Existence of multiple positive solutions for singular p-q-Laplacian problems with critical nonlinearities

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Abstract

In this article, we consider the following p-q-Laplacian system with singular and critical nonlinearity

$$\begin{cases} -\Delta_p u - \Delta_q u = \frac{h_1(x)}{u^r} + \lambda \frac{\alpha}{\alpha+\beta} u^{\alpha-1} v^{\beta} & in\ \Omega, \\ -\Delta_p v - \Delta_q v = \frac{h_2(x)}{v^r} + \lambda \frac{\beta}{\alpha+\beta} u^{\alpha} v^{\beta-1} & in\ \Omega, \\ u, v > 0 \quad in\ \Omega, \quad u = v = 0 & on\ \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. $1 < q < p < \alpha + \beta = p^*, 0 < r < 1, \alpha, \beta > 1, \lambda \in (0, \Lambda_*)$ is parameter with Λ_* is a positive constant and $h_1(x), h_2(x) \in L^\infty, h_1(x), h_2(x) > 0$. We show the existence and multiplicity of weak solution of equation above for suitable range of λ.

Keywords: p-q-Laplacian, Critical exponent, Singular nonlinearity, Weak positive solution

1 Introduction and main results

In this paper, we study the existence of positive solutions for the p-q-Laplacian problems with singular and critical nonlinearities

$$\begin{cases} -\Delta_p u - \Delta_q u = \frac{h_1(x)}{u^r} + \lambda \frac{\alpha}{\alpha+\beta} u^{\alpha-1} v^{\beta} & in\ \Omega, \\ -\Delta_p v - \Delta_q v = \frac{h_2(x)}{v^r} + \lambda \frac{\beta}{\alpha+\beta} u^{\alpha} v^{\beta-1} & in\ \Omega, \\ u, v > 0 \quad in\ \Omega, \quad u = v = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. $1 < q < p < \alpha + \beta = p^*, 0 < r < 1, \alpha, \beta > 1, \lambda \in (0, \Lambda_*)$ is parameter with Λ_* is a positive constant and $h_1(x), h_2(x) \in L^\infty, h_1(x), h_2(x) > 0$.

The singular elliptic equation has its biological, chemical and especially physical background. Up to now, the singular elliptic problem is still the research focus of the elliptic problem. Li and Gao [21] considered the following single equation with singular term

$$\begin{cases} -\Delta_p u = \frac{h}{u^r} & in\ \Omega, \\ u > 0 & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{1.2}$$

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where $\Omega \subset \mathbb{R}^n, n \geq 3$ is a bounded domain with smooth boundary $\partial\Omega$. The difficulty of such problems is the lack of integrability of singular terms, Li and Gao [21] used the variational method to restore the integrability of singular terms under suitable constraints. The problem (1.2) admitted at least one solution if and only if there exists a $u_0 \in W_0^{1,p}(\Omega)$ such that $\int_\Omega h u_0^{1-r} dx < \infty$. Moreover, Cong and Han [8] added a nonlinear term to the equation (1.2), they obtained the existence of positive solution of above problem on a compatibility condition. Sun and Wu [28] considered a class of equations with nonlinearity containing both singular and supercritical terms, the precise lower bound of parameter λ is obtained by means of the relationship between manifolds and fiber mappings, so that the above equation has a solution for all $\lambda \in (0, \lambda^*)$ and no solution for $\lambda > \lambda^*$, where λ^* is a precise number. Wang, Qin and Gang [29] also considered similar problems with [28], they proved the existence of weak positive solutions by using a priori estimation and sub-and super-solutions methods. Besides, still some general results are known in [27],[26],[5] for singular-subcritical, nonsingular-critical and singular-nonlinearities cases.

In addition, for elliptic equations with singular and critical nonlinearities, Chen and Rocha [4] considered the following equation:

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2} u = u^{2^*-1} + \mu u^{-q} & in\ \Omega\backslash\{0\}, \\ u(x) > 0 & in\ \Omega\backslash\{0\}, \\ u(x) = 0 & on\ \partial\Omega, \end{cases} \tag{1.3}$$

they showed there exists some $\mu^* > 0$ such that for any $\mu \in (0, \mu^*)$ the problem (1.3) possesses at least two positive solutions by minimizing the associated energy functional on some suitable Nehari sets. Moreover, Figueiredo and Montenegro [11] thought about a harder equation than (1.3) and established the existence of nontrivial nonnegative solution. Besides, Oliva, Sciunzi and Vaira [22] considered weak solutions to the p-Laplace equation with a critical Sobolev growth and Hardy potential. For some results of singular elliptic equations with critical exponents, we can also refer to references [13],[33],[12],[25],[18],[14] and their contained therein.

In recent years, p-Laplacian equations have been extended to the p-q-Laplacian problems with singular nonlinear term, Papageorgiou Winkert [23] studied the following equation

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda[u^{-\eta} + a(x)u^{\tau-1}] + f(x,u) & in\ \Omega \\ u|_{\partial\Omega} = 0,\ u > 0,\ \lambda > 0,\ 1 < \tau < q < p,\ 0 < \eta < 1. \end{cases} \tag{1.4}$$

They obtained positive solutions and established the accurate dependence of the set of positive solutions of (1.4) on the parameter $\lambda > 0$ as the latter changes. For a nonlinear elliptic problem involving p-q-Laplacian, it is worth referring that Sciammetta and Tornatore [24] gave a systematic research for the existence of two solutions by means of variational methods and critical points theorems. Yin and Yang [32] found multiple nontrivial weak solutions by standard variational methods and some analytical techniques. And the existence results in all of $\mathbb{R}^n$ for quasilinear problems with critical term can be seen in [20]. On the basis of the p-q-Laplacian equation with singular term, the critical term was added to above equation in Kumar, Sreenadh and Râdulescu [17]. That is to consider the equation

$$\begin{cases} -\Delta_p u - \beta\Delta_q u = \lambda u^{-\delta} + u^{r-1} & in\ \Omega, \\ u > 0 & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. $1 < q < p < r \leq p^*$, where $p^* = \frac{np}{n-p}, 0 < \delta < 1, n > p$ and $\lambda, \beta > 0$ are parameters. They proved the existence, multiplicity and regularity of weak solutions of (1.5) for suitable range of λ.

On the other hand, the p-q-Laplace systems have received much attention, it is closely related to fluid mechanics. It originates from the study of non-Newtonian fluid problems and is involved in the theory of quasi-regularization and quasi-projection mapping. There is no singular term for equation (1.1) they consider

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda|u|^{r-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & x \in \Omega, \\ -\Delta_p v - \Delta_q v = \theta|v|^{r-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \tag{1.6}$$

the main difficulty of dealing with the equation (1.6) is presence of the critical term, which makes some embedding lack compactness. Yin proved at least $cat_\Omega(\Omega)$ positive solutions of (1.6) by applying $(PS)_c$ sequence and mountain-pass theorem. For other related study, see, e.g[20]. Especially, when $p = q$, Hsu [16] showed the existence and multiplicity results. Very recently, do ó and Silva [1] studied the existence of positive ground states solutions for a large class of nonlinear terms and potential. Relying on concentration-compactness principle, mountain pass theorem, and genus theory, the existence of nontrivial weak solutions was established in the work [19].

In this line of thought, Choudhuri, Saoudi and Mouna [7] considered the following system

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(x)|u|^{r-2}u + v\frac{1-\alpha}{2-\alpha-\beta}h(x)|u|^{-\alpha}|v|^{1-\beta} & in\ \Omega, \\ -\Delta_p v - \Delta_q v = \theta g(x)|v|^{r-2}v + v\frac{1-\beta}{2-\alpha-\beta}h(x)|u|^{1-\alpha}|v|^{-\beta} & in\ \Omega, \\ u, v > 0 & in\ \Omega, \\ u = v = 0 & on\ \partial\Omega, \end{cases} \tag{1.7}$$

where $0 < \alpha < 1, 0 < \beta < 1, 2 - \alpha - \beta < q < \frac{n(p-1)}{n-p} < p < r < p^*$, with $p^* = \frac{np}{n-p}$. They studied the existence of multiple nontrivial positive weak solutions for (1.7).

Motivated by [28],[5],[19],[16],[7], in this paper we extend subcritical p-q-Laplacian systems with singular term to critical case. In order to overcome the difficult of lack of compactness, we adopt the property of cutoff function and establish the exact estimate of the upper bound of parameter λ^*. As far as we know it is entirely novel.

Our approach to this problem is based on the structure of the constraint set Λ_*, with $\Lambda_* = \left(\frac{p-1+r}{p^*-1+r}\right)\left(\frac{p^*-1+r}{p^*-p}\right)^{\frac{p-p^*}{p-1+r}}\left[|h_1|^{\frac{p}{p-1+r}}_{L^{(1-r)^*}} + |h_2|^{\frac{p}{p-1+r}}_{L^{(1-r)^*}}\right]^{\frac{p-p^*}{p}} S^{\frac{(1-r)(p^*-p)}{p(p-1+r)}} S_{\alpha,\beta}^{\frac{p^*}{p}}$. We now state the main results of this work.

Theorem 1.1. *Suppose that $\lambda \in (0, \Lambda_*)$, then the problem (1.1) has a solution $(u_0, v_0) \in X$ satisfying $I(u_0, v_0) < 0$ and $\|(u_0, v_0)\| < E_0$ (E_0 defined in Lemma 2.3).*

Theorem 1.2. *Suppose that $\lambda \in (0, \Lambda_*)$, then the problem of* (1.1) *has a solution $(U_0, V_0) \in X$ satisfying $\|(U_0, V_0)\| \geq E_\lambda > E_0$ with $E_\lambda \to +\infty$ as $\lambda \to 0$.*

In Section 2, We give some notations and preliminaries. The proof of Theorem 1.1 and 1.2 are give in Sections 3 and Sections 4.

2 Notations and Preliminaries

For a bounded domain $\Omega \subset \mathbb{R}^n$, we denote by $\|\cdot\|_p, |\cdot|_p$ the norm of $W_0^{1,p}(\Omega)$ and $L^p(\Omega)$ respectively, that is

$$\|u\|_p = (\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}}, \quad |u|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}.$$

Obviously, $X := W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ is a Banach space. Let X' be the dual of X and $\langle \cdot \rangle$ the duality pairing between X' and X. The norm on X is given by $\|(u, v)\|_p = (\|u\|_p^p + \|v\|_p^p)^{\frac{1}{p}}$. Define

$$S := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_p^p}{(|u|^{p^*} dx)^{\frac{p}{p^*}}},$$

and

$$S_{\alpha,\beta} := \inf_{(u,v) \in X \setminus \{(0,0)\}} \frac{\|(u, v)\|_p^p}{(\int_\Omega |u|^\alpha |v|^\beta dx)^{\frac{p}{\alpha+\beta}}}.$$

Then, it is easy to get that

$$\int_\Omega |u|^\alpha |v|^\beta dx \leq S_{\alpha,\beta}^{-\frac{\alpha+\beta}{p}} \|(u, v)\|_p^{\alpha+\beta}.$$

By Yin and Yang [29,Lemma 2.2], we have

$$S_{\alpha,\beta} = \left[\left(\frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}} \right] S.$$

Definition 2.1. *We say that $(u, v) \in X$ is a positive weak solution of* (1.1)*, if $(u, v) \in X$ satisifies*

$$\int_\Omega (|\nabla u|^{p-2} \nabla u \nabla \varphi_1 + |\nabla v|^{p-2} \nabla v \nabla \varphi_2) dx + \int_\Omega (|\nabla u|^{q-2} \nabla u \nabla \varphi_1 + |\nabla v|^{q-2} \nabla v \nabla \varphi_2) dx$$
$$- \int_\Omega (h_1 u^{-r} \varphi_1 + h_2 v^{-r} \varphi_2) dx - \frac{\lambda \alpha}{\alpha + \beta} \int_\Omega u^{\alpha-1} v^\beta \varphi_1 dx - \frac{\lambda \beta}{\alpha + \beta} \int_\Omega u^\alpha v^{\beta-1} \varphi_2 dx = 0 \tag{2.1}$$

for all $(\varphi_1, \varphi_2) \in X$.

Consider the Nehari manifold

$$\mathcal{N}_\lambda := \{(u, v) \in X \setminus \{(0, 0)\} : \|(u, v)\|_p^p + \|(u, v)\|_q^q - \int_\Omega (h_1 u^{-r} + h_2 v^{-r}) dx - \lambda \int_\Omega u^\alpha v^\beta dx = 0\}.$$

Note that $\mathcal{N}_\lambda$ includes all positive weak solutions of (1.1). The functional associated to (1.1) is

$$I(u, v) = \frac{1}{p} \|(u, v)\|_p^p + \frac{1}{q} \|(u, v)\|_q^q - \frac{1}{1-r} \int_\Omega (h_1 u^{1-r} + h_2 v^{1-r}) dx - \frac{\lambda}{\alpha + \beta} \int_\Omega u^\alpha v^\beta dx.$$

It is easy to know that if $(u,v)\in\mathcal{N}_\lambda$, then

$$I(u,v)=\left(\tfrac{1}{p}-\tfrac{1}{1-r}\right)\|(u,v)\|_p^p+\left(\tfrac{1}{q}-\tfrac{1}{1-r}\right)\|(u,v)\|_q^q+\lambda\left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega u^\alpha v^\beta dx$$

$$=\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\|(u,v)\|_p^p+\left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\|(u,v)\|_q^q+\left(\tfrac{1}{p^*}-\tfrac{1}{1-r}\right)\int_\Omega(h_1u^{1-r}+h_2v^{1-r})dx.$$

For $t>0$. We define the fibering maps $\phi_{u,v}:=(0,\infty)\times\{X\backslash\{(0,0)\}\}\to\mathbb{R}$

$$\phi_{u,v}(t)=I(tu,tv)=\frac{t^p}{p}\|(u,v)\|_p^p+\frac{t^q}{q}\|(u,v)\|_q^q-\frac{t^{1-r}}{1-r}\int_\Omega(h_1u^{1-r}+h_2v^{1-r})dx-\frac{\lambda t^{p^*}}{p^*}\int_\Omega u^\alpha v^\beta dx,$$

$$\phi'_{u,v}(t)=t^{p-1}\|(u,v)\|_p^p+t^{q-1}\|(u,v)\|_q^q-t^{-r}\int_\Omega(h_1u^{1-r}+h_2v^{1-r})dx-\lambda t^{p^*-1}\int_\Omega u^\alpha v^\beta dx,$$

and

$$\begin{aligned}\phi''_{u,v}(t)\ &=(p-1)t^{p-2}\|(u,v)\|_p^p+(q-1)t^{q-2}\|(u,v)\|_q^q+rt^{-r-1}\int_\Omega(h_1u^{1-r}+h_2v^{1-r})dx\\ &\quad-\lambda(p^*-1)t^{p^*-2}\int_\Omega u^\alpha v^\beta dx.\end{aligned}$$

It is easy to know that $(u,v)\in\mathcal{N}_\lambda$ if and only if $\phi'_{u,v}(1)=0$, and more generally, $(tu,tv)\in\mathcal{N}_\lambda$ if and only if $\phi'_{u,v}(t)=0$.

Therefore, we can divided the Nehari manifold of $\mathcal{N}_\lambda$ into three parts, that is

$$\mathcal{N}_\lambda^0:=\{(u,v)\in\mathcal{N}_\lambda:\phi''_{u,v}(1)=0\},$$

$$\mathcal{N}_\lambda^-:=\{(u,v)\in\mathcal{N}_\lambda:\phi''_{u,v}(1)<0\},$$

$$\mathcal{N}_\lambda^+:=\{(u,v)\in\mathcal{N}_\lambda:\phi''_{u,v}(1)>0\}.$$

corresponding to the points of inflection, the local minima, and the local maxima.

