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The mass-conserving domain decomposition method for convection diffusion equations with variable coefficients.

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Abstract In this paper, a conserved domain decomposition method for solving convection-diffusion equations with variable coefficients is analyzed. The interface fluxes over the sub-domains are firstly obtained by the explicit fluxes scheme. Secondly, the interior solutions and fluxes over each sub-domains are computed by the modified upwind implicit scheme. Then, the interface fluxes are corrected by the obtained solutions. We prove rigorously that our scheme is mass conservative, unconditionally stable and of second-order convergence in spatial step. Numerical examples test the theoretical analysis and efficiencies. Lastly, we extend our scheme to the nonlinear convection-diffusion equations and give the error estimate.

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1 Introduction

Convection-diffusion equations are widely used in science and engineering, since they describe heat and mass transfer, oil reservoir simulation, groundwater modelling, and various other physics processes(see, for example, [2–4, 9, 17, 22], etc). Due to large scale and long computing time simulations in real application, it is of great importance to develop an efficient domain decomposition method for solving the convection-diffusion equations.

Domain decomposition ([1, 5, 6, 8, 10, 12, 13, 17, 19–21, 25],etc) is an efficient and powerful tool for solving large scale simulations in applications. It is well-known that in presence of both diffusion and convection, the difference methods can become unstable and non-reliable due to spurious numerical oscillations, the characteristic ([15–18, 23, 28], etc) and upwind ([7, 24], etc) methods were used to solve the convection-dominant diffusion equations.

However, the above domain decomposition methods for solving convection-diffusion equations break mass conservation. By defining the substance fluxes which contain convection and diffusion terms on the interfaces and combining the operator splitting, papers [11, 14, 26] developed the mass-preserving and modified upwind splitting domain decomposition method(S-DDM) for solving convection-diffusion equations and multi-component contamination flows in porous media, while the scheme is unconditionally stable and of second-order convergence in space. Theoretical analysis of the scheme in paper [26] was only conducted for convection diffusion equations with constant coefficient and the analysis of the proposed scheme in paper [14] with variable coefficient cannot be obtained. By computing the interface fluxes with the local multi-point weighted average schemes on the interfaces, paper [27] analyzed an efficient mass-conserving operator splitting domain decomposition method for solving two dimensional parabolic equations with variable coefficient.

The modified upwind and mass-conserving domain decomposition method for solving convection-diffusion equations with variable coefficients is proposed in three step over each sub-domains. In the first step, the interface fluxes of are computed firstly by the modified semi-implicit flux scheme. In the second step, the interior solutions and fluxes are solved by the coupled modified upwind implicit scheme. In the third step, the interface fluxes are corrected by the obtained solutions. We prove rigorously that our proposed scheme is unconditionally stable and mass conserving. By some mathematic techniques and inequalities, we give the error estimate and prove the convergence. Numerical examples are presented. In addition, the nonlinear convection diffusion equations is developed, where the scheme, algorithm, error estimate and numerical experiments are given.

This paper is divided into five parts. In Section 2, the scheme of the convection diffusion equations with variable coefficients is proposed. The theoretical results and numerical experiments are given in Section 3 and Section 4. In Section 5, the error estimate of nonlinear convection-diffusion problems are given.

2 Scheme

The convection-diffusion equations with Neumann boundary conditions are considered as follows,

$$\begin{cases} \frac{\partial c}{\partial t} + \frac{\partial}{\partial x}(b(x,t)c - d(x,t)\frac{\partial c}{\partial x}) = f(x,t), \quad x \in I, \\ b(x,t)c - d(x,t)\frac{\partial c}{\partial x} = 0, \quad x \in \partial I, \\ c(x,0) = c_0(x). \end{cases} \tag{1}$$

where $b(x,t)$ is the velocity of the fluid and $d(x,t)$ is the diffusion coefficient. The right term is $f(x,t)$, and the initial function $c_0(x)$ is a smooth function.

The uniform conforming mesh on domain $I = [0, L]$ with $\Delta x = \frac{L}{N}$ is defined and we introduce the staggered mesh points x_i and $x_{i+\frac{1}{2}}$ as

$$x_{i+\frac{1}{2}} = i\Delta x, \quad i = 0, 1, \cdots, N, \qquad x_i = (i - \tfrac{1}{2})\Delta x, \quad i = 1, 2, \cdots, N.$$

The time interval $(0, T]$ is discretized uniformly by $t^n = n\Delta t, \quad n = 0, 1, \cdots, M$, where $\Delta t = \frac{T}{M}$. Let $\phi_i^n = \phi(x_i, t^n)$, $\phi_{i+\frac{1}{2}}^n = \phi(x_{i+\frac{1}{2}}, t^n)$ denote the function ϕ at the mesh points (x_i, t^n) and $(x_{i+\frac{1}{2}}, t^n)$. Define the following difference operators as

$$\delta_x \phi_{i+\frac{1}{2}}^n = \frac{\phi_{i+1}^n - \phi_i^n}{\Delta x}, \quad \delta_x \phi_i^n = \frac{\phi_{i+\frac{1}{2}}^n - \phi_{i-\frac{1}{2}}^n}{\Delta x}, \quad \partial_t \phi_i^{n+1} = \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t}.$$

Let

$$u(x) = b(x,t)c - d(x,t)\frac{\partial c}{\partial x}. \tag{2}$$

It is well-known that in presence of both diffusion and convection, the difference methods can become unstable and non-reliable due to spurious numerical oscillations, the modified-upwind discrete flux operators are defined for (2) as

$$u_{i+\frac{1}{2}} = b_{i+\frac{1}{2}}(\lambda(b_{i+\frac{1}{2}})c_i + (1 - \lambda(b_{i+\frac{1}{2}}))c_{i+1}) - \bar{d}_{i+\frac{1}{2}}\delta_x c_{i+\frac{1}{2}} + O((\Delta x)^2), \tag{3}$$

where

$$\bar{d}(x,t) = \frac{d(x,t)}{1 + \frac{|b(x,t)|}{2d(x,t)}\Delta x}.$$

and $\lambda(s)$ is the sign function given by

$$\lambda(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0. \end{cases}$$

Let C_i be the numerical approximation of exact solution c_i and the fluxes $U_{i+\frac{1}{2}}$ be numerical approximations of the fluxes $u_{i+\frac{1}{2}}$. The discrete fluxes can be defined as

$$U_{i+\frac{1}{2}} = b_{i+\frac{1}{2}}(\lambda(b_{i+\frac{1}{2}})C_i + (1-\lambda(b_{i+\frac{1}{2}}))C_{i+1}) - \bar{d}_{i+\frac{1}{2}}\delta_x C_{i+\frac{1}{2}}. \tag{4}$$

Assume that the domain is divided into multiple nonoverlapping sub-domains. Denote $r^{n+1}_{i_\alpha+\frac{1}{2}} = \bar{d}^{n+1}_{i_\alpha+\frac{1}{2}}\frac{\Delta t}{(\Delta x)^2}$. The domain decomposition method over each sub-domains is proposed in three step at time interval $[t^n, t^{n+1}]$ as:

Step 1. The interface fluxes $\{\tilde{U}^{n+1}_{i_\alpha+\frac{1}{2}}\}$ on $x_{i_\alpha+\frac{1}{2}} \in \Gamma_\alpha$ are obtained by the following semi-implicit schemes

$$\begin{aligned}\tilde{U}^{n+1}_{i_\alpha+\frac{1}{2}} = {} & U^n_{i_\alpha+\frac{1}{2}} + r^{n+1}_{i_\alpha+\frac{1}{2}}(U^n_{i_\alpha-\frac{1}{2}} - 2\tilde{U}^{n+1}_{i_\alpha+\frac{1}{2}} + U^n_{i_\alpha+\frac{3}{2}}) - \bar{d}^{n+1}_{i_\alpha+\frac{1}{2}}\Delta t\delta_x f^{n+1}_{i_\alpha+\frac{1}{2}} \\ & - \bar{d}^{n+1}_{i_\alpha+\frac{1}{2}}\delta_x C^n_{i_\alpha+\frac{1}{2}} + \bar{d}^n_{i_\alpha+\frac{1}{2}}\delta_x C^n_{i_\alpha+\frac{1}{2}}.\end{aligned} \tag{5}$$

Remark 1 Due to the diffusion D with the variable coefficient, the predicted scheme 5 is slightly different from the scheme in paper [11,26], which adds on term $-\bar{d}^{n+1}_{i_\alpha+\frac{1}{2}}\delta_x C^n_{i_\alpha+\frac{1}{2}} + \bar{d}^n_{i_\alpha+\frac{1}{2}}\delta_x C^n_{i_\alpha+\frac{1}{2}}$ and would be interperted as the approximation to $\frac{\partial d}{\partial t}\frac{\partial c}{\partial x}$.

Step 2. The interior solutions and fluxes $\{C^{n+1}_i, U^{n+1}_{i+\frac{1}{2}}\}$ are computed by the modified-upwind implicit solution and flux coupled scheme.

$$\begin{aligned}\frac{C^{n+1}_i - C^n_i}{\Delta t} + \delta_x U^{n+1}_i = f^{n+1}_i, &\quad x_i \in I_\alpha, \\ U^{n+1}_{i+\frac{1}{2}} = b^{n+1}_{i+\frac{1}{2}}(\lambda(b^{n+1}_{i+\frac{1}{2}})C^{n+1}_i + (1-\lambda(b^{n+1}_{i+\frac{1}{2}}))C^{n+1}_{i+1}) - \bar{d}^{n+1}_{i+\frac{1}{2}}\delta_x C^{n+1}_{i+\frac{1}{2}}, &\quad x_{i+\frac{1}{2}} \in I_\alpha, \\ U^{n+1}_{i+\frac{1}{2}} = \tilde{U}^{n+1}_{i+\frac{1}{2}}, &\quad x_{i+\frac{1}{2}} \in \Gamma_\alpha, \\ U^{n+1}_{i+\frac{1}{2}} = 0, &\quad x_{i+\frac{1}{2}} \in \partial I.\end{aligned} \tag{6}$$

Step 3. The interface fluxes $\{U^{n+1}_{i_\alpha+\frac{1}{2}}\}$ are solved as

$$U^{n+1}_{i_\alpha+\frac{1}{2}} = b^{n+1}_{i_\alpha+\frac{1}{2}}(\lambda(b^{n+1}_{i_\alpha+\frac{1}{2}})C^{n+1}_{i_\alpha} + (1-\lambda(b^{n+1}_{i_\alpha+\frac{1}{2}}))C^{n+1}_{i_\alpha+1}) - \bar{d}^{n+1}_{i_\alpha+\frac{1}{2}}\delta_x C^{n+1}_{i_\alpha+\frac{1}{2}}. \tag{7}$$

Meanwhile, the initial values are given by

$$C^0_i = c_0(x_i), \quad x_i \in I. \tag{8}$$

3 Theoretical result

In the section, without loss of generality, the domain is assumed to be divided into 2 sub-domains.

3.1 Conservation

The scheme (5)-(8) is rewritten into as

$$\begin{cases} \hat{\partial}_t C_i^{n+1} + \delta_x U_i^{n+1} = f_i^{n+1}, i = 1, 2, \cdots, i_1 - 1, i_1 + 2, \cdots, N, \\ \hat{\partial}_t C_{i_1}^{n+1} + \delta_x U_{i_1}^{n+1} = f_{i_1}^{n+1} + \frac{U_{i_1+\frac{1}{2}}^{n+1} - \tilde{U}_{i_1+\frac{1}{2}}^{n+1}}{\Delta_x}, i = i_1, \\ \hat{\partial}_t C_{i_1+1}^{n+1} + \delta_x U_{i_1+1}^{n+1} = f_{i_1+1}^{n+1} - \frac{U_{i_1+\frac{1}{2}}^{n+1} - \tilde{U}_{i_1+\frac{1}{2}}^{n+1}}{\Delta_x}, i = i_1 + 1. \end{cases} \tag{9}$$

It is easily to prove the following theorem.

Theorem 1 *The scheme (5)-(8) meet mass conservation on the global domain i.e.*

$$\sum_{i=1}^{N} C_i^n \Delta x = \sum_{i=1}^{N} C_i^0 \Delta x + \sum_{l=0}^{n-1} (\sum_{i=1}^{N} f_i^{l+1} \Delta x) \Delta t, \tag{10}$$

for $n \geq 1$.

3.2 Stability

Rigorous mathematical analysis shows that the stability of our scheme is non-obvious and more difficult to prove. For doing this, we introduce some notations and preliminary lemmas and assume that $b(x,t) \geq 0$ as well.

Let $U = \{U_i\}$, $V = \{V_i\}$, $X = \{X_{i+\frac{1}{2}}\}$, $Y = \{Y_{i+\frac{1}{2}}\}$, be discrete functions. Define the discrete inner products, and the corresponding discrete L^2-norms as follows,

$$\begin{aligned} &(U, V) = \sum_{i=1}^{N} U_i V_i \Delta x, \quad \| U \| = (U, U)^{\frac{1}{2}}, \\ &(X, Y) = \sum_{i=0}^{N-1} X_i Y_i \Delta x, \quad \| X \| = (X, X)^{\frac{1}{2}}, \\ &(X, Y)_a = \sum_{i=0}^{N-1} a_{i+\frac{1}{2}} X_{i+\frac{1}{2}} Y_{i+\frac{1}{2}} \Delta x, \quad \| X \|_a = (X, X)_a^{\frac{1}{2}}. \end{aligned}$$

Let $\Gamma_{i_\alpha+\frac{1}{2}}$ be the interface lines. We define the corresponding inner products and norms on the interface lines.

$$\begin{aligned} &< X, Y >_{\Gamma_{i_\alpha+\frac{1}{2}}} = X_{i_\alpha+\frac{1}{2}} Y_{i_\alpha+\frac{1}{2}}, \\ &< X, Y >_{a,\Gamma_{i_\alpha+\frac{1}{2}}} = a_{i_\alpha+\frac{1}{2}} X_{i_\alpha+\frac{1}{2}} Y_{i_\alpha+\frac{1}{2}}, \\ &| X |_{\Gamma_{i_\alpha+\frac{1}{2}}} = < X, X >_{\Gamma_{i_\alpha+\frac{1}{2}}}^{\frac{1}{2}}, | X |_{a,\Gamma_{i_\alpha+\frac{1}{2}}} = < X, X >_{a,\Gamma_{i_\alpha+\frac{1}{2}}}^{\frac{1}{2}}, \end{aligned}$$

where $(X, X)_a$ and $< X, Y >_a$ are defined as the inner products with the weighting function $a(x)$.

