Reconstructing the wave speed and the source

Amin Boumenir and Vu Kim Tuan

Abstract. We are concerned with the inverse problem of recovering the unknown wave speed and also the source in a multidimensional wave equation. We show that the wave speed coefficient can be reconstructed from the observations of the solution taken at a single point. For the source, we may need a sequence of observation points due to the presence of multiple spectrum and nodal lines. This new method, based on spectral estimation techniques, leads to a simple procedure that delivers both uniqueness and reconstruction of the coefficients at the same time.

1. Introduction

We consider the problem of reconstructing the wave speed coefficient $\omega(x)$ of an unknown heterogeneous medium as well as an unknown source $p(x)$ from the observations of solutions u of the wave equation at a single point $b \in \Omega$

$$\begin{cases} u_{tt}(x,t) = \dfrac{1}{\omega(x)}\Delta u(x,t) + h(t)p(x), & x \in \Omega \subset \mathbb{R}^d,\ t>0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t>0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & x \in \Omega. \end{cases} \tag{1.1}$$

We assume that we are given Ω, a connected open domain with a smooth boundary, and also the control $h \in L(\mathbb{R}_+)$. To be able to observe the solution at any fixed point $b \in \Omega$, we assume that ω is a positive continuous function and the unknown source p, together with the initial conditions f, g, have compact supports in Ω. Also the initial conditions $\{f,\ g\}$ are chosen such that the observation map at a fixed point $b \in \Omega$,

$$\Gamma_b :\ \{f,\ g\} \longrightarrow \{u(b,t)\}_{t>0} \tag{1.2}$$

contains enough spectral data to reconstruct ω, and either p or h.

Statement of the problem:
A. Reconstruct the wave speed $\omega(x)$ from the knowledge of the map Γ_b given by (1.2) only.
B. Reconstruct the source $p(x)$ given $h(t)$ and $u(x,T)$ for $x \in \Omega$.
C. Reconstruct the source $p(x)$ given $h(t)$ and a sequence of maps Γ_{β_n}.

We shall explain how to extract the spectral data from the observations $\{u(b,\ t)\}_{t>0}$ in Section 3 and how to reconstruct the wave speed $\omega(x)$ in Section 4. In Section 5 we discuss the recovery of the source $p(x)$ given the control $h(t)$ and vice versa.

The inverse problem for wave equations has important applications ranging from medical imaging, seismology, to radar, and sonar as it models the propagation of acoustic waves inside an unknown medium. For applications in medical imaging, the wave speed coefficient would reveal the different densities in human body

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which helps identify the various organs, their shapes and any foreign matter, such as tumors. In optical tomography, pressure waves are generated inside a body by a pulsed laser, which usually would target a tumor or a harder object inside a soft organ, say liver for example. The energy from the pulsed laser would force the tumor to expand quickly and to resonate at a specific frequency, and so acts as a source as it generates pulsed pressure waves. By listening to the solution, at an arbitrary point inside or on the boundary, the reconstruction of $\omega(x)$ and $p(x)$ would help identify and image the different human tissues, and organs in a non invasive way and with no radiation. By comparing them with healthy ones, one could come up with a better diagnostic.

Our method is fundamentally different from standard scattering methods or echo analysis, as it is confined to a finite domain and is based on spectral estimation techniques, Weyl's asymptotic formula for eigenfunctions and mainly Fourier analysis, [**1, 2, 3**]. It also uses smooth initial conditions to generate classical solutions, as opposed to Dirac type of initial conditions which generally lead to weak solutions. This approach was first developed for heat equations [**4**], and then extended to wave equations, where we reconstruct a potential q from one point observation, see [**5**], of the solution of the wave equation

$$v_{tt}(x,t) = \Delta v(x,t) + q(x)v(x,t). \tag{1.3}$$

Note that in (1.3), the principal operator Δ is known, and so is its space, domain and we are looking only for a perturbation by q. In contrast with (1.3), the principal part $\dfrac{1}{\omega(x)}\Delta$ of (1.1) is unknown and so are its domain, range and the inner product space.

In real life, applied control problems require measurements and data to be collected at few points only, by means of sensors, which is why the proposed method is more appropriate for medical imaging and engineering applications in general. We also believe that tools from spectral theory will make the interplay between control theory, identification problems, observability [**10**], and inverse problems more transparent. For example, in subsection 5.3 we show why the presence of a multiple spectrum, as it complicates the geometry of nodal lines, requires more observations and data processing to solve multidimensional inverse problems, compared to one dimensional inverse problems, [**4**].

2. Preliminaries

For (1.1) we assume that the wave speed coefficient is positive, and Ω is an open bounded domain in $\mathbb{R}^d$, $d > 1$, with a smooth boundary, more precisely,

$$0 < \omega_0 \le \omega(x) \in C^{\left[\frac{d}{2}\right]+1}\left(\overline{\Omega}\right), \ \partial\Omega \in C^{\left[\frac{d}{2}\right]+1}, \tag{2.1}$$

$$h \in L(\mathbb{R}_+), \ p \in H_0^{\gamma}(\Omega), \ \text{with } \gamma > \frac{3(d-1)}{2}, \tag{2.2}$$

as we shall see the dimension d plays an important role in the regularity of the eigenfunctions and solution. Under the assumptions (2.1), the operator

$$\begin{cases} A(f)(x) = \dfrac{-1}{\omega(x)}\Delta f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

is selfadjoint when acting in the weighted Hilbert space

$$L^2_\omega(\Omega) = \left\{ f \text{ measurable} : \|f\|^2_\omega = \int_\Omega |f(x)|^2 \omega(x)dx < \infty \right\},$$

and its spectrum is discrete and positive [**6, 9**].

