RELIABILITY ANALYSIS OF THE UNCERTAIN FRACTIONAL-ORDER DYNAMIC SYSTEM WITH STATE CONSTRAINT

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Abstract

Uncertain fractional-order differential equations driven by Liu process are of significance to depict the heredity and memory features of uncertain dynamical systems. This paper primarily investigates the reliability analysis of the uncertain fractional-order dynamic system with a state constraint. On the basis of the first-hitting time (FHT), a novel uncertain fractional-order dynamic system considering a state constraint is proposed. Secondly, in view of the relation between the initial state and the required standard, such uncertain fractional-order dynamic systems are subdivided into four types. The concept of reliability of proposed uncertain system with a state constraint is presented innovatively. Corresponding reliability indexes are ulteriorly formulated via FHT theorems. Lastly, the uncertain fractional-order dynamic system with a state constraint is applied to different physical and financial dynamical models. The analytic expression of the reliability index is derived to demonstrate the reasonableness of our model. Meanwhile, expected time response and American barrier option prices are calculated by using the predictor-corrector

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scheme. A sensitivity analysis is also illustrated with respect to various conditions.

Keywords: Reliability analysis; Fractional-order dynamic system; Uncertainty theory; State constraint; Caputo fractional-order derivative

1 Introduction

Reliability is an important index to measure the quality of products. Analysis of system reliability is to determine the function and reliability relationship between the system and components according to a large number of reliability data, to grasp the system's failure rule, and to find corresponding measures for the improvement and optimization of the system.

In the traditional reliability analysis, probability theory is usually used as the main mathematical tool to analyze the performance of the product by studying the characteristics of the system. Rice [1] proposed the first passage time formula, which laid the foundation of the dynamic reliability theory of the first passage damage. When applying the first passage time formula to the dynamic reliability, the integration is more complex and difficult to calculate. Therefore, the first passage time formula is only a conceptual sense of the method that can hardly be applied to the engineering practice. Subsequently, Siegert [2] proposed a new method to calculate the first passage probability for structures whose response is a continuous Markov process. Helstrom and Isley [3] deduced the analytical solution of the first passage time under the Markov envelope process. However, the above two methods are only applicable to special limit state functions. Coleman [4] obtained the Poisson approximation for the calculation of the first passage's frequency, which bridged the gap between the crossing rate and the dynamic reliability of the structure. However, the method is only accurate when the event that the structural response crosses from a safe state to a failure state is a random event, and the crossings are independent of each other. Numerical simulation is used to solve the first passage problem by Crandall et al. [5]. Spanos and Kougioumtzoglou [6] applied the first passage method to deal with a class of lightly damped nonlinear oscillators under broadband random excitations. Breitung [7], Schall et al. [8], Engelund et al. [9], Rackwitz [10] and Melchers [11] also all used the passage rate to solve the time-dependent dynamic reliability problem.

Unlike the prevailing view in the field of probability theory, it is argued that even if a large amount of data exists for study, the frequencies given by purely statistical methods are still not close to the true distribution function. In contrast, the advice given by experts through empirical and extra-field information is thought to be closer to the true pattern of the world than frequencies. Based on the above considerations, In 2007, Liu [12] established an uncertainty theory system that complements probability theory, and initially proposed uncertainty reliability analysis in the literature [12], defined reliability indexes, and gave the reliability calculation formulae for uncertain series, parallel, voting, and bridge systems. Based on the uncertainty theory, combined with the uncertainty variables, uncertainty distribution calculation method given by Liu [13], Peng [14] and others, under the assumption that the life of the system components is an uncertain variable, Gao [15] analyzed the k-out-of-n system with uncertain life. Gao et al. [16] gave the uncertainty-weighted k-out-of-n system. Gao and Yao [17] proposed the important index of the uncertain reliability system. Liu et al. [18] established some basic mathematical models of series, parallel and series-parallel systems based on the uncertain lifetime. Moveover, Liu [19] defined the first-hitting time in an uncertain system that can also measure the system reliability. Yao and Zhou [20] studied the reliability in an insurance risk process via criteria such as ruin index, ruin time, and deficit. On account of the system reliability of financial products, different kinds of barrier option pricing formulas were derived by Yang et al. [21], Tian et al. [22], and Gao et al. [23]. In the engineering field, Li et al. [24] introduced the uncertainty theory to account for such uncertainty due to small samples and build up a framework of accelerated degradation testing modeling to aid the reliability and lifetime evaluations for highly reliable products. Yu et al. [25] proposed an interest-rate model with jumps in uncertain financial markets. In 2020, Hu et al. [26] introduced a more appropriate choice for the demand in actual risk assessments [26]. Furthermore, the reliability index of financial derivatives in the uncertain market was also analyzed and calculated by Jin and Yang [27].

For ages, the classical integral theory has been a powerful tool for describing physical systems and processes. However, as mankind's understanding of nature has improved, many physical systems and processes display characteristics that cannot be explained by classical integral theory. In fact, due to special material properties or external conditions, many physical systems and processes tend to be represented dynamically in fractional-order. In recent years, fractional-order chaotic systems have attracted widespread interest and intensive research. In Chua's circuit [28], Chen system [29, 30], and Lu system [31], numerical simulations have shown that when the order of the system is fractional, the system still behaves chaotically or super chaotically and better reflects the physical phenomena presented by the system due to its genetic and memory properties. Inspired by the important theoretical significance and practical application of fractional-order control, in order to introduce randomness into fractional-order control theory and extend the fractional-order control method to randomly excited systems, scholars have turned their attention to fractional-order chaotic systems [32], and most of the relevant studies are based on probability theory for random systems. However, in cases where it is difficult for statistical methods to approach the true distribution function, professional advice from experienced experts on the degree of belief in the system is a more commonly used measure of reliability. Thus, we believe that uncertain theory is a more reasonable description of the belief degree of systems under the influence of human uncertainty and is a more appropriate theory for fractional-order control. Zhu [33] firstly defined two types of uncertain fractional-order differential equations (UFDEs), namely Riemann-Liouville and Caputo types.

Afterwards, Lu and Zhu [34] studied European option price models and derived corresponding formulas. Considering numerical methods, Lu and Zhu [35] introduced the α-path and obtaind UFDEs' numerical solutions in 2019. Considering performance analysis for the uncertain fractional-order system, Jin et al. [36] investigated the extreme value of the solutions to UFDEs for Caputo type and its application to the American options pricing problem. Meanwhile, Jin and Zhu [37] also presented first-hitting time theorems for UFDE, which had an application for the fractional risk index in 2020. Also, Lu et al.[38] and lu et al. [39] studied the the stability of uncertain fractional difference equations-order dynamic system.

In conclusion, reliability analysis is pivotal in practical engineering, while uncertain fractional-order differential equations provide a more realistic depiction of dynamic systems. In this paper, based on the previous work, we further study the performance analysis of the uncertain fractional-order dynamical system. By combing the two, we develop an uncertain fractional-order reliability index model based on a commonly used method of reliability analysis, first-hitting time, and finally applied it to both a fractional circuit system and an American option pricing problem. To our knowledge, this is the first reliability index model with such a widely used.

This paper will be composed by five sections. In Section 2, we recall a few important definitions as well as theorems on the uncertain theory. An uncertain fractional-order dynamical system with state constraint is given in Section 3, and a set of reliability indexes for the uncertain model is also developed. Specific applications of the model to both an uncertain fraction-order circuit system and the American barrier option pricing model are given in Section 4. A brief conclusion is summarized in the last section.