Lemma 1 *Let $U=\{U_i\}$ and $V=\{V_i\}$. Then it holds that*

$$(\frac{U-V}{\Delta t},U)=\frac{1}{2\Delta t}(\parallel U\parallel^2-\parallel V\parallel^2)+\frac{1}{2\Delta t}\parallel U-V\parallel^2. \tag{11}$$

Lemma 2 *If $C=\{C_i\}$ and $U=\{U_{i+\frac{1}{2}}\}$ satisfy $U_{\frac{1}{2}}=U_{I+\frac{1}{2}}=0$, it holds only if it is assumed that Eq. 7 is valid,*

$$(\delta_x U,C)=\parallel U\parallel^2_{\frac{1}{d}}-\textstyle\sum_{i=1}^{I}\frac{b_{I+\frac{1}{2}}}{d_{I+\frac{1}{2}}}U_{i+\frac{1}{2}}C_i\Delta x. \tag{12}$$

Lemma 3 *Let $C=\{C_i\}$ and $U=\{U_{i+\frac{1}{2}}\}$ be the solution of the scheme (6)-(8). We have that*

$$\begin{aligned}&(1-\beta_1\Delta t)\parallel C^{n+1}\parallel^2+\tfrac{(\Delta t)^2}{\Delta x}(\mid U^{n+1}\mid^2_{\Gamma_{i_1-\frac{1}{2}}}+\mid U^{n+1}\mid^2_{\Gamma_{i_1+\frac{3}{2}}})\\&\le(1+\alpha_1\Delta t)\parallel C^n\parallel^2+\tfrac{(\Delta t)^2}{\Delta x}(\mid U^n\mid^2_{\Gamma_{i_1-\frac{1}{2}}}+\mid U^n\mid^2_{\Gamma_{i_1+\frac{3}{2}}})\\&\quad+M\Delta t(\parallel f^{n+1}\parallel^2+\tfrac{(\Delta t)^2}{\Delta x}\sum_{i=i_1,i_1+1}\mid f^{n+1}\mid^2_{\Gamma_i}),\end{aligned} \tag{13}$$

where $M>0$ is a positive constant.

Proof Multiplying both sides of Eqns. (9) by $C_i^{n+1}\Delta x$, respectively, and summing up with i from 1 to N, we obtain that

$$(\hat{\partial}_t C^{n+1},C^{n+1})+(\delta_x U^{n+1},C^{n+1})=(f^{n+1},C^{n+1})+R, \tag{14}$$

where

$$R=(U^{n+1}_{i_1+\frac{1}{2}}-\tilde{U}^{n+1}_{i_1+\frac{1}{2}})(-\delta_x C^{n+1}_{i_1+\frac{1}{2}})\Delta x. \tag{15}$$

Applying the definition (4), it holds that

$$\begin{aligned}R&=(U^{n+1}_{i_1+\frac{1}{2}}-\tilde{U}^{n+1}_{i_1+\frac{1}{2}})\frac{-\bar{d}^{n+1}_{i_1+\frac{1}{2}}\delta_x C^{n+1}_{i_1+\frac{1}{2}}}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}}\Delta x\\&=\tfrac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}}(U^{n+1}_{i_1+\frac{1}{2}}-\tilde{U}^{n+1}_{i_1+\frac{1}{2}})(U^{n+1}_{i_1+\frac{1}{2}}-b^{n+1}_{i_1+\frac{1}{2}}C^{n+1}_{i_1})\Delta x\\&=\Delta x\langle U^{n+1}-\tilde{U}^{n+1},U^{n+1}\rangle_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}}\\&\quad-\tfrac{b^{n+1}_{i_1+\frac{1}{2}}}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}}(U^{n+1}_{i_1+\frac{1}{2}}-\tilde{U}^{n+1}_{i_1+\frac{1}{2}})C^{n+1}_{i_1}\Delta x\\&=\Delta x(\mid U^{n+1}\mid^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}}-\mid\tilde{U}^{n+1}\mid^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}})\\&\quad-\Delta x(\langle U^{n+1}-\tilde{U}^{n+1},\tilde{U}^{n+1}\rangle_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}})\\&\quad-\tfrac{b^{n+1}_{i_1+\frac{1}{2}}}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}}(U^{n+1}_{i_1+\frac{1}{2}}-\tilde{U}^{n+1}_{i_1+\frac{1}{2}})C^{n+1}_{i_1}\Delta x.\end{aligned} \tag{16}$$

Subtracting the second equation from the third equation of Eqns. (9), we have that

$$\frac{\delta_x C^{n+1}_{i_1+\frac{1}{2}}-\delta_x C^n_{i_1+\frac{1}{2}}}{\Delta t}+\frac{U^{n+1}_{i_1+\frac{3}{2}}-2\tilde{U}^{n+1}_{i_1+\frac{1}{2}}+U^{n+1}_{i_1-\frac{1}{2}}}{(\Delta x)^2}=\delta_x f^{n+1}_{i_1+\frac{1}{2}}. \tag{17}$$

Multiplying both sides of (17) with $-\bar{d}^{n+1}_{i_1+\frac{1}{2}}$, we get that

$$\begin{array}{l}\frac{-\bar{d}^{n+1}_{i_1+\frac{1}{2}}\delta_x C^{n+1}_{i_1+\frac{1}{2}}+\bar{d}^{n}_{i_1+\frac{1}{2}}\delta_x C^{n}_{i_1+\frac{1}{2}}}{\Delta t}=\frac{-\bar{d}^{n+1}_{i_1+\frac{1}{2}}\delta_x C^{n}_{i_1+\frac{1}{2}}+\bar{d}^{n}_{i_1+\frac{1}{2}}\delta_x C^{n}_{i_1+\frac{1}{2}}}{\Delta t}\\ +\bar{d}^{n+1}_{i_1+\frac{1}{2}}\frac{U^{n+1}_{i_1+\frac{3}{2}}-2\tilde{U}^{n+1}_{i_1+\frac{1}{2}}+U^{n+1}_{i_1-\frac{1}{2}}}{(\Delta x)^2}-\bar{d}^{n+1}_{i_1+\frac{1}{2}}\delta_x f^{n+1}_{i_1+\frac{1}{2}}.\end{array} \tag{18}$$

Further, we get that

$$\begin{array}{l}\frac{U^{n+1}_{i_1+\frac{1}{2}}-U^{n}_{i_1+\frac{1}{2}}}{\Delta t}-\frac{b^{n+1}_{i_1+\frac{1}{2}}C^{n+1}_{i_1}-b^{n}_{i_1+\frac{1}{2}}C^{n}_{i_1}}{\Delta t}+\bar{d}^{n+1}_{i_1+\frac{1}{2}}\delta_x f^{n+1}_{i_1+\frac{1}{2}}\\ =\frac{-\bar{d}^{n+1}_{i_1+\frac{1}{2}}\delta_x C^{n}_{i_1+\frac{1}{2}}+\bar{d}^{n}_{i_1+\frac{1}{2}}\delta_x C^{n}_{i_1+\frac{1}{2}}}{\Delta t}+\bar{d}^{n+1}_{i_1+\frac{1}{2}}\frac{U^{n+1}_{i_1+\frac{3}{2}}-2\tilde{U}^{n+1}_{i_1+\frac{1}{2}}+U^{n+1}_{i_1-\frac{1}{2}}}{(\Delta x)^2}.\end{array} \tag{19}$$

Adding both sides of Eqn. (19) with $\frac{b^{n+1}_{i_1+\frac{1}{2}}C^{n+1}_{i_1}-b^{n}_{i_1+\frac{1}{2}}C^{n}_{i_1}}{\Delta t}$ and using the definition of $U^{n+1}_{i_1+\frac{1}{2}}$, we obtain that

$$\begin{array}{l}U^{n+1}_{i_1+\frac{1}{2}}-U^{n}_{i_1+\frac{1}{2}}=b^{n+1}_{i_1+\frac{1}{2}}C^{n+1}_{i_1}-b^{n}_{i_1+\frac{1}{2}}C^{n}_{i_1}+r^{n+1}_{i_1+\frac{1}{2}}(U^{n+1}_{i_1+\frac{3}{2}}-2\tilde{U}^{n+1}_{i_1+\frac{1}{2}}+U^{n+1}_{i_1-\frac{1}{2}})\\ \quad +\bar{d}^{n}_{i_1+\frac{1}{2}}\delta_x C^{n}_{i_1+\frac{1}{2}}-\bar{d}^{n+1}_{i_1+\frac{1}{2}}\Delta t\delta_x f^{n+1}_{i_1+\frac{1}{2}}-\bar{d}^{n+1}_{i_1+\frac{1}{2}}\delta_x C^{n}_{i_1+\frac{1}{2}}.\end{array} \tag{20}$$

Subtracting Eqn. (5) from Eqn. (20), we have that

$$\begin{array}{l}U^{n+1}_{i_1+\frac{1}{2}}-\tilde{U}^{n+1}_{i_1+\frac{1}{2}}=r^{n+1}_{i_1+\frac{1}{2}}(U^{n+1}_{i_1-\frac{1}{2}}+U^{n+1}_{i_1+\frac{3}{2}})\\ \quad -r^{n+1}_{i_1+\frac{1}{2}}(U^{n}_{i_1-\frac{1}{2}}+U^{n}_{i_1+\frac{3}{2}})+b^{n+1}_{i_1+\frac{1}{2}}C^{n+1}_{i_1}-b^{n}_{i_1+\frac{1}{2}}C^{n}_{i_1}.\end{array} \tag{21}$$

Thus, it follows that

$$\begin{array}{l}-\langle U^{n+1}-\tilde{U}^{n+1},\tilde{U}^{n+1}\rangle_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}}=-\bar{d}^{n+1}_{i_1+\frac{1}{2}}\frac{\Delta t}{(\Delta x)^2}(U^{n+1}_{i_1-\frac{1}{2}}+U^{n+1}_{i_1+\frac{3}{2}})\tilde{U}^{n+1}_{i_1+\frac{1}{2}}\\ +\bar{d}^{n+1}_{i_1+\frac{1}{2}}\frac{\Delta t}{(\Delta x)^2}(U^{n}_{i_1-\frac{1}{2}}+U^{n}_{i_1+\frac{3}{2}})\tilde{U}^{n+1}_{i_1+\frac{1}{2}}+(b^{n+1}_{i_1+\frac{1}{2}}C^{n+1}_{i_1}-b^{n}_{i_1+\frac{1}{2}}C^{n}_{i_1})\tilde{U}^{n+1}_{i_1+\frac{1}{2}}.\end{array} \tag{22}$$

Applying the perfect square formula $(a-b)^2=a^2+b^2-2ab$, we obtain that

$$\begin{array}{l}-\frac{\Delta t}{(\Delta x)^2}U^{n+1}_{i_1-\frac{1}{2}}\tilde{U}^{n+1}_{i_1+\frac{1}{2}}\\ =\frac{\Delta t}{2(\Delta x)^2}(\tilde{U}^{n+1}_{i_1+\frac{1}{2}}-U^{n+1}_{i_1-\frac{1}{2}})^2-\frac{\Delta t}{2(\Delta x)^2}(\mid U^{n+1}\mid^2_{\Gamma_{i_1-\frac{1}{2}}}+\mid \tilde{U}^{n+1}\mid^2_{\Gamma_{i_1+\frac{1}{2}}}).\end{array} \tag{23}$$

For the first term of Eqn. (23), by applying the second equation of Eqns. (9), we can have that

$$\begin{array}{l}\frac{\Delta t}{2(\Delta x)^2}(\tilde{U}^{n+1}_{i_1+\frac{1}{2}}-U^{n+1}_{i_1-\frac{1}{2}})^2\\ =\frac{\Delta t}{2}(\frac{\tilde{U}^{n+1}_{i_1+\frac{1}{2}}-U^{n+1}_{i_1-\frac{1}{2}}}{\Delta x})^2=\frac{\Delta t}{2}(f^{n+1}_{i_1}-\hat{\partial}_t C^{n+1}_{i_1})^2\\ =\frac{\Delta t}{2}(\mid f^{n+1}_{i_1}\mid^2-2f^{n+1}_{i_1}\hat{\partial}_t C^{n+1}_{i_1}+\mid \hat{\partial}_t C^{n+1}_{i_1}\mid^2)\\ =\frac{\Delta t}{2}(\mid f^{n+1}_{i_1}\mid^2-2f^{n+1}_{i_1}(f^{n+1}_{i_1}+\frac{\tilde{U}^{n+1}_{i_1+\frac{1}{2}}-U^{n+1}_{i_1-\frac{1}{2}}}{\Delta x})+\mid \hat{\partial}_t C^{n+1}_{i_1}\mid^2)\\ =\frac{\Delta t}{2}((\mid \hat{\partial}_t C^{n+1}\mid^2_{\Gamma_{i_1}}-\mid f^{n+1}\mid^2_{\Gamma_{i_1}})-2\frac{\tilde{U}^{n+1}_{i_1+\frac{1}{2}}-U^{n+1}_{i_1-\frac{1}{2}}}{\Delta x}f^{n+1}_{i_1})\\ =\frac{\Delta t}{2}(\mid \hat{\partial}_t C^{n+1}\mid^2_{\Gamma_{i_1}}-\mid f^{n+1}\mid^2_{\Gamma_{i_1}})-\frac{\Delta t}{\Delta x}(\tilde{U}^{n+1}_{i_1+\frac{1}{2}}-U^{n+1}_{i_1-\frac{1}{2}})f^{n+1}_{i_1},\end{array} \tag{24}$$

where $|\hat{\partial}_t C^{n+1}|^2_{\Gamma_{i_1}} = (\hat{\partial}_t C^{n+1}_{i_1})^2, |f^{n+1}|^2_{\Gamma_{i_1}} = (\hat{f}^{n+1}_{i_1})^2$.
Substituting (24) into (23), we can obtain that

$$\begin{aligned}-\frac{\Delta t}{(\Delta x)^2}U^{n+1}_{i_1-\frac{1}{2}}\tilde{U}^{n+1}_{i_1+\frac{1}{2}} &= \frac{\Delta t}{2}(|\hat{\partial}_t C^{n+1}|^2_{\Gamma_{i_1}} - |f^{n+1}|^2_{\Gamma_{i_1}}) \\ &- \frac{\Delta t}{2(\Delta x)^2}(|U^{n+1}|^2_{\Gamma_{i_1-\frac{1}{2}}} + |\tilde{U}^{n+1}|^2_{\Gamma_{i_1+\frac{1}{2}}}) - \frac{\Delta t}{\Delta x}(\tilde{U}^{n+1}_{i_1+\frac{1}{2}} - U^{n+1}_{i_1-\frac{1}{2}})f^{n+1}_{i_1}.\end{aligned} \tag{25}$$