Let us denote the eigenvalues of (2.3), indexed in the increasing order by λ_n, taking into account their multiplicity, and associated eigenfunctions by $\varphi_n \in L^2_\omega(\Omega)$,

$$\begin{cases} -\Delta\varphi_n(x) = \lambda_n \omega(x)\varphi_n(x), & x \in \Omega, \\ \varphi_n(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

which we normalize by $\|\varphi_n\|_\omega = 1$. Recall that $\{\varphi_n\}_{n\geq 1}$ is an orthonormal basis in $L^2_\omega(\Omega)$. For the smoothness of φ_n, observe that φ_n is the solution of the Poisson equation

$$\begin{cases} -\Delta u(x) = q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.5}$$

where, from (2.4),

$$q(x) = \lambda_n \omega(x) \varphi_n(x). \tag{2.6}$$

Note that whenever $\varphi_n(x) \in H^k(\Omega)$, with $k \leq \left[\frac{d}{2}\right] + 1$, then $q(x)$ defined by (2.6) also belongs to $H^k(\Omega)$, since $\omega(x) \in C^{\left[\frac{d}{2}\right]+1}\left(\overline{\Omega}\right)$. It follows from the properties of the Laplacian operator, [**9**, Theorem 3, Section 2, Chapter IV],

$$\text{if} \quad q \in H^k(\Omega), \quad \text{then} \quad u \in H^{k+2}(\Omega). \tag{2.7}$$

Thus we deduce from (2.6) and $\varphi_n \in L^2_\omega(\Omega)$, that $q \in L^2(\Omega) = H^0(\Omega)$. By (2.7) we then have $\varphi_n = u \in H^2(\Omega)$. Since $k = 2 \leq \left[\frac{d}{2}\right] + 1$, then $q \in H^2(\Omega)$, and therefore, again by (2.7), $\varphi_n = u \in H^4(\Omega)$. We can repeat this process

$$\varphi_n(x) \in H^{2j}(\Omega) \overset{2j<\left[\frac{d}{2}\right]+1}{\Rightarrow} q(x) = \lambda_n \omega(x) \varphi_n(x) \in H^{2j}(\Omega) \overset{\Delta^{-1}}{\Rightarrow} \varphi_n(x) \in H^{2(j+1)}(\Omega), \tag{2.8}$$

to reach $\varphi_n(x) \in H^{2j}(\Omega), 2j \geq \left[\frac{d}{2}\right] + 1$. Consequently, $q \in H^{\left[\frac{d}{2}\right]+1}(\Omega)$. Together with the smoothness of the domain $\partial\Omega \in C^{\left[\frac{d}{2}\right]+1}$ it follows from [**9**, Theorem 5, Section 2, Chapter 4], that the generalized eigensolution $\varphi_n = u$ of the Poisson equation (2.5) is a classical solution, i.e.,

Lemma 2.1. *Assume that (2.1) holds, then* $\varphi_n \in C^2(\Omega) \cap C(\overline{\Omega})$.

Recall that the Weyl's law for the asymptotics of the eigenvalues λ_n has the form, [**2, 3**],

$$\lambda_n \simeq \delta n^{\frac{2}{d}}, \qquad n \to \infty, \quad \text{where} \quad \delta = \left[\frac{(2\sqrt{\pi})^{-d}}{\Gamma\left(\frac{d}{2}+1\right)} \int_\Omega \omega^{\frac{d}{2}}(x)\, dx\right]^{-\frac{2}{d}}, \tag{2.9}$$

and for the eigenfunctions $\varphi_n(x)$, the asymptotics formula

$$\sum_{|\sqrt{\lambda_n}-\lambda|\leq 1} \varphi_n^2(x) = O\left(\lambda^{d-1}\right), \quad \lambda \to \infty, \tag{2.10}$$

holds uniformly on any compact subset K of Ω, [**1**]. In particular, we deduce that

$$\varphi_n(x) = O\left(\lambda_n^{\frac{d-1}{4}}\right) = O\left(n^{\frac{d-1}{2d}}\right), \qquad n \to \infty. \tag{2.11}$$

By $\kappa_n(f)$ we denote the n^{th} Fourier coefficient of $f \in L^2_\omega(\Omega)$ in the basis $\{\varphi_n\}_{n\geq 1}$, namely

$$\kappa_n(f) = \int_\Omega f(x) \varphi_n(x) \omega(x) dx. \tag{2.12}$$

Recall that if $f \in H^\alpha_0(\Omega) \subset L^2_\omega(\Omega)$, for $\alpha \geq 0$, then its Fourier coefficient $\kappa_n(f)$ has the asymptotics [**1**]

$$\kappa_n(f) = O\left(\lambda_n^{-\frac{\alpha}{2}}\right) = O\left(n^{-\frac{\alpha}{d}}\right), \quad n \to \infty, \tag{2.13}$$

and the following convergence result will be essential for the pointwise observation of the solution of (1.1).