Lemma 2.1. *Provided that $\lambda\in(0,\Lambda_*)$, then for any $(u,v)\in X\backslash\{(0,0)\}$, $\phi_{u,v}(t)$ has exactly two zero points $t^\mp$ which satisfy*

$$0<t^-<t^+,(t^+u,t^+v)\in\mathcal{N}_\lambda^-,(t^-u,t^-v)\in\mathcal{N}_\lambda^+.$$

Proof. Since

$$\phi'_{u,v}(t)=t^{p-1}\|(u,v)\|_p^p+t^{q-1}\|(u,v)\|_q^q-t^{-r}\int_\Omega(h_1u^{1-r}+h_2v^{1-r})dx-\lambda t^{p^*-1}\int_\Omega u^\alpha v^\beta dx$$

$$=t^{p^*-1}\left[t^{p-p^*}\|(u,v)\|_p^p+t^{q-p^*}\|(u,v)\|_q^q-t^{1-r-p^*}\int_\Omega(h_1u^{1-r}+h_2v^{1-r})dx-\lambda\int_\Omega u^\alpha v^\beta dx\right].$$

Let

$$E(t)=t^{p-p^*}\|(u,v)\|_p^p+t^{q-p^*}\|(u,v)\|_q^q-t^{1-r-p^*}\int_\Omega(h_1u^{1-r}+h_2v^{1-r})dx,$$

$$E'(t) = (p-p^*)t^{p-p^*-1}\|(u,v)\|_p^p + (q-p^*)t^{q-p^*-1}\|(u,v)\|_q^q$$
$$- (1-r-p^*)t^{-r-p^*}\int_\Omega (h_1 u^{1-r} + h_2 v^{1-r})dx$$
$$= t^{-r-p^*}\Big[(p-p^*)t^{p+r-1}\|(u,v)\|_p^p + (q-p^*)t^{q+r-1}\|(u,v)\|_q^q$$
$$- (1-r-p^*)\int_\Omega (h_1 u^{1-r} + h_2 v^{1-r})dx\Big].$$

Let

$$\psi(t) = (p-p^*)t^{p+r-1}\|(u,v)\|_p^p + (q-p^*)t^{q+r-1}\|(u,v)\|_q^q - (1-r-p^*)\int_\Omega (h_1 u^{1-r} + h_2 v^{1-r})dx,$$

$$\lim_{t\to 0^+}\psi(t) = (p^*-1+r)\int_\Omega (h_1 u^{1-r} + h_2 v^{1-r})dx > 0,$$
$$\lim_{t\to +\infty}\psi(t) = -\infty,$$
$$\psi'(t) = (p-p^*)(p-1+r)t^{p+r-2}\|(u,v)\|_p^p + (q-p^*)(q-1+r)t^{q+r-2}\|(u,v)\|_q^q < 0.$$

Thus, $E(t)$ achieves its maximum at $t_{\max} = t_{\max}(u,v)$, where $t_{\max}(u,v)$ is the unique solution of the equation

$$(p-p^*)t^{p-p^*-1}\|(u,v)\|_p^p + (q-p^*)t^{q-p^*-1}\|(u,v)\|_q^q - (1-r-p^*)t^{-r-p^*}\int_\Omega (h_1 u^{1-r} + h_2 v^{1-r})dx = 0.$$

Moreover, by a direct computation, we have

$$E(t_{max}) = t_{\max}^{p-p^*}\|(u,v)\|_p^p + t_{\max}^{q-p^*}\|(u,v)\|_q^q - t_{\max}^{1-r-p^*}\int_\Omega (h_1 u^{1-r} + h_2 v^{1-r})dx$$
$$= \frac{1-p-r}{1-r-p^*}t_{\max}^{p-p^*}\|(u,v)\|_p^p + \frac{1-q-r}{1-r-p^*}t_{\max}^{q-p^*}\|(u,v)\|_q^q > 0$$

and $E'(t) > 0$, for $t \in (0, t_{\max})$, $E'(t) < 0$, for $t \in (t_{\max}, +\infty)$. So the equation $E(t) = \lambda\int_\Omega u^\alpha v^\beta dx$ has exactly two solutions $t^- < t_{\max} < t^+$ such that $E'(t^+) < 0, E'(t^-) > 0$. Further,

$$\phi''(t^+) = (t^+)^{p^*-1}E'(t^+) < 0, \phi''(t^-) = (t^-)^{p^*-1}E'(t^-) > 0.$$

This completes the proof. □

In addition, $\phi(t)$ is decreasing on $(0, t^-)$ and $(t^+, +\infty)$, increasing on (t^-, t^+). We have

$$I(t^-u, t^-v) = \inf_{0\le t\le t_{\max}} I(tu, tv), I(t^+u, t^+v) = \sup_{t\ge 0} I(tu, tv).$$

Lemma 2.2. *Provided that $\lambda \in (0, \Lambda_*)$, then $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. Arguing by way of contradiction, assume that there exists $(u_*, v_*) \in \mathcal{N}_\lambda^0, u_* \neq 0, v_* \neq 0$, it follows from $(u_*, v_*) \in \mathcal{N}_\lambda^0 \subset \mathcal{N}_\lambda$ that

$$\|(u_*, v_*)\|_p^p + \|(u_*, v_*)\|_q^q - \int_\Omega (h_1 u_*^{1-r} + h_2 v_*^{1-r}) dx - \lambda \int_\Omega u_*^\alpha v_*^\beta dx = 0,$$

and consequently

$$(p-1)\|(u_*, v_*)\|_p^p + (q-1)\|(u_*, v_*)\|_q^q + r \int_\Omega (h_1 u_*^{1-r} + h_2 v_*^{1-r}) dx - \lambda(p^* - 1) \int_\Omega u_*^\alpha v_*^\beta dx = 0,$$

we have

$$\begin{aligned}(p-1+r)\|(u_*, v_*)\|_p^p &\leq (p-1+r)\|(u_*, v_*)\|_p^p + (q-1+r)\|(u_*, v_*)\|_q^q \\ &= \lambda(p^*-1+r) \int_\Omega u_*^\alpha v_*^\beta dx,\end{aligned} \tag{2.2}$$

$$\begin{aligned}(p^*-p)\|(u_*, v_*)\|_p^p &\leq (p^*-p)\|(u_*, v_*)\|_p^p + (p^*-q)\|(u_*, v_*)\|_q^q \\ &= (p^*-1+r) \int_\Omega (h_1 u_*^{1-r} + h_2 v_*^{1-r}) dx.\end{aligned} \tag{2.3}$$

By the Hölder inequality and the Sobolev embedding inequality, we have

$$\lambda(p^*-1+r) \int_\Omega u_*^\alpha v_*^\beta dx \leq \lambda(p^*-1+r) S_{\alpha,\beta}^{-\frac{p^*}{p}} \|(u_*, v_*)\|_p^{p^*}, \tag{2.4}$$

and

$$\begin{aligned}&(p^*-1+r) \int_\Omega (h_1 u_*^{1-r} + h_2 v_*^{1-r}) dx \\ &\leq (p^*-1+r) \left[\left(\int_\Omega h_1^{\frac{p^*}{p^*-1+r}} dx \right)^{\frac{p^*-1+r}{p^*}} \left(\int_\Omega u_*^{p^*} dx \right)^{\frac{1-r}{p^*}} + \left(\int_\Omega h_2^{\frac{p^*}{p^*-1+r}} dx \right)^{\frac{p^*-1+r}{p^*}} \left(\int_\Omega v_*^{p^*} dx \right)^{\frac{1-r}{p^*}} \right] \\ &\leq (p^*-1+r) \left[|h_1|_{L^{(1-r)^*}}^{\frac{p}{p-1+r}} + |h_2|_{L^{(1-r)^*}}^{\frac{p}{p-1+r}} \right]^{\frac{p-1+r}{p}} S^{-\frac{1-r}{p}} \|(u_*, v_*)\|^{1-r},\end{aligned} \tag{2.5}$$

where $(1-r)^* = \frac{p^*}{p^*-(1-r)}$ by (2.2),(2.3),(2.4),(2.5), we get

$$\|(u_*, v_*)\|_p \geq \left(\frac{1}{\lambda} \frac{p-1+r}{p^*-1+r} S_{\alpha,\beta}^{\frac{p^*}{p}} \right)^{\frac{1}{p^*-p}},$$

and

$$\|(u_*, v_*)\|_p \leq \left(\frac{p^*-1+r}{p^*-p} \left[|h_1|_{L^{(1-r)^*}}^{\frac{p}{p-1+r}} + |h_2|_{L^{(1-r)^*}}^{\frac{p}{p-1+r}} \right]^{\frac{p-1+r}{p}} S^{-\frac{1-r}{p}} \right)^{\frac{1}{p-1+r}}.$$

It implies $\lambda > \Lambda_*$, this contradiction shows that there exists a constant $\Lambda_* > 0$ such that $\mathcal{N}_\lambda^0 = \emptyset$ for $\lambda \in (0, \Lambda_*)$. $\square$

Lemma 2.3. *Provided that* $\lambda \in (0, \Lambda_*)$, *then* $\mathcal{N}_\lambda$ *has a gap structure in the sense that* $\|(u,v)\|_p < E_0, \forall (u,v) \in \mathcal{N}_\lambda^+; \|(U,V)\|_p > E_\lambda > E_0, \forall (U,V) \in \mathcal{N}_\lambda^-$. *Clearly,* $E_\lambda \to \infty$ *as* $\lambda \to 0$.

Proof. If $(u,v) \in \mathcal{N}_\lambda^+ \in \mathcal{N}_\lambda$, then necessarily

$$0 < (p-1)\|(u,v)\|_p^p + (q-1)\|(u,v)\|_q^q + r\int_\Omega (h_1 u^{1-r} + h_2 v^{1-r})dx - \lambda(p^*-1)\int_\Omega u^\alpha v^\beta dx$$
$$= (p-p^*)\|(u,v)\|_p^p + (q-p^*)\|(u,v)\|_q^q + (r+p^*-1)\int_\Omega (h_1 u^{1-r} + h_2 v^{1-r})dx.$$

Hence, it follows from (2.5) that

$$(p^*-p)\|(u,v)\|_p^p < (p^*-p)\|(u,v)\|_p^p + (p^*-q)\|(u,v)\|_q^q$$
$$\leq (r+p^*-1)\left[|h_1|_{L^{(1-r)^*}}^{\frac{p}{p-1+r}} + |h_2|_{L^{(1-r)^*}}^{\frac{p}{p-1+r}}\right]^{\frac{p-1+r}{p}} S^{-\frac{1-r}{p}}\|(u,v)\|^{1-r},$$

which yields

$$\|(u,v)\|_p < \left(\frac{p^*-1+r}{p^*-p}\left[|h_1|_{L^{(1-r)^*}}^{\frac{p}{p-1+r}} + |h_2|_{L^{(1-r)^*}}^{\frac{p}{p-1+r}}\right]^{\frac{p-1+r}{p}} S^{-\frac{1-r}{p}}\right)^{\frac{1}{p-1+r}} \equiv E_0.$$

If $(U,V) \in \mathcal{N}_\lambda^- \in \mathcal{N}_\lambda$, then

$$0 > (p-1)\|(U,V)\|_p^p + (q-1)\|(U,V)\|_q^q + r\int_\Omega (h_1 U^{1-r} + h_2 V^{1-r})dx - \lambda(p^*-1)\int_\Omega U^\alpha V^\beta dx$$
$$= (p-1+r)\|(U,V)\|_p^p + (q-1+r)\|(U,V)\|_q^q - \lambda(p^*-1+r)\int_\Omega U^\alpha V^\beta dx.$$

Hence, it follows from (2.4) that

$$(p-1+r)\|(U,V)\|_p^p \leq (p-1+r)\|(U,V)\|_p^p + (q-1+r)\|(U,V)\|_q^q$$
$$< \lambda(p^*-1+r) S_{\alpha,\beta}^{-\frac{p^*}{p}}\|(U,V)\|_p^{p^*},$$

which yields

$$\|(U,V)\|_p > \left(\frac{1}{\lambda}\frac{p-1+r}{p^*-1+r} S_{\alpha,\beta}^{\frac{p^*}{p}}\right)^{\frac{1}{p^*-p}} \equiv E_\lambda.$$

Consequently

$$\|(U,V)\|_p < E_0, \quad \forall (u,v) \in \mathcal{N}_\lambda^+, \tag{2.6}$$

$$\|(U,V)\|_p > E_\lambda, \quad \forall (U,V) \in \mathcal{N}_\lambda^-. \tag{2.7}$$

Surprisingly enough

$$E_\lambda = E_0, \ if \ \lambda = \Lambda_*.$$

We conclude that

$$\|(U,V)\|_p > E_\lambda > E_0 > \|(u,v)\|_p, \ \forall (u,v) \in \mathcal{N}_\lambda^+, \ \forall (U,V) \in \mathcal{N}_\lambda^-,$$

for all $\lambda \in (0,\Lambda_*)$. □

Lemma 2.4. *Provided that $\lambda \in (0,\Lambda_*)$, then $\mathcal{N}_\lambda^-$ is closed set in X.*

Proof. Let $\{(U_n, V_n)\}$ is a sequence in $\mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$ with $(U_n, V_n) \to (U_0, V_0)$ in X. Then we have

$$\|(U_n, V_n)\|_p^p + \|(U_n, V_n)\|_q^q - \int_\Omega (h_1 U_n^{1-r} + h_2 V_n^{1-r})dx - \lambda \int_\Omega U_n^\alpha V_n^\beta dx = 0,$$

$$\|(U_0, V_0)\|_p^p + \|(U_0, V_0)\|_q^q = \lim_{n\to\infty} \left[\|(U_n, V_n)\|_p^p + \|(U_n, V_n)\|_q^q\right]$$

$$= \lim_{n\to\infty} \left[\int_\Omega (h_1 U_n^{1-r} + h_2 V_n^{1-r})dx - \lambda \int_\Omega U_n^\alpha V_n^\beta dx\right] = \int_\Omega (h_1 U_0^{1-r} + h_2 V_0^{1-r})dx - \lambda \int_\Omega U_0^\alpha V_0^\beta dx,$$

$$(p-1)\|(U_0, V_0)\|_p^p + (q-1)\|(U_0, V_0)\|_q^q + \int_\Omega (h_1 U_0^{1-r} + h_2 V_0^{1-r})dx - \lambda \int_\Omega U_0^\alpha V_0^\beta dx$$

$$= \lim_{n\to\infty} \left[(p-1)\|(U_n, V_n)\|_p^p + (q-1)\|(U_n, V_n)\|_q^q + \int_\Omega (h_1 U_n^{1-r} + h_2 V_n^{1-r})dx - \lambda \int_\Omega U_n^\alpha V_n^\beta dx\right] \le 0,$$

this is $(U_0, V_0) \in \mathcal{N}_\lambda^- \bigcup \mathcal{N}_\lambda^0$. Since $\{U_n, V_n\} \subset \mathcal{N}_\lambda^-$, from Lemma 2.3,

$$\|(U_0, V_0)\|_p^p = \lim_{n\to\infty} \|(U_n, V_n)\|_p^p > E_\lambda = \left(\frac{1}{\lambda} \frac{p-1+r}{p^*-1+r} S_{\alpha,\beta}^{\frac{p^*}{p}}\right)^{\frac{1}{p^*-p}} > 0,$$

we obtain $U_0, V_0 \neq 0$. It follows from Lemma 2.1, $(U_0, V_0) \notin \mathcal{N}_\lambda^0$ for any $\lambda \in (0, \Lambda_*)$.