Similarly, applying the third equation of Eqn. (9), we have that

$$\begin{aligned}-\frac{\Delta t}{(\Delta x)^2}U^{n+1}_{i_1+\frac{3}{2}}\tilde{U}^{n+1}_{i_1+\frac{1}{2}} &= \frac{\Delta t}{2}(|\hat{\partial}_t C^{n+1}|^2_{\Gamma_{i_1+1}} - |f^{n+1}|^2_{\Gamma_{i_1+1}}) \\ &- \frac{\Delta t}{2(\Delta x)^2}(|U^{n+1}|^2_{\Gamma_{i_1+\frac{3}{2}}} + |\tilde{U}^{n+1}|^2_{\Gamma_{i_1+\frac{1}{2}}}) - \frac{\Delta t}{\Delta x}(\tilde{U}^{n+1}_{i_1+\frac{3}{2}} - U^{n+1}_{i_1-\frac{1}{2}})f^{n+1}_{i_1+1}.\end{aligned} \tag{26}$$

Substituting (25) and (26) into Eqn. (22), we get that

$$\begin{aligned}&-\langle U^{n+1} - \tilde{U}^{n+1}, \tilde{U}^{n+1}\rangle_{\frac{1}{d^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}} \\ &= \tfrac{\Delta t}{2}(|\hat{\partial}_t C^{n+1}|^2_{\Gamma_{i_1}} + |\hat{\partial}_t C^{n+1}|^2_{\Gamma_{i_1+1}}) - \tfrac{\Delta t}{2}(|f^{n+1}|^2_{\Gamma_{i_1}} + |f^{n+1}|^2_{\Gamma_{i_1+1}}) \\ &-\tfrac{\Delta t}{2(\Delta x)^2}(|U^{n+1}|^2_{\Gamma_{i_1-\frac{1}{2}}} + |U^{n+1}|^2_{\Gamma_{i_1+\frac{3}{2}}} + 2|\tilde{U}^{n+1}|^2_{\Gamma_{i_1+\frac{1}{2}}}) \\ &+\tfrac{\Delta t}{\Delta x}(f^{n+1}_{i_1}(U^{n+1}_{i_1-\frac{1}{2}} - \tilde{U}^{n+1}_{i_1+\frac{1}{2}}) - f^{n+1}_{i_1+1}(U^{n+1}_{i_1+\frac{3}{2}} - \tilde{U}^{n+1}_{i_1+\frac{1}{2}})) \\ &+\tfrac{\Delta t}{(\Delta x)^2}(U^n_{i_1-\frac{1}{2}} + U^n_{i_1+\frac{3}{2}})\tilde{U}^{n+1}_{i_1+\frac{1}{2}} + (b^{n+1}_{i_1+\frac{1}{2}}C^{n+1}_{i_1} - b^n_{i_1+\frac{1}{2}}C^n_{i_1})\tilde{U}^{n+1}_{i_1+\frac{1}{2}}.\end{aligned} \tag{27}$$

For the third and fourth terms of Eqn. (27), we use the Hölder-inequality and ϵ-inequality, respectively, i.e.,

$$\tfrac{\Delta t}{(\Delta x)^2}(U^n_{i_1-\frac{1}{2}} + U^n_{i_1+\frac{3}{2}})\tilde{U}^{n+1}_{i_1+\frac{1}{2}} \le \tfrac{\Delta t}{2(\Delta x)^2}(|U^n|^2_{\Gamma_{i_1-\frac{1}{2}}} + |U^n|^2_{\Gamma_{i_1+\frac{3}{2}}} + 2|\tilde{U}^{n+1}|^2_{\Gamma_{i_1+\frac{1}{2}}}), \tag{28}$$

and

$$\begin{aligned}&\tfrac{\Delta t}{\Delta x}(f^{n+1}_{i_1}(U^{n+1}_{i_1-\frac{1}{2}} - \tilde{U}^{n+1}_{i_1+\frac{1}{2}}) - f^{n+1}_{i_1+1}(U^{n+1}_{i_1+\frac{3}{2}} - \tilde{U}^{n+1}_{i_1+\frac{1}{2}})) \\ &\le \epsilon(|U^{n+1}|^2_{\Gamma_{i_1+\frac{3}{2}}} + |U^{n+1}|^2_{\Gamma_{i_1-\frac{1}{2}}} + 2|\tilde{U}^{n+1}|^2_{\Gamma_{i_1+\frac{1}{2}}}) \\ &+M\tfrac{(\Delta t)^2}{(\Delta x)^2}\textstyle\sum_{i=i_1,i_1+1}|f^{n+1}|^2_{\Gamma_i}.\end{aligned} \tag{29}$$

Substituting (28) and (29) into Eqn. (27), we get that

$$\begin{aligned}&-\langle U^{n+1} - \tilde{U}^{n+1}, \tilde{U}^{n+1}\rangle_{\frac{1}{d^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}} \\ &\le \tfrac{\Delta t}{2}(|\hat{\partial}_t C^{n+1}|^2_{\Gamma_{i_1}} + |\hat{\partial}_t C^{n+1}|^2_{\Gamma_{i_1+1}}) + \epsilon(|U^{n+1}|^2_{\Gamma_{i_1+\frac{3}{2}}} + |U^{n+1}|^2_{\Gamma_{i_1-\frac{1}{2}}} + 2|\tilde{U}^{n+1}|^2_{\Gamma_{i_1+\frac{1}{2}}}) \\ &-\tfrac{\Delta t}{2(\Delta x)^2}(|U^{n+1}|^2_{\Gamma_{i_1-\frac{1}{2}}} + |U^{n+1}|^2_{\Gamma_{i_1+\frac{3}{2}}}) + \tfrac{\Delta t}{2(\Delta x)^2}(|U^n|^2_{\Gamma_{i_1-\frac{1}{2}}} + |U^n|^2_{\Gamma_{i_1+\frac{3}{2}}}) \\ &+M\tfrac{(\Delta t)^2}{(\Delta x)^2}\textstyle\sum_{i=i_1,i_1+1}|f^{n+1}|^2_{\Gamma_i} + (b^{n+1}_{i_1+\frac{1}{2}}C^{n+1}_{i_1} - b^n_{i_1+\frac{1}{2}}C^n_{i_1})\tilde{U}^{n+1}_{i_1+\frac{1}{2}}.\end{aligned} \tag{30}$$

Substituting (30) into (16), we obtain that

$$
\begin{array}{l}
R \leq \frac{\Delta x \Delta t}{2}(\mid \hat{\partial}_t C^{n+1} \mid^2_{\Gamma_{i_1}} + \mid \hat{\partial}_t C^{n+1} \mid^2_{\Gamma_{i_1+1}}) + \Delta x(\mid U^{n+1} \mid^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}}, \Gamma_{i_1+\frac{1}{2}}} - \mid \tilde{U}^{n+1} \mid^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}}, \Gamma_{i_1+\frac{1}{2}}}) \\
\quad -\frac{\Delta t}{2\Delta x}(\mid U^{n+1} \mid^2_{\Gamma_{i_1-\frac{1}{2}}} + \mid U^{n+1} \mid^2_{\Gamma_{i_1+\frac{3}{2}}}) + \frac{\Delta t}{2\Delta x}(\mid U^{n} \mid^2_{\Gamma_{i_1-\frac{1}{2}}} + \mid U^{n} \mid^2_{\Gamma_{i_1+\frac{3}{2}}}) \\
\quad +\epsilon \Delta x(\mid U^{n+1} \mid^2_{\Gamma_{i_1+\frac{3}{2}}} + \mid U^{n+1} \mid^2_{\Gamma_{i_1-\frac{1}{2}}} + 2 \mid \tilde{U}^{n+1} \mid^2_{\Gamma_{i_1+\frac{1}{2}}}) \\
\quad -\frac{b^{n+1}_{i_1+\frac{1}{2}}}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}} U^{n+1}_{i_1+\frac{1}{2}} C^{n+1}_{i_1} \Delta x + \frac{b^{n+1}_{i_1+\frac{1}{2}}}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}} \tilde{U}^{n+1}_{i_1+\frac{1}{2}} C^{n}_{i_1} \Delta x + M \frac{(\Delta t)^2}{\Delta x} \sum_{i=i_1, i_1+1} \mid f^{n+1} \mid^2_{\Gamma_i} \\
\quad +\Delta x (b^{n+1}_{i_1+\frac{1}{2}} C^{n+1}_{i_1} - b^{n}_{i_1+\frac{1}{2}} C^{n}_{i_1}) \tilde{U}^{n+1}_{i_1+\frac{1}{2}}.
\end{array}
\tag{31}
$$

By Lemma 1 and Lemma 2, we have that

$$
(\hat{\partial}_t C^{n+1}, C^{n+1}) = \frac{1}{2\Delta t}(\| C^{n+1} \|^2 - \| C^n \|^2) + \frac{\Delta t}{2} \| \hat{\partial}_t C^{n+1} \|^2, \tag{32}
$$

and

$$
(\delta_x U^{n+1}, C^{n+1}) = \| U^{n+1} \|^2_{\frac{1}{\bar{d}^{n+1}_{i_1}}} - \sum_{i=1}^{N-1} \frac{b^{n+1}_{i+\frac{1}{2}}}{\bar{d}^{n+1}_{i+\frac{1}{2}}} U^{n+1}_{i+\frac{1}{2}} C^{n+1}_i \Delta x. \tag{33}
$$

Substituting (31)-(33) into Eqn. (14), we can have that

$$
\begin{array}{l}
\frac{1}{2\Delta t}(\| C^{n+1} \|^2 - \| C^n \|^2) + \frac{\Delta t}{2} \| \hat{\partial}_t C^{n+1} \|^2 \\
\quad + \| U^{n+1} \|^2_{\frac{1}{\bar{d}^{n+1}}} + \frac{\Delta t}{2\Delta x}(\mid U^{n+1} \mid^2_{\Gamma_{i_1-\frac{1}{2}}} + \mid U^{n+1} \mid^2_{\Gamma_{i_1+\frac{3}{2}}}) \\
\quad -\frac{\Delta t}{2\Delta x}(\mid U^{n} \mid^2_{\Gamma_{i_1-\frac{1}{2}}} + \mid U^{n} \mid^2_{\Gamma_{i_1+\frac{3}{2}}}) \\
\leq \frac{\Delta x \Delta t}{2}(\mid \hat{\partial}_t C^{n+1} \mid^2_{\Gamma_{i_1}} + \mid \hat{\partial}_t C^{n+1} \mid^2_{\Gamma_{i_1+1}}) \\
\quad +\Delta x(\mid U^{n+1} \mid^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}}, \Gamma_{i_1+\frac{1}{2}}} - \mid \tilde{U}^{n+1} \mid^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}}, \Gamma_{i_1+\frac{1}{2}}}) \\
\quad +\epsilon \Delta x(\mid U^{n+1} \mid^2_{\Gamma_{i_1+\frac{3}{2}}} + \mid U^{n+1} \mid^2_{\Gamma_{i_1-\frac{1}{2}}} + 2 \mid \tilde{U}^{n+1} \mid^2_{\Gamma_{i_1+\frac{1}{2}}}) \\
\quad +(C, f^{n+1}) + \sum_{i=i_1, i \neq i_1} \frac{b^{n+1}_{i_1+\frac{1}{2}}}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}} U^{n+1}_{i_1+\frac{1}{2}} C^{n+1}_{i_1} \Delta x \\
\quad +\frac{b^{n+1}_{i_1+\frac{1}{2}}}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}} \tilde{U}^{n+1}_{i_1+\frac{1}{2}} C^{n}_{i_1} \Delta x + M \frac{(\Delta t)^2}{\Delta x} \sum_{i=i_1, i_1+1} \mid f^{n+1} \mid^2_{\Gamma_i} \\
\quad +\Delta x (b^{n+1}_{i_1+\frac{1}{2}} C^{n+1}_{i_1} - b^{n}_{i_1+\frac{1}{2}} C^{n}_{i_1}) \tilde{U}^{n+1}_{i_1+\frac{1}{2}}.
\end{array}
\tag{34}
$$

For the last term of (34), applying $\epsilon-$inequality, we obtain that

$$
(b^{n+1}_{i_1+\frac{1}{2}} C^{n+1}_{i_1} - b^{n}_{i_1+\frac{1}{2}} C^{n}_{i_1}) \tilde{U}^{n+1}_{i_1+\frac{1}{2}} \leq 2\epsilon \mid \tilde{U}^{n+1}_{i_1+\frac{1}{2}} \mid^2 + \frac{1}{4\epsilon}(\mid b^{n+1}_{i_1+\frac{1}{2}} \mid^2 \mid C^{n+1}_{i_1} \mid^2 + \mid b^{n}_{i_1+\frac{1}{2}} \mid^2 \mid C^{n}_{i_1} \mid^2).
$$

For the fourth, sixth and seventh terms of (34), applying the Hölder-inequality, we get that

$$
\begin{aligned}
&\tfrac{1}{2\Delta t}(\| C^{n+1} \|^2 - \| C^n \|^2) + \tfrac{\Delta t}{2} \| \hat{\partial}_t C^{n+1} \|^2 \\
&\quad + \| U^{n+1} \|^2_{\frac{1}{\bar{d}^{n+1}}} + \tfrac{\Delta t}{2\Delta x}(| U^{n+1} |^2_{\Gamma_{i_1-\frac{1}{2}}} + | U^{n+1} |^2_{\Gamma_{i_1+\frac{3}{2}}}) \\
&\quad - \tfrac{\Delta t}{2\Delta x}(| U^n |^2_{\Gamma_{i_1-\frac{1}{2}}} + | U^n |^2_{\Gamma_{i_1+\frac{3}{2}}}) \\
&\le \tfrac{\Delta x\Delta t}{2}(| \hat{\partial}_t C^{n+1} |^2_{\Gamma_{i_1}} + | \hat{\partial}_t C^{n+1} |^2_{\Gamma_{i_1+1}}) \\
&\quad + \Delta x(| U^{n+1} |^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}} - | \tilde{U}^{n+1} |^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}}) \\
&\quad + \epsilon\Delta x(| U^{n+1} |^2_{\Gamma_{i_1+\frac{3}{2}}} + | U^{n+1} |^2_{\Gamma_{i_1-\frac{1}{2}}}) \\
&\quad + 4\epsilon\Delta x \, | \tilde{U}^{n+1} |^2_{\Gamma_{i_1+\frac{1}{2}}} + \tfrac{1}{2} \| C^{n+1} \|^2 + \tfrac{1}{2} \| f^{n+1} \|^2 \\
&\quad + \tfrac{\Delta x}{4\epsilon}(| b^{n+1}_{i_1+\frac{1}{2}} |^2 | C^{n+1}_{i_1+\frac{1}{2}} |^2 + | b^n_{i_1+\frac{1}{2}} |^2 | C^n_{i_1+\frac{1}{2}} |^2) \\
&\quad + M\tfrac{(\Delta t)^2}{\Delta x} \textstyle\sum_{i=i_1,i_1+1} | f^{n+1} |^2_{\Gamma_i} \\
&\quad + \textstyle\sum_{i=i_1,i\neq i_1}^{I-1} [\frac{1}{2\bar{d}^{n+1}_{i_1+\frac{1}{2}}} | U^{n+1}_{i_1+\frac{1}{2}} |^2 + \frac{(b^{n+1}_{i_1+\frac{1}{2}})^2}{2\bar{d}^{n+1}_{i_1+\frac{1}{2}}} | C^{n+1}_{i_1} |^2]\Delta x \\
&\quad + [\tfrac{1}{2\bar{d}^{n+1}_{i_1+\frac{1}{2}}} | U^{n+1}_{i_1+\frac{1}{2}} |^2 + \tfrac{(b^{n+1}_{i_1+\frac{1}{2}})^2}{2\bar{d}^{n+1}_{i_1+\frac{1}{2}}} | C^{n+1}_{i_1} |^2]\Delta x.
\end{aligned}
\tag{35}
$$