2 Preliminary

All the required concepts as well as conclusions on uncertain theory are revisited here. For more information, please refer to [40, 41, 42].

Unless otherwise stated in the following sections, we will always assume that a real positive number p satisfies $0 \leq n-1 < p \leq n$, and C_t is a Liu process. In addition, F and G are two continuous functions on $[0,T] \times \Re$.

2.1 Uncertain fractional-order differential equation

In 2015, two styles of UFDEs are proposed by Zhu [33], while Ford and Simpson [43] and Diethelm et al. [44] indicated the fractional-order derivative in Caputo sense has more advantages than Riemann-Liouville sense for modeling the real dynamic process. Therefore, we only focus on the Caputo type UFDEs that takes the following form,

$$\begin{cases} {}^cD^pX_t = F(t, X_t) + G(t, X_t)\frac{dC_t}{dt} \\ X_t^{(l)} \mid_{t=0}= x_t, l = 0, 1, \ldots, n-1. \end{cases} \tag{1}$$

where ${}^cD^p$ is the standard Caputo type fractional derivative, which is expressed as

$${}^cD^pf(t) = \frac{1}{\Gamma(n-p)} \int_0^t (t-s)^{n-p-1} f^{(n)}(s)ds. \tag{2}$$

According to Equation (2), Lu and Zhu [34] derived the solution of Equation (1), which is an integral equation

$$\begin{aligned} X_t &= \sum_{k=0}^{n-1} \frac{x_k t^k}{\Gamma(k+1)} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} F(s, X_s)ds \\ &\quad + \frac{1}{\Gamma(p)} \int_0^t G(s, X_s)(t-s)^{p-1} dC_s, \end{aligned} \tag{3}$$

where $\Gamma(\cdot)$ is the Gamma function, $\Gamma(p) = \int_0^\infty t^{p-1}\exp(-t)dt$, and n is the smallest integer greater than or equal to p.

Specifically, let $F(t, X_t) = AX_t + B(t)$, $G(t, X_t) = \sigma(t)$, Equation (1) can be transformed into

$$\begin{cases} {}^cD^pX_t = AX_t + B(t) + \sigma(t)\frac{dC_t}{dt} \\ X_t^{(k)} \mid_{t=0}= x_t, k = 0, 1, \ldots, n-1. \end{cases} \tag{4}$$

Zhu and Lu [34] also deduced the expression of the solution of Equation (4) by Mittag-Leffler function

$$\begin{aligned} X_t &= \sum_{k=0}^{n-1} x_k t^k E_{p,(k+1)}(At^p) + \int_0^t (t-s)^{p-1} E_{p,q}(A(t-s)^p)B(s)ds \\ &\quad + \int_0^t (t-s)^{p-1} Ep,q(A(t-s)^p)\sigma(s)dC_s, \end{aligned} \tag{5}$$

where $E_{p,q}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(kp+q)}$.

Subsequently, Lu and Zhu [35] attempted to introduce the concept of α-path to build the relationship between UFDEs and FDEs to obtain the numerical solution of Equation (1). The FDEs corresponding to UDEs are

$$\begin{cases} {}^cD^pX_t^\alpha = F(t, X_t^\alpha) + \mid G(t, X_t^\alpha) \mid \Phi^{-1}(\alpha) \\ X_t^{(k)} \mid_{t=0}= x_t, k = 0, 1, \ldots, n-1, \end{cases} \tag{6}$$

where $0 < \alpha < 1, \Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$. Then, based on the fact that X_t and X_t^α are solutions and α-path of Equation (1), the equivalent relationship between the two is

$$\begin{cases} \mathcal{M}\{X_t \leq X_t^\alpha, \forall t \in [0,T]\} = \alpha \\ \mathcal{M}\{X_t > X_t^\alpha, \forall t \in [0,T]\} = 1-\alpha. \end{cases} \tag{7}$$

Lu [35] proposed the IUD of X_t as follows,

$$\Psi_t^{-1}(\alpha) = X_t^\alpha, \tag{8}$$

Particularly, when UFDEs are formed as follows,

$$\begin{cases} {}^cD^p X_t = (a - bX_t) + \sigma \frac{dC_t}{dt} \\ X_t^{(l)} \mid_{t=0} = x_t, l = 0, 1, \ldots, n-1. \end{cases} \tag{9}$$

the solution to corresponding fractional differential equations (FDEs) is given by Jin and Zhu [45],

$$X_t^\alpha = \sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}(-bt^p) + (a + \sigma\Phi^{-1}(\alpha)) t^p E_{p,p+1}(-bt^p). \tag{10}$$

2.2 First-hitting time

Pre-set a value z, the first-hitting time τ_z when X_t hits z is defined as a novel uncertainty variable. Liu [19] firstly gave the first-hitting time

$$\tau_z = \inf\{t \geq 0 \mid X_t = z\}. \tag{11}$$

The ruin time τ when the total capital X_t hits zero, actually first-hitting time was propose by Yao and Zhou [46] in 2016, of which the following uncertainty distribution for an uncertain insurance model goes

$$\Upsilon(t) = \max_{l \geq 1} \sup_{x \leq t} \Phi\left(\frac{x}{l}\right) \wedge \left(1 - \Psi\left(\frac{a+bx}{l}\right)\right)$$

where $X_t = a + bt - R_t$ represents insurance risk process, R_t denotes renewal reward process with iid uncertain interatrial times $\xi_1, \xi_2, \cdots$ and iid uncertain claim accounts $\eta_1, \eta_2, \cdots$. Meanwhile, uncertain vectors $(\xi_1, \xi_2, \cdots)$ as well as $(\eta_1, \eta_2, \cdots)$ are independent of each other, and corresponding uncertainty distributions are Φ as well as Ψ.

In 2020, the first-hitting time was introduced to the UFDE's solution (1) by the tool of the extreme value theorem [45] as well as α-path by Jin and Zhu [37]. When $J(x)$ is monotonically increasing, then the first-hitting time τ_z that $J(X_t)$ hits z takes the uncertainty distribution as follows,

$$U(s)=\begin{cases} 1-\inf\left\{\alpha\in(0,1)\mid \sup\limits_{0\le t\le s} J(X_t^\alpha)\ge z\right\}, & \textit{if } z>J(x_0)\\ \sup\left\{\alpha\in(0,1)\mid \inf\limits_{0\le t\le s} J(X_t^\alpha)\le z\right\}, & \textit{if } z<J(x_0). \end{cases} \tag{12}$$

When $J(x)$ is monotonically decreasing, the first-hitting time τ_z when $J(X_t)$ hits z subjects to the following distribution,

$$\overline{U}(s)=\begin{cases} \sup\left\{\alpha\in(0,1)\mid \sup\limits_{0\le t\le s} J(X_t^\alpha)\ge z\right\}, & \textit{if } z>J(x_0)\\ 1-\inf\left\{\alpha\in(0,1)\mid \inf\limits_{0\le t\le s} J(X_t^\alpha)\le z\right\}, & \textit{if } z<J(x_0). \end{cases} \tag{13}$$

3 Reliability of uncertain fractional-order dynamic system with state constraint

Uncertain fractional-order dynamical model (1) we studied has certain flaws in solving the problem of a system crash, so further improvement is needed. Taking the risky assets such as options as an example, no matter how the price of the underlying asset changes, the model (1) ignores the change of the nature of the option on the expiration date, and excludes the belief degree that the holder loses the ability to exercise due to the influence of some conditions. Therefore, it is urgent and meaningful for us to measure the reliability of uncertain fractional-order dynamical systems, and a novel uncertain fractional-order dynamical system considering state constraints is introduced in this section.

