D'Alembert type travelling wave solution for a wave equations on finite interval with coupled initial boundary conditions

CHEN Songlin

School of Math. and Phys., Anhui University of Technology, 243002, Ma'anshan City, Anhui

Abstract

The problem of solving equations for a class of coupled wave equations with initial-boundary conditions is discussed by using the results for the problem with initial value in this paper. A coupled wave equations which defined in semi-infinite interval and finite interval are studied respectively, the *d'Alembert* type traveling wave solutions with finite closed form of the corresponding problems are obtained and the examples are given. This research generalize the corresponding results for single wave equation and .avoid the traditional *Fourior* series solution.

Keywords Extension method; The coupled wave equations; Initial-boundary conditions; D'Alembert type travelling wave solution

1 Introduction

As the traveling wave describes the wave propagation process, therefore, solving the traveling wave solutions which can help to describe the various natural phenomena well, such as vibration, wave propagation and soliton which possess more realistic physical meaning . It is well known that the travelling wave solution for wave equation on the total infinite interval and semi-infinite interval can be given by d'Alembert formula and the solution for wave equation on the finite interval usually be given by Fourier series which is more complex to understand the solution [2,5] . The travelling wave solution of the initial-boundary value problem of single wave equation with Neumann boundary on the finite interval has been obtained[2,3]. On the basis of these results, this paper will further study the traveling wave solutions for a wave equations on finite interval with coupled initial boundary conditions which possess finite closed form, other than customary Fourier series form.

As is well-known that for the equations

$$\begin{cases} y_t + y_x = 0 \\ z_t - z_x = 0 \end{cases}, t \quad 0 \qquad (1)$$

with initial condition

$$y(x,0) = g_1(x) \qquad (2.a)$$

$$z(x,0) = g_2(x) \qquad (2.b)$$

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*the author email: slchen@ahut.edu.cn

The transformation of independent variables

$$\begin{cases} \xi = x + t \\ \eta = x - t \end{cases} \quad (3)$$

$$y_t + y_x = \frac{\partial y}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} + \frac{\partial y}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} + \frac{\partial y}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = 2 y_\xi = 0 \quad (4)$$

gives that

So y is a function of the η

$$y = \psi(\eta) = \psi(x - t) \quad (5)$$

Similarly z is a function of the ξ

$$z = \psi(\xi) = \psi(x + t) \quad (6)$$

From the initial conditions(2.a) we can obtain

$$y(x, 0) = \varphi(x - 0) = \varphi(x) = g_1(x) \quad (7)$$

so

$$y(x, t) = g_1(x - t) \quad (8)$$

From the initial conditions (2.b), the same can be obtained

$$z(x, t) = g_2(x + t) \quad (9)$$

so the traveling solution of the system (1,2) is

$$\begin{cases} y(x, t) = g_1(x - t) \\ z(x, t) = g_2(x + t) \end{cases} \quad (10)$$

The above is the d'Alembert type travelling solution for the coupled wave equations with initial data on total space. we'll use the above conclusion to discuss the solution of the initial-boundary value problem of the coupled wave equations on the semi-infinite and finite intervals respectively to obtain finite form solutions.

2 Main results

2.1 Coupled Initial boundary value problem for wave equations on semi-infinite interval

$$\begin{cases} y_t + y_x = 0 \\ z_t - z_x = 0 \end{cases}, t > 0,\ 0 < x < \infty \tag{11}$$

with initial condition

$$y(x,0) = g_1(x), 0 \le x < \infty \tag{12.a}$$

$$z(x,0) = g_2(x), 0 \le x < \infty \tag{12.b}$$

and coupled boundary condition

$$y(0,t) = k \cdot z(0,t) \tag{13}$$

Based on the results for equations on the total space, we adopt the appropriate continuation method to extend the semi-infinite interval to the total space. Here we continua the semi-infinite domain of $g_1(x), g_2(x)$ to whole spaces.