Letting $\epsilon \le \frac{1}{8}$, we get that

$$
\begin{aligned}
&\tfrac{1}{2\Delta t}(\| C^{n+1} \|^2 - \| C^n \|^2) + \tfrac{\Delta t}{2} \| \hat{\partial}_t C^{n+1} \|^2 \\
&\quad + \| U^{n+1} \|^2_{\frac{1}{\bar{d}^{n+1}}} + \tfrac{\Delta t}{2\Delta x}(| U^{n+1} |^2_{\Gamma_{i_1-\frac{1}{2}}} + | U^{n+1} |^2_{\Gamma_{i_1+\frac{3}{2}}}) \\
&\quad - \tfrac{\Delta t}{2\Delta x}(| U^n |^2_{\Gamma_{i_1-\frac{1}{2}}} + | U^n |^2_{\Gamma_{i_1+\frac{3}{2}}}) \\
&\quad - [\tfrac{1}{2} + \max_{i\neq i_1} \frac{(b^{n+1}_{i_1+\frac{1}{2}})^2}{2\bar{d}^{n+1}_{i_1+\frac{1}{2}}} + \tfrac{1}{2} | b^{n+1}_{i_1+\frac{1}{2}} |^2] \| C^{n+1} \|^2 \\
&\le \tfrac{\Delta x\Delta t}{2}(| \hat{\partial}_t C^{n+1} |^2_{\Gamma_{i_1}} + | \hat{\partial}_t C^{n+1} |^2_{\Gamma_{i_1+1}}) \\
&\quad - \Delta x(1 - \tfrac{1}{2} - 4\epsilon\bar{d}^{n+1}_{i_1+\frac{1}{2}}) \, | \tilde{U}^{n+1} |^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}} \\
&\quad + [\max_{i\neq i_1} \frac{(b^{n+1}_{i_1+\frac{1}{2}})^2}{2\bar{d}^{n+1}_{i_1+\frac{1}{2}}} + \tfrac{1}{2} | b^{n+1}_{i_1+\frac{1}{2}} |^2] \| C^n \|^2 \\
&\quad + \tfrac{1}{2} \| f^{n+1} \|^2 + M\tfrac{(\Delta t)^2}{\Delta x} \textstyle\sum_{i=i_1,i_1+1} | f^{n+1} |^2_{\Gamma_i} \\
&\quad + \Delta x \textstyle\sum_{i=i_1,i\neq i_1}^{I-1} (\frac{1}{2} + \epsilon\bar{d}^{n+1}_{i_1+\frac{1}{2}}) \, | U^{n+1} |^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}} \\
&\quad + \Delta x \, | U^{n+1} |^2_{\frac{1}{\bar{d}^{n+1}_{i_1+\frac{1}{2}}},\Gamma_{i_1+\frac{1}{2}}} .
\end{aligned}
\tag{36}
$$

Assuming that $\max\{\frac{1}{2}+4\epsilon\bar{d}^{n+1}_{i_1+\frac{1}{2}},1\}\leq 1,\quad \frac{1}{2}-4\epsilon\bar{d}^{n+1}_{i_1+\frac{1}{2}}\geq 0$, and applying the Hölder-inequality, we obtain that

$$\begin{array}{l}\frac{1}{2}\hat{\partial}_t \parallel C^{n+1}\parallel^2+\frac{\Delta t}{2\Delta x}(\mid U^{n+1}\mid^2_{\Gamma_{i_1-\frac{1}{2}}}+\mid U^{n+1}\mid^2_{\Gamma_{i_1+\frac{3}{2}}})-\frac{\Delta t}{2\Delta x}(\mid U^{n}\mid^2_{\Gamma_{i_1-\frac{1}{2}}}+\mid U^{n}\mid^2_{\Gamma_{i_1+\frac{3}{2}}})\\ \quad -[\frac{1}{2}+\max\limits_{i\neq i_1}\frac{(b^{n+1}_{i_1+\frac{1}{2}})^2}{2\bar{d}^{n+1}_{i_1+\frac{1}{2}}}+\frac{1}{2}\mid b^{n+1}_{i_1+\frac{1}{2}}\mid^2]\parallel C^{n+1}\parallel^2\\ \leq [\max\limits_{i\neq i_1}\frac{(b^{n+1}_{i_1+\frac{1}{2}})^2}{2\bar{d}^{n+1}_{i_1+\frac{1}{2}}}+\frac{1}{2}\mid b^{n+1}_{i_1+\frac{1}{2}}\mid^2]\parallel C^{n}\parallel^2+M(\parallel f^{n+1}\parallel^2+\frac{(\Delta t)^2}{\Delta x}\sum_{i=i_1,i_1+1}\mid f^{n+1}\mid^2_{\Gamma_i}).\end{array}\tag{37}$$

We complete the proof.

Then we can have the following theorem.

Theorem 2 (Stability) *The scheme (5)-(8) is unconditionally stable in the sense of discrete L^2-norm over multiple sub-domains, i.e.,*

$$|||\{C^{n+1},U^{n+1}\}|||^2\leq M(|||\{C^0,U^0\}\,|||^2+\textstyle\sum_{l=1}^{n+1}\Delta t\,|||\,f^l|||^2),\tag{38}$$

where $M>0$ is a positive constant and

$$\begin{array}{l}|||\{C^{n+1},U^{n+1}\}|||^2=\parallel C^{n+1}\parallel^2+\frac{(\Delta t)^2}{\Delta x}(\mid U^{n+1}\mid^2_{\Gamma_{i_1-\frac{1}{2}}}+\mid U^{n+1}\mid^2_{\Gamma_{i_1+\frac{3}{2}}}),\\ |||f^{n+1}|||^2=\parallel f^{n+1}\parallel^2+\frac{(\Delta t)^2}{\Delta x}\sum_{i-i_1,i_1+1}\mid f^{n+1}\mid^2_{\Gamma_i}.\end{array}\tag{39}$$

Proof From (13), we have that

$$\begin{array}{l}(1-\beta_1\Delta t)\parallel C^{n+1}\parallel^2+\frac{(\Delta t)^2}{\Delta x}(\mid U^{n+1}\mid^2_{\Gamma_{i_1-\frac{1}{2}}}+\mid U^{n+1}\mid^2_{\Gamma_{i_1+\frac{3}{2}}})\\ \leq(1+\alpha_1\Delta t)\parallel C^{n}\parallel^2+\frac{(\Delta t)^2}{\Delta x}(\mid U^{n}\mid^2_{\Gamma_{i_1-\frac{1}{2}}}+\mid U^{n}\mid^2_{\Gamma_{i_1+\frac{3}{2}}})\\ \quad +M\Delta t(\parallel f^{n+1}\parallel^2+\frac{(\Delta t)^2}{\Delta x}\sum\limits_{i=i_1,i_1+1}\mid f^{n+1}\mid^2_{\Gamma_i}).\end{array}\tag{40}$$

So we have that

$$\partial_t\,|||\,\{C^{n+1},U^{n+1}\}|||^2\leq\kappa(|||\{C^{n+1},U^{n+1}\}\,|||^2+|||\,\{C^n,U^n\}|||^2)+M\,|||\,f^l\,|||^2_{\Gamma}.\tag{41}$$

where $\kappa=\alpha>0$ is only dependent of the coefficients but independent of Δt and Δx. Summing (41) up with respect to n, and by the discrete Gronwall's lemma, we complete the proof.

3.3 Convergence

Let c^n_i be the value of the exact solution $c(x,t)$ at the point (x_i,t^n). Define the flux $u^n_{i+\frac{1}{2}}=b^n_{i+\frac{1}{2}}c^n_i-\bar{d}^n_{i+\frac{1}{2}}\delta_x c^n_{i+\frac{1}{2}}$.

In order to analyze the truncation error of the scheme (5)-(7).

Lemma 4 *Assuming that the exact solution satisfies $c \in C^0([0,T];C^4(\Omega)) \cap C^2([0,T];C^0(\Omega))$, we can have the following truncation error equations:*

$$\hat{\partial}_t c_i^{n+1} + \delta_x u_i^{n+1} = f_i^{n+1} + \theta_i^{n+1}, \tag{42}$$

where $\theta_i^{n+1} = O(\Delta t + (\Delta x)^2$.

Proof From (42), we can have that

$$\begin{aligned}\theta_i^{n+1} &= \frac{c_i^{n+1}-c_i^n}{\Delta t} + \frac{u_{i+\frac{1}{2}}^{n+1}-u_{i-\frac{1}{2}}^{n+1}}{\Delta x} - f_i^{n+1} \\ &= \frac{c_i^{n+1}-c_i^n}{\Delta t} + \frac{b_{i+\frac{1}{2}}^{n+1}c_i^{n+1}-b_{i-\frac{1}{2}}^{n}c_{i-1}^{n+1}}{\Delta x} \\ &\quad -\frac{\bar{d}_{i+\frac{1}{2}}^{n+1}\delta_x c_{i+\frac{1}{2}}^{n+1}-\bar{d}_{i-\frac{1}{2}}^{n+1}\delta_x c_{i-\frac{1}{2}}^{n+1}}{\Delta x} - f_i^{n+1}.\end{aligned} \tag{43}$$

For the first term of the right-hand side of (43), we can have that

$$\frac{c_i^{n+1}-c_i^n}{\Delta t} = (c_t)_i^{n+1} - \frac{\Delta t}{2}(c_{tt})_i^{n_1}, \tag{44}$$

For the second term of (43), we can have the following error

$$b_{i+\frac{1}{2}}^{n+1}c_i^{n+1} = b_{i+\frac{1}{2}}^{n+1}(c_{i+\frac{1}{2}}^{n+1} - \frac{\Delta x}{2}(c_x)_{i+\frac{1}{2}}^{n+1} + \frac{(\Delta x)^2}{8}(c_{xx})_{i+\frac{1}{2}}^{n+1} + O((\Delta x)^3)), \tag{45}$$

and

$$b_{i-\frac{1}{2}}^{n+1}c_{i-1}^{n+1} = b_{i-\frac{1}{2}}^{n+1}(c_{i-\frac{1}{2}}^{n+1} - \frac{\Delta x}{2}(c_x)_{i-\frac{1}{2}}^{n+1} + \frac{(\Delta x)^2}{8}(c_{xx})_{i-\frac{1}{2}}^{n+1} + O((\Delta x)^3)), \tag{46}$$

For the third term of (43), it follows that

$$\begin{aligned}\bar{d}_{i+\frac{1}{2}}^{n+1}\delta_x c_{i+\frac{1}{2}}^{n+1} &= \frac{d_{i+\frac{1}{2}}^{n+1}}{1+\frac{b_{i+\frac{1}{2}}^{n+1}}{2d_{i+\frac{1}{2}}^{n+1}}\Delta x}\delta_x c_{i+\frac{1}{2}}^{n+1} \\ &= \frac{d_{i+\frac{1}{2}}^{n+1}}{1+\frac{b_{i+\frac{1}{2}}^{n+1}}{2d_{i+\frac{1}{2}}^{n+1}}\Delta x}((c_x)_{i+\frac{1}{2}}^{n+1} + \frac{(\Delta x)^2}{24}(c_{xxx})_{i+\frac{1}{2}}^{n+1} + O((\Delta x)^3)) \\ &= d_{i+\frac{1}{2}}^{n+1}(1 - \frac{b_{i+\frac{1}{2}}^{n+1}\Delta x}{2d_{i+\frac{1}{2}}^{n+1}} + O((\Delta x)^2))((c_x)_{i+\frac{1}{2}}^{n+1} \\ &\quad + \frac{(\Delta x)^2}{24}(c_{xxx})_{i+\frac{1}{2}}^{n+1} + O((\Delta x)^3)) \\ &= d_{i+\frac{1}{2}}^{n+1}(c_x)_{i+\frac{1}{2}}^{n+1} - \frac{b_{i+\frac{1}{2}}^{n+1}\Delta x}{2}(c_x)_{i+\frac{1}{2}}^{n+1} + O((\Delta x)^2,\end{aligned} \tag{47}$$

and

$$\bar{d}_{i-\frac{1}{2}}^{n+1}\delta_x c_{i-\frac{1}{2}}^{n+1} = d_{i-\frac{1}{2}}^{n+1}(c_x)_{i-\frac{1}{2}}^{n+1} - \frac{b_{i-\frac{1}{2}}^{n+1}\Delta x}{2}(c_x)_{i-\frac{1}{2}}^{n+1} + O((\Delta x)^2). \tag{48}$$

From (45),(46) and (47), we can get that

$$b^{n+1}_{i+\frac{1}{2}} c^{n+1}_{i} - \bar{d}^{n+1}_{i+\frac{1}{2}} \delta_x c^{n+1}_{i+\frac{1}{2}} = b^{n+1}_{i+\frac{1}{2}} c^{n+1}_{i+\frac{1}{2}} - d^{n+1}_{i+\frac{1}{2}} (c_x)^{n+1}_{i+\frac{1}{2}} + O((\Delta x)^2), \tag{49}$$

and

$$b^{n+1}_{i-\frac{1}{2}} c^{n+1}_{i-1} - \bar{d}^{n+1}_{i-\frac{1}{2}} \delta_x c^{n+1}_{i-\frac{1}{2}} = b^{n+1}_{i-\frac{1}{2}} c^{n+1}_{i-\frac{1}{2}} - d^{n+1}_{i-\frac{1}{2}} (c_x)^{n+1}_{i-\frac{1}{2}} + O((\Delta x)^2). \tag{50}$$

Thus, we can have that

$$\begin{aligned} \frac{u^{n+1}_{i+\frac{1}{2}} - u^{n+1}_{i-\frac{1}{2}}}{\Delta x} &= \frac{b^{n+1}_{i+\frac{1}{2}} c^{n+1}_{i} - b^{n+1}_{i-\frac{1}{2}} c^{n+1}_{i-1}}{\Delta x} - \frac{\bar{d}^{n+1}_{i+\frac{1}{2}} \delta_x c^{n+1}_{i+\frac{1}{2}} - \bar{d}^{n+1}_{i-\frac{1}{2}} \delta_x c^{n+1}_{i-\frac{1}{2}}}{\Delta x} \\ &= \frac{b^{n+1}_{i+\frac{1}{2}} c^{n+1}_{i+\frac{1}{2}} - b^{n+1}_{i-\frac{1}{2}} c^{n+1}_{i-\frac{1}{2}}}{\Delta x} - \frac{d^{n+1}_{i+\frac{1}{2}} \delta_x (c_x)^{n+1}_{i+\frac{1}{2}} - d^{n+1}_{i-\frac{1}{2}} \delta_x (c_x)^{n+1}_{i-\frac{1}{2}}}{\Delta x} + O((\Delta x)^2). \end{aligned} \tag{51}$$