3.1 Reliability for uncertain fractional-order dynamical system for $\sup(\inf)X_t \ge L$

In practice, there is always a tendency to assume that the system in which the model is located is sufficiently reliable only if an indicator is above a certain value. For example, the temperature must be higher than the melting point of the water tap in order for the water to flow, the selling price must be higher than the cost to be profitable, the voltage must be higher than the starting voltage circuits in order for them to function properly, and so on. This involves the time at which the indicator first penetrates a given value. Based on this consideration, we define the following novel uncertain fractional-order dynamic system for $\sup(\inf)X_t \ge L$.

$$\begin{cases} {}^cD^pX_t=F(t,X_t)+G(t,X_t)\frac{dC_t}{dt}\\ X^{(k)}(0)=x_k, k=0,1,\ldots,n-1\\ \sup\limits_{0\le t\le T}(\inf)X_t\ge L \end{cases} \tag{14}$$

where L is a pre-given level, T is the maturity time, and X_t illustrates the specified performance of a system. Since the uncertain fractional-order dynamical model (14) we defined has specific conditions about uncertain variable X_t, the operation ability of our uncertain fractional-order systems is need to be discussed. Hence, the reliability index (Rel), which can measure such operation ability, will be discussed for model (14). According to the uncertain fraction-order dynamical system (14) for $\sup\limits_{0\le t\le T}(\inf) X_t \ge L$, we define reliability index of it here.

Theorem 3.1 (Reliability index) *Reliability index for the the uncertain fraction-order dynamical system* (14) *for* $\sup\limits_{0\le t\le T}(\inf) X_t \ge L$ *is*

$$Rel = \mathcal{M}\left\{\sup_{0\le t\le T} X_t \ge L\right\} = 1-\beta, \quad X_0 < L \tag{15}$$

where $\beta = \inf\left\{\alpha \mid \sup\limits_{0\le t\le T} X_t^\alpha \ge L\right\}$. *Comparably,*

$$Rel = \mathcal{M}\left\{\inf_{0\le t\le T} X_t \ge L\right\} = 1-\beta, \quad X_0 > L \tag{16}$$

where $\beta = \sup\left\{\alpha \mid \inf\limits_{0\le t\le T} X_t^\alpha \le L\right\}$.

Proof: When $X_0 < L$, according to Equation (11), it is obvious that there is an equivalence relation between $\left\{\sup\limits_{0\le t\le T} X_t \ge L\right\}$ and $\{\tau < T\}$, which can derive

$$Rel = \mathcal{M}\left\{\sup_{0\le t\le T} X_t \ge L\right\} = \mathcal{M}\{\tau < T\}. \tag{17}$$

Then, set

$$\beta = \inf\left\{\alpha \mid \sup_{0\le t\le T} X_t^\alpha \ge L\right\}.$$

By Equation (12) for first-hitting time theorems [37], it can be concluded

$$\mathcal{M}\{\tau < T\} = U(T) = 1-\beta.$$

Besides, $U(T)$ indicates DF (distribution function) for uncertain variable τ.

When $X_0 > L$, according to Equation (11), it is obvious that there is an equivalence relation between $\left\{\inf\limits_{0\le t\le T} X_t \ge L\right\}$ and $\{\tau \ge T\}$, which can derive

$$Rel = \mathcal{M}\left\{\inf_{0\le t\le T} X_t \ge L\right\} = \mathcal{M}\{\tau \ge T\}. \tag{18}$$

Then, set

$$\beta = \sup\left\{\alpha \mid \inf_{0\le t\le T} X_t^\alpha \le L\right\}.$$

By Equation (12) for first-hitting time theorems [37], it can be concluded

$$\mathcal{M}\{\tau \geq T\} = 1 - U(T) = 1 - \beta.$$

Besides, $U(T)$ indicates DF for uncertain variable τ. The proof is completed.

3.2 Reliability for uncertain fractional-order dynamical model for $\sup(\inf) X_t \leq L$

In other cases, the system where the model resides is specified as reliable enough only when the indicator is below a certain value. For example, the stress on materials should be less than the permissible stress, the circuit voltage should be less than the rated voltage, and the speed of vehicles should be less than the road speed limit, etc. This requires us to consider another kind of uncertain fractional-order dynamical model considering state constraints.

$$\begin{cases} {}^{c}D^{p} X_t = F(t, X_t) + G(t, X_t) \frac{dC_t}{dt} \\ X^{(k)}(0) = x_k, k = 0, 1, \ldots, n-1 \\ \sup\limits_{0 \leq t \leq T}(\inf) X_t \leq L \end{cases} \tag{19}$$

where L is a pre-given level, T is the maturity time. Obviously, such two kinds of the uncertain fractional-order dynamical system with state constraint we defined above reflect different conditions that the uncertain variable need to satisfy for the actual demand, are thus has guiding significance for practical applications.

Analogously, according to the uncertain fraction-order dynamical system (19) for $\sup\limits_{0 \leq t \leq T}(\inf) X_t \leq L$, we define the reliability index of it here.

Theorem 3.2 (Reliability index) *Reliability index for the uncertain fraction-order dynamical system* (19) *for* $\sup\limits_{0 \leq t \leq T}(\inf) X_t \leq L$ *is*

$$Rel = \mathcal{M}\left\{ \sup_{0 \leq t \leq T} X_t < L \right\} = \beta, \quad X_0 < L \tag{20}$$

where $\beta = \inf \left\{ \alpha \mid \sup\limits_{0 \leq t \leq T} X_t^{\alpha} \geq L \right\}$.

Comparably,

$$Rel = \mathcal{M}\left\{ \inf_{0 \leq t \leq T} X_t \leq L \right\} = \beta, \quad X_0 > L \tag{21}$$

where $\beta = \sup \left\{ \alpha \mid \inf\limits_{0 \leq t \leq T} X_t^{\alpha} \leq L \right\}$.

Proof: When $X_0 < L$, according to Equation (11), it is obvious that there is an equivalence relation between $\left\{ \sup_{0 \le t \le T} X_t < L \right\}$ and $\{\tau \ge T\}$, which can derive

$$Rel = \mathcal{M}\left\{ \sup_{0 \le t \le T} X_t < L \right\} = \mathcal{M}\left\{ \tau \ge T \right\}. \tag{22}$$

Then, set

$$\beta = \inf\left\{ \alpha \mid \sup_{0 \le t \le T} X_t^\alpha \ge L \right\}.$$

By Equation (12) for first-hitting time theorems [37], it can be concluded

$$\mathcal{M}\left\{ \tau \ge T \right\} = 1 - U(T) = \beta.$$

Besides, $U(T)$ indicates DF for uncertain variable τ.

When $X_0 > L$, according to Equation (11), it is obvious that there is an equivalence relation between $\left\{ \inf_{0 \le t \le T} X_t \le L \right\}$ and $\{\tau < T\}$, which can derive that

$$Rel = \mathcal{M}\left\{ \inf_{0 \le t \le T} X_t \le L \right\} = \beta. \tag{23}$$

Then, set

$$\beta = \sup\left\{ \alpha \mid \inf_{0 \le t \le T} X_t^\alpha \le L \right\}.$$

By Equation (12) for first-hitting time theorems [37], it can be concluded that

$$\mathcal{M}\left\{ \tau < T \right\} = U(T) = \beta.$$

Besides, $U(T)$ indicates DF for uncertain variable τ. The proof is completed.