Lemma 2.2. [**1, 7, 8**] *Let* $f \in H^\alpha_0(\Omega)$ *with* $\alpha > \frac{d}{2}$. *Then the series*

$$\sum_{n=1}^{\infty} \kappa_n(f) \varphi_n(x) \tag{2.14}$$

converges absolutely and uniformly to $f(x)$ *on any compact subset of* Ω.

The absolute convergence of (2.14) should be understood in the following unconventional way. With the presence of multiple eigenvalues, let us regroup all eigenvalues into a strictly increasing sequence $\mu_1 < \mu_2 < \ldots$ such that the sets $\{\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots\}$ and $\{\mu_1, \mu_2, \cdots, \mu_l, \cdots\}$ coincide. Then the absolute convergence of (2.14) means the convergence of the series

$$\sum_{l=1}^{\infty} \left| \sum_{\lambda_n = \mu_l} \kappa_n(f)\varphi_n(x) \right|. \tag{2.15}$$

The solution of (1.1) can be expessed as

$$u(x,t) = \sum_{n\geq 1} c_n(t)\varphi_n(x), \tag{2.16}$$

where

$$c_n(t) := \int_\Omega u(x,t)\varphi_n(x)\omega(x)dx = \kappa_n(u(.,t)). \tag{2.17}$$

To find the behaviour of $c_n(t)$, use the fact that

$$\begin{aligned} c_n''(t) &= \int_\Omega u_{tt}(x,t)\varphi_n(x)\omega(x)dx \\ &= \int_\Omega \Delta u(x,t)\varphi_n(x)dx \ + h(t)\int_\Omega p(x)\ \varphi_n(x)\,\omega(x)dx \\ &= \int_\Omega u(x,t)\Delta\varphi_n(x)dx \ + h(t)\int_\Omega p(x)\ \varphi_n(x)\,\omega(x)dx \\ &= -\lambda_n \int_\Omega u(x,t)\varphi_n(x)\omega(x)dx \ + h(t)\int_\Omega p(x)\ \varphi_n(x)\,\omega(x)dx \end{aligned}$$

to deduce the initial value problem for each $c_n(t)$,

$$\begin{cases} c_n''(t) + \lambda_n c_n(t) = h(t)\kappa_n\left(p\right), \\ c_n(0) = \kappa_n(f),\ c_n'(0) = \kappa_n(g), \end{cases} \tag{2.18}$$

whose solution is given by

$$c_n(t) = \kappa_n(f)\cos(\sqrt{\lambda_n}t) + \kappa_n(g)\frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}} + \frac{\kappa_n\left(p\right)}{\sqrt{\lambda_n}}\int_0^t \sin(\sqrt{\lambda_n}(t-\eta))h(\eta)d\eta.$$

Thus, the solution of (1.1), by (2.16), has the representation

$$u(x,t) = \sum_{n\geq 1}\left(\kappa_n(f)\cos(\sqrt{\lambda_n}t) + \frac{\kappa_n(g)}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}t) + \frac{\kappa_n(p)}{\sqrt{\lambda_n}}\int_0^t \sin(\sqrt{\lambda_n}(t-\eta))h(\eta)d\eta\right)\varphi_n(x), \tag{2.19}$$

where, at first sight, the convergence is in $L^2_\omega(\Omega)$. To obtain uniform and absolute convergence of (2.19), we use Lemma 2.2

Proposition 1. *a) Assume* (2.1)*,* (2.2)*, and*

$$f \in H_0^{\gamma+1}(\Omega), \qquad g \in H_0^{\gamma}(\Omega). \tag{2.20}$$

Then for any $t \geq 0$, the series (2.19) *converges uniformly on any compact subset of Ω, and in particular, converges at any point $b \in \Omega$.*

b) Assume (2.1) *and*

$$f,\ g,\ p\ \in H_0^\alpha(\Omega), \quad \alpha > \frac{d}{2}, \quad h \in L^{loc}[0,\infty). \tag{2.21}$$

Then for any $x \in \Omega$, the series (2.19) *converges uniformly on any compact subset of $[0,\infty)$.*

Proof. a) Let K be any compact subset of Ω. From (2.9), (2.11), and (2.13), we have

$$\kappa_n(f)\cos(\sqrt{\lambda_n}t)\varphi_n(x),\ \frac{\kappa_n(g)}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}t)\varphi_n(x),\ \frac{\kappa_n(p)}{\sqrt{\lambda_n}}\left(\int_0^t \sin(\sqrt{\lambda_n}(t-\eta))h(\eta)d\eta\right)\varphi_n(x) = O\left(n^{\frac{d-2\gamma-3}{2d}}\right) \tag{2.22}$$

on K, uniformly in $t\in[0,\infty)$. Condition $\gamma > \frac{3(d-1)}{2}$ from (2.2) yields $\frac{d-2\gamma-3}{2d} < -1$, and therefore, the series (2.19) converges uniformly on $K\times[0,\infty)$.