From (45), (46), (47), (51) and Eqn. (1), we can get that

$$\theta^{n+1}_{i} = O(\Delta t + (\Delta x)^2. \tag{52}$$

Assume that the exact solution satisfies $c \in C^0([0,T]; C^5(\Omega)) \cap C^2([0,T]; C^1(\Omega))$, we can further obtain that

$$\delta_x \theta^{n+1}_{i+\frac{1}{2}} = O(\Delta t + (\Delta x)^2. \tag{53}$$

Let

$$\begin{aligned} \rho^{n+1}_{i} = c^{n+1}_{i} - C^{n+1}_{i}, \quad \rho^{n+1}_{i+\frac{1}{2}} = c^{n+1}_{i+\frac{1}{2}} - C^{n+1}_{i+\frac{1}{2}}, \quad \forall i, \\ \zeta^{n+1}_{i+\frac{1}{2}} = u^{n+1}_{i+\frac{1}{2}} - U^{n+1}_{i+\frac{1}{2}} = b^{n+1}_{i+\frac{1}{2}} \rho^{n+1}_{i} - \bar{d}^{n+1}_{i+\frac{1}{2}} \rho^{n+1}_{i+\frac{1}{2}}, \quad i \neq i_1, \\ \tilde{\zeta}^{n+1}_{i_1+\frac{1}{2}} = u^{n+1}_{i_1+\frac{1}{2}} - \tilde{U}^{n+1}_{i_1+\frac{1}{2}}. \end{aligned} \tag{54}$$

Subtracting Eqns. (42) from Eqns. (9), we have the error equations as

$$\begin{cases} \hat{\partial}_t \rho^{n+1}_{i} + \delta_x \zeta^{n+1}_{i} = \theta^{n+1}_{i}, \quad i = 1, \cdots, i_1 - 1, i_1 + 2, \cdots, I, \\ \hat{\partial}_t \rho^{n+1}_{i_1} + \frac{\tilde{\zeta}^{n+1}_{i_1+\frac{1}{2}} - \zeta^{n+1}_{i_1-\frac{1}{2}}}{\Delta x} = \theta^{n+1}_{i_1}, \quad i = i_1, \\ \hat{\partial}_t \rho^{n+1}_{i_1+1} + \frac{\zeta^{n+1}_{i_1+\frac{3}{2}} - \tilde{\zeta}^{n+1}_{i_1+\frac{1}{2}}}{\Delta x} = \theta^{n+1}_{i_1+1}, \quad i = i_1 + 1, \end{cases} \tag{55}$$

and

$$\begin{aligned} (1 + 2r^{n+1}_{i_1+\frac{1}{2}}) \tilde{\zeta}^{n+1}_{i_1+\frac{1}{2}} &= \zeta^{n}_{i_1+\frac{1}{2}} + r^{n+1}_{i_1+\frac{1}{2}} (\zeta^{n}_{i_1-\frac{1}{2}} + \zeta^{n}_{i_1+\frac{3}{2}}) - \Delta t \bar{d}^{n+1}_{i_1+\frac{1}{2}} \delta_x \theta^{n+1}_{i_1+\frac{1}{2}} \\ &- \bar{d}^{n+1}_{i_1+\frac{1}{2}} \delta_x \rho^{n}_{i_1+\frac{1}{2}} + \bar{d}^{n}_{i_1+\frac{1}{2}} \delta_x \rho^{n}_{i_1+\frac{1}{2}} + \gamma^{n+1}_{i_1+\frac{1}{2}}. \end{aligned} \tag{56}$$

where $\gamma^{n+1}_{i_1+\frac{1}{2}} = O(\Delta t + \frac{(\Delta t)^2}{(\Delta x)^2})$ and with the boundary conditions $\zeta^{n+1}_{\frac{1}{2}} = 0$ and $\zeta^{n+1}_{I+\frac{1}{2}} = 0$.

Similar to the proof of Lemmas 3, we have the following lemmas.

Lemma 5 *Let* $\{\rho_i^{n+1}\}$ *and* $\{\zeta_{i+\frac{1}{2}}^{n+1}\}$ *be the solution of (55)-(56). It holds that*

$$\begin{aligned}&(1-\beta_1\Delta t)\parallel \rho^{n+1}\parallel^2+\tfrac{(\Delta t)^2}{\Delta x}(\mid \zeta^{n+1}\mid^2_{\Gamma_{i_1-\frac{1}{2}}}+\mid \zeta^{n+1}\mid^2_{\Gamma_{i_1+\frac{3}{2}}})\\ \leq &(1+\alpha\Delta t)\parallel \rho^{n}\parallel^2+\tfrac{(\Delta t)^2}{\Delta x}(\mid \zeta^{n}\mid^2_{\Gamma_{i_1-\frac{1}{2}}}+\mid \zeta^{n}\mid^2_{\Gamma_{i_1+\frac{3}{2}}})\\ &+M\Delta t(\parallel \theta^{n+1}\parallel^2+\tfrac{(\Delta t)^2}{\Delta x}\textstyle\sum_{i=i_1,i_1+1}\mid \theta^{n+1}\mid^2_{\Gamma_i})+M\Delta t(\Delta x\mid \gamma^{n+1}\mid^2_{\Gamma_{i_1+\frac{1}{2}}}).\end{aligned} \tag{57}$$

where $M>0$ *is a positive constant.*

Considering the estimates of θ_i and $\gamma_{i_1+\frac{1}{2}}$. Applying Lemma 5 and Theorem 2, we can obtain the following theorem.

Theorem 3 (Convergence) *Assuming that the solution* c *of (1) satisfies that*

$$c\in C^0([0,T];C^5(\Omega))\cap C^1([0,T];C^3(\Omega))\cap C^2([0,T];C^1(\Omega)).$$

Let C *be the numerical solution of the scheme (5)-(8). It holds that*

$$\parallel c^n-C^n\parallel\leq M(\Delta t+(\Delta x)^2+\frac{(\Delta t)^2}{(\Delta x)^{\frac{3}{2}}}, \tag{58}$$

for $n\geq 1$, *where* $M>0$ *is a constant.*

4 Numerical experiments

4.1 Example 1

In this example, we assume that the initial condition is $c=x^2(1-x)^2$, the velocity and diffusion coefficient are $b(x,t)=(1-x)x$, $d(x,t)=D(1+x^2)$ and the source (or sink) term $f=2e^{-t}\left(x^2(1-x)^2(1-3x)-D\left(10x^4-12x^3+9x^2-6x+1\right)\right)$. The exact solution is $c=e^{-t}x^2(1-x)^2$ over the domain [0,1].

In Table 1, the time step Δt is set 1/1000000. The space step h is taken as 1/100, 1/200, 1/400 and 1/600. We can find that our scheme is asymptotically a second order when the space h decreases.

In Table 2, we take $\Delta t=h$. The space step h is set from 1/100, 1/200, 1/300 to 1/400. We can find that the scheme is asymptotically a first order when the time step Δt becomes small.

In Table 3, the time step Δt is set 1/1000000. The space step h is taken as 1/20, 1/50, 1/100 and 1/200. The error decreases with the decrease of space step, and the mass error is kept at 10^{-13}, which satisfies mass conservation.

The efficiency of our parallel algorithm are performed on the small server which has only two CPU and each CPU (Intel(R) Xeon(R) Silver 4216) contains 16 cores. We take the space step $h=1/2000$ and time step as and $\Delta t=1/1000000$ with the diffusion D=5E-3. The speedup S_p is defined by $S_p=\frac{T_1}{T_p}$, where T_p is the total time used over multi-block sub-domains and

Table 1 The space order of convergence of the scheme

t	$D \setminus h$		1/100	1/200	1/400	1/600
0.1	1	L^2-norm	1.8568E-5	4.6201E-6	1.1332E-6	4.8769E-7
		order	-	2.0068	2.0275	2.0794
	0.1	L^2-norm	2.2867E-6	5.5247E-7	1.2029E-7	4.3829E-8
		order	-	2.0493	2.1994	2.4900
	0.01	L^2-norm	5.9887E-6	1.5678E-6	4.1068E-7	1.9287E-7
		order	-	1.9335	1.9327	1.8640
	0.001	L^2-norm	3.4603E-5	1.1330E-5	3.3635E-6	1.6008E-6
		order	-	1.6107	1.7521	1.8312
0.2	1	L^2-norm	3.4177E-5	8.5259E-6	2.1130E-6	9.2551E-7
		order	-	2.0031	2.0126	2.0360
	0.1	L^2-norm	3.8457E-6	9.4306E-7	2.1782E-7	8.4388E-8
		order	-	2.0278	2.1142	2.3387
	0.01	L^2-norm	1.0743E-5	2.8031E-6	7.2384E-7	3.3080E-7
		order	-	1.9383	1.9533	1.9313
	0.001	L^2-norm	6.5125E-5	2.1320E-5	6.3184E-6	2.9987E-6
		order	-	1.6110	1.7546	1.8381
0.5	1	L^2-norm	7.2657E-5	1.8170E-5	4.5473E-6	2.0247E-6
		order	-	1.9995	1.9985	1.9955
	0.1	L^2-norm	7.7602E-6	1.9450E-6	4.9120E-7	2.2199E-7
		order	-	1.9963	1.9854	1.9588
	0.01	L^2-norm	1.9585E-5	5.0913E-6	1.2967E-6	5.7868E-7
		order	-	1.9436	1.9732	1.9899
	0.001	L^2-norm	1.3503E-4	4.4181E-5	1.3058E-5	6.1785E-6
		order	-	1.6118	1.7585	1.8456
1	1	L^2-norm	1.1621E-4	2.9061E-5	7.2733E-6	3.2384E-6
		order	-	1.9996	1.9984	1.9956
	0.1	L^2-norm	1.2394E-5	1.9689E-6	8.7628E-7	4.9422E-7
		order	-	1.9963	1.9850	1.9577
	0.01	L^2-norm	2.2893E-5	5.9506E-6	1.5147E-6	6.7553E-7
		order	-	1.9438	1.9740	1.9915
	0.001	L^2-norm	1.9610E-4	6.3927E-5	1.8814E-5	8.8836E-6
		order	-	1.6171	1.7646	1.8507

T_1 is the time used with the second order modified upwind scheme. Let p be the number of sub-domains. E_{ff} is the parallelism efficiency, defined by $E_{ff} = \frac{S_p}{p}$. We can see that the L^2 errors are nearly same (about 10^{-7}) in Table 4, and CPU time decreases when the domain is divided into multi-block sub-domains. The speedup increases when multi-block sub-domains are used and the parallelism efficiency is about 50%.

4.2 Example 2

In the second example 2, we take the velocity $b(x,t) = \sin x$ and

$$f = e^{-t}\left(x^2(1-x)^2(\cos x - 1) + 2x\sin x(2x-1)(x-1) - 2D\left(10x^4 - 12x^3 + 9x^2 - 6x + 1\right)\right).$$

the initial solution and exact solution is same as the Example 1.

Table 2 The time order of convergence of the scheme

t	$D \setminus \Delta t$		1/100	1/200	1/300	1/400
0.1	1	L^2-norm	1.0951E-4	6.3737E-5	4.3040E-5	3.1415E-5
		order	-	0.7809	0.9684	1.0944
	0.1	L^2-norm	6.3087E-5	3.0046E-5	1.9659E-5	1.4698E-5
		order	-	1.0702	1.0462	1.0109
	0.01	L^2-norm	4.3230E-5	1.9699E-5	1.2729E-5	9.3995E-6
		order	-	1.0944	1.0770	1.0540
	0.001	L^2-norm	6.0573E-5	2.0958E-5	1.1558E-5	7.7731E-6
		order	-	1.0944	1.4678	1.3790
0.2	1	L^2-norm	1.7391E-4	7.4913E-5	4.6730E-5	3.4316E-5
		order	-	1.2151	1.1639	1.0733
	0.1	L^2-norm	7.2673E-5	3.5424E-5	2.3411E-5	1.7483E-5
		order	-	1.0367	1.0215	1.0149
	0.01	L^2-norm	5.3947E-5	2.5400E-5	1.6645E-5	1.2386E-5
		order	-	1.0867	1.0424	1.0273
	0.001	L^2-norm	7.7592E-5	2.7168E-5	1.5263E-5	1.0481E-5
		order	-	1.5140	1.4221	1.3065
0.5	1	L^2-norm	1.7573E-4	7.4084E-5	4.6382E-5	3.3700E-5
		order	-	1.2461	1.1549	1.1103
	0.1	L^2-norm	1.0553E-4	5.0890E-5	3.3509E-5	2.4975E-5
		order	-	1.0522	1.0306	1.0217
	0.01	L^2-norm	8.1443E-5	4.0537E-5	2.7095E-5	2.0365E-5
		order	-	1.0066	0.9936	0.9925
	0.001	L^2-norm	1.2784E-4	4.6462E-5	2.6905E-5	1.8967E-5
		order	-	1.4602	1.3474	1.2153
1	1	L^2-norm	2.3254E-4	8.8687E-5	5.3103E-5	3.7595E-5
		order	-	1.3907	1.2649	1.2005
	0.1	L^2-norm	1.4748E-4	7.0628E-5	4.6391E-5	3.4533E-5
		order	-	1.0622	1.0366	1.0261
	0.01	L^2-norm	1.2025E-4	6.0778E-5	4.0763E-5	3.0680E-5
		order	-	0.9844	0.9852	0.9878
	0.001	L^2-norm	1.8220E-4	6.8919E-5	4.1217E-5	2.9679E-5
		order	-	1.4025	1.2679	1.1416

In Table 5, the time step Δt is taken as 1/1000000 and the space step h is taken as 1/100, 1/200, 1/400 and 1/600, respectively. It is clear that our scheme is of second-order convergent in space.

In Table 6, we take the time step $\Delta t = h$ and the space step $h = 1/100$, 1/200, 1/300 and 1/400. We can find that the time convergence order is first when the time step Δt becomes small.

In Table 7, the time step Δt is taken as 1/1000000. And the space step h is taken as 1/20, 1/50, 1/100 and 1/200. The error decreases with the decrease of space step, and the mass error is kept at 10^{-13}.