4 Application

Since fractional-order calculus, thanks to its non-locality features to reflect genetic and memory properties, can more accurately describe the essential properties and dynamic behavior of dynamic processes. We will apply proposed uncertain fractional-order dynamical systems with state constraint into the physical and finance field. Two novel models based on uncertain fractional-order differential equations will be introduced in this section. The reliability index and some specific formulas of the model will be given accordingly.

4.1 Reliability analysis for fractional-order physical dynamical system with state contraint

As mentioned before, the regular use of a circuit considering the full response requires a voltage that is higher than the start voltage and lower than the rated voltage. At the same time, the circuit system also has two opposite trends: for the first one, the circuit goes to failure during discharge or use, and for the second, the circuit recovers during charging or maintenance. This makes it important to apply the above model to the fractional-order circuit system. The following contents give the application of the reliability analysis in fractional Resistor-Capacitance (RC) circuits.

4.1.1 Upward repairable fractional-order circuit system

First, considering the influence of uncertainties such as consistency differences in the manufacture of capacitors and resistive components, circuit connection stability, operator operation time errors, as well as external electromagnetic environment interference, we developed a state equation based on fractional-order RC circuits.

$$\begin{cases} {}^{c}D^{p}X_t = -\frac{X_t}{RC} + \frac{w}{RC} + \frac{b}{RC}\frac{dC_t}{dt} \\ X^{(k)}(0) = x_k, k = 0, 1, \ldots, n-1. \end{cases} \tag{24}$$

When $X_0 < L$, the circuit system is in non-operational condition, and it is necessary to charge the capacitor to make the capacitor voltage greater than the supply voltage, that is $\sup\limits_{0\le t\le T} X_t \ge L$, so that the fractional-order circuit can be used for its intended purpose. Hence, introducing the state equation (24) into our uncertain fractional-order dynamical system (14), which can be obtained as below

$$\begin{cases} {}^{c}D^{p}X_t = -\frac{X_t}{RC} + \frac{w}{RC} + \frac{b}{RC}\frac{dC_t}{dt} \\ X^{(k)}(0) = x_k, k = 0, 1, \ldots, n-1 \\ \sup\limits_{0\le t\le T} X_t \ge L \end{cases} \tag{25}$$

for which, according to Equation (10), the α-path of the solution can be derived as follows,

$$X_t^{\alpha} = \sum_{k=0}^{n-1} x_k \cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right) + \left(\frac{\omega}{RC} + \frac{b}{RC}\Phi^{-1}(\alpha)\right) t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right).$$

Meanwhile, since the fractional-order RC circuit is disturbed by uncertain factors. It is innovative to describe the physical process by the expected time response, where the circuit output voltage is represented by its expected value. The reliability index and expected time response are thus given in the following theorem.

Theorem 4.1 *The reliability index that the fractional-order RC system* (25) *can operate has been*

$$Rel = 1-\inf\left\{\alpha \mid \sup_{0\le t\le T}\sum_{k=0}^{n-1} x_k\cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right)+\left(\frac{\omega}{RC}+\frac{b}{RC}\Phi^{-1}(\alpha)\right)t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right)\ge L\right\},$$

and the expected time response goes

$$\int_{1-Rel}^{1}\sum_{k=0}^{n-1} x_k\cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right)+\left(\frac{\omega}{RC}+\frac{b}{RC}\Phi^{-1}(\alpha)\right)t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right)d\alpha.$$

Proof: First, according to Theorem 3.1, when $X_0 < L$,

$$\begin{aligned}\beta &= \inf\left\{\alpha \mid \sup_{0\le t\le T} X_t^\alpha \ge L\right\}\\ &= \inf\left\{\alpha \mid \sup_{0\le t\le T}\sum_{k=0}^{n-1} x_k\cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right)+\left(\frac{\omega}{RC}+\frac{b}{RC}\Phi^{-1}(\alpha)\right)t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right)\ge L\right\}.\end{aligned}$$

Thus, we can obtain that

$$\begin{aligned}Rel &= \mathcal{M}\left\{\sup_{0\le t\le T} X_t \ge L\right\}\\ &= 1-\beta\\ &= 1-\inf\left\{\alpha \mid \sup_{0\le t\le T}\sum_{k=0}^{n-1} x_k\cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right)+\left(\frac{\omega}{RC}+\frac{b}{RC}\Phi^{-1}(\alpha)\right)t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right)\ge L\right\}.\end{aligned}$$

Then, define an indicator function

$$I_L\left(\sup_{0\le t\le T} X_t\right)=\begin{cases}1, & if \quad \sup_{0\le t\le T} X_t \ge L\\ 0, & if \quad \sup_{0\le t\le T} X_t < L,\end{cases}$$

and set

$$\Lambda_1^+ = \left\{I_L\left(\sup_{0\le t\le T} X_t\right)\cdot X_t \le I_L\left(\sup_{0\le t\le T} X_t^\alpha\right)\cdot X_t^\alpha\right\},$$

$$\Lambda_1^- = \left\{I_L\left(\sup_{0\le t\le T} X_t\right)\cdot X_t > I_L\left(\sup_{0\le t\le T} X_t^\alpha\right)\cdot X_t^\alpha\right\}.$$

It is obviously that

$$\Lambda_1^+ \supset \left\{\sup_{0\le t\le T} X_t \le \sup_{0\le t\le T} X_t^\alpha, X_t \le X_t^\alpha\right\}\supset\left\{X_t \le X_t^\alpha, \forall t\right\}. \tag{26}$$

and

$$\Lambda_1^- \supset \left\{\sup_{0\le t\le T} X_t > \sup_{0\le t\le T} X_t^\alpha, X_T > X_T^\alpha\right\}\supset\left\{X_t > X_t^\alpha, \forall t\right\}. \tag{27}$$

Following from Equation (7) and

$$\mathcal{M}\left\{\Lambda_1^+\right\}+\mathcal{M}\left\{\Lambda_1^-\right\}=1, \tag{28}$$

we derive that

$$\mathcal{M}\left\{\Lambda_1^+\right\}=\alpha.$$

That means, the uncertain variable $I_L\left(\sup\limits_{0\le t\le T} X_t\right)\cdot X_t$ has an IUD

$$I_L\left(\sup_{0\le t\le T} X_t^\alpha\right)\cdot X_t^\alpha. \tag{29}$$

Finally, since

$$E(\xi)=\int_0^1 \Gamma^{-1}(\alpha)d\alpha$$

we can obtain that

$$\begin{aligned}
E(X_t) &= E\left[I_L\left(\sup_{0\le t\le T} X_t\right)\cdot X_t\right]\\
&= \int_0^1 I_L\left(\sup_{0\le t\le T} X_t^\alpha\right)\cdot X_t^\alpha d\alpha\\
&= \int_{1-Rel}^1 X_t^\alpha d\alpha\\
&= \int_{1-Rel}^1 \sum_{k=0}^{n-1} x_k\cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right)+\left(\frac{\omega}{RC}+\frac{b}{RC}\Phi^{-1}(\alpha)\right)t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right)d\alpha.
\end{aligned}$$

The proof is complete.

Jin et al. [36] pointed out that predictor-corrector scheme [47] has been a useful as well as effective scheme to solve FDEs. Hence, by using it, we can calculate the reliability index and expected time response for the uncertain fractional-order RC circuit model (25).