b) Under assumptions (2.21) we can apply Lemma 2.2 to (2.19) to obtain

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty}\left(\kappa_n(f)\cos(\sqrt{\lambda_n}t) + \frac{\kappa_n(g)}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}t) + \frac{\kappa_n(p)}{\sqrt{\lambda_n}}\int_0^t \sin(\sqrt{\lambda_n}(t-\eta))h(\eta)d\eta\right)\varphi_n(x)\right| \\
&\le \sum_{l=1}^{\infty}\left|\sum_{\lambda_n=\mu_l}\kappa_n(f)\varphi_n(x)\cos(\sqrt{\mu_l}t)\right| + \sum_{l=1}^{\infty}\left|\sum_{\lambda_n=\mu_l}\kappa_n(g)\varphi_n(x)\frac{\sin(\sqrt{\mu_l}t)}{\sqrt{\mu_l}}\right| \\
&\quad + \sum_{l=1}^{\infty}\left|\sum_{\lambda_n=\mu_l}\kappa_n(p)\varphi_n(x)\frac{1}{\sqrt{\mu_l}}\int_0^t \sin(\sqrt{\mu_l}(t-\eta))h(\eta)d\eta\right| \\
&\le \sum_{l=1}^{\infty}\left|\sum_{\lambda_n=\mu_l}\kappa_n(f)\varphi_n(x)\right| + \frac{1}{\sqrt{\lambda_1}}\sum_{l=1}^{\infty}\left|\sum_{\lambda_n=\mu_l}\kappa_n(g)\varphi_n(x)\right| + \frac{\|h\|_{L(0,t)}}{\sqrt{\lambda_1}}\sum_{l=1}^{\infty}\left|\sum_{\lambda_n=\mu_l}\kappa_n(p)\varphi_n(x)\right| < \infty.
\end{aligned} \tag{2.23}$$

Thus the Weierstrass M test yields the uniform convergence of (2.19) in any compact subset of $[0,\infty)$. □

3. Extracting spectral data

To reconstruct a function, we only need a basis and its coordinates in the given basis. In our case we are interested in reconstructing the weight ω and the source p and as we shall see in the next section, the coordinates of $\varphi_1(x)\omega(x)$ in the basis $\{\psi_n\}_{n\ge1}$ are nothing else than $\{\kappa_1(\psi_n)\}_{n\ge1}$, while $\{\kappa_n(p)\}_{n\ge1}$ are coordinates of the source p in the basis $\{\varphi_n\}_{n\ge1}$. Thus we need to extract the coefficients $\{\kappa_1(\psi_n)\}_{n\ge1}$ and $\{\kappa_n(p)\}_{n\ge1}$ from the series (2.19), which is the observation of the solution of (1.1) at a single point $b\in\Omega$.

To do so, recall that $C_0^\infty(\Omega)$, the space of infinitely differentiable functions with compact support in Ω, is dense in $L^2(\Omega)$, and so we can choose an orthonormal basis $\{\psi_n\}_{n\ge1}$ of $L^2(\Omega)$ from $C_0^\infty(\Omega)$ as initial conditions and so conditions (2.20) or (2.21) can hold. Denote then the observation of the solution at the fixed point $b\in\Omega$, by $u_{f,g}(b,t)$ when its initial conditions are $\{f,g\}=\{\psi_{2k-1},\psi_{2k}\}, k=1,2,\cdots$, or $\{0,0\}$. Proposition 1 allows us to express the observed solutions in the form

$$\begin{aligned}
u_{\psi_{2k-1},\psi_{2k}}(b,t) &= \sum_{n\ge1}\left(\kappa_n(\psi_{2k-1})\cos(\sqrt{\lambda_n}t) + \frac{\kappa_n(\psi_{2k})}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}t) + \frac{\kappa_n(p)}{\sqrt{\lambda_n}}\int_0^t \sin(\sqrt{\lambda_n}(t-\eta))h(\eta)d\eta\right)\varphi_n(b), \\
u_{0,0}(b,t) &= \sum_{n\ge1}\frac{\kappa_n(p)\varphi_n(b)}{\sqrt{\lambda_n}}\int_0^t \sin(\sqrt{\lambda_n}(t-\eta))h(\eta)d\eta,
\end{aligned} \tag{3.1}$$

where the convergence is uniform in $[0,\infty)$. Consequently,

$$u_{\psi_{2k-1},\psi_{2k}}(b,t) - u_{0,0}(b,t) = \sum_{n\ge1}\left(\kappa_n(\psi_{2k-1})\cos(\sqrt{\lambda_n}t) + \frac{\kappa_n(\psi_{2k})}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}t)\right)\varphi_n(b), \qquad k=1,2\cdots, \tag{3.2}$$

converges uniformly on $[0,\infty)$, which allows to apply the Laplace transform

$$F(s) = (\mathcal{L}f)(s) = \int_0^\infty e^{-st}f(t)\,dt$$

to (3.1) and (3.2) termwise, to deduce, with $U = \mathcal{L}u, H = \mathcal{L}h$,

$$U_{\psi_{2k-1},\ \psi_{2k}}(b,s) = \sum_{n\geq 1} \frac{s\kappa_n(\psi_{2k-1}) + \kappa_n(\psi_{2k}) + H(s)\kappa_n(p)}{s^2+\lambda_n}\ \varphi_n(b),$$

$$U_{0,0}(b,s) = H(s)\sum_{n\geq 1} \frac{\kappa_n(p)}{s^2+\lambda_n}\ \varphi_n(b), \tag{3.3}$$

$$U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s) = \sum_{n\geq 1} \frac{s\kappa_n(\psi_{2k-1}) + \kappa_n(\psi_{2k})}{s^2+\lambda_n}\ \varphi_n(b) \tag{3.4}$$

$$= \sum_{l\geq 1}\sum_{\lambda_n=\mu_l} \frac{s\kappa_n(\psi_{2k-1}) + \kappa_n(\psi_{2k})}{s^2+\lambda_n}\ \varphi_n(b), \quad k = 1, 2\cdots. \tag{3.5}$$