In turn, $(U_0, V_0) \in \mathcal{N}_\lambda^-$. Therefore, $\mathcal{N}_\lambda^-$ is closed set. □

3 The proof of Theorem 1.1

Proof. For any $(u_0, v_0) \in \mathcal{N}_\lambda$, we have

$$I(u_0, v_0) = \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right) \|(u_0, v_0)\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{p^*}\right) \|(u_0, v_0)\|_q^q - \left(\tfrac{1}{1-r} - \tfrac{1}{p^*}\right) \int_\Omega (h_1 u_0^{1-r} + h_2 v_0^{1-r})dx$$

$$\ge \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right) \|(u_0, v_0)\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{p^*}\right) \|(u_0, v_0)\|_q^q$$

$$- \left(\tfrac{1}{1-r} - \tfrac{1}{p^*}\right) \left[|h_1|_{L^{(1-r)^*}}^{\frac{p}{p-1+r}} + |h_2|_{L^{(1-r)^*}}^{\frac{p}{p-1+r}}\right]^{\frac{p-1+r}{p}} S^{-\frac{1-r}{p}} \|(u_0, v_0)\|^{1-r}.$$

Therefore I is coercive and bounded below in X. From Lemma 2.4, $\mathcal{N}_\lambda^+ \bigcup \mathcal{N}_\lambda^0$ and $\mathcal{N}_\lambda^-$ are two closed sets in X provided that $\lambda \in (0, \Lambda_*)$. This allows us to extract "best" minimizing sequences by means of Ekeland variational principle.

First, consider $z_n := (u_n, v_n) \subset \mathcal{N}_\lambda^+ \bigcup \mathcal{N}_\lambda^0$ with the following properties:

(i) $I(u_n, v_n) < \inf_{\mathcal{N}_\lambda^+ \bigcup \mathcal{N}_\lambda^0} I(u, v) + \frac{1}{n}$;

(ii) $I(u, v) \ge I(u_n, v_n) - \frac{1}{n}\|(u - u_n, v - v_n\|, \forall (u, v) \in \mathcal{N}_\lambda^+ \bigcup \mathcal{N}_\lambda^0$.

From $I(|u|, |v|) = I(u, v)$, we may assume that $u_n, v_n \ge 0$, since I is bounded below on $\mathcal{N}_\lambda$, clearly, z_n is bounded in X (denoted by $\|(u_n, v_n)\| \le C_1$,$C_1 > 0$ is a constant). Going if necessary to a sequence, we can suppose that

$$u_n \rightharpoonup u, v_n \rightharpoonup v \qquad in\ \ W_0^{1,p}(\Omega),$$

$$u_n \to u, v_n \to v \qquad in\ \ L^s(\Omega), 1 \le s < p^*,$$

$$u_n(x) \to u(x), v_n(x) \to v(x) \qquad \textit{almost everywhere in } \Omega.$$

For any $z := (u,v) \in \mathcal{N}_\lambda^+ (\subset \mathcal{N}_\lambda)$, we have

$$I(u,v) = \left(\tfrac{1}{p} - \tfrac{1}{1-r}\right)\|(u,v)\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{1-r}\right)\|(u,v)\|_q^q - \lambda\left(\tfrac{1}{p^*} - \tfrac{1}{1-r}\right)\int_\Omega u^\alpha v^\beta dx$$

$$\leq \left(\tfrac{1}{p} - \tfrac{1}{1-r}\right)\|(u,v)\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{1-r}\right)\|(u,v)\|_q^q - \lambda\left(\tfrac{1}{p^*} - \tfrac{1}{1-r}\right)\left[\tfrac{p-1+r}{p^*-1+r}\|(u,v)\|_p^p + \tfrac{q-1+r}{p^*-1+r}\|(u,v)\|_q^q\right]$$

$$= \tfrac{(p-1+r)(p-p^*)}{pp^*(1-r)}\|(u,v)\|_p^p + \tfrac{(q-1+r)(p-p^*)}{qp^*(1-r)}\|(u,v)\|_q^q < 0.$$

It means that $\inf\limits_{\mathcal{N}_\lambda^+} I < 0$ for $\lambda \in (0, \Lambda_*)$, from Lemma 2.1 we get that $\mathcal{N}_\lambda^0 = \{(0,0)\}$. Together, these imply that $(u_n, v_n) \in \mathcal{N}_\lambda^+$ for n largely and

$$\inf_{\mathcal{N}_\lambda^+ \bigcup \mathcal{N}_\lambda^0} I = \inf_{\mathcal{N}_\lambda^+} I < 0.$$

Thus, by the weak lower semi-continuity of $\|\cdot\|_p, \|\cdot\|_q$

$$I(u_0, v_0) \leq \lim_{n\to\infty}\inf I(u_n, v_n) = \inf_{\mathcal{N}_\lambda^+ \bigcup \mathcal{N}_\lambda^0} I < 0,$$

we see that $u_0 \not\equiv 0, v_0 \not\equiv 0$ and $(u_n, v_n) \subset \mathcal{N}_\lambda^+$.

Proposition 3.1. *There exists $\varepsilon > 0$ such that $u_0 \geq \varepsilon e_1, v_0 \geq \varepsilon e_1$, for $\forall x \in \Omega$.*

Proof. First we will show that when $\lambda \in (0, \Lambda_*)$

$$\lim_{n\to\infty}\inf\left[(p^*-p)\|(u_n,v_n)\|_p^p + (p^*-q)\|(u_n,v_n)\|_q^q\right] < (r+p^*-1)\int_\Omega (h_1 u_0^{1-r} + h_2 v_0^{1-r})dx, \tag{3.1}$$

since $(u_n, v_n) \subset \mathcal{N}_\lambda^+$, then

$$\lim_{n\to\infty}\inf\left[(p^*-p)\|(u_n,v_n)\|_p^p + (p^*-q)\|(u_n,v_n)\|_q^q\right] \leq (r+p^*-1)\int_\Omega (h_1 u_0^{1-r} + h_2 v_0^{1-r})dx,$$

by $\lambda \in (0, \Lambda_*)$, we have

$$(p^*-p)\|(u_n,v_n)\|_p^p > (r+p^*-1)\int_\Omega (h_1 u_n^{1-r} + h_2 v_n^{1-r})dx,$$

so

$$\lim_{n\to\infty}\inf\left[(p^*-p)\|(u_n,v_n)\|_p^p + (p^*-q)\|(u_n,v_n)\|_q^q\right] = (r+p^*-1)\int_\Omega (h_1 u_0^{1-r} + h_2 v_0^{1-r})dx,$$

which is clearly impossible. By (3.1), we may choose a subsequence such that

$$(p^*-p)\|(u_n,v_n)\|_p^p + (p^*-q)\|(u_n,v_n)\|_q^q - (r+p^*-1)\int_\Omega (h_1 u_n^{1-r} + h_2 v_n^{1-r})dx \leq -C_2 \tag{3.2}$$

for n large enough and a appropriate positive constant C_2.

Fix $(\varphi_1, \varphi_2) \in X$ with $\varphi_1, \varphi_2 \geq 0$. We use the implicit function theorem at the point $(0,1)$, there exist a continuous function $f_n(t)$ such that $f_n(0) = 1$ and $(u_n, v_n), f_n(t)(u_n +$

$t\varphi_1, v_n + t\varphi_2) \subset \mathcal{N}_\lambda^+(\subset \mathcal{N}_\lambda)$ for all sufficiently small $t \geq 0$. It follows from $(u_n, v_n) \subset \mathcal{N}_\lambda$ and $f_n(t)(u_n + t\varphi_1, v_n + t\varphi_2) \subset \mathcal{N}_\lambda$ that

$$\|(u_n, v_n)\|_p^p + \|(u_n, v_n)\|_q^q - \int_\Omega (h_1 u_n^{1-r} + h_2 v_n^{1-r}) dx - \int_\Omega u_n^\alpha v_n^\beta dx = 0,$$

and

$$f_n^p(t)\|(u_n + t\varphi_1, v_n + t\varphi_2)\|_p^p - f_n^{1-r}(t)\int_\Omega (h_1(u_n + t\varphi_1)^{1-r} + h_2(v_n + t\varphi_2)^{1-r}) dx$$

$$+ f_n^q(t)\|(u_n + t\varphi_1, v_n + t\varphi_2)\|_q^q - \lambda f_n^{p^*}(t)\int_\Omega (u_n + t\varphi_1)^\alpha (v_n + t\varphi_2)^\beta dx = 0.$$

So we have that

$$0 \leq [f_n^p(t) - 1]\|(u_n + t\varphi_1, v_n + t\varphi_2)\|_p^p + (\|(u_n + t\varphi_1, v_n + t\varphi_2)\|_p^p - \|(u_n, v_n)\|_p^p)$$

$$+ [f_n^q(t) - 1]\|(u_n + t\varphi_1, v_n + t\varphi_2)\|_q^q + (\|(u_n + t\varphi_1, v_n + t\varphi_2)\|_q^q - \|(u_n, v_n)\|_q^q)$$

$$- [f_n^{1-r}(t) - 1]\int_\Omega (h_1(u_n + t\varphi_1)^{1-r} + h_2(v_n + t\varphi_2)^{1-r}) dx$$

$$- \lambda[f_n^{p^*}(t) - 1]\int_\Omega (u_n + t\varphi_1)^\alpha (v_n + t\varphi_2)^\beta dx - \lambda \int_\Omega \left[(u_n + t\varphi_1)^\alpha (v_n + t\varphi_2)^\beta - u_n^\alpha v_n^\beta\right] dx.$$

Dividing by $t > 0$ and taking the limit for $t \to 0$, we derive that

$$0 \leq p f_n'(0)\|(u_n, v_n)\|_p^p + p\int_\Omega (\nabla u_n^{p-1} \nabla \varphi_1) dx + p\int_\Omega (\nabla v_n^{p-1} \nabla \varphi_2) dx + q f_n'(0)\|(u_n, v_n)\|_q^q$$

$$+ q\int_\Omega (\nabla u_n^{q-1} \nabla \varphi_1) dx + q\int_\Omega (\nabla v_n^{q-1} \nabla \varphi_2) dx - (1-r) f_n'(0)\int_\Omega (h_1 u_n^{1-r} + h_2 v_n^{1-r}) dx$$

$$- \lambda p^* f_n'(0) \textstyle\int_\Omega u_n^\alpha v_n^\beta dx - \lambda \displaystyle\int_\Omega (\alpha u_n^{\alpha-1} v_n^\beta \varphi_1 + \beta u_n^\alpha v_n^{\beta-1} \varphi_2) dx$$

$$= f_n'(0)\left[p\|(u_n, v_n)\|_p^p + q\|(u_n, v_n)\|_q^q - (1-r)\int_\Omega (h_1 u_n^{1-r} + h_2 v_n^{1-r}) dx - \lambda p^* \int_\Omega u_n^\alpha v_n^\beta dx\right]$$

$$+ p\int_\Omega (\nabla u_n^{p-1} \nabla \varphi_1 + \nabla v_n^{p-1} \nabla \varphi_2) dx + q\int_\Omega (\nabla u_n^{q-1} \nabla \varphi_1 + \nabla v_n^{q-1} \nabla \varphi_2) dx$$

$$- \lambda \int_\Omega (\alpha u_n^{\alpha-1} v_n^\beta \varphi_1 + \beta u_n^\alpha v_n^{\beta-1} \varphi_2) dx$$

$$= f_n'(0)[(p - p^*)\|(u_n, v_n)\|_p^p + (q - p^*)\|(u_n, v_n)\|_q^q - (1 - r - p^*)\int_\Omega (h_1 u_n^{1-r} + h_2 v_n^{1-r}) dx]$$

$$+ p\int_\Omega (\nabla u_n^{p-1} \nabla \varphi_1 + \nabla v_n^{p-1} \nabla \varphi_2) dx + q\int_\Omega (\nabla u_n^{q-1} \nabla \varphi_1 + \nabla v_n^{q-1} \nabla \varphi_2) dx$$

$$- \lambda \int_\Omega \left(\alpha u_n^{\alpha-1} v_n^\beta \varphi_1 + \beta u_n^\alpha v_n^{\beta-1} \varphi_2\right) dx. \tag{3.3}$$

From (3.2) and (3.3), we know immediately that $f_n'(0) \neq -\infty$.

Now we show that $f_n'(0) \neq +\infty$. Arguing that contradiction, we assume that $f_n'(0) = +\infty$. Since

$$
\begin{aligned}
&|f_n(t)-1|\|(u_n,v_n)\| + tf_n(t)\|(\varphi_1,\varphi_2)\| \\
&\geq \| ([f_n(t)-1]u_n + tf_n(t)\varphi_1, [f_n(t)-1]v_n + tf_n(t)\varphi_2) \| \\
&= \| (f_n(t)(u_n+t\varphi_1) - u_n, f_n(t)(v_n+t\varphi_2) - v_n) \|,
\end{aligned}
\tag{3.4}
$$

and

$$
f_n(t) > f_n(0) = 1,
$$

for n sufficiently large, from the definition of derivative $f_n'(0)$, applying condition (ii) with $u = f_n(t)(u_n+t\varphi_1), v = f_n(t)(v_n+t\varphi_2)$ and $z=(u,v) \in \mathcal{N}_\lambda^+$, we clearly have that

$$
\begin{aligned}
&|f_n(t)-1|\frac{\|(u_n,v_n)\|}{n} + tf_n(t)\frac{\|(\varphi_1,\varphi_2)\|}{n} \geq \frac{\|(f_n(t)(u_n+t\varphi_1)-u_n, f_n(t)(v_n+t\varphi_2)-v_n)\|}{n} \\
&\geq I(u_n,v_n) - I\Big(f_n(t)(u_n+t\varphi_1), f_n(t)(v_n+t\varphi_2)\Big) \\
&= \left(\tfrac{1}{p} - \tfrac{1}{1-r}\right)\|(u_n,v_n)\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{1-r}\right)\|(u_n,v_n)\|_q^q - \lambda\left(\tfrac{1}{p^*} - \tfrac{1}{1-r}\right)\int_\Omega u_n^\alpha v_n^\beta dx \\
&-\left(\tfrac{1}{p} - \tfrac{1}{1-r}\right) f_n^p(t)\|(u_n+t\varphi_1, v_n+t\varphi_2)\|_p^p - \left(\tfrac{1}{q} - \tfrac{1}{1-r}\right) f_n^q(t)\|(u_n+t\varphi_1, v_n+t\varphi_2)\|_q^q \\
&+\lambda\left(\tfrac{1}{p^*} - \tfrac{1}{1-r}\right) f_n^{p^*}(t)\int_\Omega (u_n+t\varphi_1)^\alpha (v_n+t\varphi_2)^\beta dx \\
&= \left(\tfrac{1}{1-r} - \tfrac{1}{p}\right)(\|(u_n+t\varphi_1, v_n+t\varphi_2)\|_p^p - \|(u_n,v_n)\|_p^p) \\
&+ \left(\tfrac{1}{1-r} - \tfrac{1}{p}\right)[f_n^p(t)-1]\|(u_n+t\varphi_1, v_n+t\varphi_2)\|_p^p \\
&+ \left(\tfrac{1}{1-r} - \tfrac{1}{q}\right)(\|(u_n+t\varphi_1, v_n+t\varphi_2)\|_q^q - \|(u_n,v_n)\|_q^q) \\
&+ \left(\tfrac{1}{1-r} - \tfrac{1}{q}\right)[f_n^q(t)-1]\|(u_n+t\varphi_1, v_n+t\varphi_2)\|_q^q - \lambda\left(\tfrac{1}{1-r} - \tfrac{1}{p^*}\right)[f_n^{p^*}(t)-1]\int_\Omega u_n^\alpha v_n^\beta dx \\
&-\lambda\left(\tfrac{1}{1-r} - \tfrac{1}{p^*}\right) f_n^{p^*}(t)\int_\Omega \Big[(u_n+t\varphi_1)^\alpha (v_n+t\varphi_2)^\beta - u_n^\alpha v_n^\beta\Big] dx.
\end{aligned}
$$