Example 4.1 *Assume an uncertain fractional-order RC circuit model* (25) *has current voltage* $x_0=0$*,* $x_1=1$*, resistance* $R=5$*, and capacitance* $C=1$*. Furthermore, the log-diffusion* $\sigma=0.5$ *and* $p=1.6$*. Consider the upper bound of maturity time is* $T=5$ *and the pre-given level* $L=4$*.*

Table 1 indicates that when pre-given level L is lower (expiration time T is longer), then Rel is bigger. The above phenomenon conforms to the fact: For a lower pre-given level L (longer expiration time T). The pre-given level has more possibilities to be hit, Rel is thus higher. Meanwhile, Figure 1 shows that the expected time response is increasing with respect to time t.

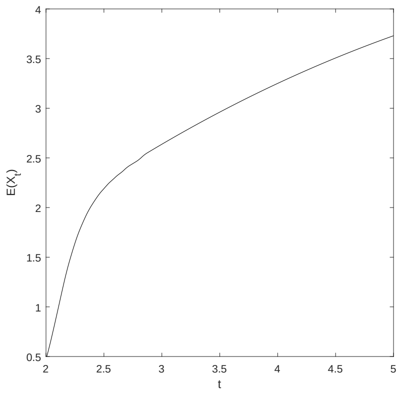

Figure 1: The expected time response X_t with uncertain factors (upward repairable)

Downward repairable fractional-order circuit system

Analogously, when $X_0 > L$, the circuit system is also in non-operational condition, and it is necessary to discharge the capacitor to make the capacitor voltage smaller than the pre-given voltage, that is $\inf_{0\leq t\leq T} X_t \leq L$, so that the fractional-order circuit can be used for its intended purpose. Hence, introducing the state equation (24) into our fractional-order dynamical system (19), we obtain that

$$\begin{cases} {}^cD^pX_t = -\frac{X_t}{RC} + \frac{w}{RC} + \frac{b}{RC}\frac{dC_t}{dt} \\ X^{(k)}(0) = x_k, k = 0, 1, \ldots, n-1 \\ \inf\limits_{0\leq t\leq T} X_t \leq L. \end{cases} \tag{30}$$

Table 1: Sensitivity analysis of the reliability index (upward repairable)

T	1.3	1.6	1.9	2.2	2.5	2.8	3.1	3.4	3.7	4.0	4.3
Rel	0.00	0.0294	0.1715	0.4598	0.7444	0.8978	0.9534	0.9733	0.9819	0.9873	0.9900
L	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
Rel	0.9900	0.9844	0.9468	0.8148	0.5648	0.2824	0.0912	0.0164	0.0016	0.0000	0.0000

Theorem 4.2 *The reliability index that the first-hitting time model* (30) *can operate has been*

$$Rel = \sup\left\{\alpha \mid \inf_{0\leq t\leq T}\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right) + \left(\frac{\omega}{RC} + \frac{b}{RC}\Phi^{-1}(\alpha)\right) t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right) \leq L\right\},$$

and the expected time response goes

$$\int_0^{Rel}\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right) + \left(\frac{\omega}{RC} + \frac{b}{RC}\Phi^{-1}(\alpha)\right) t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right) d\alpha.$$

Proof: First, according to Theorem 3.2, when $X_0 > L$,

$$\begin{aligned}
\beta &= \sup\left\{\alpha \mid \inf_{0\le t\le T} X_t^\alpha \le L\right\} \\
&= \sup\left\{\alpha \mid \inf_{0\le t\le T} \sum_{k=0}^{n-1} x_k \cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right) + \left(\frac{\omega}{RC} + \frac{b}{RC}\Phi^{-1}(\alpha)\right) t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right) \le L\right\},
\end{aligned}$$

thus we can obtain

$$\begin{aligned}
Rel &= \mathcal{M}\left\{\inf_{0\le t\le T} X_t \le L\right\} \\
&= \beta \\
&= \sup\left\{\alpha \mid \inf_{0\le t\le T} \sum_{k=0}^{n-1} x_k \cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right) + \left(\frac{\omega}{RC} + \frac{b}{RC}\Phi^{-1}(\alpha)\right) t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right) \le L\right\}.
\end{aligned}$$

Then, define another indicator function

$$I_L\left(\inf_{0\le t\le T} X_t\right) = \begin{cases} 1, & if \quad \inf_{0\le t\le T} X_t \le L \\ 0, & if \quad \sup_{0\le t\le T} X_t > L, \end{cases}$$

and set

$$\Lambda_2^+ = \left\{I_L\left(\inf_{0\le t\le T} X_t\right) \cdot X_t \le I_L\left(\inf_{0\le t\le T} X_t^\alpha\right) \cdot X_t^\alpha\right\},$$

$$\Lambda_2^- = \left\{I_L\left(\inf_{0\le t\le T} X_t\right) \cdot X_t > I_L\left(\inf_{0\le t\le T} X_t^\alpha\right) \cdot X_t^\alpha\right\}.$$

It is obviously that

$$\Lambda_2^+ \supset \left\{\inf_{0\le t\le T} X_t \le \inf_{0\le t\le T} X_t^\alpha, X_t \le X_t^\alpha\right\} \supset \left\{X_t \le X_t^\alpha, \forall t\right\}. \tag{31}$$

and

$$\Lambda_2^- \supset \left\{\inf_{0\le t\le T} X_t > \inf_{0\le t\le T} X_t^\alpha, X_T > X_T^\alpha\right\} \supset \left\{X_t > X_t^\alpha, \forall t\right\}. \tag{32}$$

Following from Equation (7) and

$$\mathcal{M}\left\{\Lambda_2^+\right\} + \mathcal{M}\left\{\Lambda_2^-\right\} = 1, \tag{33}$$

we derive that

$$\mathcal{M}\left\{\Lambda_2^+\right\} = \alpha.$$

That means, the uncertain variable $I_L\left(\inf_{0\le t\le T} X_t\right)\cdot X_t$ has an IUD

$$I_L\left(\inf_{0\le t\le T} X_t^\alpha\right)\cdot X_t^\alpha. \tag{34}$$

Finally, since

$$E(\xi)=\int_0^1 \Gamma^{-1}(\alpha)d\alpha,$$

we can obtain that

$$\begin{aligned}
E(X_t) &= E\left[I_L\left(\inf_{0\le t\le T} X_t\right)\cdot X_t\right]\\
&= \int_0^1 I_L\left(\inf_{0\le t\le T} X_t^\alpha\right)\cdot X_t^\alpha d\alpha\\
&= \int_0^{Rel} X_t^\alpha d\alpha\\
&= \int_0^{Rel} \sum_{k=0}^{n-1} x_k\cdot t^k E_{p,(k+1)}\left(-\frac{t^p}{RC}\right)+\left(\frac{\omega}{RC}+\frac{b}{RC}\Phi^{-1}(\alpha)\right)t^p E_{p,p+1}\left(-\frac{t^p}{RC}\right)d\alpha
\end{aligned}$$

The proof is completed.

Example 4.2 *Assume an uncertain fractional-order RC circuit model* (30) *has current voltage* $x_0=5$, $x_1=-1$, *resistance* $R=4$, *and capacitance* $C=2$. *Furthermore, the log-diffusion* $\sigma=0.5$ *and* $p=1.6$. *Consider the upper bound of maturity time is* $T=2$ *and the pre-given level* $L=2$.