Let $\mathbb{K}$ be any compact subset of the complex plane $\mathbb{C}$. Then there exists a positive real number $R > 1$ such that $|s| < R$ if $s \in \mathbb{K}$. Since $\lambda_n \to \infty$, there exists N such that $\lambda_n > 2R^2$ as $n \geq N$. Thus, for any $s \in \mathbb{K}$ and $n \geq N$,

$$\left|\frac{s}{s^2+\lambda_n}\right| < \frac{R}{2R^2-R^2} = \frac{1}{R}, \quad \left|\frac{1}{s^2+\lambda_n}\right| < \frac{1}{2R^2-R^2} = \frac{1}{R^2} < \frac{1}{R}. \tag{3.6}$$

From (2.11) we have

$$\varphi_n(b) = O\left(n^{\frac{d-1}{2d}}\right). \tag{3.7}$$

As $\psi_k \in C_0^\infty(\Omega) \subset H_0^\alpha(\Omega)$ for any $\alpha > 0$, from (2.13) it follows

$$\kappa_n(\psi_k) = O\left(n^{-\alpha}\right) \tag{3.8}$$

for any $\alpha > 0$. Combining (3.6), (3.7), and (3.8), we obtain that for any $\epsilon > 0$ there exists M such that for any $m \geq M$ and any $s \in \mathbb{K}$

$$\left|\sum_{n\geq m} \frac{s\kappa_n(\psi_{2k-1}) + \kappa_n(\psi_{2k})}{s^2+\lambda_n}\ \varphi_n(b)\right| \leq C\sum_{n\geq m} n^{-2} < \epsilon. \tag{3.9}$$

Thus, the series (3.4) converges uniformly on any compact $\mathbb{K}$, not containing $\{\pm i\sqrt{\lambda_n}\}_{n\geq 1}$, and therefore, $U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)$ is a meromorphic function with possible poles at $\{\pm i\sqrt{\mu_l}\}_{l\geq 1}$. However, it is possible that $\varphi_n(b) = 0$ for all n such that $\lambda_n = \mu_l$ (b is a nodal point of the eigenfunction $\varphi_n(x)$), in that case $\pm i\sqrt{\mu_l}$ are not poles of $U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)$, and therefore, that eigenvalue μ_l does not appear in the observations $u_{f,g}(b,t)$. Nevertheless, we have

Proposition 2. *We can determine uniquely* λ_1 *and* $\kappa_1(\psi_k)\varphi_1(b)$, *for* $k = 1, 2, \cdots$, *from the observations* $u_{0,0}(b,t)$ *and* $u_{\psi_{2k-1},\psi_{2k}}(b,t)$, $k = 1, 2, \cdots$.

Proof. By taking the Laplace transforms of observations we get the meromorphic functions $U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)$, $k = 1, 2, \cdots$, and therefore, we can find their poles uniquely. Denote the set of poles of $U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)$, $k = 1, 2, \cdots$, by $\{\tau_1, \tau_2, \cdots\}$. From (3.5) it is clear that the poles of the meromorphic functions $U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)$ are inside the set $\{\pm i\sqrt{\mu_l}\}_{l\geq 1}$

$$\{\tau_1, \tau_2, \cdots\} \subset \{\pm i\sqrt{\mu_l}\}_{l\geq 1}.$$

Then we compute the residues of $U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)$ at $\pm i\sqrt{\mu_m}$

$$\lim_{s\to\pm i\sqrt{\mu_m}} \left(s \mp i\sqrt{\mu_m}\right) \left(U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)\right) \tag{3.10}$$

$$= \lim_{s\to\pm i\sqrt{\mu_m}} (s \mp i\sqrt{\mu_m}) \sum_{l\geq 1} \sum_{\lambda_n=\mu_l} \frac{s\kappa_n(\psi_{2k-1}) + \kappa_n(\psi_{2k})}{\left(s + i\sqrt{\lambda_n}\right)\left(s - i\sqrt{\lambda_n}\right)} \varphi_n(b)$$

$$= \sum_{\lambda_n=\mu_m} \frac{\pm i\sqrt{\mu_m}\kappa_n(\psi_{2k-1}) + \kappa_n(\psi_{2k})}{\pm 2i\sqrt{\mu_m}} \varphi_n(b) \tag{3.11}$$

$$= \sum_{\lambda_n=\mu_m} \frac{\varphi_n(b)}{2} \left(\kappa_n(\psi_{2k-1}) \mp \frac{i\kappa_n(\psi_{2k})}{\sqrt{\mu_m}}\right). \tag{3.12}$$