Diving by $t>0$ and passing to the limit as $t \to 0$, we can obtain that

$$
\begin{aligned}
&f_n'(0)\frac{\|(u_n,v_n)\|}{n} + \frac{\|(\varphi_1,\varphi_2)\|}{n} \\
&\geq \tfrac{p-(1-r)}{1-r}\left[\int_\Omega (\nabla u_n^{p-1}\nabla\varphi_1 + \nabla v_n^{p-1}\nabla\varphi_2)dx\right] + \tfrac{p-(1-r)}{1-r} f_n'(0)\|(u_n,v_n)\|_p^p \\
&+\tfrac{q-(1-r)}{1-r}\left[\int_\Omega (\nabla u_n^{q-1}\nabla\varphi_1 + \nabla v_n^{q-1}\nabla\varphi_2)dx\right] + \tfrac{q-(1-r)}{1-r} f_n'(0)\|(u_n,v_n)\|_q^q
\end{aligned}
$$

$$-\lambda\tfrac{p^*-(1-r)}{p^*(1-r)}\int_\Omega(\alpha u_n^{\alpha-1}v_n^\beta\varphi_1+\beta u_n^\alpha v_n^{\beta-1}\varphi_2)dx-\lambda\frac{p^*-(1-r)}{p^*(1-r)}f_n'(0)\int_\Omega u_n^\alpha v_n^\beta dx$$

$$=\tfrac{f_n'(0)}{1-r}\left[(p-1+r)\|(u_n,v_n)\|_p^p+(q-1+r)\|(u_n,v_n)\|_q^q-\lambda(p^*-1+r)\int_\Omega u_n^\alpha v_n^\beta dx\right]$$

$$+\tfrac{p-1+r}{1-r}\left[\int_\Omega\left(\nabla u_n^{p-1}\nabla\varphi_1+\nabla v_n^{p-1}\nabla\varphi_2\right)dx\right]+\tfrac{q-1+r}{1-r}\left[\int_\Omega\left(\nabla u_n^{q-1}\nabla\varphi_1+\nabla v_n^{q-1}\nabla\varphi_2\right)dx\right]$$

$$-\lambda\tfrac{p^*-1+r}{(1-r)p^*}\int_\Omega\left(\alpha u_n^{\alpha-1}v_n^\beta\varphi_1+\beta u_n^\alpha v_n^{\beta-1}\varphi_2\right)dx,$$

that is

$$\frac{\|(\varphi_1,\varphi_2)\|}{n}\geq\frac{f_n'(0)}{1-r}\Big[(p-p^*)\|(u_n,v_n)\|_p^p+(q-p^*)\|(u_n,v_n)\|_q^q-\frac{(1-r)\|(u_n,v_n)\|}{n}$$

$$-(1-r-p^*)\int_\Omega(h_1u_n^{1-r}+h_2v_n^{1-r})dx\Big]+\frac{p-1+r}{1-r}\left[\int_\Omega(\nabla u_n^{p-1}\nabla\varphi_1+\nabla v_n^{p-1}\nabla\varphi_2)dx\right]$$

$$+\tfrac{q-1+r}{1-r}\left[\int_\Omega(\nabla u_n^{q-1}\nabla\varphi_1+\nabla v_n^{q-1}\nabla\varphi_2)dx\right]-\lambda\tfrac{p^*-1+r}{(1-r)p^*}\int_\Omega(\alpha u_n^{\alpha-1}v_n^\beta\varphi_1+\beta u_n^\alpha v_n^{\beta-1}\varphi_2)dx,\tag{3.5}$$

which is impossible, because $f_n'(0)=+\infty$ and

$$(p-p^*)\|(u_n,v_n)\|_p^p+(q-p^*)\|(u_n,v_n)\|_q^q-(1-r-p^*)\int_\Omega(h_1u_n^{1-r}+h_2v_n^{1-r})dx$$

$$-\tfrac{(1-r)\|(u_n,v_n)\|}{n}\geq C_2-\tfrac{(1-r)C_1}{n}>0.$$

In conclusion, $|f_n'(0)|\neq+\infty$. Furthermore, (3.2) with $\|(u_n,v_n)\|\leq C_1(n=1,2,\cdots)$ and the two inequalities (3.3) and (3.5) also imply that $|f_n'(0)|\leq C_3$ for n sufficiently large and a proper constant $C_3>0$.

Now, using (3.3) and condition (ii) again, we infer that

$$\tfrac{1}{n}\Big[|f_n(t)-1|\|(u_n,v_n)\|+tf_n(t)\|(\varphi_1,\varphi_2)\|\Big]$$

$$\geq\tfrac{1}{n}\|(f_n(t)(u_n+t\varphi_1)-u_n,f_n(t)(v_n+t\varphi_2)-v_n)\|$$

$$\geq I(u_n,v_n)-I(f_n(t)(u_n+t\varphi_1),f_n(t)(v_n+t\varphi_2))$$

$$\geq-\tfrac{f_n^p(t)-1}{p}\|(u_n,v_n)\|_p^p-\tfrac{f_n^p(t)}{p}(\|(u_n+t\varphi_1,v_n+t\varphi_2)\|_p^p-\|(u_n,v_n)\|_p^p)$$

$$-\tfrac{f_n^q(t)-1}{q}\|(u_n,v_n)\|_q^q-\tfrac{f_n^q(t)}{q}(\|(u_n+t\varphi_1,v_n+t\varphi_2)\|_q^q-\|(u_n,v_n)\|_q^q)$$

$$+\tfrac{f_n^{1-r}(t)-1}{1-r}\int_\Omega\left(h_1(u_n+t\varphi_1)^{1-r}+h_2(v_n+t\varphi_2)^{1-r}\right)dx$$

$$+\tfrac{1}{1-r}\int_\Omega\left(h_1[(u_n+t\varphi_1)^{1-r}-u_n^{1-r}]+h_2[(v_n+t\varphi_2)^{1-r})-v_n^{1-r}]\right)dx$$

$$+\lambda\tfrac{f_n^{p^*}-1}{p^*}\int_\Omega(u_n+t\varphi_1)^\alpha(v_n+t\varphi_2)^\beta dx+\frac{\lambda}{p^*}\int_\Omega\left[(u_n+t\varphi_1)^\alpha(v_n+t\varphi_2)^\beta-u_n^\alpha v_n^\beta\right]dx.$$

Dividing by $t > 0$ and passing to the limit for $t \to 0^+$, we can get that

$$\begin{aligned}
&\tfrac{1}{n}[|f_n'(0)|\|(u_n,v_n)\| + \|(\varphi_1,\varphi_2)\|] \\
&\geq -f_n'(0)\left[\|(u_n,v_n)\|_p^p + \|(u_n,v_n)\|_q^q - \int_\Omega \left(h_1 u_n^{1-r} + h_2 v_n^{1-r}\right)dx - \lambda \int_\Omega u_n^\alpha v_n^\beta dx\right] \\
&-\int_\Omega \left(\nabla u_n^{p-1}\nabla\varphi_1 + \nabla v_n^{p-1}\nabla\varphi_2\right)dx - \int_\Omega \left(\nabla u_n^{q-1}\nabla\varphi_1 + \nabla v_n^{q-1}\nabla\varphi_2\right)dx \\
&+\tfrac{\lambda}{p^*}\int_\Omega \left(\alpha u_n^{\alpha-1}v_n^\beta\varphi_1 + \beta u_n^\alpha v_n^{\beta-1}\varphi_2\right)dx \\
&+\lim_{t\to 0^+}\inf \tfrac{1}{1-r}\int_\Omega \frac{h_1[(u_n+t\varphi_1)^{1-r} - u_n^{1-r}] + h_2[(v_n+t\varphi_2)^{1-r} - v_n^{1-r}]}{t}dx \\
&= -\int_\Omega \left(\nabla u_n^{p-1}\nabla\varphi_1 + \nabla v_n^{p-1}\nabla\varphi_2\right)dx - \int_\Omega \left(\nabla u_n^{q-1}\nabla\varphi_1 + \nabla v_n^{q-1}\nabla\varphi_2\right)dx \\
&+\tfrac{\lambda}{p^*}\int_\Omega \left(\alpha u_n^{\alpha-1}v_n^\beta\varphi_1 + \beta u_n^\alpha v_n^{\beta-1}\varphi_2\right)dx \\
&+\lim_{t\to 0^+}\inf \tfrac{1}{1-r}\int_\Omega \frac{h_1[(u_n+t\varphi_1)^{1-r} - u_n^{1-r}] + h_2[(v_n+t\varphi_2)^{1-r} - v_n^{1-r}]}{t}dx,
\end{aligned}$$

which gives,

$$\begin{aligned}
&\lim_{t\to 0^+}\inf \tfrac{1}{1-r}\int_\Omega \frac{h_1[(u_n+t\varphi_1)^{1-r} - u_n^{1-r}] + h_2[(v_n+t\varphi_2)^{1-r} - v_n^{1-r}]}{t}dx \\
&\leq \int_\Omega \left(\nabla u_n^{p-1}\nabla\varphi_1 + \nabla v_n^{p-1}\nabla\varphi_2\right)dx + \int_\Omega \left(\nabla u_n^{q-1}\nabla\varphi_1 + \nabla v_n^{q-1}\nabla\varphi_2\right)dx \\
&-\tfrac{\lambda}{p^*}\int_\Omega \left(\alpha u_n^{\alpha-1}v_n^\beta\varphi_1 + \beta u_n^\alpha v_n^{\beta-1}\varphi_2\right)dx + \frac{1}{n}[|f_n'(0)|\|(u_n,v_n)\| + \|(\varphi_1,\varphi_2)\|],
\end{aligned}$$

since $h_1\frac{(u_n+t\varphi_1)^{1-r}-u_n^{1-r}}{t} + h_2\frac{(v_n+t\varphi_2)^{1-r}-v_n^{1-r}}{t} \geq 0, \forall t > 0.$

By Fatou's Lemma we have that

$$\lim_{t\to 0^+}\inf \frac{1}{1-r}\int_\Omega \frac{h_1[(u_n+t\varphi_1)^{1-r} - u_n^{1-r}] + h_2[(v_n+t\varphi_2)^{1-r} - v_n^{1-r}]}{t}dx$$

is integrable and

$$\begin{aligned}
&\int_\Omega \lim_{t\to 0^+}\inf \frac{1}{1-r}\left(h_1\frac{(u_n+t\varphi_1)^{1-r} - u_n^{1-r}}{t} + h_2\frac{(v_n+t\varphi_2)^{1-r} - v_n^{1-r}}{t}\right)dx \\
&\leq \int_\Omega \left(\nabla u_n^{p-1}\nabla\varphi_1 + \nabla v_n^{p-1}\nabla\varphi_2\right)dx + \int_\Omega \left(\nabla u_n^{q-1}\nabla\varphi_1 + \nabla v_n^{q-1}\nabla\varphi_2\right)dx \qquad (3.6) \\
&-\tfrac{\lambda}{p^*}\int_\Omega \left(\alpha u_n^{\alpha-1}v_n^\beta\varphi_1 + \beta u_n^\alpha v_n^{\beta-1}\varphi_2\right)dx + \frac{1}{n}[|f_n'(0)|\|(u_n,v_n)\| + \|(\varphi_1,\varphi_2)\|].
\end{aligned}$$

Note that

$$\frac{1}{1-r}\left(h_1\frac{(u_n+t\varphi_1)^{1-r} - u_n^{1-r}}{t} + h_2\frac{(v_n+t\varphi_2)^{1-r} - v_n^{1-r}}{t}\right) \quad \xrightarrow{t\to 0^+}$$

$$\begin{cases} 0, & if\ at\ least\ one\ of\ the\ u_n(x), v_n(x)\ is\ zero, \varphi_1(x), \varphi_2(x) = 0, \\ +\infty, & if\ at\ least\ one\ of\ the\ u_n(x), v_n(x)\ is\ zero, \varphi_1(x), \varphi_2(x) > 0, \\ \frac{h_1(x)}{u^r}\varphi_1 + \frac{h_2(x)}{v^r}\varphi_2, & if\ u_n(x), v_n(x) > 0, \varphi_1(x), \varphi_2(x) \geq 0, \end{cases}$$

we consider $u_n(x), v_n(x) > 0, \varphi_1(x), \varphi_2(x) = e_1$ as a test function in (3.6), applying the Fatou's Lemma once more, we infer that

$$\int_\Omega (h_1 u_0^{-r}\varphi_1 + h_2 v_0^{-r}\varphi_2)dx \leq \lim_{t\to 0^+} \inf \int_\Omega \left(h_1 u_n^{-r}\varphi_1 + h_2 v_n^{-r}\varphi_2\right) dx$$

$$\leq \lim_{t\to 0^+} \inf \left[\int_\Omega (\nabla u_n^{p-1}\nabla\varphi_1 + \nabla v_n^{p-1}\nabla\varphi_2)dx + \int_\Omega \left(\nabla u_n^{q-1}\nabla\varphi_1 + \nabla v_n^{q-1}\nabla\varphi_2\right) dx\right.$$

$$\left.-\tfrac{\lambda}{p^*}\int_\Omega \left(\alpha u_n^{\alpha-1} v_n^\beta \varphi_1 + \beta u_n^\alpha v_n^{\beta-1}\varphi_2\right) dx + \frac{C_1 C_3 + \|(\varphi_1, \varphi_2)\|}{n}\right]$$

$$= \int_\Omega (\nabla u_0^{p-1}\nabla\varphi_1 + \nabla v_0^{p-1}\nabla\varphi_2)dx + \int_\Omega (\nabla u_0^{q-1}\nabla\varphi_1 + \nabla v_0^{q-1}\nabla\varphi_2)dx$$

$$-\tfrac{\lambda}{p^*}\int_\Omega (\alpha u_0^{\alpha-1} v_0^\beta \varphi_1 + \beta u_0^\alpha v_0^{\beta-1}\varphi_2)dx.$$

Note that in particular, $\int_\Omega (h_1 u_0^{-r}\varphi_1 + h_2 v_0^{-r}\varphi_2)dx < \infty$. Which guarantees that $\exists\ \varepsilon_0 > 0$, s.t. $u_0 \geq \varepsilon e_1, v_0 \geq \varepsilon e_1$ *a.e. in* Ω. □

In other words, $(u, v) \in X, u > 0, v > 0$ for any $(\varphi_1, \varphi_2) \in X, \varphi_1 > 0, \varphi_2 > 0$, there holds

$$\begin{aligned} &\int_\Omega (\nabla u_0^{p-1}\nabla\varphi_1 + \nabla v_0^{p-1}\nabla\varphi_2)dx + \int_\Omega (\nabla u_0^{q-1}\nabla\varphi_1 + \nabla v_0^{q-1}\nabla\varphi_2)dx \\ &-\int_\Omega (h_1 u_0^{-r}\varphi_1 + h_2 v_0^{-r}\varphi_2)dx - \frac{\lambda}{p^*}\int_\Omega (\alpha u_n^{\alpha-1} v_0^\beta\varphi_1 + \beta u_n^\alpha v_0^{\beta-1}\varphi_2)dx \geq 0, \end{aligned} \tag{3.7}$$

using (3.6) with $(\varphi_1, \varphi_2) = (u, v)$, we get that

$$\|(u_0, v_0)\|_p^p + \|(u_0, v_0)\|_q^q - \int_\Omega (h_1 u_0^{1-r} + h_2 v_0^{1-r})dx - \lambda\int_\Omega u_0^\alpha v_0^\beta dx \geq 0. \tag{3.8}$$

Proposition 3.2. $(u_0, v_0) \in \mathcal{N}_\lambda$ *with* $\lambda \in (0, \Lambda_*)$.