In the above ,through the variables substitution and initial conditions, we get

$$y(x,t) = g_1(x-t) \tag{14}$$

$$z(x,t) = g_2(x+t) \tag{15}$$

Next it is obtained by boundary conditions (13)

$$y(0,t) = g_1(0-t) = k \cdot z(0,t) = k \cdot g_2(0+t) \tag{16}$$

$$g_1(x) = k \cdot g_2(-x) \tag{17}$$

$$g_2(x) = \frac{1}{k} \cdot g_1(-x) \tag{18}$$

Therefore, we can obtain the following continuations for

$$\tilde{g}_1(x) = \begin{cases} g_1(x), & x \in [0, +\infty) \\ k \cdot g_2(-x), & x \in (-\infty, 0) \end{cases} \tag{19}$$

$$\tilde{g}_2(x) = \begin{cases} g_2(x), & x \in [0, +\infty) \\ \frac{1}{k} \cdot g_1(-x), & x \in (-\infty, 0) \end{cases} \tag{20}$$

So, the d'Alembert type travelling solution of the system of equations on a semi-infinite interval is

$$\begin{cases} y(x,t) = \tilde{g}_1(x-t) \\ z(x,t) = \tilde{g}_2(x+t) \end{cases} \tag{21}$$

Example 1

$$\begin{cases} y_t + y_x = 0 \\ z_t - z_x = 0 \end{cases}, t \quad 0$$

initial condition

$$\begin{cases} y(x,0) = \sin x \\ z(x,0) = 2\sin x \cdot \cos x \end{cases}$$

boundary condition

$$y(0,t) = 4 \cdot z(0,t)$$

At $x = 3$,the follows are the figures of the solutions varying with time

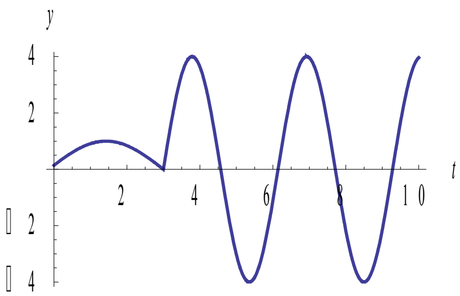


Fig. 1 *y*-variable with time at $x = 3$

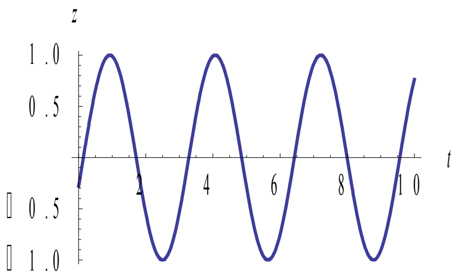


Fig. 2 *z*-variable with time at $x = 3$

The following figures are the ones for the solution of x in $[0,10]$ and time t in $[0,10]$

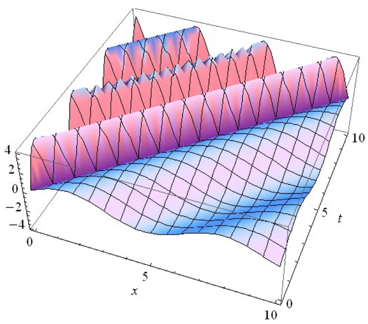


Fig. 3 y- figure

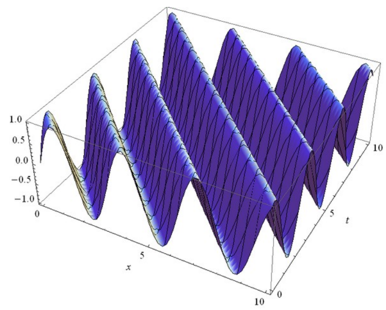


Fig. 4 *z*- figure

The above is the result of the initial boundary value problem of the system defined in the semi-infinite interval. We further discuss the initial boundary value problem of the system in the finite interval $[0,l]$ by appropriate continuation method.

2.2 Coupled Initial boundary value problem for wave equations on finite interval

$$\begin{cases} y_t + y_x = 0 \\ z_t - z_x = 0 \end{cases}, 0 \le t, 0 \le x \le l \tag{22}$$

with initial condition

$$\begin{cases} y(x,0) = g_1(x) \\ z(x,0) = g_2(x) \end{cases}, 0 \le x \le l \tag{23}$$

and coupled boundary condition

$$y(0,t) = k_1 \cdot z(0,t) \tag{24.a}$$

$$z(l,t) = k_2 \cdot y(l,t) \tag{24.b}$$

In the above, through the variables substitution and initial conditions, we get

$$y(x,t) = g_1(x-t) \tag{25}$$

$$z(x,t) = g_2(x+t) \tag{26}$$

it is obtained by boundary conditions (24.a)