Consequently, the Fourier coefficients $\sum_{\lambda_n=\mu_m} \kappa_n(\psi_k)\varphi_n(b)$ of $\sum_{\lambda_n=\mu_m} \varphi_n(b)\varphi_n(x)\omega(x)$ in the basis $\{\psi_k\}_{k\geq1}$ can be computed through the residues

$$\sum_{\lambda_n=\mu_m} \kappa_n(\psi_{2k-1})\varphi_n(b) = Res_{i\sqrt{\mu_m}} \left(U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)\right) + Res_{-i\sqrt{\mu_m}} \left(U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)\right),$$

$$\sum_{\lambda_n=\mu_m} \kappa_n(\psi_{2k})\varphi_n(b) = i\sqrt{\mu_m} Res_{i\sqrt{\mu_m}} \left(U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)\right) - i\sqrt{\mu_m} Res_{-i\sqrt{\mu_m}} \left(U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)\right),$$

$$k = 1, 2, \cdots. \tag{3.13}$$

Recall that the first eigenvalue $\lambda_1 = \mu_1$ is simple [**6**], and therefore, the formula for the first Fourier coefficient has the form

$$\kappa_1(\psi_{2k-1})\varphi_1(b) = Res_{i\sqrt{\lambda_1}} \left(U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)\right) + Res_{-i\sqrt{\lambda_1}} \left(U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)\right),$$

$$\kappa_1(\psi_{2k})\varphi_1(b) = i\sqrt{\lambda_1} Res_{i\sqrt{\lambda_1}} \left(U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)\right) - i\sqrt{\lambda_1} Res_{-i\sqrt{\lambda_1}} \left(U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)\right),$$

$$k = 1, 2, \cdots. \tag{3.14}$$

If $\varphi_n(b) = 0$ for all $\lambda_n = \mu_m$, then $\pm i\sqrt{\mu_m}$ are not poles of $U_{f,g}(b,s)$ for any initial conditions $\{f,g\}$, and therefore, cannot be detected from observations $u_{f,g}(b,t)$. Recall that the first eigenfunction $\varphi_1(x)$ does not change sign in Ω, therefore, $\varphi_1(b) \neq 0$, [**6**]. Since $\{\kappa_1(\psi_k)\}_{k\geq1}$ are the Fourier coefficients of $\omega(x)\varphi_1(x) \not\equiv 0$ in the basis $\{\psi_k\}_{k\geq1}$, at least one of them is not zero. In other words, there exists $k \geq 1$ such that $\left(U_{\psi_{2k-1},\psi_{2k}}(b,s) - U_{0,0}(b,s)\right)$ has a pole at $\pm i\sqrt{\lambda_1}$. Consequently, the first eigenvalue λ_1 can be found uniquely from the set of poles

$$\lambda_1 := \min\{|\tau_1|^2, |\tau_2|^2, \cdots\}, \tag{3.15}$$

and the Fourier coefficients $\kappa_1(\psi_k)\varphi_1(b)$, $k = 1, 2, \cdots$, from (3.14). □

4. Reconstructing $\omega(x)$

We now can determine $\omega(x)$ in Ω and to do so we recall formula (2.12)

$$\kappa_1(\psi_k) = \int_\Omega \omega(x)\varphi_1(x)\psi_k(x)\,dx,$$

and since $\{\psi_k\}_{k\geq1}$ is an orthonormal basis of $L^2(\Omega)$, we obtain

$$\varphi_1(b)\varphi_1(x)\omega(x) = q(x) := \sum_{k\geq1} \kappa_1(\psi_k)\varphi_1(b)\psi_k(x) \quad \text{in } L^2(\Omega). \tag{4.1}$$

In case $\{\psi_k\}_{k\geq1}$ were eigenfunctions of some self-adjoint positive elliptic differential operator of the second order, we would have pointwise or even uniform convergence in (4.1) under some extra assumptions. Observe that although the series in (4.1) converges almost everywhere in Ω to $\varphi_1(b)\varphi_1(x)\omega(x)$ in the L^2 sense, we recall that it was shown by Lemma 2.1 that $\varphi_1(x) \in C^2(\Omega) \cap C(\overline{\Omega})$. We explain now how to find the continuous weight $\omega(x)$.

Consider the Poisson equation with the Dirichlet boundary condition

$$\begin{cases} \Delta u(x) = -\lambda_1 q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $q(x) = \varphi_1(b)\varphi_1(x)\omega(x)$ is given by (4.1) in Ω. By Poisson's formula [**6, 9**], $u(x)$ is uniquely determined in $\overline{\Omega}$ and since $u(x) = \varphi_1(x)\varphi_1(b)$, we can then find the first eigenfunction $\varphi_1(x) = \frac{u(x)}{\sqrt{u(b)}}$ and the sought wave speed coefficient

$$\omega(x) = \frac{q(x)}{u(x)} \quad \text{in } \Omega.$$

Proposition 3. *Assume that conditions* (2.1) *and* (2.2) *hold, and* $\{\psi_k\}_{k\geq 1} \subset C_0^\infty(\Omega)$ *is a basis of* $L^2(\Omega)$. *Then we can reconstruct* ω *on* Ω *from a sequence of measurements at one point* $b \in \Omega$ *which are generated by the initial conditions* $\{0,0\}, \{\psi_{2k-1}, \psi_{2k}\}_{k\geq 1}$.

5. Reconstructing the source $p(x)$ or the control $h(t)$.

Once ω has been recovered by Proposition 3, then $\{\varphi_n\}_{n\geq 1}$ can also be reconstructed from (2.4). These $\{\varphi_n\}_{n\geq 1}$ provide a new basis that helps determine either $h(t)$ or $p(x)$ respectively from the knowledge of $p(x)$ or $h(t)$.