Proof. Denote

$$a_0 = \|(u_0, v_0)\|_p^p + \|(u_0, v_0)\|_q^q - \int_\Omega (h_1 u_0^{1-r} + h_2 v_0^{1-r})dx - \lambda\int_\Omega u_0^\alpha v_0^\beta dx.$$

By (3.8), we know $a_0 \geq 0$. Let us argue by contradiction and assume that $a_0 > 0$. In the following we will concentrate on a contradiction.

By the assumption $a_0 > 0$, there exists a unique $C_0 > 0$ such that

$$C_0^p B_1 + C_0^q B_2 - \lambda C_0^{p^*} A_1 = -a_0,$$

where

$$\|(\alpha^{\frac{1}{p}} U_{\varepsilon,a}, \beta^{\frac{1}{p}} U_{\varepsilon,a})\|_p^p = (\alpha+\beta)\|U_{\epsilon,a}\|_p^p = B_1 + o(\varepsilon^{\frac{n-p}{p}}),$$

$$\int_\Omega (\alpha^{\frac{1}{p}} U_{\varepsilon,a})^\alpha (\beta^{\frac{1}{p}} U_{\varepsilon,a})^\beta dx = \alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} \int_\Omega U_{\varepsilon,a}^{p^*} dx = A_1 + o(\varepsilon^{\frac{n-p}{p}}),$$

$$S = \frac{\|U_{\epsilon,a}\|_p^p}{\left(\int_\Omega U_{\varepsilon,a}^{p^*} dx\right)^{\frac{p}{p^*}}} = \frac{B^p}{A^{\frac{p}{p^*}}},$$

$$S_{\alpha,\beta} = \frac{\|(\alpha^{\frac{1}{p}} U_{\varepsilon,a}, \beta^{\frac{1}{p}} U_{\varepsilon,a})\|_p^p}{\left(\int_\Omega (\alpha^{\frac{1}{p}} U_{\varepsilon,a})^\alpha (\beta^{\frac{1}{p}} U_{\varepsilon,a})^\beta dx\right)^{\frac{p}{p^*}}} = \frac{(\alpha+\beta) B^p}{\alpha^{\frac{\alpha}{p^*}} \beta^{\frac{\beta}{p^*}} A^{\frac{p}{p^*}}} = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{p^*}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{p^*}}\right] S,$$

$$\|(\alpha^{\frac{1}{p}} U_{\varepsilon,a}, \beta^{\frac{1}{p}} U_{\varepsilon,a})\|_q^q = (\alpha^{\frac{q}{p}} + \beta^{\frac{q}{p}}) \|U_{\epsilon,a}\|_q^q = (\alpha^{\frac{q}{p}} + \beta^{\frac{q}{p}}) B^q + o(\varepsilon^{\frac{n-p}{p}}),$$

$$(\alpha+\beta) B^p := B_1, (\alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}}) A := A_1, (\alpha^{\frac{q}{p}} + \beta^{\frac{q}{p}}) B^q := B_2.$$

But as $I(u_n, v_n) \to \mu_0 := \inf\limits_{\mathcal{N}_\lambda^+ \bigcup \mathcal{N}_\lambda^0} I = \inf\limits_{\mathcal{N}_\lambda^+} I$ with $(u_n, v_n) \in \mathcal{N}_\lambda^+ (\subset \mathcal{N}_\lambda)$, by the Brezis-Lieb Lemma we have

$$\begin{aligned}
&\mu_0 + o(1) = I(u_n, v_n) \\
&= \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right) \|(u_0, v_0)\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{p^*}\right) \|(u_0, v_0)\|_q^q - \left(\tfrac{1}{1-r} - \tfrac{1}{p^*}\right) \int_\Omega (h_1 u_0^{1-r} + h_2 v_0^{1-r}) dx \\
&+ \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right) \|(\widetilde{u_n}, \widetilde{v_n})\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{p^*}\right) \|(\widetilde{u_n}, \widetilde{v_n})\|_q^q + o(1),
\end{aligned}$$

where $\widetilde{u_n} = u_n - u, \widetilde{v_n} = v_n - v$, and

$$\begin{aligned}
0 &= \|(u_n, v_n)\|_p^p + \|(u_n, v_n)\|_q^q - \textstyle\int_\Omega (h_1 u^{1-r} + h_2 v^{1-r}) dx - \lambda \displaystyle\int_\Omega u_n^\alpha v_n^\beta dx \\
&= a_0 + \|(\widetilde{u_n}, \widetilde{v_n})\|_p^p + \|(\widetilde{u_n}, \widetilde{v_n})\|_q^q - \lambda \int_\Omega \widetilde{u_n}^\alpha \widetilde{v_n}^\beta dx + o(1) \\
&\geq a_0 + S_{\alpha,\beta} \left(\int_\Omega \widetilde{u_n}^\alpha \widetilde{v_n}^\beta\right)^{\frac{p}{p^*}} + S_{\alpha,\beta}^{\frac{q}{p}} \left(\int_\Omega \widetilde{u_n}^\alpha \widetilde{v_n}^\beta\right)^{\frac{q}{p^*}} - \lambda \int_\Omega \widetilde{u_n}^\alpha \widetilde{v_n}^\beta dx + o(1),
\end{aligned}$$

which would imply that $\lim\limits_{n\to\infty} \int_\Omega \widetilde{u_n}^\alpha \widetilde{v_n}^\beta dx$ exists and

$$\lim_{n\to\infty} \int_\Omega \widetilde{u_n}^\alpha \widetilde{v_n}^\beta dx \geq \alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} C_0^{p^*} A = C_0^{p^*} A_1.$$

In other words, (u, v) satisfies

$$\begin{aligned}
\mu_0 \geq \; &\left(\tfrac{1}{p} - \tfrac{1}{p^*}\right) \|(u_0, v_0)\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{p^*}\right) \|(u_0, v_0)\|_q^q - \left(\tfrac{1}{1-r} - \tfrac{1}{p^*}\right) \int_\Omega (h_1 u_0^{1-r} + h_2 v_0^{1-r}) dx \\
&+ \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right) C_0^p B_1 + \left(\tfrac{1}{q} - \tfrac{1}{p^*}\right) C_0^q B_2.
\end{aligned} \tag{3.9}$$

On the other hand, for any $(u, v) \in X$ with

$$a_{u,v} = \|(u, v)\|_p^p + \|(u, v)\|_q^q - \int_\Omega (h_1 u^{1-r} + h_2 v^{1-r}) dx - \lambda \int_\Omega u^\alpha v^\beta dx.$$

We can find $R>0$ such that

$$\|(u,v)\|_p^p+\|(u,v)\|_q^q-\int_\Omega(h_1u^{1-r}+h_2v^{1-r})dx-\lambda\int_\Omega u^\alpha v^\beta dx+R^pB_1+R^qB_2-\lambda R^{p^*}A_1<0,$$

and thus

$$\begin{aligned}&\|(u+R\alpha^{\frac{1}{p}}U_{\varepsilon,a},v+R\beta^{\frac{1}{p}}U_{\varepsilon,a})\|_p^p+\|(u+R\alpha^{\frac{1}{p}}U_{\varepsilon,a},v+R\beta^{\frac{1}{p}}U_{\varepsilon,a})\|_q^q\\&-\int_\Omega\left[h_1(u+R\alpha^{\frac{1}{p}}U_{\varepsilon,a})^{1-r}+h_2(v+R\beta^{\frac{1}{p}}U_{\varepsilon,a})^{1-r}\right]dx-\lambda\int_\Omega(u+R\alpha^{\frac{1}{p}}U_{\varepsilon,a})^\alpha(v+R\beta^{\frac{1}{p}}U_{\varepsilon,a})^\beta dx\\&=a_{u,v}+R^pB_1+R^qB_2-\lambda R^{p^*}A_1<0.\end{aligned}$$

This allow us to take $0<C_{0,\epsilon}<R$ to satisfy

$$\begin{aligned}&\|(u+C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a},v+C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a})\|_p^p+\|(u+C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a},v+C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a})\|_q^q\\&-\int_\Omega\left[h_1(u+C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a})^{1-r}+h_2(v+C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a})^{1-r}\right]dx\\&-\lambda\int_\Omega(u+C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a})^\alpha(v+C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a})^\beta dx=0,\end{aligned}\tag{3.10}$$

that is $\left(u+C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a},v+C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a}\right)\in\mathcal{N}_\lambda$.

Furthermore, since $a_{u,v}>0$, let $C_{u,v}>0$ is the unique that

$$C_{u,v}^pB_1+C_{u,v}^qB_2-\lambda C_{u,v}^{p^*}A_1=-a_{u,v},$$

then

$$C_{u,v}B_1>\left(\frac{B_1}{\lambda A_1}\right)^{\frac{1}{p^*-p}}.$$

From (3.10) it follows that $0=a_{u,v}+C_{u,v}^pB_1+C_{u,v}^qB_2-\lambda C_{u,v}^{p^*}A_1+o(1)$ and hence

$$C_{0,\varepsilon}\to C_{u,v},\quad as\ \varepsilon\to 0.\tag{3.11}$$

Which yield

$$\begin{aligned}&\|(u+C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a},v+C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a})\|_p^p\\&>C_{u,v}^p(\alpha+\beta)B^p>\left(\tfrac{B_1}{\lambda A_1}\right)^{\frac{p}{p^*-p}}B_1=\left(\tfrac{1}{\lambda}\right)^{\frac{n-p}{p}}S_{\alpha,\beta}^{\frac{n}{p}},\qquad for\ \varepsilon>0\ small,\end{aligned}$$

where

$$\frac{p}{p^*-p}=\frac{n-p}{p},\ \frac{p^*}{p^*-p}=\frac{n}{p}.$$

Necessarily

$$\begin{aligned}&\|(u+C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a},v+C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a})\|_p\\&>\left(\tfrac{1}{\lambda}\right)^{\frac{n-p}{p^2}}S_{\alpha,\beta}^{\frac{n}{p^2}}>\left(\tfrac{p-1+r}{p^*-1+r}\tfrac{1}{\lambda}S_{\alpha,\beta}^{\frac{p^*}{p}}\right)^{\frac{1}{p^*-p}}\equiv E_\lambda.\end{aligned}$$

The gap structure of $\mathcal{N}_\lambda$ guarantees $(u+C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a}, v+C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a}) \in \mathcal{N}_\lambda^-$ in view of the fact $\inf\limits_{\mathcal{N}_\lambda^+} I = \inf\limits_{\mathcal{N}_\lambda} I$ we derive that

$$\mu_0 \leq I\left(u+C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a}, v+C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a}\right)$$

$$= \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\|(u,v)\|_p^p + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\|(u,v)\|_q^q - \left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega (h_1u^{1-r}+h_2v^{1-r})dx$$

$$+\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)C_{0,\varepsilon}^pB_1 + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)C_{0,\varepsilon}^qB_2 + o(1),$$

that is

$$\mu_0 \leq \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\|(u,v)\|_p^p + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\|(u,v)\|_q^q - \left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega (h_1u^{1-r}+h_2v^{1-r})dx$$

$$+\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)C_{u,v}^pB_1 + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)C_{u,v}^qB_2 + o(1). \tag{3.12}$$

Now, putting together (3.9) and (3.12), we see that

$$\mu_0 = \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\|(u_0,v_0)\|_p^p + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\|(u_0,v_0)\|_q^q - \left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega (h_1u_0^{1-r}+h_2v_0^{1-r})dx$$

$$+\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)C_0^pB_1 + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)C_0^qB_2 + o(1). \tag{3.13}$$

This implies that, necessarily (u_0,v_0) is a local minimizer for the functional:

$$\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\|(u_0,v_0)\|_p^p + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\|(u_0,v_0)\|_q^q - \left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega (h_1u_0^{1-r}+h_2v_0^{1-r})dx$$

$$+\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)C_0^pB_1 + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)C_0^qB_2 = 0. \tag{3.14}$$

For the function $C_{u,v}$, let $\varphi_1,\varphi_2 \in C_0^\infty(\Omega)$ and evaluate $g(t) := C_{u_0+t\varphi_1, v_0+t\varphi_2}$ in a small neighborhood of $t=0$ that is

$$[g(t)]^pB_1 + [g(t)]^qB_2 - \lambda[g(t)]^{p^*}A_1$$

$$= -\Big[\|(u_0+t\varphi_1, v_0+t\varphi_2)\|_p^p + \|(u_0+t\varphi_1, v_0+t\varphi_2)\|_q^q$$

$$-\int_\Omega \left(h_1(u_0+t\varphi_1)^{1-r} + h_2(v_0+t\varphi_2)^{1-r}\right)dx - \lambda\int_\Omega (u+t\varphi_1)^\alpha(v+t\varphi_2)^\beta dx\Big].$$

By $a_0>0$, we know that $g(t)$ exists, with $g(0)=C_0$, moreover, since $u_0 \geq \varepsilon_0e_1, v_0 \geq \varepsilon_0e_1$ in Ω, by Dominated convergence Theorem

$$\frac{\displaystyle\int_\Omega h_1(u_0+t\varphi_1)^{1-r}dx - \int_\Omega h_1u_0^{1-r}dx}{t} = \int_{supp\ \varphi_1}(1-r)h_1(u_0+\theta t\varphi_1)^{-r}dx$$

$$\xrightarrow{t\to 0}\int_{supp\ \varphi_1}(1-r)h_1u_0^{-r}\varphi_1dx = \int_\Omega (1-r)h_1u_0^{-r}\varphi_1dx,$$

and consequently

$$\Big\{[g(0)+\theta(g(t)-g(0))]^{p-1}B_1+[g(0)+\theta(g(t)-g(0))]^{q-1}B_2$$

$$-\lambda[g(0)+\theta(g(t)-g(0))]^{p^*-1}A_1\Big\}\frac{g(t)-g(0)}{t}$$

$$=\frac{[g(t)]^pB_1+[g(t)]^qB_2-\lambda[g(t)]^{p^*}A_1-[g(0)]^pB_1+[g(0)]^qB_2}{t}$$

$$=-\frac{1}{t}\Big\{\|(u_0+t\varphi_1,v_0+t\varphi_2)\|_p^p+\|(u_0+t\varphi_1,v_0+t\varphi_2)\|_q^q-\lambda\int_\Omega(u_0+t\varphi_1)^\alpha(v_0+t\varphi_2)^\beta dx$$

$$-\int_\Omega(h_1(u_0+t\varphi_1)^{1-r}+h_2(v_0+t\varphi_2)^{1-r})dx-\|(u_0,v_0)\|_p^p+\|(u_0,v_0)\|_q^q$$

$$-\int_\Omega(h_1u_0^{1-r}+h_2v_0^{1-r})dx-\lambda\int_\Omega u_0^\alpha v_0^\beta dx\Big\}$$

$$\xrightarrow{t\to0}-\Big[p\int_\Omega(\nabla u_0^{p-1}\nabla\varphi_1+\nabla v_0^{p-1}\nabla\varphi_2)dx+q\int_\Omega(\nabla u_0^{q-1}\nabla\varphi_1+\nabla v_0^{q-1}\nabla\varphi_2)dx$$

$$-(1-r)\int_\Omega(h_1u_0^{-r}\varphi_1+h_2v_0^{-r}\varphi_2)dx-\lambda\alpha\int_\Omega u_0^{\alpha-1}v_0^\beta\varphi_1dx-\lambda\beta\int_\Omega u_0^\alpha v_0^{\beta-1}\varphi_2dx\Big],$$

which implies that $g'(0)$ exists and

$$g'(0)=\frac{-1}{C_0^{p-1}B_1+C_0^{q-1}B_2-\lambda C_0^{p^*-1}A_1}\Big[p\int_\Omega(\nabla u_0^{p-1}\nabla\varphi_1+\nabla v_0^{p-1}\nabla\varphi_2)dx$$

$$+q\int_\Omega(\nabla u_0^{q-1}\nabla\varphi_1+\nabla v_0^{q-1}\nabla\varphi_2)dx-(1-r)\int_\Omega(h_1u_0^{-r}\varphi_1+h_2v_0^{-r}\varphi_2)dx$$

$$-\lambda\alpha\int_\Omega u_0^{\alpha-1}v_0^\beta\varphi_1dx-\lambda\beta\int_\Omega u_0^\alpha v_0^{\beta-1}\varphi_2dx\Big].$$