$$g_1(-x) = k_1 \cdot g_2(x) \tag{27}$$

and it is obtained by boundary conditions (24.b)

$$g_2(l+x) = k_2 \cdot g_1(l-x) \tag{28}$$

Set $w = l + x$

$$g_2(w) = k_2 \cdot g_1(2l - w) \tag{29}$$

When $x \in [-l,0)$, $-x \in (0,l]$

$$g_1(x) = k_1 \cdot g_2(-x) \tag{30}$$

When $x \in (l,2l]$, $(2l-x) \in [0,l)$

$$g_2(x) = k_2 \cdot g_1(2l-x) \tag{31}$$

When $x \in (2l,3l]$, $(2l-x) \in [-l,0)$

$$g_2(x) = k_2 \cdot g_1(2l-x) = k_2 \cdot k_1 \cdot g_2(x-2l) \tag{32}$$

When $x \in [-2l,-l)$, $-x \in (l,2l]$

$$g_1(x) = k_1 \cdot g_2(-x) = k_1 \cdot k_2 \cdot g_1(2l+x) \tag{33}$$

Keeping the calculations going , we can get

When $x \in (0,+\infty)$

$$\tilde{g}_2(x)=\begin{cases} k_1^n \cdot k_2^n \cdot g_2(x-2nl), x\in(2nl,(2n+1)l] \\ k_1^n \cdot k_2^{n+1} \cdot g_1(2(n+1)l-x), x\in((2n+1)l,2(n+1)l] \end{cases} \quad (34)$$

When $x\in(-\infty,0)$

$$\tilde{g}_1(x)=\begin{cases} k_1^{n+1} \cdot k_2^n \cdot g_2(-x-2nl), x\in[-(2n+1)l,-2nl) \\ k_1^{n+1} \cdot k_2^{n+1} \cdot g_1(2(n+1)l+x), x\in[-2(n+1)l,-(2n+1)l) \end{cases} \quad (35)$$

$n=0,1,2,3\cdots$。

In addition,(27) and (29) Can be deformed into

$$g_2(x)=\frac{1}{k_1}g_1(-x) \quad (36)$$

$$g_1(x)=\frac{1}{k_2}\cdot g_2(2l-x) \quad (37)$$

Similarly we can get
When $x\in(-\infty,0)$

$$\tilde{g}_2(x)=\begin{cases} \frac{1}{k_1^{n+1}k_2^n}\cdot g_1(-x-2nl), x\in[-(2n+1)l,-2nl) \\ \frac{1}{k_1^{n+1}k_2^{n+1}}\cdot g_2(2(n+1)l+x), x\in[-2(n+1)l,-(2n+1)l) \end{cases} \quad (38)$$

When $x\in(0,+\infty)$

$$\tilde{g}_1(x)=\begin{cases} \frac{1}{k_1^n k_2^n} g_1(x-2nl), x\in[2nl,(2n+1)l) \\ \frac{1}{k_1^n k_2^{n+1}}\cdot g_2(2(n+1)-x), x\in((2n+1)l,2(n+1)l] \end{cases} \quad (39)$$

$n=0,1,2,3\cdots$.

So, the d'Alembert type travelling solution of the system of equations on the finite interval $[0,l]$ is

$$\begin{cases} y(x,t)=\tilde{g}_1(x-t) \\ z(x,t)=\tilde{g}_2(x+t) \end{cases} \quad (40)$$

Remark . The formula (40) shows that for given (x,t), the solution is of the finite form compared with the form of customary Fourier infinite series.

Example 2

$$\begin{cases} y_t + y_x = 0 \\ z_t - z_x = 0 \end{cases}, t > 0,\ \ 0 < x < \pi$$

with initial condition

$$\begin{cases} y(x,0) = \sin x \cdot x \cdot (x-\pi) \\ z(x,0) = \cos x \cdot x \cdot (x-\pi) \end{cases}$$

and boundary condition

$$\begin{cases} y(0,t) = 3 \cdot z(0,t) \\ z(\pi,t) = 3 \cdot y(\pi,t) \end{cases}$$

At $x = \frac{\pi}{4}$, the follows are the figures of the solutions varying with time