5.1. Determination of $h(t)$. Assume $p(x) \in H_0^\gamma(\Omega)$ is given, then from (3.3) we have

$$H(s) = \frac{U_{0,0}(b,s)}{\sum_{n\geq 1} \frac{\kappa_n(p)\varphi_n(b)}{s^2 + \lambda_n}}, \tag{5.1}$$

and so $h(t)$ can be determined uniquely by taking the Laplace inverse of $H(s)$ given by (5.1).

5.2. Reconstructing $p(x)$ from $u_{0,0}(x,T)$. We now examine the reconstruction of the source p if we have more data, for example $u_{0,0}(x,T)$ for all $x \in \Omega$, where $T > 0$ is a certain fixed positive time. We also assume that $h(t)$ is given in $L^2(0,\ T)$. From the given data we deduce

$$\int_\Omega u_{0,0}(x,T)\varphi_n(x)\omega(x)dx = \frac{\kappa_n(p)}{\sqrt{\lambda_n}} \int_0^T \sin(\sqrt{\lambda_n}(T-\eta))h(\eta)d\eta. \tag{5.2}$$

It is readily seen that to solve (5.2) for $\kappa_n(p)$ we need to choose the control h such that

$$\psi(\sqrt{\lambda_n}) := \int_0^T \sin(\sqrt{\lambda_n}(T-\eta))h(\eta)d\eta \neq 0 \quad \text{for all} \quad \lambda_n. \tag{5.3}$$

To this end it is readily seen that $h(t) = t \in L(0,T)$, yields

$$\int_0^T \eta \sin(\sqrt{\lambda_n}(T-\eta))\, d\eta = \frac{T\sqrt{\lambda_n} - \sin\left(T\sqrt{\lambda_n}\right)}{\lambda_n} > 0 \qquad \text{for all} \quad \lambda_n > 0. \tag{5.4}$$

Consequently, we can find the complete sequence $\{\kappa_n(p)\}_{n\geq 1}$

$$\kappa_n(p) = \frac{(\lambda_n)^{3/2}}{T\sqrt{\lambda_n} - \sin\left(T\sqrt{\lambda_n}\right)} \int_\Omega u_{0,0}(x,T)\varphi_n(x)\omega(x)dx, \tag{5.5}$$

and so reconstruct the source p in $L^2_\omega(\Omega)$

$$p(x) = \sum_{n\geq 1} \kappa_n(p)\, \varphi_n(x). \tag{5.6}$$

Since $p \in H_0^\gamma(\Omega)$ with $\gamma > \frac{3(d-1)}{2} > \frac{d}{2}$, Lemma 2.2 yields the uniform convergence of (5.6) on any compact subset of Ω. Thus we have proved

PROPOSITION 4. *Given the overdetermination* $u_{0,0}(x,T)$ *for all* $x \in \Omega$, *we can reconstruct the source* p *by choosing the control* $h(t) = t$ *on* $(0,T)$.

In practice it is difficult, if not impossible, to have the data, $u_{0,0}(x,T)$ for all $x \in \Omega$. We now show a method that recovers $p(x)$, by observing the solution at most over a sequence of points.

5.3. Reconstructing p from measurements at countable set of points. We start with the following lemma that helps us extract data in the presence of multiple eigenvalues.

LEMMA 5.1. *Let* $f_1(x), f_2(x), \cdots, f_n(x) \in C(\Omega)$, $n > 1$, *be linearly independent. Then there exist points* $x_1, x_2, \cdots, x_n \in \Omega$ *such that the matrix*

$$(f_i(x_j))_{i,j=1,\cdots,n} \tag{5.7}$$

is nonsingular.

PROOF. Assume the contrary, that the matrix (5.7) is singular for any set of points $\{x_1, \cdots, x_n\}$. Without loss of generality we can assume that the matrix (5.7) has rank $k \leq n-1$, and that the submatrix $(f_i(x_j))_{i,j=1,\cdots,k}$ is nonsingular. Since the matrix $(f_i(x_j))_{i,j=1,\cdots,k+1}$ is singular, its row vectors are linearly dependent, and the last row vector can be expressed as a linear combination of the other rows

$$f_{k+1}(x_j) = \alpha_1 f_1(x_j) + \alpha_2 f_2(x_j) + \cdots + \alpha_k f_k(x_j), \quad j = 1, 2, \cdots, k+1. \tag{5.8}$$

In particular,

$$\alpha_1 f_1(x_j) + \alpha_2 f_2(x_j) + \cdots + \alpha_k f_k(x_j) = f_{k+1}(x_j), \quad j = 1, 2, \cdots, k. \tag{5.9}$$

Consider (5.9) as a $k \times k$ system of linear equations with the unknown vector $(\alpha_1, \cdots, \alpha_k)^T$. Since $\det (f_i(x_j))_{i,j=1,\cdots,k} \neq 0$, the linear system (5.9) has the unique solution $(\alpha_1, \cdots, \alpha_k)^T$. In other words, the constants $\alpha_1, \cdots, \alpha_k$, depend only on $(f_i(x_j))_{i=1,\cdots,k+1}^{j=1,\cdots,k}$, but are independent of $(f_1(x_{k+1}), \cdots, f_{k+1}(x_{k+1}))$, i.e., independent of x_{k+1}. In particular, denote $x = x_{k+1}$, equation (5.8) for $j = k+1$ becomes

$$f_{k+1}(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_k f_k(x), \quad k+1 \leq n, \tag{5.10}$$

for any $x \in \Omega$, and $\alpha_1, \cdots, \alpha_k$, are independent of x, that contradicts the linear independence of $f_1, \cdots, f_n$ on Ω. $\square$