Resuming from (3.14) we see that

$$\frac{d}{dt}\Big\{\left(\frac{1}{p}-\frac{1}{p^*}\right)\|(u_0+t\varphi_1,v_0+t\varphi_2)\|_p^p+\left(\frac{1}{q}-\frac{1}{p^*}\right)\|(u_0+t\varphi_1,v_0+t\varphi_2)\|_q^q$$

$$-\left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega\left(h_1(u_0+t\varphi_1)^{1-r}+h_2(v_0+t\varphi_2)^{1-r}\right)dx$$

$$+\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)[g(t)]^pB_1+\left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)[g(t)]^qB_2\Big\}\Big|_{t=0}=0$$

for all $\varphi_1,\varphi_2\in C_0^\infty(\Omega)$. Since from Theorem 3.1 follows immediately that $h_1u_0^{-r},h_2v_0^{-r}\in L^\infty(\Omega)$. Hence, for all $(\varphi_1,\varphi_2)\in X$, we conclude that

$$0=\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)p\int_\Omega(\nabla u_0^{p-1}\nabla\varphi_1+\nabla v_0^{p-1}\nabla\varphi_2)dx+\left(\frac{1}{q}-\frac{1}{p^*}\right)q\int_\Omega(\nabla u_0^{q-1}\nabla\varphi_1+\nabla v_0^{q-1}\nabla\varphi_2)dx$$

$$-\left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)(1-r)\int_\Omega(h_1u_0^{-r}\varphi_1+h_2v_0^{-r}\varphi_2)dx$$

$$+\Big\{\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)pC_0^{p-1}B_1+\left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)qC_0^{q-1}B_2\Big\}g'(0). \tag{3.15}$$

We use the famous estimates as follows

$$\int_{\Omega} \nabla u_0 \nabla U_{\varepsilon,a} dx = 0,$$

$$\int_{\Omega} \nabla u_0^{p-1} \nabla U_{\varepsilon,a} dx = o\left(\varepsilon^{\frac{n-p}{p(p-1)}}\right),$$

$$\int_{\Omega} \nabla U_{\varepsilon,a}^{p-1} \nabla u_0 dx = o\left(\varepsilon^{\frac{n-p}{p}}\right),$$

$$\int_{\Omega} \nabla U_{\varepsilon,a}^{p^*-1} \nabla u_0 dx = u_0(a)\varepsilon^{\frac{n-p}{p}} \int_{\mathbb{R}^n} \frac{1}{(1+|x|^{\frac{p}{p-1}})^{\frac{np-n+p}{p}}} dx + o\left(\varepsilon^{\frac{n-p}{p}}\right),$$

$$\int_{\Omega} \nabla u_0^{p^*-1} \nabla U_{\varepsilon,a} dx = \varepsilon^{\frac{n-p}{p}} \int_{\mathbb{R}^n} \frac{u_0^{p-1}\eta}{(|x-a|^{\frac{p}{p-1}})^{\frac{n-p}{p}}} dx + o\left(\varepsilon^{\frac{n-p}{p}}\right).$$

In particular, as $h_1 u_0^{-r}, h_2 v_0^{-r} \in L^\infty(\Omega)$, we can reevaluate

$$\begin{aligned}
&\int_{\Omega} h_1(u_0 + C_{0,\varepsilon}\alpha^{\frac{1}{p}} U_{\varepsilon,a})^{1-r} dx - \int_{\Omega} h_1 u_0^{1-r} dx \\
&= \int_{\Omega} h_1(1-r)(u_0 + \theta\alpha^{\frac{1}{p}} C_{0,\varepsilon} U_{\varepsilon,a})^{-r} \alpha^{\frac{1}{p}} C_{0,\varepsilon} U_{\varepsilon,a} dx \\
&= \varepsilon^{\frac{n-p}{p}} \left[(1-r)\alpha^{\frac{1}{p}} C_0 \int_{\Omega} h_1 u_0^{-r} \frac{\eta}{(|x-a|^{\frac{p}{p-1}})^{\frac{n-p}{p}}} dx + o(1)\right],
\end{aligned}$$

write $C_{0,\varepsilon} = C_0 + \delta_\varepsilon$, by $C_{0,\varepsilon} \to C_0$ as $\varepsilon \to 0$, we get $\delta_\varepsilon \to 0$.

Inserting all the above estimates into (3.10), we obtain

$$\begin{aligned}
0 = &-\left[C_0^p B_1 + C_0^q B_2 - \lambda C_0^{p^*} A_1\right] + C_{0,\varepsilon}^p B_1 + C_{0,\varepsilon}^q B_2 - \lambda C_{0,\varepsilon}^{p^*} A_1 \\
&+ pC_{0,\varepsilon}\left[\int_{\Omega} (\alpha \nabla u_0^{p-1} \nabla U_{\varepsilon,a} + \beta \nabla v_0^{p-1} \nabla U_{\varepsilon,a}) dx\right] + qC_{0,\varepsilon}\left[\int_{\Omega} (\alpha^{\frac{q}{p}} \nabla u_0^{q-1} \nabla U_{\varepsilon,a}\right. \\
&\left.+ \beta^{\frac{q}{p}} \nabla v_0^{q-1} \nabla U_{\varepsilon,a}) dx\right] - \varepsilon^{\frac{n-p}{p}} (1-r) C_0 \int_{\Omega} (h_1 u_0^{-r} + h_2 v_0^{-r}) \frac{\eta}{\left(|x-a|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}} dx + o\left(\varepsilon^{\frac{n-p}{p}}\right),
\end{aligned}$$

which gives

$$\begin{aligned}
&\left[pC_0^{p-1} B_1 + qC_0^{q-1} B_2 - \lambda p^* C_0^{p^*-1} A_1\right](-\delta_\varepsilon) \\
&= pC_{0,\varepsilon}\left[\int_{\Omega} (\alpha \nabla u_0^{p-1} \nabla U_{\varepsilon,a} + \beta \nabla v_0^{p-1} \nabla U_{\varepsilon,a}) dx\right] + qC_{0,\varepsilon}\left[\int_{\Omega} (\alpha^{\frac{q}{p}} \nabla u_0^{q-1} \nabla U_{\varepsilon,a}\right. \\
&\left.+ \beta^{\frac{q}{p}} \nabla v_0^{p-1} \nabla U_{\varepsilon,a}) dx\right] - (1-r) C_0 \int_{\Omega} (h_1 u_0^{-r} + h_2 v_0^{-r}) dx + o\left(\varepsilon^{\frac{n-p}{p}}\right) - \lambda p^* C_0^{p^*-1} A_1.
\end{aligned}$$

Furthermore, from (3.15) follows that

$$(-\delta_\varepsilon) = C_0 \frac{\left(\frac{1}{p}-\frac{1}{p^*}\right)p\int_\Omega (\nabla u_n^{p-1}\nabla\varphi_1 + \nabla v_n^{p-1}\nabla\varphi_2 dx)}{\left(\frac{1}{p}-\frac{1}{p^*}\right)pC_0^{p-1}B_1+\left(\frac{1}{q}-\frac{1}{p^*}\right)qC_0^{q-1}B_2}$$

$$-C_0 \frac{\left(\frac{1}{1-r}-\frac{1}{p^*}\right)\int_\Omega \left(h_1 u_0^{-r}\varphi_1 + h_2 v_0^{-r}\varphi_2\right)dx + \left(\frac{1}{q}-\frac{1}{p^*}\right)q\int_\Omega (\nabla u_n^{q-1}\nabla\varphi_1 + \nabla v_n^{q-1}\nabla\varphi_2)dx}{\left(\frac{1}{p}-\frac{1}{p^*}\right)pC_0^{p-1}B_1+\left(\frac{1}{q}-\frac{1}{p^*}\right)qC_0^{q-1}B_2}$$

$$-\frac{\lambda p^* C_0^{p^*-1}A_1}{pC_0^{p-1}B_1+qC_0^{q-1}B_2-\lambda p^* C_0^{p^*-1}A_1} + o\left(\varepsilon^{\frac{n-p}{p}}\right). \tag{3.16}$$

Also, $-\delta_\varepsilon = o\left(\varepsilon^{\frac{n-p}{p}}\right)$.

Now, we can proceed to get the contradiction. Since $a_0 > 0$, clearly,

$$pC_0^{p-1}B_1 + qC_0^{q-1}B_2 - \lambda p^* C_0^{p^*-1}A_1 < \tfrac{p^*}{C_0}\left[C_0^p B_1 + C_0^q B_2 - \lambda C_0^{p^*}A_1\right] = -\tfrac{p^*}{C_0}a_0 < 0,$$

subsequently, in virtue of $\left(u + C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a}, v + C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a}\right) \in \mathcal{N}_\lambda$, applying (3.14) and (3.16), we obtain

$$\begin{aligned}
&I(u + C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a}, v + C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a})\\
&= \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\|(u + C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a}, v + C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a})\|_p^p\\
&+ \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\|(u + C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a}, v + C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a})\|_q^q\\
&- \left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega \left(h_1(u + C_{0,\epsilon}\alpha^{\frac{1}{p}}U_{\varepsilon,a})^{1-r} + h_2(v + C_{0,\epsilon}\beta^{\frac{1}{p}}U_{\varepsilon,a})^{1-r}\right)dx\\
&= \mu_0 + \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)pC_{0,\epsilon}^{p-1}\delta_\varepsilon B_1 + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)qC_{0,\varepsilon}^{q-1}\delta_\varepsilon B_2\\
&+pC_{0,\varepsilon}\int_\Omega (\alpha^{\frac{1}{p}}\nabla u_0^{p-1}\nabla U_{\varepsilon,a} + \beta^{\frac{1}{p}}\nabla v_0^{p-1}\nabla U_{\varepsilon,a})dx\\
&+qC_{0,\varepsilon}\int_\Omega (\alpha^{\frac{1}{p}}\nabla u_0^{q-1}\nabla U_{\varepsilon,a} + \beta^{\frac{1}{p}}\nabla v_0^{q-1}\nabla U_{\varepsilon,a})dx\\
&- \left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)(1-r)C_0\int_\Omega (h_1 u_0^{-r} + h_2 v_0^{-r})dx\\
&= \mu_0 + \left[\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)pC_0^{p-1}\delta_\varepsilon B_1 + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)qC_0^{q-1}\delta_\varepsilon B_2\right]\tfrac{\lambda p^* C_0^{p^*-1}A_1}{pC_0^{p-1}B_1+qC_0^{q-1}B_2-\lambda p^* C_0^{p^*-1}A_1}\\
&+o\left(\varepsilon^{\frac{n-p}{p}}\right) < \mu_0,
\end{aligned}$$

which is clearly impossible. This ends the proof of Proposition 3.2. □

Proposition 3.3. (u_0, v_0) *is a solution of* (1.1) .

Proof. For $\varphi_1, \varphi_2 \in W_0^{1,p}(\Omega), \varepsilon > 0$. Define

$$\Psi_1 := (u_0 + \varepsilon\varphi_1)^+ \in W_0^{1,p}(\Omega), \Psi_2 := (v_0 + \varepsilon\varphi_2)^+ \in W_0^{1,p}(\Omega),$$

$$\Omega_1 = \{x | u_0 + \varepsilon\varphi_1 > 0\}, \Omega_1 = \{x | v_0 + \varepsilon\varphi_2 > 0\}, \Omega^- = \Omega - \Omega_1 \cap \Omega_2, \Omega^+ = \Omega_1 \cap \Omega_2.$$

Using Proposition 3.2 and inserting Ω_1, Ω_2 into (3.7), we see that

$$\begin{aligned}
0 \leq & \int_\Omega (\nabla u_0^{p-1} \nabla \Psi_1 + \nabla v_0^{p-1} \nabla \Psi_2) dx + \int_\Omega (\nabla u_0^{q-1} \nabla \Psi_1 + \nabla v_0^{q-1} \nabla \Psi_2) dx \\
& - \int_\Omega (h_1 u_0^{-r} \Psi_1 + h_2 v_0^{-r} \Psi_2) dx - \frac{\lambda}{p^*} \int_\Omega (\alpha u_0^{\alpha-1} v_0^\beta \Psi_1 + \beta u_0^\alpha v_0^{\beta-1} \Psi_2) dx \\
= & \int_{\Omega^+} \left(\nabla u_0^{p-1} \nabla (u_0 + \varepsilon\varphi_1) + \nabla v_0^{p-1} \nabla (v_0 + \varepsilon\varphi_2) \right) dx \\
& + \int_{\Omega^+} \left(\nabla u_0^{q-1} \nabla (u_0 + \varepsilon\varphi_1) + \nabla v_0^{q-1} \nabla (v_0 + \varepsilon\varphi_2) \right) dx \\
& - \int_{\Omega^+} \left(h_1 u_0^{-r} (u_0 + \varepsilon\varphi_1) + h_2 v_0^{-r} (v_0 + \varepsilon\varphi_2) \right) dx \\
& - \tfrac{\lambda}{p^*} \int_{\Omega^+} (\alpha u_0^{\alpha-1} v_0^\beta (u_0 + \varepsilon\varphi_1) + \beta u_0^\alpha v_0^{\beta-1} (v_0 + \varepsilon\varphi_2)) dx \\
= & \, \varepsilon \Bigg[\int_\Omega (\nabla u_0^{p-1} \nabla \varphi_1 + \nabla v_0^{p-1} \nabla \varphi_2) dx + \int_\Omega (\nabla u_0^{q-1} \nabla \varphi_1 + \nabla v_0^{q-1} \nabla \varphi_2) dx \\
& - \int_\Omega (h_1 u_0^{-r} \varphi_1 + h_2 v_0^{-r} \varphi_2) dx - \frac{\lambda}{p^*} \int_\Omega \left(\alpha u_n^{\alpha-1} v_0^\beta \varphi_1 + \beta u_n^\alpha v_0^{\beta-1} \varphi_2 \right) dx \Bigg] \\
& - \int_{\Omega^-} (\nabla u_0^{p-1} \nabla (u_0 + \varepsilon\varphi_1) + \nabla v_0^{p-1} \nabla (v_0 + \varepsilon\varphi_2)) dx \\
& - \int_{\Omega^-} (\nabla u_0^{q-1} \nabla (u_0 + \varepsilon\varphi_1) + \nabla v_0^{q-1} \nabla (v_0 + \varepsilon\varphi_2)) dx \\
& + \int_{\Omega^-} \left(h_1 u_0^{-r} (u_0 + \varepsilon\varphi_1) + h_2 v_0^{-r} (v_0 + \varepsilon\varphi_2) \right) dx \\
& + \tfrac{\lambda}{p^*} \int_{\Omega^-} \left(\alpha u_0^{\alpha-1} v_0^\beta (u_0 + \varepsilon\varphi_1) + \beta u_0^\alpha v_0^{\beta-1} (v_0 + \varepsilon\varphi_2) \right) dx \\
\leq & \, \varepsilon \Bigg[\int_\Omega (\nabla u_0^{p-1} \nabla \varphi_1 + \nabla v_0^{p-1} \nabla \varphi_2) dx + \int_\Omega (\nabla u_0^{q-1} \nabla \varphi_1 + \nabla v_0^{q-1} \nabla \varphi_2) dx \\
& - \int_\Omega (h_1 u_0^{-r} \varphi_1 + h_2 v_0^{-r} \varphi_2) dx - \frac{\lambda}{p^*} \int_\Omega (\alpha u_n^{\alpha-1} v_0^\beta \varphi_1 + \beta u_n^\alpha v_0^{\beta-1} \varphi_2) dx \Bigg] \\
& - \int_{\Omega^-} (\nabla u_0^{p-1} \nabla (u_0 + \varepsilon\varphi_1) + \nabla v_0^{p-1} \nabla (v_0 + \varepsilon\varphi_2)) dx \\
& - \int_{\Omega^-} (\nabla u_0^{q-1} \nabla (u_0 + \varepsilon\varphi_1) + \nabla v_0^{q-1} \nabla (v_0 + \varepsilon\varphi_2)) dx.
\end{aligned}$$