Table 3 The error and mass errors of the scheme

t	$D \setminus h$		1/20	1/50	1/100	1/200
0.1	1	L^2-norm	4.6446E-4	7.4349E-5	1.8568E-5	4.6201E-6
		Masserror	9.8887E-14	2.1965E-13	3.4212E-14	1.1049E-13
	0.1	L^2-norm	5.7306E-5	9.2159E-6	2.2867E-6	5.5247E-7
		Masserror	2.2177E-13	3.6174E-14	6.5523E-14	6.3232E-14
	0.01	L^2-norm	1.1393E-4	2.2188E-5	5.9887E-6	1.5678E-6
		Masserror	1.2411E-13	1.5550E-13	5.9483E-14	5.2941E-14
	0.001	L^2-norm	2.9985E-4	9.4188E-5	3.4603E-5	1.1330E-5
		Masserror	2.8627E-14	1.8106E-14	4.1340E-14	9.4625E-14
0.5	1	L^2-norm	1.8148E-3	2.9058E-4	7.2657E-5	1.8170E-5
		Masserror	4.0842E-13	9.1076E-13	1.4161E-13	4.5687E-13
	0.1	L^2-norm	1.9363E-4	3.1020E-5	7.7602E-6	1.9450E-6
		Masserror	9.2033E-13	1.4992E-13	2.7087E-13	2.6139E-13
	0.01	L^2-norm	3.7116E-4	7.2663E-5	1.9585E-5	5.0913E-6
		Masserror	5.1320E-13	6.4202E-13	1.4580e-13	2.1906E-13
	0.001	L^2-norm	1.1401E-3	3.6587E-4	1.3503E-4	4.4181E-5
		Masserror	1.1575E-13	3.6587E-04	1.7074E-13	4.4181E-05
1	1	L^2-norm	2.9028E-3	4.6477E-4	1.1621E-4	2.9061E-5
		Masserror	6.5747E-13	1.4677E-12	2.2827E-13	7.3471E-13
	0.1	L^2-norm	3.0927E-4	4.9538E-5	1.2394E-5	3.1065E-6
		Masserror	1.4835E-12	2.4081E-13	4.3507E-13	4.1927E-13
	0.01	L^2-norm	4.3048E-4	8.4856E-5	2.2893E-5	5.9506E-6
		Masserror	8.2302E-13	1.0319E-12	3.9390E-13	3.5081E-13
	0.001	L^2-norm	1.6213E-3	5.3079E-4	1.9610E-4	6.3927E-5
		Masserror	1.8559E-13	1.2723E-13	2.7589E-13	6.2725E-13
2	1	L^2-norm	3.9655E-3	6.3493E-4	1.5876E-4	3.9701E-5
		Masserror	9.0390E-13	2.0214E-12	3.1442E-13	1.0066E-12
	0.1	L^2-norm	4.2723E-4	6.8426E-5	1.7119E-5	4.2914E-6
		Masserror	2.0453E-12	3.2876E-13	5.9493E-13	5.7399E-13
	0.01	L^2-norm	2.6950E-4	5.3638E-5	1.4530E-5	3.7818E-6
		Masserror	1.1249E-12	1.4091E-12	5.4015E-13	4.8152E-13
	0.001	L^2-norm	1.6645E-3	5.7959E-4	2.1583E-4	6.9971E-5
		Masserror	2.5711E-13	1.7607E-13	3.8015E-13	8.5872E-13

5 Nonlinear convection diffusion equations

In this section, we extend our scheme to nonlinear convection-diffusion equations. The nonlinear convection diffusion equations are considered as follows,

$$\begin{aligned} \frac{\partial\theta}{\partial t} + \frac{\partial}{\partial z}(B(\theta)\theta - D(\theta)\frac{\partial\theta}{\partial z}) = f, (z,t) \in (0,L)\times(0,T), \\ B(\theta)\theta - D(\theta)\frac{\partial\theta}{\partial z} = 0, if \quad z = 0\, or\, z = L, \quad t > 0, \\ \theta(z,0) = \theta_0(z), \qquad z \in (0,L), \end{aligned} \tag{59}$$

where the initial function $\theta_0(z)$ is a smooth function. Let $q(z,t)$ be the flux of problem (59) defined as $q = B(\theta)\theta - D(\theta)\frac{\partial\theta}{\partial z}$ which ensure that our proposed scheme is mass conservative.The numerical solutions $\{\Theta_i^n\}$ and numerical fluxes $\{Q^n_{i+\frac{1}{2}}\}$ to denote the numerical approximations to solution $\theta(z_i,t^n)$ and fluxes $q(z_{i+\frac{1}{2}},t^n)$, respectively. Let $\Theta_{i+\frac{1}{2}} = \frac{\Theta_i+\Theta_{i+1}}{2}$.

Table 4 Speedup and efficiency over multi-block sub-domains

		1	2	4	8	10
t=0.1	e_h	5.5543E-8	5.5544E-8	5.5543E-8	5.5543E-8	5.5543E-8
	Masserr	2.6754E-15	2.6103E-15	2.6784E-15	1.44286E-15	1.88521E-15
	T_p(sec)	4.9301	3.0488	2.0474	1.5897	1.4438
	S_p	1.0000	1.6171	2.4080	3.1013	3.4147
	E_{ff}(%)	100.00	80.86	60.20	38.77	34.15
t=0.2	e_h	7.5750E-8	7.5751E-8	7.5750E-8	7.5749E-8	7.5749E-8
	Masserr	5.1339E-15	5.0055E-15	5.2102E-15	2.8259E-15	3.6438E-15
	T_p(sec)	9.1878	5.7497	3.6867	2.5771	2.3824
	S_p	1.0000	1.5980	2.4921	3.5652	3.8565
	E_{ff}(%)	100.00	79.90	62.30	44.56	38.57
t=0.5	e_h	1.1217E-7	1.1217E-7	1.1216E-7	1.1216E-7	1.1216E-7
	Masserr	1.0950E-14	1.1090E-14	1.1276E-14	5.9709E-15	7.8843E-15
	T_p(sec)	23.8091	14.6632	8.2321	5.3486	4.7635
	S_p	1.0000	1.6237	2.8922	4.4515	4.9982
	E_{ff}(%)	100.00	81.19	72.30	55.64	49.98
t=1	e_h	1.4487E-7	1.4487E-7	1.4487E-7	1.4487E-7	1.4487E-7
	Masserr	1.8579E-14	1.7923E-14	1.8024E-14	9.2704E-15	1.2278E-14
	T_p(sec)	45.5337	28.5940	17.2753	10.6398	9.4153
	S_p	1.0000	1.5924	2.6358	4.2796	4.8361
	E_{ff}(%)	100.00	79.62	65.89	53.49	48.36

5.1 Scheme

Our mass-conserving domain decomposition method is described as follows.

Step 1: The predicted interface fluxes $\{\tilde{Q}^{n+1}_{i_\alpha+\frac{1}{2}}\}$ are computed as

$$\frac{\tilde{Q}^{n+1}_{i_\alpha+\frac{1}{2}}-Q^{n}_{i_\alpha+\frac{1}{2}}}{\Delta t}=\bar{D}^{n+1}_{i_\alpha+\frac{1}{2}}\frac{Q^{n}_{i_\alpha+\frac{3}{2}}-2\tilde{Q}^{n+1}_{i_\alpha+\frac{1}{2}}+Q^{n}_{i_\alpha-\frac{1}{2}}}{(\Delta z)^2}-\bar{D}^{n+1}_{i_\alpha+\frac{1}{2}}\delta_z f^{n+1}_{i_\alpha+\frac{1}{2}}-\frac{\bar{D}^{n+1}_{i_\alpha+\frac{1}{2}}-\bar{D}^{n}_{i_\alpha+\frac{1}{2}}}{\Delta t}\delta_x C^{n}_{i_\alpha+\frac{1}{2}}, \quad (60)$$

where i_α is the location index number of the interface and $D^{n}_{i_\alpha+\frac{1}{2}}=D(\Theta^{n}_{i_\alpha+\frac{1}{2}})$ and

$$\bar{D}^{n}_{i+\frac{1}{2}}=\frac{D^{n}_{i+\frac{1}{2}}}{1+\frac{|B^{n}_{i+\frac{1}{2}}|}{2D^{n}_{i+\frac{1}{2}}}\Delta z}.$$

Step 2: The intermediate solutions $\{\Theta^{n+1}_{i}\}$ are computed by the coupled implicit scheme.

$$\begin{aligned}\frac{\Theta^{n+1}_i-\Theta^{n}_i}{\Delta t}&=-\delta_z Q^{n+1}_i+f^{n+1}_i, && z_i\in\Omega_\alpha,\\ Q^{n+1}_{i+\frac{1}{2}}&=B^{n+1}_{i+\frac{1}{2}}(\lambda(B^{n+1}_{i+\frac{1}{2}})\Theta^{n+1}_i+(1-\lambda(B^{n+1}_{i+\frac{1}{2}}))\Theta^{n+1}_{i+1})-\bar{D}^{n+1}_{i+\frac{1}{2}}\delta_z\Theta^{n+1}_{i+\frac{1}{2}}, && z_{i+\frac{1}{2}}\in\Omega_\alpha,\\ Q^{n+1}_{i_\alpha+\frac{1}{2}}&=\tilde{Q}^{n+1}_{i_\alpha+\frac{1}{2}}, && z_{i_\alpha+\frac{1}{2}}\in\Gamma_\alpha.\end{aligned} \quad (61)$$

where $D^{n+1}_{i+\frac{1}{2}}=D(\Theta^{n+1}_{i+\frac{1}{2}})$ and $B^{n+1}_{i+\frac{1}{2}}=B(\Theta^{n+1}_{i+\frac{1}{2}})$.

Step 3: The interface fluxes $\{\tilde{Q}^{n+1}_{i_\alpha+\frac{1}{2}}\}$ are recomputed by

$$Q^{n+1}_{i_\alpha+\frac{1}{2}}=B^{n+1}_{i_\alpha+\frac{1}{2}}(\lambda(B^{n+1}_{i_\alpha+\frac{1}{2}})\Theta^{n+1}_{i_\alpha}+(1-\lambda(B^{n+1}_{i_\alpha+\frac{1}{2}}))\Theta^{n+1}_{i_\alpha+1})-\bar{D}^{n+1}_{i_\alpha+\frac{1}{2}}\delta_z\Theta^{n+1}_{i_\alpha+\frac{1}{2}}. \quad (62)$$

Table 5 The space order of convergence of the scheme

t	$D\setminus h$		1/100	1/200	1/400	1/600
0.1	1	L^2-norm	1.7843E-5	4.4391E-6	1.0881E-6	4.6776E-7
		order	-	2.0070	2.0285	2.0821
	0.1	L^2-norm	3.3944E-6	8.5137E-7	2.1201E-7	9.6216E-8
		order	-	1.9953	2.0057	1.9485
	0.01	L^2-norm	4.9170E-5	1.3527eE-5	3.5602E-6	1.6150E-6
		order	-	1.8619	1.9258	1.9496
	0.001	L^2-norm	2.4633E-4	1.0696E-4	3.9381E-5	2.0353E-5
		order	-	1.2035	1.4415	1.6279
0.2	1	L^2-norm	3.3009E-5	8.2343eE-6	2.0404E-6	8.9350E-7
		order	-	2.0031	2.0128	2.0366
	0.1	L^2-norm	4.3698E-6	1.0937E-6	2.6916E-7	1.1769E-7
		order	-	1.9984	2.0227	2.0403
	0.01	L^2-norm	9.5280E-5	2.6369E-5	6.9490E-6	3.1473E-6
		order	-	1.8533	1.9240	1.9534
	0.001	L^2-norm	5.9878E-4	2.7535-4	1.0443E-4	5.4445E-5
		order	-	1.1208	1.3987	1.6064
0.5	1	L^2-norm	7.0267E-5	1.7572E-5	4.3980E-6	1.9583E-6
		order	-	1.9996	1.9984	1.9954
	0.1	L^2-norm	5.7084E-6	1.4371E-6	3.6431E-7	1.6524E-7
		order	-	1.9899	1.9799	1.9499
	0.01	L^2-norm	1.5968E-4	4.4633E-5	1.1805E-5	5.3472E-6
		order	-	1.8390	1.9187	1.9532
	0.001	L^2-norm	1.3326E-3	6.3852E-4	2.4680E-4	1.2937E-4
		order	-	1.0614	1.3714	1.5930
1	1	L^2-norm	1.1231E-4	2.8087E-5	7.0297E-6	3.1302E-6
		order	-	1.9995	1.9984	1.9954
	0.1	L^2-norm	8.3130E-6	2.0913E-6	5.3228E-7	2.4328E-7
		order	-	1.9910	1.9741	1.9310
	0.01	L^2-norm	9.2289E-5	2.6174E-5	6.9880E-6	3.1810E-6
		order	-	1.8180	1.9052	1.9410
	0.001	L^2-norm	8.3056E-4	4.0300E-4	1.5725E-4	8.2785E-5
		order	-	1.0433	1.3577	1.5824

The boundary conditions are approximated by

$$Q^{n+1}_{\frac{1}{2}} = q^{n+1}_{\frac{1}{2}}, \quad Q^{n+1}_{I+\frac{1}{2}} = q^{n+1}_{I+\frac{1}{2}}. \tag{63}$$

The initial values are computed by

$$\Theta^0_i = \theta_0(z_i). \tag{64}$$

Next, we will give the theoretical analysis when $Q^{n+1}_{\frac{1}{2}} = Q^{n+1}_{I+\frac{1}{2}} = 0$, and it does extend to more general boundary problems.