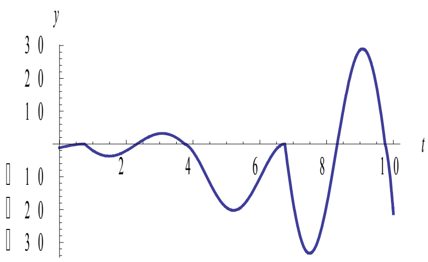


Fig. 5 *y*-variable with time at $x = \frac{\pi}{4}$

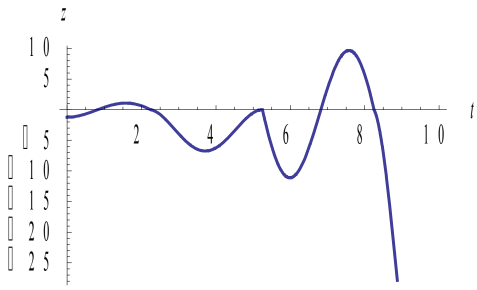
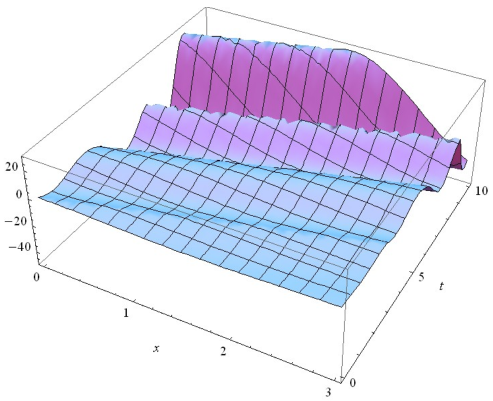


Fig. 6 *z*-variable with time at $x = \frac{\pi}{4}$

The following figures are the ones for the solution of x in $[0,\pi]$ and time t in $[0,10]$



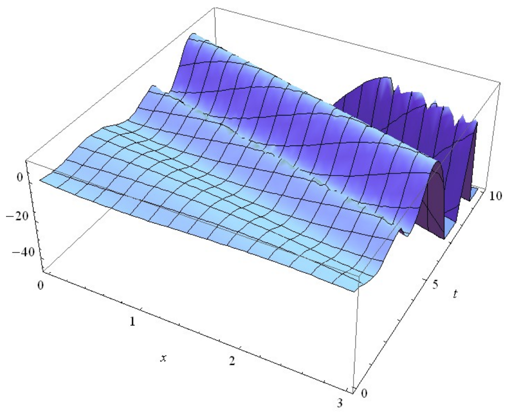


Fig. 7 y- figure **Fig. 8** z- figure

3. Conclusions

Wave equations with coupled initial-boundary conditions, which defined in the semi-infinite interval and finite interval, are studied respectively, by the continuation method. The d'Alembert type travelling solution of the corresponding problems are obtained and the examples are given. It is worthy to remind that for the problem defined on finite interval, the classic representation of the solution is of the form of the Fourier infinite series, but the results obtained in the present paper such as (21) and (40) is of the finite form, it is superior to the form of customary infinite series when are applied. This research generalize the corresponding results for wave equation[2,5].

4. Remarks and the scope of future research

Based on the above discussions, we want to generalize further the method to other problems with coupled wave equations. However, this method will not always successful. For example, for the following system

$$\begin{cases} y_t + y_x = 0 \\ z_t + z_x = 0 \end{cases}, t \quad 0, 0 \le x < \infty \tag{41}$$

with initial condition

$$y(x,0) = g_1(x), 0 \le x < \infty \tag{42.a}$$

$$z(x,0) = g_2(x), 0 \le x < \infty \tag{42.b}$$

and boundary condition

$$y(0,t) = k \cdot z(0,t) \tag{43}$$

The traveling wave solution for (41)~(42)

$$y(x,t) = g_1(x-t) \tag{44}$$

$$z(x,t) = g_2(x-t) \tag{45}$$

can't be used to complete the continuation by using the boundary conditions(43) to obtain the solution of the total problem (41)~(43), thus further study of the methods will be expected.

Declarations

Availability of data and material: (not applicable)

Competing interest: (not applicable)

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Biographical notes

CHEN Songlin, born in 1964, is currently a professor in School of Math. & Phys., Anhui University of Technology, Anhui, China. He has published more than 30 papers. His research interests include mathematical physics, asymptotic approximation. Tel: +86-555-2315570; E-mail: slchen@ahut.edu.cn