Since the measure of domain of integration $[u_0 + \varepsilon\varphi_1 \leq 0], [v_0 + \varepsilon\varphi_2 \geq 0]$ tend to zero as $\varepsilon \to 0$, it follows that

$$\int_{\Omega^-} \left(\nabla u_0^{p-1}\nabla(u_0 + \varepsilon\varphi_1) + \nabla v_0^{p-1}\nabla(v_0 + \varepsilon\varphi_2)\right) dx$$
$$+ \int_{\Omega^-} \left(\nabla u_0^{q-1}\nabla(u_0 + \varepsilon\varphi_1) + \nabla v_0^{q-1}\nabla(v_0 + \varepsilon\varphi_2)\right) dx \to 0.$$

Dividing by ε and letting $\varepsilon \to 0$ therefore shows

$$\int_\Omega (\nabla u_0^{p-1}\nabla\varphi_1 + \nabla v_0^{p-1}\nabla\varphi_2)dx + \int_\Omega (\nabla u_0^{q-1}\nabla\varphi_1 + \nabla v_0^{q-1}\nabla\varphi_2)dx$$
$$-\int_\Omega (h_1 u_0^{-r}\varphi_1 + h_2 v_0^{-r}\varphi_2)dx - \frac{\lambda}{p^*}\int_\Omega (\alpha u_0^{\alpha-1}v_0^{\beta}\varphi_1 + \beta u_0^{\alpha}v_0^{\beta-1}\varphi_2)dx \geq 0,$$

and since this holds equally well for $-\varphi_1, -\varphi_2$, it follows that (u_0, v_0) is a solution of the (3.7). $\square$

Since $u_n \rightharpoonup u_0, v_n \rightharpoonup v_0$ weakly in $W_0^{1,p}(\Omega)$, by the weak lower semi-continuity of norm, we conclude that $\|(u_0, v_0)\| = \lim_{n\to\infty}\inf \|(u_n, v_n)\| \leq E_0$, Proposition 3.2 and the gap structure of $\mathcal{N}_\lambda$ in turn imply that $(u_0, v_0) \in \mathcal{N}_\lambda^+$. At this point, from $I(u_n, v_n) \to \inf_{\mathcal{N}_\lambda^+} I$, we see that

$$\inf_{\mathcal{N}_\lambda^+} I \geq \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right)\|(u_0, v_0)\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{p^*}\right)\|(u_0, v_0)\|_q^q - \left(\tfrac{1}{1-r} - \tfrac{1}{p^*}\right)\int_\Omega (h_1 u_0^{1-r} + h_2 v_0^{1-r})dx$$
$$= I(u_0, v_0),$$

that is $I(u_0, v_0) = \inf_{\mathcal{N}_\lambda^+} I(u, v)$. This ends the proof of Theorem 1.1. $\square$

4 The proof of Theorem 1.2

Proof. We provide only a sketch, as the arguments are by now familiar. Then consider $(U_n, V_n) \subset \mathcal{N}_\lambda^-$ the best minimizing sequence for $\inf_{\mathcal{N}_\lambda^-} I$. Since (U_n, V_n) is bounded in X, after passing to a subsequence, we may assume that $U_n \rightharpoonup U_0, V_n \rightharpoonup V_0$ weakly in $W_0^{1,p}(\Omega)$. The result

$$\lim_{n\to\infty}\inf[(p^* - p)\|(U_n, V_n)\|_p^p + (p^* - q)\|(U_n, V_n)\|_q^q] > (p^* + r - 1)\int_\Omega (h_1 U_0^{1-r} + h_2 V_0^{1-r})dx,$$

follows easily with an argument by contradiction. In fact provided that

$$\lim_{n\to\infty}\inf\,[(p^* - p)\|(U_n, V_n)\|_p^p + (p^* - q)\|(U_n, V_n)\|_q^q] = (p^* + r - 1)\int_\Omega (h_1 U_0^{1-r} + h_2 V_0^{1-r})dx,$$

there exists a subsequence of (U_n, V_n), called (U_{nk}, V_{nk}) and recalling $\|(U, V)\| > E_\lambda$ for all $(U, V) \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$, we have

$$E_\lambda < \|(U_{nk}, V_{nk})\|_p,$$

$$i.e.\ \ (p-1+r)\|(U_{nk},V_{nk})\|_p^p < \lambda(p^*-1+r)\int_\Omega U_{nk}^\alpha V_{nk}^\beta dx,$$

and

$$\|(U_{nk},V_{nk})\|_p^p + \|(U_{nk},V_{nk})\|_q^q - \int_\Omega (h_1 U_{nk}^{1-r} + h_2 V_{nk}^{1-r})dx - \lambda\int_\Omega U_{nk}^\alpha V_{nk}^\beta dx = 0,$$

$$\|(p-p^*)(U_{nk},V_{nk})\|_p^p + (q-p^*)\|(U_{nk},V_{nk})\|_q^q - (p^*-1+r)\int_\Omega (h_1 U_{nk}^{1-r} + h_2 V_{nk}^{1-r})dx < 0.$$

Consequently

$$0 > \left(1 - \frac{p^*-p}{p^*+r-1} - \frac{p-1+r}{p^*-1+r}\right)\|(U_{nk},V_{nk})\|_p^p + \left(1-\frac{p^*-q}{p^*+r-1}\right)\|(U_{nk},V_{nk})\|_q^q,$$

which is clearly impossible.

Thus, we can proceed as in the proof of Theorem 1.1 to obtain $U_0(x) > \varepsilon_0 e_1(x), U_0(x) > \varepsilon_0 e_1(x), \forall x \in \Omega$. And

$$\int_\Omega (\nabla U_0^{p-1}\nabla\varphi_1 + \nabla V_0^{p-1}\nabla\varphi_2)dx + \int_\Omega (\nabla U_0^{q-1}\nabla\varphi_1 + \nabla V_0^{q-1}\nabla\varphi_2)dx$$

$$-\int_\Omega (h_1 U_0^{-r}\varphi_1 + h_2 V_0^{-r}\varphi_2)dx - \frac{\lambda}{p^*}\int_\Omega (\alpha U_0^{\alpha-1}V_0^\beta\varphi_1 + \beta U_0^\alpha V_0^{\beta-1}\varphi_2)dx \geq 0.$$

By taking $\varphi_1 = U_0, \varphi_2 = V_0$, we know that

$$\|(U_0,V_0)\|_p^p + \|(U_0,V_0)\|_q^q - \int_\Omega (h_1 U_0^{1-r} + h_2 V_0^{1-r})dx - \lambda\int_\Omega U_0^\alpha V_0^\beta dx = 0,$$

all that remains is to prove that $(U_0,V_0) \in \mathcal{N}_\lambda$.

Arguing by contradiction and assume that

$$A_0 = \|(U_0,V_0)\|_p^p + \|(U_0,V_0)\|_q^q - \int_\Omega (h_1 U_0^{1-r} + h_2 V_0^{1-r})dx - \lambda\int_\Omega U_0^\alpha V_0^\beta dx > 0,$$

then there would exist a unique point $\widetilde{C_0} > 0$ such that

$$\widetilde{C_0}^p D_1 + \widetilde{C_0}^q D_2 - \lambda\widetilde{C_0}^{p^*} E_1 = -\widetilde{a_0}.$$

Since $I(U_n,V_n) \to \pi_0 := \inf_{\mathcal{N}_\lambda^-} I$ with $(U_n,V_n) \in \mathcal{N}_\lambda^- (\subset \mathcal{N}_\lambda)$, we have

$$\begin{aligned}
&\pi_0 + o(1) = I(U_n,V_n)\\
&= \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right)\|(U_n,V_n)\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{p^*}\right)\|(U_n,V_n)\|_q^q - \left(\tfrac{1}{1-r} - \tfrac{1}{p^*}\right)\int_\Omega (h_1 U_n^{1-r} + h_2 V_n^{1-r})dx\\
&= \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right)\|(\widetilde{U_n},\widetilde{V_n})\|_p^p + \left(\tfrac{1}{q} - \tfrac{1}{p^*}\right)\|(\widetilde{U_n},\widetilde{V_n})\|_q^q + o(1),
\end{aligned}$$

where $\widetilde{U_n} = U_n - U_0, \widetilde{V_n} = V_n - V_0$, and

$$\begin{aligned}
0 &= \|(U_n,V_n)\|_p^p + \|(U_n,V_n)\|_q^q - \int_\Omega (h_1 U^{1-r} + h_2 V^{1-r})dx - \lambda\int_\Omega U_n^\alpha V_n^\beta dx\\
&= A_0 + \|(\widetilde{U_n},\widetilde{V_n})\|_p^p + \|(\widetilde{U_n},\widetilde{V_n})\|_q^q - \lambda\int_\Omega \widetilde{U_n}^\alpha\widetilde{V_n}^\beta dx + o(1)\\
&\geq A_0 + S_{\alpha,\beta}\left(\int_\Omega \widetilde{U_n}^\alpha\widetilde{V_n}^\beta dx\right)^{\frac{p}{p^*}} + S_{\alpha,\beta}^{\frac{q}{p}}\left(\int_\Omega \widetilde{U_n}^\alpha\widetilde{V_n}^\beta dx\right)^{\frac{q}{p^*}} - \lambda\int_\Omega \widetilde{U_n}^\alpha\widetilde{V_n}^\beta dx + o(1),
\end{aligned}$$

which would imply that $\lim_{n\to\infty}\int_\Omega \widetilde{U_n}^\alpha \widetilde{V_n}^\beta dx$ exists and

$$\lim_{n\to\infty}\int_\Omega \widetilde{U_n}^\alpha \widetilde{V_n}^\beta dx \geq \alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}}\widetilde{C_0}^{p^*}D = \widetilde{C_0}^{p^*}D_1,$$

and sequently

$$\begin{aligned}\pi_0 \geq \quad & \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\|(U_0,V_0)\|_p^p + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\|(U_0,V_0)\|_q^q - \left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega (h_1U_0^{1-r}+h_2V_0^{1-r})dx \\ & + \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\widetilde{C_0}^p D_1 + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\widetilde{C_0}^q D_2.\end{aligned} \tag{4.1}$$

As shown in the proof of Theorem 1.1 for any $(U,V)\in X$ with

$$A_{u,v} = \|(U,V)\|_p^p + \|(U,V)\|_q^q - \int_\Omega (h_1U^{1-r}+h_2V^{1-r})dx - \lambda\int_\Omega U^\alpha V^\beta dx,$$

we can alway find $0<\widetilde{C_{0,\varepsilon}}<\widetilde{R_{u,v}}$ such that $\left(U+\widetilde{C_{0,\varepsilon}}\alpha^{\frac{1}{p}}U_{\varepsilon,a}, V+\widetilde{C_{0,\varepsilon}}\beta^{\frac{1}{p}}U_{\varepsilon,a}\right)\in\mathcal{N}_\lambda^-$ for $\varepsilon>0$ small. Sequently

$$\begin{aligned}&\pi_0 \leq I(U+\widetilde{C_{0,\epsilon}\alpha}^{\frac{1}{p}}U_{\varepsilon,a}, V+\widetilde{C_{0,\epsilon}\alpha}\beta^{\frac{1}{p}}U_{\varepsilon,a}) \\ &= \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\|(U,V)\|_p^p + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\|(U,V)\|_q^q - \left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega (h_1U^{1-r}+h_2V^{1-r})dx \\ &+ \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\widetilde{C_{0,\epsilon}}^p D_1 + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\widetilde{C_{0,\epsilon}}^q D_2 + o(1).\end{aligned} \tag{4.2}$$

Putting together (4.1) and (4.2), we get that

$$\begin{aligned}\pi_0 = & \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\|(U_0,V_0)\|_p^p + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\|(U_0,V_0)\|_q^q - \left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega (h_1U_0^{1-r}+h_2V_0^{1-r})dx \\ & + \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\widetilde{C_0}^p D_1 + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\widetilde{C_0}^q D_2 + o(1),\end{aligned}$$

and that for every $\varphi_1,\varphi_2\in C_0^\infty(\Omega)$

$$\begin{aligned}&\tfrac{d}{dt}\Big\{\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\|(U,V)\|_p^p + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)\|(U,V)\|_q^q - \left(\tfrac{1}{1-r}-\tfrac{1}{p^*}\right)\int_\Omega (h_1U^{1-r}+h_2V^{1-r})dx \\ &+ \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)[G(t)]^pD_1 + \left(\tfrac{1}{q}-\tfrac{1}{p^*}\right)[G(t)]^qD_2\Big\}\Big|_{t=0} = 0,\end{aligned}$$

where

$$\begin{aligned}&[G(t)]^pD_1 + [G(t)]^qD_2 - \lambda[G(t)]^{p^*}E_1 \\ &= -\Big[\|(U+t\varphi_1, V+t\varphi_2)\|_p^p + \|(U+t\varphi_1,V+t\varphi_2)\|_q^q \\ &-\int_\Omega (h_1(U+t\varphi_1)^{1-r} + h_2(V+t\varphi_2)^{1-r})dx - \lambda\int_\Omega (U+t\varphi_1)^\alpha(V+t\varphi_2)^\beta dx\Big],\end{aligned}$$

and we can proceed as in (3.15), (3.16) to reach a contradiction. The desire result that (U_0,V_0) is a solution of the problem (1.1).

Still no location information can be obtained for (U_0,V_0). Consequently, we prove that $(U_0,V_0)\in\mathcal{N}_\lambda^-$.

Proposition 4.1. *Suppose that $\lambda \in (0, \Lambda_*)$, then the solution $(U_0, V_0) \in \mathcal{N}_\lambda^-$.*

Proof. Now consider

$$W_1 = \left\{(u,v) \in X\backslash(0,0) \Big| \frac{1}{\|(u,v)\|} t^- \left(\frac{(u,v)}{\|(u,v)\|}\right) > 1\right\} \cup (0,0),$$
$$W_2 = \left\{(u,v) \in X\backslash(0,0) \Big| \frac{1}{\|(u,v)\|} t^- \left(\frac{(u,v)}{\|(u,v)\|}\right) < 1\right\}.$$

Then $\mathcal{N}_\lambda^-$ disconnect X in two connected components W_1 and W_2 and $X\backslash\mathcal{N}_\lambda = W_1 \bigcup W_2$, for each $(u,v) \in \mathcal{N}_\lambda^+$, we have $1 < t_{\max}((u,v)) < t^-((u,v))$, since $t^-((u,v)) = \frac{1}{\|(u,v)\|} t^- \left(\frac{(u,v)}{\|(u,v)\|}\right)$, then $\mathcal{N}_\lambda^+ \subset W_1$. In particular, $(u_0, v_0) \in W_1$.