Theorem 4 *The scheme (60)-(64) satisfy mass conservation over the global domain with $f = 0$, i.e.,*

$$\textstyle\sum_{i=1}^{I} \Theta^n_i \Delta z = \sum_{i=1}^{I} \Theta^0_i \Delta z, \quad n = 1, \cdots, M. \tag{65}$$

Table 6 The time order of convergence of the scheme

t	$\Delta t \setminus h$		1/100	1/200	1/300	1/400
0.1	1	L^2-norm	1.9612E-4	1.1937E-4	8.1425E-5	5.9695E-5
		order	-	0.7163	0.9435	1.0791
	0.1	L^2-norm	1.1427E-4	5.4797E-5	3.5953E-5	2.6930E-5
		order	-	1.0603	1.0394	1.0045
	0.01	L^2-norm	8.0336E-5	3.3090E-5	2.0806E-5	1.5222E-5
		order	-	1.2797	1.1443	1.0863
	0.001	L^2-norm	2.5328E-4	1.0869E-4	6.1642E-5	3.9834E-5
		order	-	1.2205	1.3988	1.5177
0.2	1	L^2-norm	3.1139E-4	1.3871E-4	8.7350E-5	6.4475E-5
		order	-	1.1667	1.1406	1.0555
	0.1	L^2-norm	1.2328E-4	6.1046E-5	4.0558E-5	3.0368E-5
		order	-	1.0140	1.0085	1.0058
	0.01	L^2-norm	1.1247E-4	4.0920E-5	2.4424E-5	1.7474E-5
		order	-	1.4587	1.2727	1.1640
	0.001	L^2-norm	5.9542E-4	2.7523E-4	1.6022E-4	1.0481E-4
		order	-	1.1133	1.3344	1.4752
0.5	1	L^2-norm	2.4972E-4	1.1596E-4	7.5145E-5	5.5579E-5
		order	-	1.1067	1.0699	1.0484
	0.1	L^2-norm	1.6636E-4	8.2069E-5	5.4461E-5	4.0751E-5
		order	-	1.0194	1.0114	1.0081
	0.01	L^2-norm	2.0606E-4	7.3143E-5	4.1859E-5	2.8931E-5
		order	-	1.4943	1.3765	1.2840
	0.001	L^2-norm	1.3602E-3	6.6634E-4	3.9837E-4	2.6544E-4
		order	-	1.0295	1.2687	1.4113
1	1	L^2-norm	2.5895E-4	1.0538E-4	6.5179E-5	4.7036E-5
		order	-	1.2971	1.1849	1.1340
	0.1	L^2-norm	2.2155E-4	1.0868E-4	7.1980E-5	5.3808E-5
		order	-	1.0275	1.0162	1.0114
	0.01	L^2-norm	3.2282E-4	1.4510E-4	9.2435E-5	6.7621E-5
		order	-	1.1537	1.1121	1.0866
	0.001	L^2-norm	1.2028E-3	6.3124E-4	4.0330E-4	2.8580E-4
		order	-	0.9301	1.1049	1.1971

Proof Multiplying the first equation of (83) with Δz and summing i from 1 to I, respectively, we can have that

$$\sum_{i=1}^{I} \frac{\Theta_i^{n+1}-\Theta_i^n}{\Delta t}\Delta z = \sum_{i=1}^{I} \delta_z Q_i^{n+1}\Delta z. \tag{66}$$

Applying the boundary condition $Q_{\frac{1}{2}}^{n+1} = Q_{I+\frac{1}{2}}^{n+1} = 0$, it holds that

$$\sum_{i=1}^{I} \delta_z Q_i^{n+1} = 0. \tag{67}$$

Substituting (67) into (66), we obtain that

$$\sum_{i=1}^{I} \Theta_i^{n+1}\Delta z - \sum_{i=1}^{I} \Theta_i^{n}\Delta z = 0. \tag{68}$$

We complete the proof.

Table 7 The error and mass errors of the scheme

t	$D \setminus h$		1/20	1/50	1/100	1/200
0.1	1	L^2−norm	4.4641E-4	7.1453E-5	1.7843E-5	4.4391E-6
		Masserror	2.7086E-13	1.0094E-13	8.6157E-14	3.5036E-14
	0.1	L^2−norm	7.9558E-5	1.3385E-5	3.3944E-6	8.5137E-7
		Masserror	4.3108E-15	1.4119E-13	9.5466E-15	2.1038E-14
	0.01	L^2−norm	6.5579E-4	1.6458E-4	4.9170E-5	1.3527E-5
		Masserror	6.5219E-14	3.3466E-14	2.7756E-14	1.6400E-14
	0.001	L^2−norm	1.1986E-3	5.0673E-4	2.4633E-4	1.0696E-4
		Masserror	1.5593E-13	2.1219E-14	3.2715E-14	2.5764E-14
0.5	1	L^2−norm	1.7553E-3	2.8102E-4	7.0267E-5	1.7572E-5
		Masserror	1.1481E-12	4.1885E-13	3.5588E-13	1.4531E-13
	0.1	L^2−norm	1.3842E-4	2.2640E-5	5.7084E-6	1.4371E-6
		Masserror	1.8862E-14	5.8428E-13	3.9108E-14	8.6289E-14
	0.01	L^2−norm	1.8155E-3	5.1442E-4	1.5968E-4	4.4633E-5
		Masserror	2.5483E-13	1.4192E-13	1.1558E-13	6.7564E-14
	0.001	L^2−norm	4.0892E-3	2.3293E-3	1.3326E-3	6.3852E-4
		Masserror	6.1763E-13	9.0266E-14	1.3458E-13	1.0576E-13
1	1	L^2−norm	2.8057E-3	4.4919E-4	1.1231E-4	2.8087E-5
		Masserror	1.9064E-12	6.7633E-13	5.7333E-13	2.3411E-13
	0.1	L^2−norm	2.0439E-4	3.3087E-5	8.3130E-6	2.0913E-6
		Masserror	3.0753E-14	9.3947E-13	6.3435E-14	1.3884E-13
	0.01	L^2−norm	1.0113E-3	2.9102E-4	9.2289E-5	2.6174E-5
		Masserror	4.0142E-13	2.3241E-13	1.8639E-13	1.0821E-13
	0.001	L^2−norm	2.4745E-3	1.4293E-3	8.3056E-4	4.0300E-4
		Masserror	9.7590E-13	1.5031E-13	2.1555E-13	1.6974E-13
2	1	L^2−norm	3.8310E-3	6.1334E-4	1.5336E-4	3.8351E-5
		Masserror	2.8083E-12	9.3320E-13	7.8938E-13	3.2077E-13
	0.1	L^2−norm	3.1039E-4	4.9951E-5	1.2521E-5	3.1468E-6
		Masserror	4.1966E-14	1.2858E-12	8.6774E-14	1.8832E-13
	0.01	L^2−norm	7.3458E-4	1.4279E-4	3.6996E-5	9.2541E-6
		Masserror	5.5041E-13	3.1929E-13	2.5390E-13	1.4930E-13
	0.001	L^2−norm	1.8149E-3	7.4523E-4	3.3285E-4	1.2156E-4
		Masserror	1.3589E-12	2.1200E-13	2.9425E-13	2.3329E-13

5.2 Error estimates

In this subsection, we will prove error estimate of the mass conserved DDM scheme in discrete L^2-norm. Let $\theta_i^n = \theta(z_i, t^n), \bar{\theta}^n_{i+\frac{1}{2}} = \frac{\theta_i^n + \theta_{i+1}^n}{2}$ and the fluxes $q^n_{i+\frac{1}{2}} = B(\bar{\theta}^n_{i+\frac{1}{2}})(\lambda(B(\bar{\theta}^n_{i+\frac{1}{2}}))\theta_i^n + (1-\lambda(B(\bar{\theta}^n_{i+\frac{1}{2}})))\theta^n_{i+1}) - \bar{D}(\bar{\theta}^n_{i+\frac{1}{2}})\delta_z\theta^n_{i+\frac{1}{2}}$. Meanwhile, we first give the following three assumptions as:

(I) The problem has a unique smooth solution $\theta(z,t)$ and satisfy the regularity condition, i.e.,

$$\theta \in C^0\left([0,T]; C^5(\Omega)\right) \cap C^2\left([0,T]; C^1(\Omega)\right). \tag{69}$$

(II) There has a positive constant D_0, such that for any ξ and θ,

$$(\xi, D(z,t,\theta)\xi) \geq D_0 \mid \xi \mid^2 . \tag{70}$$

(III) The coefficient $D(z,t,\theta)$, $B(z,t,\theta)$. and $f(z,t,\theta)\xi$ are continuous with respect to x,t, and continuously differentiable with respect to θ, i.e.,

$$\max\{\mid D_{\theta\theta}\mid, \mid B_{\theta\theta}\mid, \mid f_{\theta}\mid\} \leq G. \tag{71}$$

When the conditions (I)-(III) are hold, we only provide the analysis about error estimate, where we assume that the solutions of the scheme (60)-(64) are exist and unique.

Lemma 6 *Assuming that the exact solutions $\theta \in C^0\left([0,T];C^4(\Omega)\right)\cap C^2\left([0,T];C^0(\Omega)\right)$. We have the following truncation error equations as*

$$\tfrac{\theta_i^{n+1}-\theta_i^n}{\Delta t} = \delta_z q_i^{n+1} + f_i^{n+1} + o_i^{n+1}, \tag{72}$$

where, $o = O(\Delta t + (\Delta z)^2)$.

Proof From (72), it holds that

$$o_i^{n+1} = \tfrac{\theta_i^{n+1}-\theta_i^n}{\Delta t} - \tfrac{q_{i+\frac{1}{2}}^{n+1}-q_{i-\frac{1}{2}}^{n+1}}{\Delta z} - f_i^{n+1}. \tag{73}$$

Applying the Taylor formula for the first term of the right-hand side of (73), i.e.,

$$\tfrac{\theta_i^{n+1}-\theta_i^n}{\Delta t} = (\theta_t)_i^{n+1} - \tfrac{\Delta t}{2}(\theta_{tt})_i^{n_1}. \tag{74}$$

For the second term of (73), it holds that

$$\begin{aligned}
q_{i+\frac{1}{2}}^{n+1} &= B(\bar{\theta}_{i+\frac{1}{2}}^{n+1})(\lambda(B(\bar{\theta}_{i+\frac{1}{2}}^{n+1}))\theta_i^{n+1} + (1-\lambda(B(\bar{\theta}_{i+\frac{1}{2}}^{n+1})))\theta_{i+1}^{n+1}) - \bar{D}(\bar{\theta}_{i+\frac{1}{2}}^{n+1})\tfrac{\theta_{i+1}^{n+1}-\theta_i^{n+1}}{\Delta z} \\
&= B(\bar{\theta}_{i+\frac{1}{2}}^{n+1})(\theta_{i+\frac{1}{2}}^{n+1} + \tfrac{1-2\lambda(B(\bar{\theta}_{i+\frac{1}{2}}^{n+1}))}{2}\Delta z(\theta_z)_{i+\frac{1}{2}}^{n+1} + O(\Delta z)^2) \\
&\quad -(D(\bar{\theta}_{i+\frac{1}{2}}^{n+1}) - \tfrac{2\lambda(B(\bar{\theta}_{i+\frac{1}{2}}^{n+1}))-1}{2}\Delta z + O(\Delta z)^2)((\theta_z)_{i+\frac{1}{2}}^{n+1} + \tfrac{(\Delta z)^2}{24}(\theta_{zzz})_{t_0}^{n+1}) \\
&= D(\bar{\theta}_{i+\frac{1}{2}}^{n+1})((\theta_z)_{i+\frac{1}{2}}^{n+1} + \tfrac{(\Delta z)^2}{24}(\theta_{zzz})_{t_0}^{n+1}) - B(\bar{\theta}_{i+\frac{1}{2}}^{n+1}) \\
&= B(\theta_{i+\frac{1}{2}}^{n+1})\theta_{i+\frac{1}{2}}^{n+1} - D(\theta_{i+\frac{1}{2}}^{n+1})(\theta_z)_{i+\frac{1}{2}}^{n+1} + O((\Delta z)^2).
\end{aligned} \tag{75}$$

Similarly, we obtain that

$$q_{i-\frac{1}{2}}^{n+1} = B(\theta_{i-\frac{1}{2}}^{n+1})\theta_{i-\frac{1}{2}}^{n+1} - D(\theta_{i-\frac{1}{2}}^{n+1})(\theta_z)_{i-\frac{1}{2}}^{n+1} + O((\Delta z)^2). \tag{76}$$

Further, we obtain that

$$\tfrac{q_{i+\frac{1}{2}}^{n+1}-q_{i-\frac{1}{2}}^{n+1}}{\Delta z} = \left((B\theta)_z - \tfrac{\partial}{\partial z}(D(\theta)\tfrac{\partial\theta}{\partial z})\right)_i^{n+1} + O((\Delta z)^2). \tag{77}$$

We obtain that

$$\begin{aligned}
o_i^{n+1} &= (\theta_t - \tfrac{\partial}{\partial z}(D(\theta)\tfrac{\partial\theta}{\partial z}) + \tfrac{\partial(B\theta)}{\partial z} - f)_i^{n+1} + O(\Delta t + (\Delta z)^2) \\
&= O(\Delta t + (\Delta z)^2).
\end{aligned} \tag{78}$$

Similarly, if $\theta \in C^0\left([0,T];C^5(\Omega)\right) \cap C^2\left([0,T];C^1(\Omega)\right)$, we can obtain that $\delta_z o_{i+\frac{1}{2}}^{n+1} = O(\Delta t + (\Delta z)^2)$.

Similar as the proof of Theorem 3, we can obtain the error estimate as

Theorem 5 *Assume that* $\theta \in C^0\left([0,T]; C^5(\Omega)\right) \cap C^2\left([0,T]; C^1(\Omega)\right)$*, it holds that*

$$\parallel \theta^n - \Theta^n \parallel \leq M(\Delta t + (\Delta z)^2 + \frac{(\Delta t)^2}{(\Delta z)^{\frac{3}{2}}}), \tag{79}$$

for $n \geq 1$*, where* $M > 0$ *is a constant.*

Remark 2 From Theorem 5, we give the error estimate. Because of the nonlinearity of B and D, we need give the assumption (I)-(III).

5.3 Algorithm

In this subsection, we propose the algorithm for (59). Because of it's strong nonlinearity, we describe the iterative approach by using the extrapolation technique. Let $k \geq 0$ be the index of the linearization number, and for $k = 0$, we define

$$\bar{w}^{n,k} = \begin{cases} w^{n-1}, & n = 1, \\ 2w^{n-1} - w^{n-2}, & 2 \leq n \leq M. \end{cases} \tag{80}$$

and for $k \geq 1$,

$$\bar{w}^{n,k} = \begin{cases} w^{n,0}, & k = 1, \\ (1-\theta)w^{n,k-1} + \theta w^{n,k-2}, & k \geq 2. \end{cases} \tag{81}$$

where $-1 < \theta < 1$ is a weighting parameter for the iterative procedure. Now, we describe the algorithm of our iterative approach over each sub-domain as below,

A. Initialization:
The content θ is given at initial time t_0 and set $\Theta_i^0 = \theta(t_0, z_i)$.
B. Time Stepping Procedure:
for the time steps t^n $n = 0, 1, \cdots, M-1$ **do**
B1. Set the linearization index $k = 1$.
B2. Compute the interface solutions $\{\tilde{Q}^{n+1,k}_{i_\alpha+\frac{1}{2}}\}$ by as

$$\frac{\tilde{Q}^{n+1,k}_{i_\alpha+\frac{1}{2}} - Q^n_{i_\alpha+\frac{1}{2}}}{\Delta t} = \bar{D}^{n+1,k-1}_{i_\alpha+\frac{1}{2}} \frac{Q^n_{i_\alpha+\frac{3}{2}} - 2\tilde{Q}^{n+1,k}_{i_\alpha+\frac{1}{2}} + Q^n_{i_\alpha-\frac{1}{2}}}{(\Delta z)^2} - \bar{D}^{n+1,k-1}_{i_\alpha+\frac{1}{2}} \delta_z f^{n+1}_{i_\alpha+\frac{1}{2}} - \frac{\bar{D}^{n+1,k-1}_{i_\alpha+\frac{1}{2}} - \bar{D}^n_{i_\alpha+\frac{1}{2}}}{\Delta t} \delta_x C^n_{i_\alpha+\frac{1}{2}}. \tag{82}$$

B3. Compute solutions and fluxes $\{\Theta^{n+1,k}_i, Q^{n+1,k}_{i+\frac{1}{2}}\}$ by the coupled implicit scheme.

$$\begin{gathered} \frac{\Theta^{n+1,k}_i - \Theta^n_i}{\Delta t} + \delta_z Q^{n+1,k}_i = f^{n+1}_i, \\ Q^{n+1,k}_{i+\frac{1}{2}} = B^{n+1,k-1}_{i+\frac{1}{2}}(\lambda(B^{n+1,k-1}_{i+\frac{1}{2}})\Theta^{n+1,k}_i + (1 - \lambda(B^{n+1,k-1}_{i+\frac{1}{2}}))\Theta^{n+1,k}_{i+1}) - D^{n+1,k-1}_{i+\frac{1}{2}} \delta_z \Theta^{n+1,k}_{i+\frac{1}{2}}, \\ Q^{n+1,k}_{i_\alpha+\frac{1}{2}} = \tilde{Q}^{n+1,k}_{i_\alpha+\frac{1}{2}}, \qquad z_{i_\alpha+\frac{1}{2}} \in \Gamma_\alpha. \end{gathered} \tag{83}$$

B4. Set the linearization index $k = k + 1$.