Let $\lambda_{n-1} < \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1} = \mu_l < \lambda_{n+m}$, i.e. the eigenvalue μ_l has the multiplicity $m > 1$. According to Lemma 5.1 there exist points $\beta_n, \cdots, \beta_{n+m-1}$ from Ω such that the matrix $(\varphi_i(\beta_j))_{i,j=n,\cdots,n+m-1}$ is nonsingular. If λ_n is a simple eigenvalue, $\lambda_{n-1} < \lambda_n < \lambda_{n+1}$, then take β_n such that $\varphi_n(\beta_n) \neq 0$. Choose $\beta_1, \beta_2, \cdots$, in that way. Let $0 \not\equiv h(t) \in L(\mathbb{R}_+)$ be given. From the observations $u_{0,0}(b,t), u_{\psi_{2k-1},\psi_{2k}}(b,t)$, $k = 1, 2, \cdots$, we can find $\{\varphi_n\}_{n\geq 1}$ as above. Assume further that we observe the solution at points $x = \beta_j$, $j = 1, 2, \cdots$, with the homogeneous initial conditions $\{f, g\} = \{0, 0\}$, i.e. $u_{0,0}(\beta_j, t)$, $j = 1, 2, \cdots$. Then (3.3) with b replaced by β_j yields

$$U_{0,0}(\beta_j, s) = H(s) \sum_{n\geq 1} \frac{\kappa_n(p)}{s^2 + \lambda_n} \varphi_n(\beta_j), \quad j = 1, 2, \cdots.$$

Thus the ratios

$$\frac{U_{0,0}(\beta_j, s)}{H(s)} = \sum_{n\geq 1} \frac{\kappa_n(p)\,\varphi_n(\beta_j)}{s^2 + \lambda_n} = \sum_{l\geq 1} \frac{1}{s^2 + \mu_l} \sum_{\lambda_n = \mu_l} \kappa_n(p)\,\varphi_n(\beta_j), \quad j = 1, 2, \cdots,$$

are meromorphic functions. Finding the residues of $\dfrac{U_{0,0}(\beta_j, s)}{H(s)}$ at $i\sqrt{\mu_l}$, we obtain

$$\sum_{i=n}^{n+m-1} \kappa_i(p)\,\varphi_i(\beta_j) = \sum_{\lambda_i = \mu_l} \kappa_i(p)\,\varphi_i(\beta_j) = 2i\sqrt{\mu_l}\; Res_{s=i\sqrt{\mu_l}} \frac{U_{0,0}(\beta_j, s)}{H(s)}, \quad j = n, \cdots, n+m-1, \quad m \geq 1. \tag{5.11}$$

Since the matrix $(\varphi_i(\beta_j))_{i,j=n,\cdots,n+m-1}$ is nonsingular, from (5.11) we can find $\kappa_i(p)$, $i = n, \cdots, n+m-1$, uniquely, and p can be reconstructed by (5.6).

Proposition 5. *Assume that the observed points $\beta_1, \beta_2, \cdots$, are chosen as above. Then from observations $u_{\psi_{2k-1},\psi_{2k}}(b,t), u_{0,0}(b,t), u_{0,0}(\beta_k,t)$, $k = 1, 2, \cdots$, we can reconstruct p by solving* (5.11) *and* (5.6).

Remark 5.2. Clearly in Proposition 5 one can take $\beta_1 = b$.

Assume after observing the solution at $x = b$ we discover that all the eigenvalues $\mu_1, \mu_2, \cdots$, are simple. As each nodal set $\varphi_n^{-1}(0)$ has measure zero, the set

$$\mathcal{N} = \bigcup_{n\geq 1} \varphi_n^{-1}(0), \tag{5.12}$$

which is their countable union also has measure zero. In other words we can find $\beta \in \Omega - \mathcal{N}$ such that $\varphi_n(\beta) \neq 0$ for all $n \geq 1$. In formula (5.11) now we have $m = 1$, and therefore,

$$\kappa_n(p) = \frac{2i\sqrt{\lambda_n}}{\varphi_n(\beta)}\ Res_{s=i\sqrt{\lambda_n}} \frac{U_{0,0}(\beta,s)}{H(s)}, \qquad n = 1, 2, \cdots. \tag{5.13}$$

So the observation $u_{0,0}(\beta,t)$ with $\beta \in \Omega - \mathcal{N}$ will determine p uniquely, and we arrive at

Proposition 6. *Assume that from the observations $u_{0,0}(b,t), u_{\psi_{2k-1},\psi_{2k}}(b,t)$, $k = 1, 2, \cdots$, it is found that all the eigenvalues of* (2.3) *are simple. If $b \in \Omega - \mathcal{N}$ then we can reconstruct p uniquely by* (5.13) *(where $\beta = b$) and* (5.6). *If $b \in \mathcal{N}$, then one more observation $u_{0,0}(\beta,t)$ with $\beta \in \Omega - \mathcal{N}$ will determine p uniquely.*

In other words, if after reconstructing ω, one discovers that the spectrum is simple, then at most one more observed point is enough to determine the source $p(x)$.

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Department of mathematics, University of West Georgia
E-mail address: boumenir@westga.edu

Department of mathematics, University of West Georgia
E-mail address: vu@westga.edu