Analogously, by using predictor-corrector numerical method [47], we calculate *Rel* and expected time response for the uncertain fractional-order RC circuit model (30). Table 2 indicates that when barrier level L (expiration time T) is higher, *Rel* is bigger. The above phenomenon conforms to the fact: For a bigger pre-given level L (longer expiration time T). The pre-given level has more possibilities to be hit, *Rel* is thus higher. Meanwhile, Figure 2 shows that the expected time response is changing with respect to time t.

Table 2: Sensitivity analysis of the reliability index (downward repairable)

T	1.90	1.96	2.02	2.08	2.14	2.20	2.26	2.32	2.38	2.44	2.50
Rel	0.0100	0.0540	0.1584	0.3488	0.5820	0.7720	0.8844	0.9412	0.9684	0.9808	0.9900
L	1.80	1.90	2.00	2.10	2.20	2.30	2.40	2.50	2.60	2.70	2.80
Rel	0.0100	0.0540	0.1584	0.3488	0.5820	0.7720	0.8844	0.9412	0.9684	0.9808	0.9900

Remark 4.1 *The resulting* $E(X_t)$ *is the weight mean of* X_t *and is a more accurate indicator in practical applications of characterizing the voltage of a circuit system under the influence of uncertainty.*

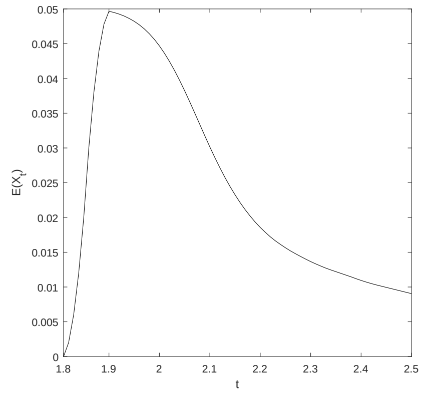

Figure 2: The expected time response X_t with uncertain factors (downward repairable)

Remark 4.2 *We find that fractional derivative p affects reliability index and expected time response formulas for the fractional-order circuit systems* (25) *and* (30)*.*

Remark 4.3 *We extend the reliability analysis of the uncertain second-order circuit system (Jin et al. [48]) to the case of fractional-order circuit with a wider application range.*

4.2 Reliability analysis for fractional-order financial dynamical system with state constraint

Barrier options, which are financial derivatives with high popularity in the financial market, stems from the demand for traders to delineate risks, alleviate market speculation, avoid bid-up stock prices, and so on. In 2020, Yao and Qin [49] took the structure of the solutions of uncertain differential equations as a basis, constructed an uncertain stock model to describe barrier options. However, since the validity of the option is in connection with whether the price of the underlying asset reaches the pre-set barrier (ceiling or floor) during the renewal period, the system will only be activated or extinguished when the barrier mechanism acts. Thus, it is meaningful for us to apply the uncertain fractional-order dynamical system and reliability index in the area of barrier options.

4.2.1 Reliability Analysis of American Barrier Call Option

Typically, discuss several class American call options whose execution is subjected to the barrier mechanism $\sup\limits_{0\le t\le T}(\inf)X_t \ge L$. In uncertain financial markets, investors give priority to purchasing such options in anticipation of the uncertain events $\{\sup\limits_{0\le t\le T}(\inf)X_t \ge L\}$ happen. The above uncertain fraction-order dynamical system (14) can be further transformed into

$$\begin{cases} dY_t = rY_t dt \\ {}^cD^p X_t = aX_t + bX_t \frac{dC_t}{dt} \\ X^{(k)}(0) = x_k, k = 0, 1, \ldots, n-1 \\ \sup\limits_{0 \le t \le T}(\inf) X_t \ge L. \end{cases} \tag{35}$$

Where Y_t is used to represent the bond price, and X_t is set as an uncertain process to describe the dynamic change of the underlying asset price, C_t denotes the Liu process, r, a, and b demonstrate the interest rate, diffusion, and drift, respectively.

To verify the applicability and feasibility of the uncertain fraction-order model (35) in the financial field, the following content selects American up-and-in call options and American down-and-out call options as the main analysis objects, respectively derives corresponding reliability indicators and supplements the corresponding pricing formula.

up-and-in call option

Analyze the reliability of American up-and-in call options. According to the barrier mechanism, only when the price of the underlying asset exceeds the predetermined ceiling, the option can realize the transition from invalid to valid state, and then has the possibility of being executable, which means that the operation of the uncertain fractional-order dynamical system (35) is required to meet the conditions that $\{\sup\limits_{0 \le t \le T} X_t \ge L\}$. Then, the corresponding reliability index of American up-and-in call option can be deduced as follows.

Theorem 4.3 *Consider an uncertain fraction-order dynamical system* (35)*, which has a strike price K and a barrier price L. Set $X_0 < L$, then the reliability index for American up-and-in call option is*

$$Rel = 1 - \inf\left\{\alpha \mid \sup_{0 \le t \le T}\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) \ge L\right\}. \tag{36}$$

and the price for the American up-and-in call option goes

$$f_{ui}^c = \int_{1-Rel}^{1} \sup_{0 \le t \le T} \exp\left(-rt\right)\left(\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) - K\right)^+ d\alpha. \tag{37}$$

Proof: Jin and Zhu [36] derived the α-path for solution (35) as below,

$$X_t^\alpha = \sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right). \tag{38}$$

According to Theorem 3.1, when $X_0 < L$, let

$$\begin{aligned}\beta &= \inf\left\{\alpha \mid \sup_{0\le t\le T} X_t^\alpha \ge L\right\}\\ &= \inf\left\{\alpha \mid \sup_{0\le t\le T}\left(\sum_{k=0}^{n-1} x_k\cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) \ge L\right\}\end{aligned}$$

thus we can obtain

$$\begin{aligned}Rel &= \mathcal{M}\left\{\sup_{0\le t\le T} X_t \ge L\right\}\\ &= 1-\beta\\ &= 1-\inf\left\{\alpha \mid \sup_{0\le t\le T}\left(\sum_{k=0}^{n-1} x_k\cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) \ge L\right\}.\end{aligned}$$

Then, like the proof of Theorem in Gao et al. [23]. the price for the American up-and-in call option in the uncertain fraction-order dynamical system (35) is that

$$f_{ui}^c = \int_{Rel}^1 \sup_{0\le t\le T} \exp\left(-rt\right)\left(\left(\sum_{k=0}^{n-1} x_k\cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) - K\right)^+ d\alpha.$$

The proof is complete.

Parameter estimation is an important issue in the wide applications of uncertain differential equations. By setting the empirical moments of the functions of the parameters and the observed data equal to the moments of the standard normal uncertainty distribution, Yao and Liu [50] obtained a system of equations of the parameters whose solutions are essentially the estimates of the parameters. Hence, all the parameters which will appear in the following models are estimated according to the real market data of the Great Wall of China stock between January 2, 2018 and June 30, 2020 and the proposed parameter estimation method.

Example 4.3 *Suppose an investor signs an American up-and-in call option, which stipulates that the maturity date is* $T = 10$*, the barrier level is* $B = 10$*, and the strike price is* $K = 8$*. Then, the fixed parameters of the uncertain fraction-order dynamical system* (35) *are estimated such that the initial price* $X_0 = 8.1100$*, the instantaneous growth rate* $X_1 = 0.19$*,* $a = 0.0029$*,* $b = 0.0084$*, and the constant interest rate* $r = 0.2966$*. Beside, Caputo fractional-order derivative* $p = 1.5$*.*

By using predictor-corrector numerical method [47], we calculate *Rel* and American up-and-in call option formulas for uncertain fraction-order dynamical system (35). Table 3 indicates that when barrier level B is lower or maturity date T is longer, *Rel* is bigger. The above phenomenon conforms to the fact: For a lower pre-given level B (longer expiration time T). The pre-given level has more possibilities to be hit, *Rel* is thus higher. Meanwhile, the first half of the table shows that the price increases with respect to T. The second half of the table indicates that the price decreases with respect to K. The above phenomenon conforms to the fact: for a smaller K, the option has more possibilities to be carried out, then prices are bigger.

