0$, $\exists\,\varepsilon$ and $\delta>0$ s.t. $\varepsilon<r$.
Then by setting

$$\Omega_n(z,r) = \Big\{n: x_m<n\leq y_m \text{ and } e_{y_m-n}f_nr^{y_m-n}P(|\mathcal{Y}_n(f,z)-f(z)|\geq r)\Big\}\geq\delta.$$

For $i=\mathbb{N}\cup\{0\}$, we have

$$\begin{aligned}\Omega_{i,n}(z,r) &= \Big\{n: x_m<n\leq y_m \text{ and } e_{y_m-n}f_nr^{y_m-n}P\Big(|\mathcal{Y}_n(f_i,z)-f_i(z)|\\ &\qquad \geq\frac{r-\varepsilon}{3\mathcal{K}}\Big)\Big\}\geq\delta,\end{aligned}$$

such that

$$\Omega_n(z,r) \leq \sum_{i=0}^{2} \Omega_{i,n}(z,r).$$

Clearly, we have

$$\frac{||\Omega_n(z,r)||_{\mathcal{C}(Z)}}{\mathcal{R}_m} \leq \sum_{i=0}^{2} \frac{||\Omega_{i,n}(z,r)||_{\mathcal{C}(Z)}}{\mathcal{R}_m}. \tag{3.13}$$

Next, by using the above assumption about the implications in equations (3.2), (3.3), (3.4) and by definition 2.2, R.H.S. of (3.13) $\to 0$ as $m \to \infty$. Therefore, we get

$$\frac{||\Omega_n(z,r)||_{\mathcal{C}(Z)}}{\mathcal{R}_m} \quad = \quad 0, \quad (\delta, r > 0).$$

Hence, (3.1) is true. □

Corollary 3.2. *Let $\mathcal{Y}_n : \mathcal{C}(Z) \to \mathcal{C}(Z)$ be the sequence of random variables of positive linear operators. Suppose that $f \in \mathcal{C}(Z)$, then*

$$St_{D(NE)} \lim_{n\to\infty} ||\mathcal{Y}_n(f,z) - f(z)||_\infty = 0 \tag{3.14}$$

iff

$$St_{D(NE)} \lim_{n\to\infty} ||\mathcal{Y}_n(1,z) - 1||_\infty = 0, \tag{3.15}$$

$$St_{D(NE)} \lim_{n\to\infty} ||\mathcal{Y}_n(z,z) - z||_\infty = 0, \tag{3.16}$$

$$St_{D(NE)} \lim_{n\to\infty} ||\mathcal{Y}_n(z^2,z) - z^2||_\infty = 0. \tag{3.17}$$

Let us recall an operator used by Al-Salam in [1] as

$$z(1+zD) \quad \Big(D = \frac{d}{dz}\Big).$$

Example 3: Let $\mathcal{M}_m(f,z)$ be a Meyer-König and Zeller operators on $\mathcal{C}[0,1]$ and $Z = [0,1]$ as defined in [1] as

$$\mathcal{M}_m(f,z) = (1-z)^{m+1} \sum_{t=0}^{\infty} f\Big(\frac{t}{t+m+1}\Big) \binom{m+t}{t} z^t.$$

Further, let us consider a sequence of operators $\mathcal{Y}_n : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ and (Z_n) as defined in example 2.4 such that

$$\mathcal{Y}_n(f,z) = [1+Z_n]z(1+zD)\mathcal{M}_n(f), \tag{3.18}$$

where $f \in \mathcal{C}[0,1]$. Now we observe that

$$\mathcal{Y}_n(1,z) = [1+Z_n]z(1+zD)1 = [1+Z_n]z,$$

$$\mathcal{Y}_n(u,z) = [1+Z_n]z(1+zD)z = [1+Z_n]z(1+z)$$

and

$$\begin{aligned}\mathcal{Y}_n(u^2,z) &= [1+Z_n]z(1+zD)\Big\{z^2\Big(\frac{m+2}{m+1}\Big)+\frac{z}{m+1}\Big\}\\ &= [1+f_m(z)]\Big\{z^2\Big[\Big(\frac{m+2}{m+1}\Big)z+2\Big(\frac{1}{m+1}\Big)+2z\Big(\frac{m+2}{m+1}\Big)\Big]\Big\}.\end{aligned}$$

Therefore, we have

$$St_{(D(NE)P)}\lim_{n\to\infty}||\mathcal{Y}_n(1,z)-1||_\infty=0,$$

$$St_{(D(NE)P)}\lim_{n\to\infty}||\mathcal{Y}_n(z,z)-z||_\infty=0,$$

$$St_{(D(NE)P)}\lim_{n\to\infty}||\mathcal{Y}_n(z^2,z)-z^2||_\infty=0.$$

Hence, $\mathcal{Y}_n(f,z)$ fulfills (3.2), (3.3) and (3.4). Thus, from Theorem 3.1

$$St_{(D(NE)P)}\lim_{n\to\infty}||\mathcal{Y}_n(f,z)-f||_\infty=0.$$

Hence, it is $(D(NE)P)-$convergent. However, (Z_m) is neither $(DN)-$statistical convergent nor $(D(NE)S)-$convergent. Thus, we can exhibit that the work in [23] does not hold for our operators described in (3.18). Hence, our Theorem 3.1 is stronger than the theorem proved in [23].

Compliance with ethical standards

Availability of data and material: Not applicable.

Conflict of interest: The authors declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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