B5. Iterations. Repeat B2-B4 at current time step until $Error <$ tolerance or $k > k_0$.

$$Error = \| \Theta^{n+1,k} - \Theta^{n+1,k-1} \| .$$

end do

5.4 Numerical examples

Let $B(\theta) = e^{-t} - \theta$, $D(\theta) = D(1+\theta^2)$ and the initial solutions as $\theta_0 = cos(\pi x)$. The right source(or sink) term is defined as

$$f = e^{-3t}\left[D\pi^2 \cos\pi x\left(3\cos^2\pi x - 2\right) + \pi e^t \sin\pi x(2\cos\pi x - 1) + e^{2t}\cos\pi x\left(D\pi^2 - 1\right)\right].$$

The domain is set as $[0, 2]$ and we can solve the exact solution $\theta = e^{-t}cos(\pi x)$.

Table 8 The space order of convergence of the scheme

t	$D \setminus h$		1/300	1/400	1/500	1/600
0.1	1	L^2-norm	3.1227E-5	1.7635E-5	1.1343E-5	7.9252E-6
		order	-	1.9862	1.9776	1.9666
	0.1	L^2-norm	1.3348E-4	7.5593E-5	4.8525E-5	3.3728E-5
		order	-	1.9764	1.9865	1.9951
	0.01	L^2-norm	1.1763E-3	7.1186E-4	4.7720E-4	3.4216E-4
		order	-	1.7458	1.7923	1.8246
	0.001	L^2-norm	3.3593E-3	2.3671E-3	1.7845E-3	1.4054E-3
		order	-	1.2169	1.2661	1.3099
0.2	1	L^2-norm	3.2837E-5	1.8556E-5	1.1946E-5	8.3551E-6
		order	-	1.9840	1.9736	1.9610
	0.1	L^2-norm	1.8041E-4	1.0207E-4	6.5450E-5	4.5438E-5
		order	-	1.9799	1.9914	2.0016
	0.01	L^2-norm	1.9477E-3	1.1759E-3	7.8693E-4	5.6352E-4
		order	-	1.7541	1.8000	1.8316
	0.001	L^2-norm	5.8870E-3	4.1465E-3	3.1231E-3	2.4570E-3
		order	-	1.2183	1.2702	1.3157
0.5	1	L^2-norm	2.5234E-5	1.4278E-5	9.2064E-6	6.4516E-6
		order	-	1.9795	1.9665	1.9502
	0.1	L^2-norm	1.5512E-4	8.7470E-5	5.5890E-5	3.8645E-5
		order	-	1.9914	2.0073	2.0237
	0.01	L^2-norm	2.3976E-3	1.4348E-3	9.5454E-4	6.8060E-4
		order	-	1.7848	1.8264	1.8553
	0.001	L^2-norm	1.0517E-2	7.3995E-3	5.5569E-3	4.3566E-3
		order	-	1.2221	1.2834	1.3347
1	1	L^2-norm	1.5229E-5	8.6349E-6	5.5829E-6	3.9253E-6
		order	-	1.9723	1.9544	1.9321
	0.1	L^2-norm	9.9089E-5	5.5688E-5	3.5439E-5	2.4388E-5
		order	-	2.0031	2.0254	2.0498
	0.01	L^2-norm	1.5229E-3	8.9917E-4	5.9305E-4	4.2020E-4
		order	-	1.8315	1.8651	1.8898
	0.001	L^2-norm	1.2458E-2	8.5180E-3	6.2445E-3	4.7986E-3
		order	-	1.3215	1.3914	1.4446

In Table 8, the time step Δt is taken as 1/100000 and the space step h is set from 1/300, 1/400, 1/500 to 1/600. It is clear that our scheme is asymptotically a second order in space when the space step h is smaller.

Table 9 The time order of convergence of the scheme

t	$D \setminus h$		1/300	1/400	1/600	1/800
0.1	1	L^2-norm	9.1204E-3	7.5249E-3	5.3776E-3	4.0395E-3
		order	-	0.6684	0.8286	0.9946
	0.1	L^2-norm	4.4661E-3	3.2708E-3	2.1274E-3	1.5926E-3
		order	-	1.0827	1.0608	1.0064
	0.01	L^2-norm	1.5892E-3	1.0954E-3	6.8164E-4	5.0043E-4
		order	-	1.2935	1.1699	1.0742
	0.001	L^2-norm	4.1004E-3	2.6066E-3	1.4122E-3	9.1920E-4
		order	-	1.5748	1.5116	1.4926
0.2	1	L^2-norm	1.1875E-2	8.7030E-3	5.6092E-3	4.1771E-3
		order	-	1.0802	1.0833	1.0247
	0.1	L^2-norm	3.9113E-3	2.9432E-3	1.9651E-3	1.4750E-3
		order	-	0.9885	0.9963	0.9972
	0.01	L^2-norm	1.8978E-3	1.2288E-3	7.0015E-4	4.9109E-4
		order	-	1.5109	1.3873	1.2328
	0.001	L^2-norm	5.6069E-3	3.8766E-3	2.2707E-3	1.5255E-3
		order	-	1.2828	1.3192	1.3827
0.5	1	L^2-norm	9.9674E-3	7.4332E-3	4.9271E-3	3.6848E-3
		order	-	1.0197	1.0142	1.0099
	0.1	L^2-norm	2.6351E-3	1.9833E-3	1.3267E-3	9.9670E-4
		order	-	0.9878	0.9916	0.9942
	0.01	L^2-norm	2.0684E-3	1.3496E-3	7.9438E-4	5.7537E-4
		order	-	1.4842	1.3071	1.1212
	0.001	L^2-norm	9.0352E-3	6.4498E-3	3.8708E-3	2.6278E-3
		order	-	1.1717	1.2593	1.3463
1	1	L^2-norm	5.1933E-3	3.8843E-3	2.5822E-3	1.9338E-3
		order	-	1.0095	1.0070	1.0051
	0.1	L^2-norm	1.5896E-3	1.2010E-3	8.0653E-4	6.0707E-4
		order	-	0.9744	0.9820	0.9875
	0.01	L^2-norm	1.8261E-3	1.3974E-3	9.7634E-4	7.5781E-4
		order	-	0.9301	0.8843	0.8808
	0.001	L^2-norm	8.5774E-3	5.9614E-3	3.4497E-3	2.3118E-3
		order	-	1.2647	1.3491	1.3913

Take $\Delta t = h$ and choose $h = 1/300$, $1/400$, $1/600$ and $1/800$ in Table 9. It is clear that the time convergence order is first when the time step Δt becomes small.

In Table 10, the time step Δt is taken as $1/100000$. And the space step h is taken as $1/20$, $1/50$, $1/100$ and $1/200$. The L^2-error decreases with the decrease of space step and the scheme preserve mass.

In Table 11 shows the error and mass error when the domain is divided into multiple sub-domains. Take time step $\Delta t = 1/10000$ and the space step $h = 1/400$. We can find the L^2-norm is almost same and mass is conservative when the domain is divided into 2, 5, 10 and 20 sub-domains.

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Table 10 The error and mass errors of the scheme

t	$D \setminus h$		1/20	1/50	1/100	1/200
0.1	1	L^2-norm	7.0585E-3	1.1212E-3	2.7998E-4	7.0070E-5
		Masserror	5.2548E-15	1.9022E-15	1.1494E-15	7.5839E-16
	0.1	L^2-norm	2.0435E-2	4.1223E-3	1.1279E-3	2.9573E-4
		Masserror	2.9070E-15	5.6303E-15	2.0592E-15	2.1959E-15
	0.01	L^2-norm	5.0403E-2	1.7232E-2	6.7290E-3	2.3177E-3
		Masserror	7.0990E-16	4.9840E-16	1.6581E-15	1.4051E-16
	0.001	L^2-norm	5.7501E-2	2.3408E-2	1.1454E-2	5.3758E-3
		Masserror	1.0110E-15	1.6151E-15	1.2408E-15	4.7040E-16
0.2	1	L^2-norm	7.4041E-3	1.1778E-3	2.9416E-4	7.3646E-5
		Masserror	4.3279E-15	8.5013E-16	7.1828E-16	2.1799E-15
	0.1	L^2-norm	2.8132E-2	5.6118E-3	1.5298E-3	4.0017E-4
		Masserror	6.7642E-15	6.7314E-15	4.3157E-15	4.7690E-15
	0.01	L^2-norm	8.2196E-1	2.8637E-2	1.1220E-2	3.8505E-3
		Masserror	1.1141E-14	3.5341E-15	3.5167E-15	8.5725E-17
	0.001	L^2-norm	9.5617E-2	4.0160E-2	1.9922E-2	9.4118E-3
		Masserror	1.9602E-15	2.1446E-15	4.4036E-15	2.1482E-15
0.5	1	L^2-norm	5.6772E-3	9.0328E-4	2.2570E-4	5.6543E-5
		Masserror	1.2723E-14	5.6242E-15	6.2310E-15	5.6602E-16
	0.1	L^2-norm	2.5641E-2	4.9323E-3	1.3288E-3	3.4529E-4
		Masserror	1.4883E-15	5.9071E-15	8.2621E-15	3.7117E-15
	0.01	L^2-norm	1.1192E-1	3.8043E-2	1.4465E-2	4.8118E-3
		Masserror	1.0704E-15	4.3369E-15	1.2085E-15	4.8261E-17
	0.001	L^2-norm	1.4404E-1	6.5860E-2	3.4512E-2	1.6741E-2
		Masserror	1.6371E-15	1.1375E-15	2.2404E-15	2.7546E-16
1	1	L^2-norm	3.4127E-3	5.4340E-4	1.3585E-4	3.4071E-5
		Masserror	1.8416E-14	4.6822E-15	5.6733E-15	3.3378E-15
	0.1	L^2-norm	1.6700E-2	3.1853E-3	8.5433E-4	2.2122E-4
		Masserror	3.9061E-15	6.9211E-15	1.0974E-14	7.9761E-16
	0.01	L^2-norm	1.0069E-1	2.9449E-2	1.0080E-2	3.1341E-3
		Masserror	7.6241E-16	1.2407E-15	1.7852E-15	8.6033E-16
	0.001	L^2-norm	1.6030E-1	8.1647E-2	4.3447E-2	2.0472E-2
		Masserror	1.4946E-14	6.2165E-15	3.2762E-15	3.7681E-17

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Table 11 The error and mass errors over multiple subdomains

t	$D \setminus subd$		2	5	10	20
0.1	1	L^2-norm	1.8474E-5	1.3430E-5	7.9914E-6	1.2503E-5
		Masserror	4.6680E-15	1.0935E-15	1.9102E-15	7.4391E-16
	0.1	L^2-norm	7.2882E-5	7.2943E-5	7.2808E-5	7.2443E-5
		Masserror	3.0354E-16	8.8162E-17	7.8725E-17	6.2940E-16
	0.01	L^2-norm	7.0898E-4	7.0909E-4	7.0909E-4	7.0910E-4
		Masserror	9.2661E-17	3.9925E-17	1.9480E-16	4.3747E-16
	0.001	L^2-norm	2.3636E-3	2.3637E-3	2.3637E-3	2.3638E-3
		Masserror	5.2186E-16	2.9526E-16	9.9860E-17	3.2147E-16
0.2	1	L^2-norm	1.9743E-5	1.3714E-5	7.2790E-6	1.2789E-5
		Masserror	3.2715E-15	1.4385E-15	1.6478E-15	2.2619E-15
	0.1	L^2-norm	9.7132E-5	9.7042E-5	9.6893E-5	9.6409E-5
		Masserror	8.8814E-17	3.0976E-16	3.4416E-16	6.9333E-16
	0.01	L^2-norm	1.1701E-3	1.1702E-3	1.1703E-3	1.1703E-3
		Masserror	4.3831E-16	1.4465E-16	3.4185E-17	3.5393E-16
	0.001	L^2-norm	4.1390E-3	4.1392E-3	4.1393E-3	4.1393E-3
		Masserror	1.0250E-15	3.4998E-16	4.0816E-17	2.3899E-16
0.5	1	L^2-norm	1.5828E-5	1.0582E-5	5.1503E-5	9.1816E-6
		Masserror	1.3072E-15	1.1418E-15	5.8724E-16	6.0763E-15
	0.1	L^2-norm	7.9223E-5	7.9071E-5	7.8609E-5	7.7676E-5
		Masserror	6.3118E-16	5.6009E-16	2.0485E-16	1.0965E-18
	0.01	L^2-norm	1.4200E-3	1.4197E-3	1.4198E-3	1.4199E-3
		Masserror	2.5860E-16	4.0404E-16	8.0816E-16	5.3893E-16
	0.001	L^2-norm	7.3742E-3	7.3740E-3	7.3746E-3	7.3755E-3
		Masserror	1.5848E-16	9.0209E-17	3.8053E-16	3.6110E-16
1	1	L^2-norm	1.0122E-5	6.4924E-6	2.9342E-6	5.6282E-6
		Masserror	2.6202E-15	1.4867E-16	1.7307E-15	3.2975E-15
	0.1	L^2-norm	4.7677E-5	4.7654E-5	4.6989E-5	4.5822E-5
		Masserror	8.3854E-16	5.1010E-16	2.6178E-16	7.9635E-16
	0.01	L^2-norm	8.7611E-4	8.7612E-4	8.7577E-4	8.7518E-4
		Masserror	5.6419E-16	4.0820E-16	7.6292E-16	2.3695E-16
	0.001	L^2-norm	8.4596E-3	8.4594E-3	8.4600E-3	8.4628E-3
		Masserror	1.2312E-16	6.8267E-16	4.7224E-16	5.9496E-16

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