Table 3: Sensitivity analysis of Rel (up-and-in call option)

T	2.00	3.75	5.50	7.25	9.00	10.75	12.50	14.25	16.00	17.75	19.50
Rel	0.0000	0.0120	0.1780	0.4336	0.5944	0.6776	0.7224	0.7488	0.7640	0.7740	0.7800
B	8.0	10.2	12.4	14.6	16.8	19.0	21.2	23.4	25.6	27.8	30.0
Rel	0.9900	0.5040	0.3824	0.2820	0.2088	0.1540	0.1140	0.0840	0.0624	0.0500	0.0400

Table 4: Sensitivity analysis of the option price f^c_{ui}

T	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
f^c_{ui}	0.8177	1.4391	2.0769	2.7310	3.3034	3.8103	4.2191	4.5789	4.8733	5.1186	5.3148
K	6.5	6.7	6.9	7.1	7.3	7.5	7.7	7.9	8.1	8.3	8.5
f^c_{ui}	77.7893	68.1260	58.4628	48.7995	39.1362	29.4730	19.8097	10.3447	3.5000	0.4214	0.0191

down-and-out call option

Analyze the reliability of American down-and-out call options. According to the barrier mechanism, the option takes effect on the premise that the price of the underlying asset remains above the floor. In this case, the operation of the uncertain fractional-order dynamical system (35) is required to meet the conditions that $\{\inf_{0\le t\le T} X_t > L\}$. Then, the corresponding reliability index of American down-and-out call option can be deduced as follows.

Theorem 4.4 *Consider an uncertain fraction-order dynamical system* (35)*, which has a strike price K and a barrier price L. Set $X_0 > L$, then the reliability index for American down-and-out call option is*

$$Rel = 1 - \sup\left\{\alpha \mid \inf_{0\le t\le T}\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) \le L\right\}. \tag{39}$$

and the price for the American down-and-out call option goes

$$f^c_{do} = \int_{1-Rel}^{1} \sup_{0\le t\le T} \exp(-rt)\left(\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) - K\right)^+ d\alpha. \tag{40}$$

Proof: According to Theorem 3.2, when $X_0 > L$, let

$$\begin{aligned}\beta &= \sup\left\{\alpha \mid \inf_{0\le t\le T} X_t^\alpha \le L\right\}\\ &= \sup\left\{\alpha \mid \inf_{0\le t\le T}\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) \le L\right\}\end{aligned}$$

thus we can obtain

$$\begin{aligned} Rel &= \mathcal{M}\left\{\inf_{0\leq t\leq T} X_t > L\right\} \\ &= 1-\beta \\ &= 1-\sup\left\{\alpha \mid \inf_{0\leq t\leq T}\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) \leq L\right\}. \end{aligned}$$

Then, like the proof of Theorem in Gao et al. [23]. The price for the American down-an-out call option in the uncertain fraction-order dynamical system (35) has been

$$f_{do}^c = \int_{1-Rel}^1 \sup_{0\leq t\leq T} \exp(-rt)\left(\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) - K\right)^+ d\alpha.$$

The proof is complete.

Example 4.4 *Suppose an investor signs an American down-an-out call option, which stipulates that the maturity date is* $T = 10$*, the barrier level is* $B = 3$*, and the strike price is* $K = 8$*. Then, the fixed parameters of the uncertain fraction-order dynamical system* (35) *are estimated such that the initial price* $X_0 = 6.05$*, the instantaneous growth rate* $X_1 = -0.60$*,* $a = 0.0016$*,* $b = 0.0350$*, and the constant interest rate* $r = 0.5492$*. Beside, Caputo fractional-order derivative* $p = 1.5$*.*

By using predictor-corrector numerical method [47], we calculate Rel and American down-an-out call option formulas of uncertain fraction-order dynamical system (35). Table 5 indicates that when pre-given level B (expiration time T) is smaller, Rel is bigger. The above phenomenon conforms to the fact: For a lower pre-given level B (expiration time T). The pre-given level has more possibilities to be hit, Rel is thus higher.

Table 5: Sensitivity analysis of Rel (down-and-out call option)

T	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
Rel	0.5400	0.4656	0.4020	0.3532	0.3156	0.2868	0.2656	0.2504	0.2384	0.2283	0.2200
B	1.0	1.4	1.8	2.2	2.6	3.0	3.4	3.8	4.2	4.6	5.0
Rel	0.9600	0.9244	0.8636	0.7768	0.6664	0.5412	0.4204	0.3116	0.2240	0.1604	0.1100

Table 6: Sensitivity analysis of the option price f_{do}^c

T	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
f_{do}^c	1.5590	1.3442	1.1606	1.0197	0.9112	0.8280	0.7668	0.7229	0.6883	0.6590	0.6352
K	2.0	2.4	2.8	3.2	3.6	4.0	4.4	4.8	5.2	5.6	6.0
f_{do}^c	126.2824	113.8101	101.3377	88.8654	76.3931	63.9207	51.4484	38.9760	26.5037	14.0314	1.5590

4.2.2 Reliability Analysis of American barrier put option

Similarly, discuss several class American put options whose execution is subjected to the barrier mechanism $\sup\limits_{0\le t\le T}(\inf)X_t \le L$. In uncertain financial markets, investors give priority to purchasing such options in anticipation of the uncertain events $\{\sup\limits_{0\le t\le T}(\inf)X_t \le L\}$ happen. The above uncertain fraction-order dynamical system (19) is further transformed into

$$\begin{cases} dY_t = rY_t dt \\ {}^cD^p X_t = aX_t + bX_t \frac{dC_t}{dt} \\ X^{(k)}(0) = x_k, l = 0, 1, \dots, n-1 \\ \sup\limits_{0\le t\le T}(\inf)X_t \le L \end{cases} \tag{41}$$

where Y_t is used to represent the bond price and X_t is set as an uncertain process to describe dynamic change for price. C_t denotes the Liu process, r, a and b demonstrates the interest rate, diffusion and drift, respectively.

To verify the applicability and feasibility of the uncertain fraction-order dynamical model (41) in the financial field, the following content selects American down-and-in put options and American up-and-out put options as the main analysis objects, respectively derives corresponding reliability indicators and supplements the corresponding pricing formula.

down-and-in put option

Analyze the reliability of American down-and-in put options. The barrier of the down-and-in put option is the floor of the underlying asset price fluctuation, which can be triggered to change the option from invalid to valid. In this case, the operation of the uncertain fractional-order dynamical system (35) is required to meet the conditions that $\{\inf\limits_{0\le t\le T} X_t > L\}$. Then, the corresponding reliability index of American down-and-in put option can be deduced as follows.