Now we claim that there exists $R_0 > 0$ such that $\left(u_0 + R_0 \alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R_0 \beta^{\frac{1}{p}} U_{\varepsilon,a}\right) \in W_2$. First, we find a constant $C > 0$ such that

$$0 < t^- \left(\left(\frac{(u_0 + R\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R\beta^{\frac{1}{p}} U_{\varepsilon,a})}{\|(u_0 + R\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R\beta^{\frac{1}{p}} U_{\varepsilon,a})\|} \right) \right) < C, \ \forall R > 0.$$

Otherwise, there exists a sequence $\{R_k\}$ such that, as $k \to \infty, R_k \to \infty$ and

$$t^- \left(\left(\frac{(u_0 + R\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R\beta^{\frac{1}{p}} U_{\varepsilon,a})}{\|(u_0 + R\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R\beta^{\frac{1}{p}} U_{\varepsilon,a})\|} \right) \right) \longrightarrow \infty.$$

Let

$$(\overline{u_k}, \overline{v_k}) = \frac{(u_0 + R_k\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R_k\beta^{\frac{1}{p}} U_{\varepsilon,a})}{\|(u_0 + R_k\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R_k\beta^{\frac{1}{p}} U_{\varepsilon,a})\|}.$$

Since $t^-((\overline{u_k}, \overline{v_k}))(\overline{u_k}, \overline{v_k}) \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$ and by the Dominated convergence Theorem,

$$\begin{aligned}
&\lim_{k\to\infty} \int_\Omega |\overline{u_k}(x,0)|^\alpha |\overline{v_k}(x,0)|^\beta dx \\
&= \lim_{k\to\infty} \frac{\displaystyle\int_\Omega |(u_0 + R_k\alpha^{\frac{1}{p}} U_{\varepsilon,a})(x,0)|^\alpha |v_0 + R_k\beta^{\frac{1}{p}} U_{\varepsilon,a})|^\beta dx}{\|(u_0 + R_k\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R_k\beta^{\frac{1}{p}} U_{\varepsilon,a})\|} \\
&= \lim_{k\to\infty} \frac{\displaystyle\int_\Omega |(\frac{u_0}{R_k} + \alpha^{\frac{1}{p}} U_{\varepsilon,a})(x,0)|^\alpha |(\frac{v_0}{R_k} + \beta^{\frac{1}{p}} U_{\varepsilon,a})(x,0)|^\beta dx}{\|(\frac{v_0}{R_k} + \alpha^{\frac{1}{p}} U_{\varepsilon,a}, \frac{v_0}{R_k} + \beta^{\frac{1}{p}} U_{\varepsilon,a})\|} \\
&= \frac{\displaystyle\int_\Omega \alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} U_{\varepsilon,a}^{p^*} dx}{\|(\alpha^{\frac{1}{p}} U_{\varepsilon,a}, \beta^{\frac{1}{p}} U_{\varepsilon,a})\|^{p^*}}.
\end{aligned}$$

Now

$$\begin{aligned}
&I(t^-((\overline{u_k}, \overline{v_k}))(\overline{u_k}, \overline{v_k})) \\
&= \tfrac{1}{p}(t^-((\overline{u_k}, \overline{v_k}))^p \|(\overline{u_k}, \overline{v_k})\|_p^p) + \tfrac{1}{q}(t^-((\overline{u_k}, \overline{v_k}))^q \|(\overline{u_k}, \overline{v_k})\|_q^q) \\
&- \tfrac{(t^-((\overline{u_k}, \overline{v_k})))^{1-r}}{1-r} \int_\Omega (h_1 \overline{u_k}^{1-r} + h_2 \overline{v_k}^{1-r}) dx - \frac{(t^-((\overline{u_k}, \overline{v_k})))^{p^*}}{p^*} \int_\Omega |\overline{u_k}(x,0)|^\alpha |\overline{v_k}(x,0)|^\beta dx \\
&\longrightarrow -\infty \quad as\ k \to \infty.
\end{aligned}$$

this contradicts that I is bounded below on $\mathcal{N}_\lambda$.

Let

$$R_0 = \frac{|c^p - \|(u_0, v_0)\|^p|^{\frac{1}{p}}}{\|(\alpha^{\frac{1}{p}} U_{\varepsilon,a}, \beta^{\frac{1}{p}} U_{\varepsilon,a})\|_p} + 1,$$

then

$$\begin{aligned}
&\|(\alpha^{\frac{1}{p}} U_{\varepsilon,a}, \beta^{\frac{1}{p}} U_{\varepsilon,a})\|_p^p \\
&> \|(u_0, v_0)\|^p + |c^p - \|(u_0, v_0)\|^p| + pR_0 \int_\Omega (\alpha^{\frac{1}{p}} \nabla u_0^{p-1} \nabla U_{\varepsilon,a} + \beta^{\frac{1}{p}} \nabla v_0^{p-1} \nabla U_{\varepsilon,a}) dx \\
&> c^p > t^- \left(\left(\frac{(u_0 + R\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R\beta^{\frac{1}{p}} U_{\varepsilon,a})}{\|(u_0 + R\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R\beta^{\frac{1}{p}} U_{\varepsilon,a})\|} \right) \right)^p,
\end{aligned}$$

that is $\left(u_0 + R_0 \alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R_0 \beta^{\frac{1}{p}} U_{\varepsilon,a}\right) \in W_2$. □

In this part, we assume

$$n > p, q < \frac{n(p-1)}{n-1}.$$

Proposition 4.2. *There exists $\varepsilon_3 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_3)$ there holds*

$$I\left(u_0 + tR\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + tR\beta^{\frac{1}{p}} U_{\varepsilon,a}\right) < I(u_0, v_0) + \frac{1}{n} (\frac{1}{\lambda})^{\frac{n-p}{p}} S_{\alpha,\beta}^{\frac{n}{p}}, \forall t \in [0, 1]$$

.

Proof. (u_0, v_0) is a solution of (1.1), we derive

$$\begin{aligned}
&I\left(u_0 + tR\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + tR\beta^{\frac{1}{p}} U_{\varepsilon,a}\right) \\
&= I(u_0, v_0) + \tfrac{1}{p}(tR)^p B_1 + \tfrac{1}{q}(tR)^q B_2 - \tfrac{\lambda}{p^*}(tR)^{p^*} A_1 \\
&-\tfrac{\lambda}{p^*} \alpha^{\frac{p+\alpha-1}{p}} \beta^{\frac{\beta}{p}} (tR)^{p^*-1} u_0(a) D\varepsilon^{\frac{n-p}{p}} - \tfrac{\lambda}{p^*} \alpha^{\frac{p+\alpha-1}{p}} \beta^{\frac{\beta}{p}} (tR)^{p^*-1} v_0(a) D\varepsilon^{\frac{n-p}{p}} + o\left(\varepsilon^{\frac{n-p}{p}}\right),
\end{aligned}$$

with $\displaystyle\int_{\mathbb{R}^n} \frac{u_0^{p-1}\eta}{\left(|x-a|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}} dx$ and

$$\begin{aligned}
&\int_\Omega \left[h_1(u_0 + tR\alpha^{\frac{1}{p}} U_{\varepsilon,a})^{1-r} + h_2(v_0 + tR\beta^{\frac{1}{p}} U_{\varepsilon,a})^{1-r}\right] dx - \int_\Omega (h_1 u_0^{1-r} + h_2 v_0^{1-r}) dx \\
&= (1-r)(tR_0) \int_\Omega (h_1 u_0^{-r} + h_2 v_0^{-r}) \frac{u_0^{p-1}\eta}{(|x-a|^{\frac{p}{p-1}})^{\frac{n-p}{p}}} dx \varepsilon^{\frac{n-p}{p}}.
\end{aligned}$$

Define

$$\begin{aligned}
q_\varepsilon(s) &= \tfrac{1}{p} s^p B_1 + \tfrac{1}{q} s^q B_2 - \tfrac{\lambda}{p^*} s^{p^*} A_1 \\
&-\lambda \alpha^{\frac{p+\alpha-1}{p}} \beta^{\frac{\beta}{p}} s^{p^*-1} u_0(a) D\varepsilon^{\frac{n-p}{p}} - \lambda \alpha^{\frac{\alpha}{p}} \beta^{\frac{\beta}{p}} s^{p^*-1} v_0(a) D\varepsilon^{\frac{n-p}{p}},
\end{aligned}$$

when $\varepsilon = 0$, q_0 attains its maximum in $[0, \infty]$ at R_0

$$q_0(s) = \frac{1}{p} s^p B_1 + \frac{1}{q} s^q B_2 - \frac{\lambda}{p^*} s^{p^*} A_1.$$

Let $s_0 = \left(\frac{B_1}{\lambda A_1}\right)^{\frac{1}{p^*-p}}$ and $s_\varepsilon > 0$ is the unique, such that $q_{s_\varepsilon} = \max\limits_{\forall s \geq 0} q(s)$, clearly, $s_\varepsilon \to s_0$ *as* $\varepsilon \to 0$. write $s_\varepsilon = s_0 + l_\varepsilon$ with $l_\varepsilon \to 0$, since $q'(s_\varepsilon) = 0$ it follows that

$$(s_0 + l_\varepsilon)^{p+1-p^*} B_1 + (s_0 + l_\varepsilon)^{q+1-p^*} B_2 - \lambda(s_0 + l_\varepsilon) A_1 + s_0^{p-1+p^*} B_1 - \lambda s_0 A_1$$

$$= \lambda \frac{p^*-1}{p^*} \alpha^{\frac{\alpha+p-1}{p}} \beta^{\frac{\beta}{p}} u_0(a) D \varepsilon^{\frac{n-p}{p}} - \lambda \frac{p^*-1}{p^*} \alpha^{\frac{\alpha}{p}} \beta^{\frac{p+\beta-1}{p}} v_0(a) D \varepsilon^{\frac{n-p}{p}},$$

so $l_\varepsilon = o\left(\varepsilon^{\frac{n-p}{p}}\right)$, consequently,

$$q(tR_0) \leq q(s_\varepsilon) \leq \frac{1}{p} s_0^p B_1 - \frac{\lambda}{p^*} s_0^{p^*} A_1 + o\left(\varepsilon^{\frac{n-p}{p}}\right) - \frac{1}{p}\left(\frac{B_1}{\lambda A_1}\right)^{\frac{p}{p^*-p}} B_1 - \frac{\lambda}{p^*}\left(\frac{B_1}{\lambda A_1}\right)^{\frac{p}{p^*-p}} A_1$$

$$= \frac{1}{n}\left(\frac{1}{\lambda}\right)^{\frac{n-p}{p}} S_{\alpha,\beta}^{\frac{n}{p}} + o\left(\varepsilon^{\frac{n-p}{p}}\right).$$

Therefore, for all $t \in [0,1]$, we have

$$I\left(u_0 + tR\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + tR\beta^{\frac{1}{p}} U_{\varepsilon,a}\right) < I(u_0, v_0) + \frac{1}{n}\left(\frac{1}{\lambda}\right)^{\frac{n-p}{p}} S_{\alpha,\beta}^{\frac{n}{p}} + o\left(\varepsilon^{\frac{n-p}{p}}\right),$$

then exists $0 < \varepsilon_3 (< \varepsilon_2)$ such that $\forall \varepsilon \in (0, \varepsilon_3)$

$$I\left(u_0 + tR\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + tR\beta^{\frac{1}{p}} U_{\varepsilon,a}\right) < I(u_0, v_0) + \frac{1}{n}\left(\frac{1}{\lambda}\right)^{\frac{n-p}{p}} S_{\alpha,\beta}^{\frac{n}{p}}, \quad \forall t \in [0,1].$$

□

Now,we locate (U_0, V_0). Since from Theorem 1.1 and Proposition 4.1 we have $(u_0, v_0) \in \mathcal{N}_\lambda^+ \subset W_1$ and $\left(u_0 + R_0\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R_0\beta^{\frac{1}{p}} U_{\varepsilon,a}\right) \in W_2$, there must exists $t_\varepsilon \in (0,1)$ such that $\left(u_0 + R_0\alpha^{\frac{1}{p}} U_{\varepsilon,a}, v_0 + R_0\beta^{\frac{1}{p}} U_{\varepsilon,a}\right) \in \mathcal{N}_\lambda^-$, and from Proposition 4.2 we derive that

$$\inf_{\mathcal{N}_\lambda^-} I < I(u_0, v_0) + \frac{1}{n}\left(\frac{1}{\lambda}\right)^{\frac{n-p}{p}} S_{\alpha,\beta}^{\frac{n}{p}}. \tag{4.3}$$

Moreover, since $(U_n, V_n), (U_0, V_0) \in \mathcal{N}_\lambda$, we clearly have

$$\begin{aligned} 0 = &\|(U_0, V_0)\|_p^p + \|(U_0, V_0)\|_q^q - \int_\Omega (h_1 U_0^{1-r} + h_2 V_0^{1-r}) dx - \lambda \int_\Omega U_0^\alpha V_0^\beta dx \\ &+ \|(\widetilde{U_n}, \widetilde{V_n})\|_p^p + \|(\widetilde{U_n}, \widetilde{V_n})\|_q^q - \lambda \int_\Omega \widetilde{U_n}^\alpha \widetilde{V_n}^\beta dx + o(1). \end{aligned} \tag{4.4}$$

The desire result $(U_n, V_n) \to (U_0, V_0)$ strong in X. Now follows with an argument by contradiction. In fact, suppose that there exists a subsequence $\{(U_{nk}, V_{nk})\}$ with $\|(\widetilde{U_{nk}}, \widetilde{V_{nk}})\| \geq C_3 > 0$ and from (4.4) that $\int_\Omega \widetilde{U_{nk}}^\alpha \widetilde{V_{nk}}^\beta dx \geq C$. Then (4.4) yields

$$\int_\Omega \widetilde{U_{nk}}^\alpha \widetilde{V_{nk}}^\beta dx > \left[\frac{S_{\alpha,\beta}}{\lambda} + o(1)\right]^{\frac{p^*}{p^*-p}}, \tag{4.5}$$

combining (4.3) and (4.5) we get

$$
\begin{aligned}
& I(u_0,v_0)+\frac{1}{n}(\frac{1}{\lambda})^{\frac{n-p}{p}}S_{\alpha,\beta}^{\frac{n}{p}} \\
& > I(U_{nk},V_{nk}) \qquad \left(since \quad I(U_n,V_n)\to \inf_{\mathcal{N}_\lambda^-} I \right) \\
& = I(U_0,V_0)+\left(\frac{1}{p}-\frac{1}{p^*}\right)\|(\widetilde{U_n},\widetilde{V_n})\|_p^p+\left(\frac{1}{q}-\frac{1}{p^*}\right)\|(\widetilde{U_n},\widetilde{V_n})\|_q^q+o(1) \\
& \geq \inf_{\mathcal{N}_\lambda} I+\left(\frac{1}{p}-\frac{1}{p^*}\right)\|(\widetilde{U_n},\widetilde{V_n})\|_p^p+\left(\frac{1}{q}-\frac{1}{p^*}\right)\|(\widetilde{U_n},\widetilde{V_n})\|_q^q+o(1) \\
& = I(u_0,v_0)+\left(\frac{1}{p}-\frac{1}{p^*}\right)\|(\widetilde{U_n},\widetilde{V_n})\|_p^p+\left(\frac{1}{q}-\frac{1}{p^*}\right)\|(\widetilde{U_n},\widetilde{V_n})\|_q^q+o(1) \\
& \geq I(u_0,v_0)+\frac{1}{n}\left(\frac{1}{\lambda}\right)^{\frac{n-p}{p}}S_{\alpha,\beta}^{\frac{n}{p}}+o(1),
\end{aligned}
$$

this is a contradiction. The gap structure of $\mathcal{N}_\lambda$ ensures that $(U_0,V_0)\in\mathcal{N}_\lambda^-$, therefore, (u_0,v_0) and (U_0,V_0) denote two different solutions for the problem (1.1).

This completes the proof of Theorem 1.2. □

Acknowledgments.

This research was supported by the Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi Province (No. 201802085), Innovative Research Team of North University of China (No. TD201901), Program for the Innovative Talents of Higher Education Institutions of Shanxi Province, the Fund for Shanxi 1331KIRT..

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