Theorem 4.5 *Consider an uncertain fraction-order dynamical system* (41)*, which has a strike price K and a barrier price L. Set $X_0 > L$, then the reliability index for American down-and-in put option is*

$$Rel = \sup\left\{\alpha \mid \inf_{0\le t\le T}\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) \le L\right\}. \tag{42}$$

and the price for the American down-an-in put option goes

$$f_{di}^p = \int_0^{Rel} \sup_{0\le t\le T} \exp(-rt)\left(K - \left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right)\right)^+ d\alpha. \tag{43}$$

Proof: Similar to Theorem 4.4, the reliability index can be obtained as follows,

$$\begin{aligned} Rel &= \mathcal{M}\left\{\inf_{0\le t\le T} X_t \le L\right\} \\ &= \beta \\ &= \sup\left\{\alpha \mid \inf_{0\le t\le T}\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) \le L\right\}. \end{aligned}$$

Then, like the proof of Theorem in Gao et al. [23]. The price for the American down-an-in put option in the uncertain fraction-order model (35) has been

$$f_{di}^p = \int_0^{Rel} \sup_{0\le t\le T} \exp(-rt)\left(K - \left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right)\right)^+ d\alpha.$$

The proof is complete.

Example 4.5 *Suppose an investor signs an American down-an-in put option, which stipulates that the maturity date is $T = 5$, the barrier level is $B = 3$, and the strike price is $K = 8$. Then, the fixed parameters of the uncertain fraction-order dynamical system* (41) *are estimated such that the initial price $X_0 = 7.55$, the instantaneous growth rate $X_1 = -0.71$, $a = 0.0027$, $b = 0.0322$, and the constant interest rate $r = 0.3623$. Beside, Caputo fractional-order derivative* $p = 1.5$.

By using predictor-corrector numerical method [47], we calculate Rel and American down-an-in put option formulas for the uncertain fraction-order dynamical system (41). Table 7 indicates that when barrier level B is higher or maturity date T is longer, Rel is bigger. The above phenomenon conforms to the fact: For a higher pre-given level B (longer expiration time T). The pre-given level has more possibilities to be hit, Rel is thus higher.

Table 7: Sensitivity analysis of Rel (down-and-in put option)

T	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
Rel	0.2100	0.3005	0.3856	0.4612	0.5244	0.5756	0.6160	0.6484	0.6752	0.6957	0.7100
B	1.0	1.6	2.2	2.8	3.4	4.0	4.6	5.2	5.8	6.4	7.0
Rel	0.0100	0.0385	0.0912	0.1800	0.3056	0.4560	0.6056	0.7332	0.8272	0.8928	0.9472

Table 8: Sensitivity analysis of the option price f_{di}^p

T	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
f_{di}^p	6.5779	9.4137	12.0783	14.4463	16.4259	18.0297	19.2952	20.3100	21.1495	21.7927	22.2395
K	8.0	8.4	8.8	9.2	9.6	10.0	10.4	10.8	11.2	11.6	12.0
f_{di}^p	6.5779	12.4249	18.2719	24.1189	29.9660	35.8130	41.6600	47.5070	53.3540	59.2011	65.0481

up-and-out put option

Analyze the reliability of American up-and-out put options. Similar to down-and-out call option, only when the price of the underlying asset never exceeds the ceiling, the up-and-out put option can remain effective. In other words, the operation of the uncertain fractional-order dynamical system (35) is required to meet the conditions that $\{\sup_{0\le t\le T} X_t < L\}$. Then, the corresponding reliability index of American up-and-out put option can be deduced as follows.

Theorem 4.6 *Consider an uncertain fraction-order dynamical system* (41)*, which has a strike price K and a barrier price L. Set $X_0 < L$, then the reliability index for American up-and-out put option is*

$$Rel = \inf\left\{\alpha \mid \sup_{0\le t\le T}\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) \ge L\right\}. \tag{44}$$

and the price for the American up-and-out put option goes

$$f_{uo}^p = \int_0^{Rel} \sup_{0\le t\le T} \exp(-rt)\left(K - \left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right)\right)^+ d\alpha. \tag{45}$$

Proof: Similar to Theorem 4.3, the reliability index can be obtained as follows,

$$\begin{aligned} Rel &= \mathcal{M}\left\{\sup_{0\le t\le T} X_t < L\right\} \\ &= \beta \\ &= \inf\left\{\alpha \mid \sup_{0\le t\le T}\left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right) \ge L\right\}. \end{aligned}$$

Then, like the proof of Theorem in Gao et al. [23]. The price for the American up-and-out put option in uncertain fraction-order model (41) has been

$$f_{uo}^p = \int_0^{Rel} \sup_{0\le t\le T} \exp(-rt)\left(K - \left(\sum_{k=0}^{n-1} x_k \cdot t^k E_{p,k+1}\left(at^p + b\Phi^{-1}(\alpha)t^p\right)\right)\right)^+ d\alpha.$$

The proof is complete.

Example 4.6 *Suppose an investor signs an American up-and-out put option, which stipulates that the maturity date is $T = 5$, the barrier level is $B = 10$, and the strike price is $K = 8$. Then, the fixed parameters of the uncertain fraction-order dynamical system* (41) *are estimated such that the initial price $X_0 = 7.85$, the instantaneous growth rate $X_1 = 0.35$, $a = 0.0038$, $b = 0.0124$, and the constant interest rate $r = 0.3980$. Beside, Caputo fractional-order derivative $p = 1.5$.*

By using predictor-corrector numerical method [47], we calculate *Rel* and American up-and-out put option formulas for uncertain fraction-order dynamical system (41). Table 9 indicates that when barrier level B is higher or maturity date T is smaller, Rel is bigger. The above phenomenon conforms to the fact: For a higher pre-given level B (smaller expiration time T). The pre-given level has more possibilities to be hit, Rel is thus higher.

Table 9: Sensitivity analysis of Rel (up-and-out put option)

T	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
Rel	0.5700	0.4665	0.3894	0.3319	0.2921	0.2626	0.2386	0.2203	0.2080	0.1980	0.1900
B	8.0	8.4	8.8	9.2	9.6	10.0	10.4	10.8	11.2	11.6	12.0
Rel	0.0100	0.0504	0.1072	0.2132	0.3744	0.5644	0.7344	0.8552	0.9260	0.9648	0.9900

Table 10: Sensitivity analysis of the option price f_{uo}^{p}

T	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0
f_{uo}^{p}	5.7427	4.6995	3.9236	3.3441	2.9427	2.6453	2.4043	2.2197	2.0956
K	8.0	8.4	8.8	9.2	9.6	10.0	10.4	10.8	11.2
f_{uo}^{p}	5.7427	21.0566	36.3705	51.6844	66.9983	82.3122	97.6261	112.9400	128.2539

Remark 4.4 *It is obvious that the conclusions for the case $p = 1$ in Theorems 4.3, 4.4, 4.5 and 4.6 are the same as those in Gao et al. [23] in 2019, and we extend American barrier option prices to the case of UFDEs.*

Remark 4.5 *We derive American barrier option prices via first-hitting time theorems, which improves the proof for UDEs' model.*

5 Conclusion

Considering there are obstacles for the UFDE model, this paper refined the UFDE model we studied before and introduced a novel uncertain fractional-order model with state constraint. The reliability index theorem of the proposed model was derived by the FHT theorem for two cases that $\sup(\inf)X_t > L$ as well as $\sup(\inf)X_t < L$, respectively. As the application of the uncertain fractional-order dynamical system with state constraint, the fractional-order circuit model and American barrier option model were proposed. The analytic expressions of the reliability index for such two models were obtained by using the proposed theorem. Expected time response and American barrier option prices were also calculated by the predictor-corrector method. Furthermore, we discussed the fluctuation of the reliability index concerning different parameters. In the future, we will analyze the survivability of our fractional-order dynamical system and find corresponding kernel.

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